On $p$-Bergman kernel for bounded domains in $\mathbb{C}^n$

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In this paper, we obtain some properties of the $p$-Bergman kernels by applying $L^p$ extension theorem. We prove that for any bounded domain in $\mathbb{C}^n$, it is pseudoconvex if and only if its $p$-Bergman kernel is an exhaustion function, for any $p \in (0, 2)$. As an application, we give a negative answer to a conjecture of Tsuji.

1. Introduction

T. Ohsawa and K. Takegoshi [16] proved the Ohsawa-Takegoshi $L^2$ extension theorem, which turns out to be useful in several complex variables and complex geometry. B. Berndtsson and M. P˘aun [2] proved the $L^{2/m}$ version of Ohsawa-Takegoshi theorem for $m \in \mathbb{N}$. Recently, Qi’an Guan and Xiangyu Zhou [9] obtained optimal estimate for $L^p$ ($0 < p \leq 2$) extension as an application of their solution of a sharp $L^2$ extension problem.

In the present paper, we study the $p$-Bergman kernels for bounded domains in $\mathbb{C}^n$, and apply $L^p$ extension theorem to give some properties of $p$-Bergman kernels.

The definition of $p$-Bergman kernel is as follows:

**Definition 1.1.** For a domain $\Omega \subseteq \mathbb{C}^n$ and $p \in (0, 2]$, the $p$-Bergmann kernel $K_{\Omega,p}$ is denoted by

$$K_{\Omega,p}(z) = \sup_{f \in A^p(\Omega)} \frac{|f(z)|^p}{\int_{\Omega} |f|^p},$$

where

$$A^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) : \int_{\Omega} |f|^p < +\infty \right\}$$

(the integral is w.r.t. Lebesgue measure).

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According to the extreme property, the usual Bergman kernel is just 2-Bergman kernel for the case \( p = 2 \) in the above definition, which has been studied for years.

Let \( S \) be a closed complex subvariety of a domain \( U \subset \mathbb{C}^n \). It’s known that one has the same Bergman kernels on \( U \) and \( U \setminus S \), since for any \( f \in A^2(U \setminus S) \), one can holomorphically extend \( f \) to \( U \). That is to say, one can not distinguish \( U \) and \( U \setminus S \) by the Bergman kernel.

However, the \( p \)-Bergman kernel may give some distinction. We will prove that for a bounded domain, it is pseudoconvex if and only if its \( p \)-Bergman kernel is an exhaustion function for any \( p \in (0, 2) \). Besides, the \( p \)-Bergman kernel is interesting per se. We’ll also give estimate about the boundary behavior of the \( p \)-Bergman kernel for a bounded pseudoconvex domain. In the last section, we’ll answer negatively a conjecture of H. Tsuji in [20].

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2. The \( p \)-Bergman kernel

Note that when \( p = 2 \), the \( p \)-Bergmann kernel is just the usual Bergman kernel. For simplicity, we write \( K_\Omega \) for \( K_{\Omega, 2} \). The \( p \)-Bergmann kernel has some properties similar to the usual Bergman kernel, for example, it is easy to see that \( K_{\Omega_1, p}(z) \geq K_{\Omega_2, p}(z) \) for \( z \in \Omega_1 \) and two domains \( \Omega_1 \subseteq \Omega_2 \), and the \( p \)-Bergmann kernels are plurisubharmonic.

We will study some more properties of \( K_{\Omega, p} \).

**Proposition 2.1.** Let \( \Omega_1 \subset \mathbb{C}^n \) be simply connected domain and \( \Omega_2 \subset \mathbb{C}^n \) be a domain. Then for any \( \phi : \Omega_1 \to \Omega_2 \) biholomorphism, we have

\[
K_{\Omega_1, p}(z) = K_{\Omega_2, p}(\phi(z)) |J\phi(z)|^2,
\]

where \( J\phi \) is the determinant of Jacobian of \( \phi \). In particular, if \( p = \frac{2}{m} \), where \( m \in \mathbb{N} \), there is no need for the condition that \( \Omega_1 \) is simply connected.

**Proof.** As \( \Omega_1 \) is simply connected and \( J\phi \) is nonvanishing, we can choose a single valued holomorphic function of \( \log J\phi \).

