Holomorphic automorphisms of the loop space of $\mathbb{P}^n$

Ning Zhang

The loop space $L\mathbb{P}^n$ of the complex projective space $\mathbb{P}^n$ consisting of all $C^k$ or Sobolev $W^{k,p}$ maps $S^1 \to \mathbb{P}^n$ is an infinite dimensional complex manifold. We identify a class of holomorphic self-maps of $L\mathbb{P}^n$, including all automorphisms.

1. Introduction

Let $M$ be a finite dimensional complex manifold. We fix a smoothness class $C^k$, $k = 1, 2, \ldots, \infty$, or Sobolev $W^{k,p}$, $k = 1, 2, \ldots, 1 \leq p < \infty$, and consider the loop space $LM = L_k M$, or $L_{k,p} M$, of all maps $S^1 \to M$ with the given regularity. It carries a natural complex Banach/Fréchet manifold structure (see [L]). In this paper, we identify a class of holomorphic self-maps of the loop space $L\mathbb{P}^n$ of the complex projective space $\mathbb{P}^n$. As a consequence, we compute the group $\text{Aut}(L\mathbb{P}^n)$ of holomorphic automorphisms of $L\mathbb{P}^n$. This work was directly motivated by [MZ, Z1, LZ, Z2], in which certain subgroups of $\text{Aut}(L\mathbb{P}^1)$ play a key role to study Dolbeault cohomology groups with values in line bundles over $L\mathbb{P}^1$.

There are two simple ways to construct holomorphic self-maps of a given loop space $LM$. First, such a map can be obtained from a family of holomorphic self-maps of $M$ smoothly parameterized by $t \in S^1$. For example, let $G \simeq PGL(n+1, \mathbb{C})$ be the group of holomorphic automorphisms of $\mathbb{P}^n$. Its loop space $LG$ with pointwise group operation is again a complex Lie group and acts on $L\mathbb{P}^n$ holomorphically; thus any element of $LG$ can be considered as a holomorphic automorphism of $L\mathbb{P}^n$. Second, let $\mathcal{T}(S^1) = \mathcal{T}_k(S^1)$ resp. $\mathcal{T}_{k,p}(S^1)$ be the space of maps $\phi : S^1 \to S^1$ with

---

2010 Mathematics Subject Classification: 32H02, 58B12, 46G20, 58D15, 58C10.

Key words and phrases: Loop space, Holomorphic self-map, Automorphism group, Projective space.

The author is grateful to L. Lempert and the referee for their very helpful comments. This research was partially supported by the National Natural Science Foundation of China grant 10871002.
the following properties: for any $x$ in $L^C = L^C_k$ resp. $L^C_{k,p}$, the pull back \( \phi^*x = x \circ \phi \) is still in $L^C = L^C_k$ resp. $L^C_{k,p}$, and the complex linear operator $L^C \ni x \mapsto x \circ \phi \in L^C$ is continuous. If we set the loop $x$ above to be the inclusion $S^1 \to \mathbb{C}$, we see that any $\phi$ in $T^k(S^1)$ resp. $T^k_{k,p}(S^1)$ is a $C^k$ resp. $W^{k,p}$ map. It is easy to verify that $T^k(S^1) = C^k(S^1, S^1)$. Now any $\phi \in \mathcal{T}(S^1)$ induces a holomorphic map

\[ f_{\phi, LM} : LM \ni x \mapsto x \circ \phi \in LM \]

(see Proposition 2.1). We shall write $f_\phi$ for $f_{\phi, L^P_n}$.

Recall that $H^2(L^P_n, \mathbb{Z}) \cong \mathbb{Z}$ (see [CJ, Part II, Proposition 15.33] and [P, Theorem 13.14]). Any holomorphic self-map $f$ of $L^P_n$ induces a homomorphism $[\omega] \mapsto m_f[\omega]$, where $[\omega] \in H^2(L^P_n, \mathbb{Z})$, and $m_f$ is a non-negative integer. Our main result is

**Theorem 1.1.** Let $f$ be a holomorphic self-map of $L^P_n$. Then $m_f = 1$ if and only if there exist $\gamma \in LG$ and $\phi \in \mathcal{T}(S^1)$ such that

\[ f = \gamma \circ f_\phi. \]

Note that all constant loops are fixed points of $f_\phi$, and $\gamma$ does not share this property unless it is the identity. Hence the decomposition of $f$ as in (1) is unique.

