Renormalization ideas in conformal dynamics

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1 Introduction

How to look at a dynamical system f at a small scale? You should take a small piece of the phase space, consider the *first return map* to this piece, and then rescale it to "the original size". The new dynamical system is called the *renormalization* Rf of the original one. It may happen that Rf looks "similar" to f, and then you can try to repeat this procedure, and construct the second renormalization $R^2 f$, etc. Asymptotic properties of this sequence of renormalizations reflect micro-structure of the original system. For example, convergence of the sequence $R^n f$ to a map f_* independent of f (from some class of similar maps) means that all maps of this class have in small scales a universal geometry represented by f_*.

A striking phenomenon of this kind is the Feigenbaum-Coullet-Tresser Universality Law ([CT, F], see [McM1], §6). It deals with the class of sufficiently smooth unimodal maps of an interval I with the critical point 0 of a given type $|x|^d$ ("unimodal" means: "with one critical point"). Under some combinatorial assumptions on the positions of the first four iterates of the critical point, the interval $J = [-f^2 0, f^2 0]$ turns out to be invariant under f^2. Moreover $f^2|J$ is again a unimodal map of the same class. Rescaling J to the original size, we obtain the "doubling renormalization" Rf of f. A map f of such kind can be called "renormalizable". If it happens that this procedure can be repeated, we have twice renormalizable maps, etc. The Universality Law asserts that the renormalizations $R^n f$ of infinitely renormalizable maps converge to a map f_* independent of f. Thus all infinitely renormalizable unimodal maps with a given type of the critical point have asymptotically the same geometry in small scales. A similar picture is observed not only for the doubling renormalization but for other periods as well.

We have here a kind of the rigidity phenomenon: Combinatorics of an object determines its geometry. Compare it with the Rigidity Conjecture discussed by McMullen [McM1]. The latter is concerned with a finitely dimensional family of globally defined objects, rational maps. The rigidity conclusion is also global: the geometry of the whole Julia set is determined by combinatorics. In the Feigenbaum-Coullet-Tresser situation we deal with an infinitely dimensional family of "partially defined" maps (think of polynomial-like maps). The Julia set is not rigid any more (as one can vary the multipliers of periodic points), but its most important part (the post-critical set) is still rigid!

The Universality Law was backed by numerous computer experiments and then by a computer assisted proof (Lanford [La]) - see the book [CE] and the survey [VSK] for that stage of events). In mid 80th Sullivan suggested a program of conceptual understanding of this phenomenon [S1-S3]. It included three big steps:

- Geometric *a priori* bounds;
- From geometric bounds to quasi-rigidity and further to global rigidity of polynomials;
- Contracting property of the renormalization transformation R with respect to an appropriate Teichmüller metric.

The first step of this program motivated by the work of Douady and Hubbard on polynomial-like maps [DH2] proved to be a hard analytical issue. Sullivan resolved it for real infinitely renormalizable maps of "bounded type". In §4 we will discuss, along with this work, the further development which settled the problem for all real quadratic maps and many complex ones [L5, GS2, LS, LY].

The second step has been resolved by a nice geometric argument based on the theory of quasi-conformal maps (see §5). The ideas for this part introduced into dynamics by Sullivan and Thurston can be tracked back to the Mostow Rigidity.

For the last step, Sullivan developed a sophisticated Teichmüller theory of "Riemann surface laminations" (see [S3] and the book of de Melo & van Stiren [MvS]). A different approach was suggested by McMullen who introduced a global dynamical object called a *tower*, and reduced the universality law to the quasi-conformal rigidity of towers [McM3].

What we have described above is the universality phenomenon in the *dynamical* plane. Not less intriguing is the *parameter* universality, which was actually discovered first by Feigenbaum, Coullet and Tresser. They gave an explanation of this phenomenon based upon conjectural hyperbolicity of the renormalization transformation R at the fixed point f_*. Proofs of this conjecture for period doubling case were given by Lanford [La] and Eckmann-Epstein [EE]. Recently the author proved it for all real combinatorial types [L8]. The proof is based upon a Rigidity Theorem for quadratic-like maps with *a priori bounds* [L7]. We will discuss this issue in §6.