Then

\[
\Phi : A^p(\Omega_2) \to A^p(\Omega_1)
\]

\[
f \mapsto f \circ \phi e^{\frac{2}{p} \log J\phi}
\]

is isometric, since

\[
\int_{\Omega_2} |f|^p = \int_{\Omega_1} |f \circ \phi|^p |J\phi|^2 = \int_{\Omega_1} |f \circ \phi e^{\frac{2}{p} \log J\phi}|^p.
\]
When \( p = \frac{2}{m}, m \in \mathbb{N} \), we take

\[
\Phi : A^p(\Omega_2) \rightarrow A^p(\Omega_1) \\
f \mapsto f \circ \phi (J\phi)^m,
\]
in this case, the simply connected condition is not needed any more.

By definition,

\[
K_{\Omega_2,p}(\phi(z)) = \sup_{f \in A^p(\Omega_2)} \frac{|f(\phi(z))|^p}{\int_{\Omega_2} |f|^p} \\
= \sup_{f \in A^p(\Omega_2)} \frac{|f(\phi(z))|^p}{\int_{\Omega_1} |f \circ \phi|^p |J\phi|^2} \\
= \frac{1}{|J\phi(z)|^2} \sup_{f \in A^p(\Omega_2)} \frac{|f \circ \phi(z) e^{\frac{2}{p} \log(J\phi(z))}|^p}{\int_{\Omega_1} |f \circ \phi e^{\frac{2}{p} \log(J\phi)}|^p} \\
= K_{\Omega_1,p}(z) \frac{|J\phi(z)|}{|J\phi(z)|^2}.
\]

\[\square\]

It’s easy to see that, if \( J\phi \) is constant, then the above proposition is still true without the assumption that \( \Omega_1 \) is simply connected. For example, if the domain \( \Omega \) is a \( G \)-invariant domain w.r.t. a linear action of a semisimple Lie group \( G \), then the \( p \)-Bergmann kernel is \( G \)-invariant.

The condition that \( \Omega_1 \) is simply connected is necessary for some \( p \in (0, 2) \) (see Remark 2.3).

Similar to the usual Bergman kernel, the following proposition holds for the \( p \)-Bergman kernel.

**Proposition 2.2.** Suppose that \( \Omega_j \subset \mathbb{C}^n \) are bounded domains and \( \Omega_j \subset \Omega_{j+1} \) for \( j \geq 1 \), \( \cup_{j=1}^{\infty} \Omega_j = \Omega \), where \( \Omega \) is a bounded domain in \( \mathbb{C}^n \). Then for \( 0 < p \leq 2 \),

\[
\lim_{j \to \infty} K_{\Omega_j,p}(z) = K_{\Omega,p}(z),
\]
and the convergence is uniform on compact subsets of \( \Omega \).

**Proof.** As \( K_{\Omega_j,p}(z) \) is decreasing,

\[
\lim_{j \to \infty} K_{\Omega_j,p}(z)
\]
exists and \( \geq K_{\Omega,p}(z) \).
For fixed $z \in \Omega$, we may assume $z \in \Omega_{j_0}$. There is $f_j \in \mathcal{O}(\Omega_j)$ such that
\[
\int_{\Omega_j} |f_j|^p = 1
\]
and
\[
|f_j(z)|^p = K_{\Omega_j,p}(z)
\]
for each $j \geq j_0$.

By the Montel theorem, there is a subsequence of $j_k$ such that
\[
\lim_{k \to \infty} f_{j_k}
\]
is uniformly convergent to $f \in \mathcal{O}(\Omega)$.

It is easy to check that
\[
\int_{\Omega} |f|^p \leq 1.
\]

By the definition, we have
\[
K_{\Omega,p}(z) \geq |f(z)|^p = \lim_{j \to \infty} K_{\Omega_j,p}(z).
\]

As $K_{\Omega_p}(z)$ is continuous and $K_{\Omega_j,p}(z)$ is decreasing, it follows that $K_{\Omega_j,p}(z)$ converges uniformly to $K_{\Omega,p}(z)$ on compact subsets of $\Omega$.