It is straightforward to verify that

\[ f_\phi \circ \gamma = f_{\phi, LG}(\gamma) \circ f_\phi, \quad \gamma \in LG, \, \phi \in \mathcal{T}(S^1). \]

Let $\mathcal{D}(S^1) = \mathcal{D}_k(S^1)$ resp. $\mathcal{D}_{k,p}(S^1)$ be the space of bijections $\phi : S^1 \to S^1$ such that both $\phi$ and $\phi^{-1}$ are in $\mathcal{T}(S^1) = \mathcal{R}(S^1)$ resp. $\mathcal{R}_{k,p}(S^1)$. Then $\mathcal{D}_k(S^1)$ is the space of $C^k$ diffeomorphisms of $S^1$. If $k = 2, 3, \ldots, \text{ or } k = p = 1$, then $\mathcal{D}_{k,p}(S^1)$ is the space of bijections $\phi : S^1 \to S^1$ such that $\phi, \phi^{-1} \in W^{k,p}(S^1, S^1)$; and $\mathcal{D}_{1,p}(S^1)$, where $p > 1$, is the space of bi-Lipschitz maps (see [V, Theorem 4] and [HS, Corollary 20.5]). Note that $f_\phi \in Aut(L^P_n)$ if and only if $\phi \in \mathcal{D}(S^1)$, and $\mathcal{D}(S^1)$ can be considered as a subgroup of $Aut(L^P_n)$.

Combination of Theorem 1.1 and (2) gives

**Corollary 1.2.** The group $Aut(L^P_n)$ is the semidirect product $LG \rtimes \mathcal{D}(S^1)$.

The group of holomorphic automorphisms of a compact complex manifold is a complex Lie group. We would like to know whether $Aut(L^P_n)$
can be endowed with a natural complex Lie group structure. Recall that $LG$ is a complex Lie group. If $D(S^1)$ can be endowed with a manifold structure such that $\text{Aut}(L\mathbb{P}^n) = LG \times D(S^1)$ becomes a complex Lie group, then $D(S^1) \cong \text{Aut}(L\mathbb{P}^n)/LG$ is a complex Lie group. The space $\mathcal{D}_k(S^1)$ can be considered as the open subset of $L_kS^1$ consisting of embedded loops. With this manifold structure, $\mathcal{D}_\infty(S^1)$ is a Lie group (and $\mathcal{D}_k(S^1)$, where $k < \infty$, is not a Lie group). It follows from [PS, Proposition 3.3.2] that we cannot endow $D(S^1)$ with a complex Lie group structure such that the inclusion $D_\infty(S^1) \hookrightarrow D(S^1)$ is a Lie group homomorphism.

Let $\mathcal{C} = \mathcal{C}(L\mathbb{P}^n)$ be the set \{\(\mu(L^1)\}\) of curves in $L\mathbb{P}^n$, where $\mu$ ranges over all holomorphic embeddings $\mathbb{P}^1 \rightarrow L\mathbb{P}^n$ such that $\mu^* : H^2(L\mathbb{P}^n, \mathbb{Z}) \rightarrow H^2(\mathbb{P}^1, \mathbb{Z})$ is an isomorphism, and let $\mathcal{H} = \mathcal{H}(L\mathbb{P}^n)$ be the space of holomorphic self-maps $f$ of $L\mathbb{P}^n$ with the following property: for any curve $C \in \mathcal{C}$, there exists a curve $C' \in \mathcal{C}$ such that $f(C) \subset C'$. It turns out that any holomorphic self-map $f$ of $L\mathbb{P}^n$ with $m_f = 1$ is in $\mathcal{H}$ (see Section 3). To prove Theorem 1.1, we only need to study maps in $\mathcal{H}$.

This paper is organized as follows. In Section 2, we recall some relevant facts about loop spaces and prove two propositions which will be needed later on. Section 3 contains a complete proof of Theorem 1.1. In the final Section 4, we try to classify maps in $\mathcal{H}$.