We see that the global rigidity problem is an intimate part of the universality phenomenon. There has been recently several big breakthroughs in this problem which first looked complementary to the renormalization theory, but then were linked to it. The combinatorial game called "puzzle" appeared in the work of Branner, Hubbard and Yoccoz [BH, H] and allowed one to settle the rigidity problem for all maps with one non-escaping critical point of quadratic type, which are "at most finitely renormalizable" (see §4.4). Further contribution to the rigidity problem has been made by McMullen [McM2], Swiatek [Sw] and the author [L5]-[L7], which, in particular, settled it for real quadratic maps.

The notion which links "non-renormalizable" and "infinitely renormalizable" cases is "generalized renormalization". It allows one to embed the non-renormalizable maps into the renormalization theory, and to handle a number of geometric and measure-theoretical problems of real and complex dynamics [LM], [L3]-[L5], [SN]. We will particularly emphasize renormalization in the family of Fibonacci maps. A new curious phenomenon enlightened by this family is dependence of geometric and measure-theoretic properties of the map on the degree. In particular, the quadratic Fibonacci map has the Julia set of measure zero (Lyubich-Shishikura [L3]), while the maps of sufficiently high degree have positive measure Julia sets (Nowicki-van Strien [SN]).

This paper is linked to McMullen's paper [McM1] in this volume: concepts and results discussed in [McM1] may be used here without extra comments.

2 Combinatorics of complex unimodal maps

2.1 Renormalization in the sense of Douady and Hubbard

Polynomial-like maps were introduced by Douady and Hubbard in order to explain partial self-similarity of the Mandelbrot set. It is important to realize that a "polynomial-like map" actually means a *germ* near the filled Julia sets, so that there is a flexibility in the choice of the domain and range. Thus referring to a *conjugacy* between two polynomial-like maps we mean conjugacy near the filled Julia sets. In particular, the germ of a polynomial f near its filled Julia set $K(f)$ is polynomial-like. In this sense polynomials are also considered as polynomial-like maps. Polynomial-like maps with a single critical point (maybe degenerate) will be called *unimodal*, or *complex unimodal maps.*

Besides topological/quasi-conformal/conformal/affine categories of conjugacies, there is one more category called *hybrid.* Two polynomial-like maps are hybrid equivalent if they are conjugate by a quasi-conformal map h such that $\bar{\partial} h = 0$ almost everywhere on the filled Julia set. Let $\mathcal{H}(f)$ denote the hybrid class of a polynomial-like map f modulo conformal equivalence. The following basic result explains the importance of the hybrid category:

Straightening Theorem [DH2]. *Every hybrid class $\mathcal{H}(f)$ contains a polynomial. This polynomial is unique (modulo affine conjugacy) provided the Julia set $J(f)$ is connected.*

Sullivan views these hybrid classes as infinitely dimensional Teichmüller spaces [S1]. The Teichmüller pseudo-metric on this space is defined as follows:

$$\mathrm{dist}_T(f,g) = \inf \log K_h,$$

where h runs over the hybrid conjugacies between f an g, and K_h stands for its dilatation. It is not obvious but turns out to be true that this pseudo-metric is actually a metric provided $J(f)$ is connected. Note that unlike the classical situation, by the Straightening Theorem this Teichmüller space has a *preferred point.*

Let M_d be the connectedness locus of the family $z \mapsto z^d + c$, that is, the set of parameter values c for which the Julia set $J(z^d + c)$ is connected. Let Γ_n be the cyclic group of rotations $z \mapsto e^{2\pi i m/n} z$ of order n. By the Straightening Theorem, for any complex unimodal map f of degree d with connected Julia set, the hybrid class $\mathcal{H}(f)$ contains a unique polynomial $z \mapsto z^d + c(f)$ (modulo conjugacy by rotations $\gamma \in \Gamma_{d-1}$), where $c(f)$ belongs to M_d. In particular, hybrid classes of quadratic-like maps with connected Julia set are labeled by the points of the Mandelbrot set $M \equiv M_2$.