\begin{theorem}
Let $\Omega$ be one of the classical domains (see [11], [12], [13]):
\begin{align*}
\mathcal{R}_1 & := \{ Z \in M(m, n) : I^{(m)} - ZZ' > 0 \}, \\
\mathcal{R}_2 & := \{ Z \in M(n, n) : I^{(n)} - ZZ' > 0, Z = Z' \}, \\
\mathcal{R}_3 & := \{ Z \in M(n, n) : I^{(n)} - ZZ' > 0, Z = -Z' \}, \\
\mathcal{R}_4 & := \{ Z \in M(1, n) : |ZZ'| + 1 - 2ZZ' > 0, |ZZ'| < 1 \}.
\end{align*}

Then
\[
K_{\Omega,p}(Z) = K_{\Omega,2}(Z)
\]
for $Z \in \Omega$ for $p > 0$.
\end{theorem}

\begin{proof}
For $Z \in \Omega$ and $|t| \leq 1$, we have $tZ \in \Omega$.

For any $f \in \mathcal{O}(\Omega)$, we have
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}Z)|^p d\theta \geq |f(0)|^p.
\]

Then by the Fubini Theorem,
\[
\int_{\Omega} |f|^p = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{\Omega} |f(e^{i\theta}Z)|^p dV_Z d\theta
\]

\[
= \int_{\Omega} dV_Z \frac{1}{2\pi} \int_{0}^{2\pi} |f(e^{i\theta}Z)|^p d\theta \geq |f(0)|^p Vol(\Omega),
\]

we have

\[
K_{\Omega,p}(0) = \frac{1}{Vol(\Omega)}.
\]

As \( \Omega \) is homogenous, it is well known that \( \Omega \) is also simply connected, combining with the above proposition, we have \( K_{\Omega,p}(Z) = K_{\Omega,2}(Z) \) for \( Z \in \Omega \).

\[ \square \]

**Remark 2.1.** The above result is true for any complete circular and bounded homogeneous domain. It’s known that any bounded symmetric domain is such a domain.

For a general bounded homogenous domain \( \Omega \), we have \( K_{\Omega,p}(z) \geq K_{\Omega,2}(z) \). It is well known that \( K_\Omega(z,w) \) is zero free and \( \Omega \) is simply connected, we can define a holomorphic function \( \log K_\Omega(z,w) \) for \( z \in \Omega \) and fixed \( w \in \Omega \). Then \( e^{2/p} \log K_\Omega(z,w) \in A^p(\Omega) \), and it is easy to get \( K_{\Omega,p}(z) \geq K_{\Omega,2}(z) \).

It seems to be strange that the \( p \)-Bergmann kernel may be independent of \( p \) for some domains. From the following theorem, we can deduce that, in general, \( K_{\Omega,p} \) is dependent on \( p \).

**Lemma 2.4.** For \( \Omega \subset \mathbb{C}^n \), we have

\[ K_{\Omega,p_m}(z) \geq K_{\Omega,p}(z) \]

for any \( p \in (0,2) \) and \( m \in \mathbb{N} \).

**Proof.** If \( f \in A^p(\Omega) \), then

\[ f^m \in A^{\frac{p}{m}}(\Omega), \]

and

\[ \int_{\Omega} |f|^p = \int_{\Omega} |f^m|^{\frac{p}{m}}. \]

By the definition of \( p \)-Bergman kernel, we have

\[ K_{\Omega,p_m}(z) \geq K_{\Omega,p}(z). \]

\[ \square \]
The next theorem needs the $L^p$ extension theorem. We state it in the following. For the proof, one can refer to [2] or [9].

**Theorem 2.5.** (see [2] or [9]) Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $L$ be a complex affine line in $\mathbb{C}^n$, and $\Omega \cap L \neq \emptyset$. For $0 < p \leq 2$, then for any $f \in A^p(\Omega \cap L)$, there is $F \in A^p(\Omega)$, such that $F|_{\Omega \cap L} = f$ and

$$\int_{\Omega} |F|^p \leq C \int_{\Omega \cap L} |f|^p,$$

where $C$ is a constant depending only on diam $\Omega$ and $n$.

**Theorem 2.6.** Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $p \in (0, 2)$ and $l = \max\{s \in \mathbb{N}_+ : s < \frac{2}{p}\}$. Then we have

$$K_{\Omega, p}(z) \geq \frac{c}{\delta(z)^{pl}},$$

where $\delta(z) = \inf_{w \in \partial \Omega} d(z, w)$ and $c$ is a constant positive number.