We refer to [D, H] for the fundamentals of infinite dimensional holomorphy.

2. Preliminaries

Let $\Psi : M \rightarrow M'$ be a holomorphic map between finite dimensional complex manifolds. Then $L\Psi : LM \ni x \mapsto \Psi \circ x \in LM'$ is holomorphic, and $L$ is functorial. For any $t \in S^1$, the evaluation map $E_t : LM \ni x \mapsto x(t) \in M$ is holomorphic (see [L]). The constant loops form a submanifold of $LM$, which can be identified with $M$. Note that all elements of $LM$ can be represented by absolutely continuous maps.

Let $G \simeq PGL(n+1, \mathbb{C})$ be as in Section 1. Applying the functor $L$ to the holomorphic action $G \times \mathbb{P}^n \rightarrow \mathbb{P}^n$, we obtain a holomorphic action $LG \times L\mathbb{P}^n \rightarrow L\mathbb{P}^n$.

**Proposition 2.1.** Suppose $M$ is an $n$-dimensional complex manifold, where $0 < n < \infty$, and $\phi : S^1 \rightarrow S^1$ is a map. Then $\phi \in \mathcal{F}(S^1)$ if and only if for any $x \in LM$, $x \circ \phi$ is still in $LM$, and the map $f_{\phi, LM} : LM \ni x \mapsto x \circ \phi \in LM$ is holomorphic.
Proof. First we show that if $\phi \in \mathcal{T}(S^1)$ and $x \in LM$, then $x \circ \phi \in LM$. Let $(U, \Phi)$ be a coordinate chart of $M$, where $U \subset M$ is open, and $\Phi$ is a biholomorphic map from $U$ to an open subset of $\mathbb{C}^n$. Then $LU \subset LM$ is open, and $L\Phi$ is a biholomorphic map from $LU$ to an open subset of $L\mathbb{C}^n$. If $x \in LU$, then
\[
x \circ \phi = (L\Phi)^{-1} \circ f_{\phi, \mathbb{C}^n} \circ L\Phi(x) \in LU \subset LM.
\]

For a general loop $x \in LM$ and any $t_0 \in S^1$, there exist a neighborhood $V' \subset S^1$ of $\phi(t_0)$, a coordinate chart $(U, \Phi)$ of $M$ and $\tilde{x} \in LU$ such that $x(t) = \tilde{x}(t)$, $t \in V'$. Note that $\phi$ is continuous. Choose a neighborhood $V \subset S^1$ of $t_0$ such that $\phi(V) \subset V'$, then $x \circ \phi(t) = \tilde{x} \circ \phi(t)$, $t \in V$. So $x \circ \phi$ is $C^k$ resp. $W^{k, p}$, i.e. $x \circ \phi \in LM$, and the map $f_{\phi, LM}$ is well-defined.

Next we investigate the relationship between maps $f_{\phi, LM}$ and $f_{\phi, \mathbb{C}^n}$. Recall the complex structure of $LM$ as constructed in [LS, Subsection 1.1], where an open neighborhood of $y \in LM$ is mapped by the local chart $\varphi_y$ to an open subset of $C^k(y^*TM)$ resp. $W^{k, p}(y^*TM)$, the space of $C^k$ resp. $W^{k, p}$ sections of the pull back of the tangent bundle of $M$ by $y$. It is straightforward to verify that $\varphi_y \circ f_{\phi, LM} \circ \varphi_y^{-1}$ is precisely (the restriction of) the pull back
\[
\phi^* : C^k(y^*TM) \to C^k(\phi^* y^*TM) \text{ resp. } W^{k, p}(y^*TM) \to W^{k, p}(\phi^* y^*TM).
\]

The bundle $y^*TM$ over $S^1$ is always trivial, and the above map can be considered as $f_{\phi, \mathbb{C}^n}$. Now we have $\varphi_y \circ f_{\phi, LM} \circ \varphi_y^{-1} = f_{\phi, \mathbb{C}^n}$. Thus $\phi \in \mathcal{T}(S^1)$ if and only if the map $f_{\phi, LM}$ is well-defined and holomorphic. \qed

Let $i : \mathbb{P}^n \to \mathbb{P}^n$ be the inclusion of the submanifold of constant loops. Since $E_t \circ f_{\phi} \circ i$ is the identity, the induced maps $i^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathbb{Z}), f_{\phi}^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathbb{Z})$, and $E_t^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathbb{Z})$ are all isomorphisms.

The following simple proposition will be very useful:

**Proposition 2.2.** Let $\mu : \mathbb{P}^1 \to \mathbb{P}^n$ and $\tau : \mathbb{P}^n \to \mathbb{P}^n$ be holomorphic maps such that $\mu^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^1, \mathbb{Z})$ and $\tau^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathbb{Z})$ are isomorphisms. Then:

(a) For any $t \in S^1$, the map $E_t \circ \mu : \mathbb{P}^1 \to \mathbb{P}^n$ is an embedding whose image is a projective line. In particular, $\mu$ itself is an embedding.

(b) There exists $\gamma \in LG$ (considered as an automorphism of $\mathbb{P}^n$) such that $\tau = \gamma|_{\mathbb{P}^n}$. 
Proof. (a) Note that \((E_t \circ \mu)^* = \mu^* \circ E_t^* : H^2(\mathbb{P}^n, \mathbb{Z}) \rightarrow H^2(\mathbb{P}^1, \mathbb{Z})\) is an isomorphism, which maps the first Chern class of the hyperplane section bundle over \(\mathbb{P}^n\) to the first Chern class of the hyperplane section bundle over \(\mathbb{P}^1\). Therefore the inverse image of any hyperplane of \(\mathbb{P}^n\) under \(E_t \circ \mu\) is either a hyperplane (i.e. a single point) or the entire \(\mathbb{P}^1\). The proof will be by induction on \(n\). If \(n = 1\), then \(E_t \circ \mu\) is conformal. If \(n > 1\), take a hyperplane \(H \subset \mathbb{P}^n\) containing two different points in \(E_t \circ \mu(\mathbb{P}^1)\). Then we must have \(E_t \circ \mu(\mathbb{P}^1) \subset H \cong \mathbb{P}^{n-1}\).

(b) The restriction of \(\tau\) to any projective line in \(\mathbb{P}^n\) can be considered as the map \(\mu\). In view of part (a), for any \(t \in S^1\), \(E_t \circ \tau : \mathbb{P}^n \rightarrow \mathbb{P}^n\) is injective; hence \(E_t \circ \tau \in G\). Define \(\gamma : S^1 \ni t \mapsto E_t \circ \tau \in G\). Then

\[(3) \quad \gamma(t)(\zeta) = \tau(\zeta)(t), \quad \zeta \in \mathbb{P}^n, t \in S^1.\]

Note that \(\gamma\) is \(C^k\) resp. \(W^{k,p}\) (i.e. \(\gamma \in LG\)) if \(\gamma(t_j), j = 1, 2, \ldots, n+2\), are \(C^k\) resp. \(W^{k,p}\) maps of \(t\), where \(\{\zeta_1, \zeta_2, \ldots, \zeta_{n+2}\} \subset \mathbb{P}^n\) is any given set of \(n+2\) points in general position. It follows from (3) that \(\gamma \in LG\) and \(\tau = \gamma|_{\mathbb{P}^n}\). □

3. Proof of Theorem 1.1

We begin with two results concerning holomorphic self-maps of loop spaces of the type \(f_{\phi,LM}\).

**Proposition 3.1.** Let \(f\) be a holomorphic self-map of \(LM\). Then \(f = f_{\phi,LM}\) for some \(\phi \in \mathcal{T}(S^1)\) if and only if for any \(t \in S^1\), there exists \(t' \in S^1\) such that \(E_t \circ f = E_{t'}\).