**Proof.** For any complex line $L$, after a unitary transform, we may assume $L = \{z_2 = \cdots = z_n = 0\}$.

Let $z^0 = (z^0_1, 0, \ldots, 0) \in \partial \Omega \cap L$, take

$$f = \frac{1}{(z_1 - z^0_1)^l} \in A^p(\Omega \cap L).$$

From the $L^p$ extension theorem 2.5, we get $F \in A^p(\Omega)$ such that $F|_{\Omega \cap L} = f$, and

$$\int_{\Omega} |F|^p \leq C \int_{\Omega \cap L} |f|^p \leq 1/c$$

for some constant $c > 0$, $c$ depends only on diam $\Omega$ and $n$.

Then

$$K_{\Omega, p}(z)|_{\Omega \cap L} \geq \frac{c}{|z_1 - z^0_1|^{pl}}.$$

As we can choose arbitrary complex line and boundary points, we get

$$K_{\Omega, p}(z) \geq \frac{c}{\delta(z)^{pl}}.$$

According to the above theorem and the fact that the $p$-Bergman kernel is plurisubharmonic, we can easily get the following interesting theorem.
**Theorem 2.7.** For any bounded domain $\Omega$ in $\mathbb{C}^n$, $\Omega$ is pseudoconvex if and only if $K_{\Omega,p}(z)$ is an exhaustion function for $p \in (0, 2)$.

**Remark 2.2.** The condition that $\Omega$ is bounded is necessary. If we consider $\Omega = \mathbb{C} \setminus \Delta$, then $K_{\Omega,p}(z)$ is bounded near $\infty$ for $0 < p < 2$.

**Theorem 2.8.** Let $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $1 \leq p < 2$, then we have $K_{\Delta^*,p}(z) = O(1/|z|^p)$.

**Proof.** For any $f \in A^p(\Delta^*)$, we have $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, then $g(z) := \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

In the present proof, we denote by $\|f\|_p = (\int_{\Delta^*} |f|^p)^{\frac{1}{p}}$ for $f \in A^p(\Delta^*)$.

Obviously, $\int_{\Delta^*} |g(z)|^p < \infty$, where $\Delta^*_\tau = \{z \in \mathbb{C} : 0 < |z| < \tau\}$ and $0 < \tau < 1$.

It’s easy to see that

$$\int_{\Delta^*} \left| \frac{1}{z} \right|^p dxdy = \int_0^1 \int_0^{2\pi} r^{1-p} d\theta dr = \frac{2\pi}{2-p}.$$  

From

$$\|g + h\|_p \leq \|g\|_p + \|h\|_p,$$

we get

$$h(z) := \sum_{n=-\infty}^{2} a_n z^n \in A^p(\Delta^*_\tau).$$

We want to prove $h = 0$.

$$\int_{\Delta^*_\tau} |h(z)|^p dxdy = \int_{C \setminus \Delta^*_\tau} \left| h \left( \frac{1}{z} \right) \right|^p \frac{dxdy}{|z|^4} = \int_{\frac{1}{\tau}}^{\infty} \int_0^{2\pi} \left| h \left( \frac{e^{i\theta}}{r} \right) \right|^p dr d\theta.$$  

Let $\tilde{h}(z) = h(1/z)$, then $\tilde{h}$ is holomorphic on $\mathbb{C} \setminus \Delta^*_\frac{1}{\tau}$ and

$$\tilde{h}(z) = \sum_{n=2}^{\infty} a_{-n} z^n.$$  

If $\tilde{h}$ is not 0, then there is $n_0 > 1$ such that $a_{-n_0} \neq 0$ and $a_{-n} = 0$ for $1 < n < n_0$. Write $\tilde{h}(z) = z^{n_0} f_1(z)$, where $f_1(z) = \sum_{n=n_0}^{\infty} a_{-n} z^{n-n_0}$.
By the submean property
\[ \int_0^{2\pi} |f_1 \left( \frac{e^{i\theta}}{r} \right) |^p d\theta \geq 2\pi |a_{-n_0}|^p, \]
and \( n_0p - 3 > -1 \), it follows that
\[ \int_{\Delta^*_r} |h(z)|^p dxdy \geq 2\pi |a_{-n_0}|^p \int_1^\infty r^{n_0p-3} dr = \infty. \]

Therefore, \( h = 0 \). That is to say, for any \( f \in A^p(\Delta^*) \), we have \( f(z) = \sum_{n=-1}^\infty a_n z^n \).