**Proof.** If \(f = f_{\phi,LM}\), then \(E_t \circ f = E_{\phi(t)}\). For the other direction, define the map \(\phi : S^1 \ni t \mapsto t' \in S^1\). Then \(f(x) = x \circ \phi\). It follows from Proposition 2.1 that \(\phi \in \mathcal{T}(S^1)\). □

**Lemma 3.2.** Let \(f\) be a self-map of \(L\mathbb{C}^n\). Then \(f = f_{\phi,LC^n}\) for some \(\phi \in \mathcal{T}(S^1)\) if and only if \(f\) is continuous complex linear and

\[(4) \quad f(x)(S^1) \subset x(S^1)\]

for all \(x \in L\mathbb{C}^n\).

**Proof.** One direction being trivial, we shall only verify the sufficiency part of the claim. Let \(e_1, e_2, \ldots, e_n\) be the standard basis of \(\mathbb{C}^n \subset L\mathbb{C}^n\). For any
Proposition 3.3. Let \( (6) \) \( \mu \mapsto \text{exists a curve } \\) where \( \text{we obtain that } (5) \) \( E \) \( \) is a projective line in \( \text{proof of [LS, Lemma 2.2] also works for } \) holomorphic map \( L\pi \) \( \) such that \( \tilde{x} \) \( \notin \) \( t \) \( \} \) \( E \) \( \) \( x \) \( ∈ \) \( \} \) \( f \) \( \) \( tin\text{uous complex linear maps } f_j : L\mathbb{C} \ni x_j \mapsto y_j \in \mathbb{C}, j = 1, 2, \ldots, n. \) Now \( f(x) = \sum_{j=1}^{n} f_j(x_j)e_j. \) Let \( \mathbb{C}^* = \mathbb{C} \setminus \{0\}. \) Then \( f_j(L\mathbb{C}^*) \subset L\mathbb{C}^* \) and \( f_j(1) = 1. \) It follows from [Z1, Lemma 3.1] that for any \( t \in S^1, \) there exists \( t \in S^1 \) such that \( E_t \circ f_j = E_t. \) Therefore

\[
E_t \circ f(x) = \sum_{j=1}^{n} x_j(t_j)e_j.
\]

Let \( x_0 \in L\mathbb{C} \) be an embedded loop. Setting \( x = \sum_{j=1}^{n} x_0e_j \) in (4) and (5), we obtain that \( x_0(t_1) = x_0(t_2) = \cdots = x_0(t_n). \) Hence \( t_1 = t_2 = \cdots = t_n \) and \( E_t \circ f = E_t. \) By Proposition 3.1, \( f = f_{\phi, L\mathbb{C}^n} \) for some \( \phi \in \mathcal{T}(S^1). \)

**Proposition 3.3.** Let \( x \) and \( y \) be two different points of \( L\mathbb{P}^n. \) Then there exists a curve \( C \in \mathcal{C} \) through both \( x \) and \( y \) if and only if \( x(t) \neq y(t) \) for all \( t \in S^1. \)

**Proof.** The “only if” direction follows from Proposition 2.2(a). To show the “if” direction, consider the natural projection \( \pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n \) and the holomorphic map \( L\pi : L(\mathbb{C}^{n+1} \setminus \{0\}) \to L\mathbb{P}^n. \) Let \( \tilde{x}, \tilde{y} \in L(\mathbb{C}^{n+1} \setminus \{0\}) \) be such that \( \tilde{x}(t) \) and \( \tilde{y}(t) \) are linearly independent for all \( t \in S^1. \) Then the map

\[
(6) \quad \mu = \mu_{\tilde{x}, \tilde{y}} : \mathbb{P}^1 \ni [Z_0, Z_1] \mapsto L\pi (Z_0\tilde{x} + Z_1\tilde{y}) \in L\mathbb{P}^n,
\]

where \( Z_0, Z_1 \) are homogeneous coordinates on \( \mathbb{P}^1, \) is well-defined and holomorphic. For any \( t \in S^1, \) the map \( E_t \circ \mu \) is an embedding whose image is a projective line in \( \mathbb{P}^n; \) thus \( \mu \) is an embedding and \( \mu^* : H^2(L\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^1, \mathbb{Z}) \) is an isomorphism.

The map \( L\pi \) is surjective (for \( C^k \) loops see [LS, Lemma 2.2], and the proof of [LS, Lemma 2.2] also works for \( W^{k,p} \) loops). Take \( \tilde{x} \in (L\pi)^{-1}(x) \) and \( \tilde{y} \in (L\pi)^{-1}(y) \) in (6), then the image of \( \mu \) is a curve \( C \in \mathcal{C} \) through both \( x \) and \( y. \)

From now on, we shall concentrate on holomorphic self-maps \( f \) of \( L\mathbb{P}^n, \) and curves in \( \mathcal{C} \) will be needed throughout the rest of the paper.

**Proposition 3.4.** If \( m_f = 0, \) then \( f \) is constant.

**Proof.** If \( \eta : \mathbb{P}^1 \to \mathbb{P}^n \) is a holomorphic map such that \( \eta^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^1, \mathbb{Z}) \) is zero, then \( \eta \) is constant. So for any curve \( C \in \mathcal{C} \) and any \( t \in S^1, \)
Holomorphic automorphisms

the map $E_t \circ f|_C : C \to \mathbb{P}^n$ is constant; hence $f|_C$ is constant. It follows from Proposition 3.3 that for any $x \in L\mathbb{P}^n$ and any constant loop $\zeta \in \mathbb{P}^n \setminus x(S^1)$, there exists a curve $C \in \mathcal{C}$ through both $x$ and $\zeta$; thus $f$ is constant.

Recall the mapping space $\mathcal{H}$ as defined in Section 1. If $f \in \mathcal{H}$, then the topological degree of $f|_C : C \to C'$ is $m_f$. In particular, $f(C) = C'$ if $m_f \geq 1$, and $f|_C$ is one-to-one if $m_f = 1$. By Proposition 2.2(a), any holomorphic self-map $f$ of $L\mathbb{P}^n$ with $m_f = 1$ is in $\mathcal{H}$. Hence $LG \subset \mathcal{H}$, and $f_\phi \in \mathcal{H}$ for any $\phi \in \mathcal{T}(S^1)$.

**Proposition 3.5.** Suppose $f \in \mathcal{H}$, $m_f \geq 1$, $x \in L\mathbb{P}^n$, and $\zeta \in \mathbb{P}^n \setminus x(S^1)$ is a constant loop. If either $m_f = 1$ or $f(x) \not\in f(\mathbb{P}^n)$, then $f(x)(t) \neq f(\zeta)(t)$ for all $t \in S^1$.

**Proof.** By Proposition 3.3, there exists a curve $C \in \mathcal{C}$ through both $x$ and $\zeta$. If either $m_f = 1$ or $f(x) \not\in f(\mathbb{P}^n)$, then $f(x) \neq f(\zeta)$. Since both $f(x)$ and $f(\zeta)$ are in $f(C) \in \mathcal{C}$, it follows from Proposition 3.3 that $f(x)(t) \neq f(\zeta)(t)$ for all $t \in S^1$.

**Theorem 3.6.** Let $f$ be a holomorphic self-map of $L\mathbb{P}^n$. Then $f = f_\phi$ for some $\phi \in \mathcal{T}(S^1)$ if and only if every constant loop is a fixed point of $f$.

**Proof.** The necessity is obvious. Regarding sufficiency, note that $f|_{\mathbb{P}^n}$ is the identity; hence $m_f = 1$ and $f \in \mathcal{H}$. It follows from Proposition 3.5 that

$$f(x)(S^1) \subset x(S^1)$$

for all $x \in L\mathbb{P}^n$. Let

$$U_0 = \{ [Z_0, Z_1, \ldots, Z_n] \in \mathbb{P}^n : Z_0 \neq 0 \},$$

where $Z_0, Z_1, \ldots, Z_n$ are homogeneous coordinates on $\mathbb{P}^n$. Now consider $U_0$ as $\mathbb{C}^n$. It follows from (7) that $f(L\mathbb{C}^n) \subset L\mathbb{C}^n$. Next we show that $f|_{L\mathbb{C}^n}$ is complex linear.