Note that
\[ K_{\Delta^*,p}(z) \geq \frac{1}{|z|^p} \frac{|a + f(z)|^p}{\int_{\Delta^*} |z + f(z)|^p dxdy} \]

From (1), for \( z \) near 0, we may take \( a = 1 \). For \( f \in A^p(\Delta) \)

(a) If \( \|f\|_p^p > 2^p \frac{2\pi}{2-p} \), then \( \|f(z) + \frac{1}{z}\|_p \geq \|f(z)\|_p - \|\frac{1}{z}\|_p > \frac{1}{2} \|f(z)\|_p \), so
\[ \frac{|1 + zf(z)|^p}{\int_{\Delta^*} |\frac{1}{z} + f(z)|^p dxdy} < \frac{2^p(1 + |zf(z)|^p)}{(1/2^p) \int_{\Delta^*} |f|_p^p} < 2^{2p} \left( \frac{2 - p}{2p+1}\pi + |z|^p K_{\Delta,p}(z) \right). \]

(b) If \( \|f\|_p^p \leq 2^p \frac{2\pi}{2-p} \), then \( |f(z)| \leq C \) for all \( z \) near 0, where \( C \) is a positive constant independent on \( f \).

Since
\[ \int_{\Delta^*} \left| \frac{1}{z} + f(z) \right|^p dxdy = \int_0^1 r^{1-p} dr \int_0^{2\pi} |1 + re^{i\theta} f(re^{i\theta})|^p d\theta \]
\[ \geq 2\pi \int_0^1 r^{1-p} dr = \frac{2\pi}{2 - p}, \]
then
\[ \frac{|1 + zf(z)|^p}{\int_{\Delta^*} |\frac{1}{z} + f(z)|^p dxdy} < \frac{(2 - p)(1 + |z|^p)}{2\pi}. \]

According to (a) and (b), we get that \( |z|^p K_{\Delta^*,p}(z) \) is bounded near 0.
From the above theorem, we know the lower bounds of Theorem 2.6 is optimal.

**Remark 2.3.** Let \( D = \{ z \in \mathbb{C} : |z| > 1 \} \), for \( p \in (1, 2) \), there is \( c = c(p) > 0 \) such that
\[
K_{D,p}(z) \leq \frac{c}{|z|^{2p}}
\]
for \( |z| \gg 1 \).

Let \( \varphi : \Delta^* \to D, z \mapsto 1/z \). For \( p \in (4/3, 2) \),
\[
K_{\Delta^*,p}(z) \neq K_{D,p}(1/z) \frac{1}{|z|^4}.
\]

Proof of the Remark:
For any \( f \in A^p(D) \), we have
\[
f(z) = \sum_{n=-1}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^{-n}.
\]
Let \( f_1(z) = \sum_{n=-1}^{\infty} a_n z^n \) and \( f_2(z) = \sum_{n=2}^{\infty} b_n z^{-n} \).

It is easy to check that there is \( r \gg 1 \) such that \( \int_{\{|z|>r\}} |f_2|^p < \infty \) holds.

Hence \( \int_{\{|z|>r\}} |f_1|^p < \infty \).

If \( f_1 \) is not 0, we may choose \( k \) to be the integer such that \( a_n = 0 \) for \( n < k \), \( a_k \neq 0 \), then
\[
\int_{\{|z|>r\}} |f_1|^p = \int_{\{|z|>r\}} \left| \sum_{n=k}^{\infty} a_n z^n \right|^p = \int_r^\infty \rho d\rho \int_0^{2\pi} (\rho)^{kp} \left| \sum_{n=k}^{\infty} a_n z^{-k-n} \right|^p \geq 2\pi |a_k|^p \int_r^\infty (\rho)^{1+kp} = \infty.
\]

Therefore, \( f_1 = 0 \).

We get \( K_{D,p}(z) \leq \frac{c}{|z|^{2p}} \) for \( |z| \gg 1 \).

By the above theorem, \( K_{\Delta^*,p}(z) = O\left(\frac{1}{|z|^p}\right) \).