Let $x = (x_1, \ldots, x_n) \in L\mathbb{C}^n$ and $y = (y_1, \ldots, y_n) \in L(\mathbb{C}^n \setminus \{0\})$. If we choose $\tilde{x} = (1, x_1, \ldots, x_n)$ and $\tilde{y} = (0, y_1, \ldots, y_n)$ in (6), then the image of $\mu$ is a curve $C_{x,y} \in \mathcal{C}$, and

$$C_{x,y} \cap L\mathbb{C}^n = C_{x,y} \setminus \{ L\pi(\tilde{y}) \} = \{ x + \lambda y \in L\mathbb{C}^n : \lambda \in \mathbb{C} \},$$

where $L\pi(\tilde{y}) \in L(\mathbb{P}^n \setminus \mathbb{C}^n)$. Note that $f(C_{x,y})$ is also a curve in $\mathcal{C}$, and $f$ maps $C_{x,y}$ conformally onto $f(C_{x,y})$. By Proposition 2.2(a), for any $t \in S^1$, for all...
$E_t \circ f(C_{x,y})$ is a projective line in $\mathbb{P}^n$. In view of (7), we have

\begin{equation}
E_t \circ f(C_{x,y}) \cap \mathbb{C}^n = \{E_t \circ f(x + \lambda y) : \lambda \in \mathbb{C}\}.
\end{equation}

The above set must be an affine line of $\mathbb{C}^n$. As a function of $\lambda$, $E_t \circ f(x + \lambda y)$ maps $\mathbb{C}$ bijectively onto the affine line in (8); therefore it is a polynomial of degree one. So

\[ E_t \circ f(x + \lambda y) = [f(x + y)(t) - f(x)(t)] \lambda + f(x)(t) \]

for all $t \in S^1$. Thus

\[ f(x + \lambda y) = [f(x + y) - f(x)] \lambda + f(x), \]

i.e. $\mathbb{C} \ni \lambda \mapsto f(x + \lambda y) \in L\mathbb{C}^n$ is a polynomial of degree one, where $x \in L\mathbb{C}^n$ and $y \in L(\mathbb{C}^n \setminus \{0\})$. Since $L(\mathbb{C}^n \setminus \{0\})$ is dense in $L\mathbb{C}^n$, $f(x + \lambda y)$ is a polynomial of $\lambda$ of degree less than or equal to one for all $x,y \in L\mathbb{C}^n$. Hence $f|_{L\mathbb{C}^n}$ is a polynomial of degree one (see [H, Section 2.2]). As $f(0) = 0$, $f|_{L\mathbb{C}^n}$ is complex linear.

By (7) and Lemma 3.2, we have $f|_{L\mathbb{C}^n} = f_{\phi, L\mathbb{C}^n}$ for some $\phi \in \mathcal{T}(S^1)$; thus $f = f_{\phi}$ on the connected manifold $L\mathbb{P}^n$. $\Box$

**Proof of Theorem 1.1.** The sufficiency is obvious. Regarding necessity, by Proposition 2.2(b), there exists $\gamma \in LG$ such that $f|_{\mathbb{P}^n} = \gamma|_{\mathbb{P}^n}$. Then every constant loop is a fixed point of $\gamma^{-1} \circ f$. It follows from Theorem 3.6 that $\gamma^{-1} \circ f = f_{\phi}$ for some $\phi \in \mathcal{T}(S^1)$, i.e. $f = \gamma \circ f_{\phi}$. $\Box$

### 4. The mapping space $\mathcal{H}$

In this section, we continue to study maps in $\mathcal{H}$. Recall that all holomorphic self-maps $f$ of $L\mathbb{P}^n$ with $m_f \leq 1$ are in $\mathcal{H}$.

**Theorem 4.1.** Let $f \in \mathcal{H} = \mathcal{H}(L\mathbb{P}^n)$.

(a) If $n \geq 2$, then $m_f \leq 1$.

(b) If $n = 1$ and $m_f \geq 2$, then $f(L\mathbb{P}^1) = f(\mathbb{P}^1)$ (i.e. $f(C) = f(\mathbb{P}^1)$ for all $C \in \mathcal{C}$).

**Proof.** (a) If $f$ is constant, then $m_f = 0$. If $f$ is non-constant, by Proposition 3.4 we have $m_f \geq 1$. Fix $t \in S^1$ and define $\alpha = E_t \circ f|_{\mathbb{P}^n} : \mathbb{P}^n \to \mathbb{P}^n$. Then $\alpha$ induces a homomorphism $[\omega] \mapsto m_f[\omega]$, where $[\omega] \in H^2(\mathbb{P}^n, \mathbb{Z})$. So
Holomorphic automorphisms

the topological degree of $\alpha$ is $m_f \neq 0$. Let $\zeta^* \in \mathbb{P}^n$ be a regular value of $\alpha$. Take $\zeta \in \alpha^{-1}(\zeta^*)$ and choose a hyperplane $H \subset \mathbb{P}^n$ such that $\zeta \notin H$. For any $w \in H$, the projective line $P_{\zeta, w}$ through both $\zeta$ and $w$ is in $\mathcal{C}$; thus $f(P_{\zeta, w}) \in \mathcal{C}$. In view of Proposition 2.2(a), $\alpha(P_{\zeta, w})$ is a projective line of $\mathbb{P}^n$. The topological degree of the map $P_{\zeta, w} \xrightarrow{\alpha} \alpha(P_{\zeta, w})$ is $m_f$. If $m_f \geq 2$, then $P_{\zeta, w} \setminus \{\zeta\}$ must contain at least one point in $\alpha^{-1}(\zeta^*)$. For different $w \in H$, the sets $P_{\zeta, w} \setminus \{\zeta\}$ are disjoint; hence $\alpha^{-1}(\zeta^*)$ is not a finite set, which is a contradiction.

(b) Note that $f(\mathbb{P}^1) \in \mathcal{C}$. By Proposition 2.2(b) (in which we choose $\tau$ to be a suitable holomorphic embedding $\mathbb{P}^1 \to L\mathbb{P}^1$ with image $f(\mathbb{P}^1)$), there exists $\gamma \in LG$ such that $\gamma(\mathbb{P}^1) = f(\mathbb{P}^1)$; thus $\gamma^{-1} \circ f(\mathbb{P}^1) = \mathbb{P}^1$. Without loss of generality, we may assume that $f(\mathbb{P}^1) = \mathbb{P}^1$.

Let $\rho = f|_{\mathbb{P}^1}$. Then $\rho : \mathbb{P}^1 \to \mathbb{P}^1$ is an $m_f$-sheeted branched covering. Choose a non-empty open subset $W \subset \mathbb{P}^1$ such that $\rho|_W$ is one-to-one. Then $LW \subset L\mathbb{P}^1$ is a non-empty open subset. We claim that

$$f(x) \in \mathbb{P}^1 \text{ for } x \in LW. \quad (9)$$

Otherwise there would exist $x_0 \in LW$ such that $y_0 = f(x_0) \notin \mathbb{P}^1$. The set $y_0(S^1)$ is not finite; thus we can find a regular value $\zeta^*$ of $\rho$ in $y_0(S^1)$. Take $\zeta_1, \zeta_2 \in \rho^{-1}(\zeta^*)$, $\zeta_1 \neq \zeta_2$. It follows from Proposition 3.5 that $\zeta_1, \zeta_2 \in x_0(S^1) \subset W$; then $\rho|_W$ is not one-to-one, which is a contradiction.

By (9), maps $E_t \circ f|_{LW}$ are independent of $t \in S^1$; hence maps $E_t \circ f$ are independent of $t$ on the connected manifold $L\mathbb{P}^1$, i.e. $f(L\mathbb{P}^1) \subset \mathbb{P}^1$. \[ \square \]

The maps as in Theorem 4.1(b) do exist: Let $\rho : \mathbb{P}^1 \to \mathbb{P}^1 \subset L\mathbb{P}^1$ be a holomorphic map with topological degree $m \geq 2$, $t \in S^1$ and $\gamma \in LG$. Then $f = \gamma \circ \rho \circ E_t \in \mathcal{H}(L\mathbb{P}^1)$ and $m_f = m$.

References


School of Mathematical Sciences, Peking University, Beijing, 100871, China

*Current address:*

School of Mathematical Sciences, Ocean University of China, Qingdao, 266100, China

*E-mail address:* nzhang@ouc.edu.cn

Received September 21, 2012