As
\[
K_{D,p}(1/z) \frac{1}{|z|^4} \leq \frac{c}{|z|^{4-2p}}
\]
for \( |z| \ll 1 \),
if \( p > 4/3 \), then
\[
K_{\Delta^*, p}(z) \neq K_{D, p}(z) \frac{1}{|z|^4}.
\]

We have finished the proof of the remark.

### 3. A conjecture of H. Tsuji

We first recall a definition for complex manifolds, see H. Tsuji [20].

**Definition 3.1.** Let \( M \) be a complex manifold with the canonical line bundle \( K_M \), for every positive integer \( m \), we set
\[
Z_m := \left\{ \sigma \in \Gamma(M, \mathcal{O}_M(mK_M)) \left| \int_M (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| < +\infty \right\}
\]
and
\[
K_{M, m} := \sup \left\{ |\sigma|^{\frac{2}{m}} ; \sigma \in \Gamma(M, \mathcal{O}_M(mK_M)) \left| \int_M (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| \leq 1 \right\},
\]
where the sup denotes the pointwise supremum.

Then let
\[
K_{M, \infty} := \limsup_{m \to \infty} K_{M, m}
\]
and \( h_{\text{can}, M} := \text{the lower envelope of } \frac{1}{K_{M, \infty}} \).

**Lemma 3.1.** For \( \Omega \subset \mathbb{C}^n \), we have
\[
\sup_{m \in \mathbb{N}} K_{\Omega, \frac{2}{m}}(z) = \sup_{p \in (0, 2]} K_{\Omega, p}(z).
\]

**Proof.** By Lemma 2.4, we have
\[
\sup_{m \in \mathbb{N}} K_{\Omega, \frac{2}{m}}(z) = \sup_{p \in (0, 2]} K_{\Omega, p}(z).
\]

If \( f \in \mathcal{O}(\Omega) \) and \( \int_\Omega |f|^p < \infty \), then
\[
\lim_{q \to p, q < p} \int_\Omega |f|^q = \int_\Omega |f|^p.
\]

So
\[
\sup_{p \in (0, 2]} K_{\Omega, p}(z) = \sup_{p \in (0, 2]} K_{\Omega, p}(z)
\]
and the lemma follows. \( \square \)
For $\Delta^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$, since the canonical bundle $K_{\Delta^*}$ is trivial, so when we consider $K_{\Delta^*,\infty}$ and $h_{\text{can},\Delta^*}^{-1}$, we can omit the form $dt$.

H. Tsuji [20] proposed the following conjecture (see Conjecture 2.16 in [20]):

$$h_{\text{can},\Delta^*}^{-1} = O \left( \frac{1}{|z|^2(\log|z|)^2} \right)$$

holds.

However, we get the following theorem:

**Theorem 3.2.** One has

$$h_{\text{can},\Delta^*}^{-1}(z) \geq K_{\Delta^*,\infty}(z) \geq \frac{1}{2\pi e} \frac{1}{|z|^2|\log|z||}$$

for $0 < |z| < e^{-1}$.

**Proof.** Since

$$\int_{\Delta^*} \left| \frac{1}{z} \right|^p = \frac{2\pi}{2 - p},$$

by Lemma 2.4 and Lemma 3.1, we get

$$K_{\Delta^*,\infty}(z) = \limsup_{m \to \infty} K_{\Delta^*,m}(z) = \sup_{m \geq 1} K_{\Delta^*,m}(z)$$

$$= \sup_{p \in (0,2]} K_{\Delta^*,p}(z) \geq \sup_{p \in (0,2]} \frac{2 - p}{2\pi} \frac{1}{|z|^p}.$$

For $0 < |z| < e^{-1}$, let

$$p = 2 + \frac{1}{\log|z|} \in [1, 2],$$

therefore

$$\frac{2 - p}{2\pi} \frac{1}{|z|^p} = \frac{1}{2\pi e} \frac{1}{|z|^2|\log|z||},$$

so

$$K_{\Delta^*,\infty}(z) \geq \frac{1}{2\pi e} \frac{1}{|z|^2|\log|z||}.$$

Hence

$$h_{\text{can},\Delta^*}^{-1}(z) \geq K_{\Delta^*,\infty}(z) \geq \frac{1}{2\pi e} \frac{1}{|z|^2|\log|z||}.$$

\[ \square \]

From the above theorem, we know that $h_{\text{can},\Delta^*}^{-1}$ is not integrable near 0.
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