Renormalization (in the sense of Douady and Hubbard) of a complex unimodal map f means extracting from it a complex unimodal map $Rf = f^p$ of the same degree (see [McM1], §6). If this is possible, the map is usually called *renormalizable.* Notice that Rf is not a polynomial even when f is, so that the renormalization procedure automatically leads to the class of polynomial-like maps.

However this procedure respects the hybrid equivalence, so that it induces a map σ_d from a part of the connectedness locus M_d/Γ_{d-1}, where the map is renormalizable, into M_d/Γ_{d-1}. Douady and Hubbard have proved [D, DH2] that in degree two this map gives a homeomorphism of appropriate pieces of the Mandelbrot set onto the whole M. These pieces are exactly "small copies of M".

Different copies specify different "combinatorial types" of the renormalization. Let us consider the family $\mathcal{M}$ of *maximal* copies of M, that is, the copies which are not contained in any other copies. We have a map $\sigma \equiv \sigma_2 : \cup_{M'_l \in \mathcal{M}} M'_l \to M$ from the union of these copies onto M. If a parameter value $c \in M$ is infinitely renormalizable then we can apply this map infinitely many time. Let us keep track of the combinatorial types of the corresponding renormalizations by looking how the trajectory $c, \sigma c, \sigma^2 c, \dots$ travels through the copies: let $\sigma^n c \in M'_{l(n)}$, $n = 0, 1, \dots$. Let us call the sequence $\tau(f) = \{l(0), l(1), \dots\}$ the *combinatorial type* of f. The combinatorial class $Com(f)$ of an infinitely renormalizable quadratic-like map is the set of maps with the same combinatorial type. (One can show that this definition fits to the definition in terms of rational laminations, see [McM1], §5).

2.2 Generalized renormalization

So, there are "non-renormalizable" maps. How does it fit to the general idea of renormalization indicated in the introduction? The answer is hidden in the word "similar": renormalization Rf is supposed to be similar to f. In the above discussion the criterion for this similarity was the polynomial-like property in the sense of Douady and Hubbard. But why should we stick to it? It turns out that there is a fruitful extension of the class of polynomial-like maps which allows us to apply the renormalization ideas to "non-renormalizable" maps as well.

Let U_i be a family of disjoint topological disks compactly contained in another topological disk V. A generalized polynomial-like map $f : \cup U_i \to V$ is a branched covering which is univalent on all U_i except at most finitely many. If such a map has a single critical point, it is called (generalized) unimodal. If this point is non-degenerate, f is also called a (generalized) quadratic-like map. The filled Julia set $K(f)$ is again defined as the set of non-escaping points, and the Julia set $J(f)$ is defined as its boundary. In this setting, we have the following Straightening Theorem: *Any generalized polynomial-like map is hybrid equivalent to a polynomial with the same number of non-escaping critical points.*

We can now try to renormalize a complex unimodal polynomial $z \mapsto z^d + c$ in the class of generalized complex unimodal maps with non-escaping critical point. We will see that this is indeed possible for all "combinatorially recurrent" polynomials. However it requires a careful selection of the disk V. Indeed, if you take a random disk V and pull it back along an orbit $z \in V, fz, \dots, f^n z \in V$, you may well obtain a domain U which intersects V in a crazy way. We will discuss two good ways to select the domain: as a Yoccoz puzzle piece and (for real maps) just as a Euclidean disk.

2.3 Yoccoz puzzle

The puzzle provides us with a family of topological disks which always intersect nicely. The idea is to cut a neighborhood of the filled Julia set by a *forward invariant* family of curves, and then pull the corresponding domains back. A nice selection of the neighborhood is a topological disk D bounded by some equipotential E (note that fE encloses E). A nice selection of the cuts is the union of several rational external rays R_i (see [McM1], §5 for the definition).

(The most popular choice in the quadratic case is the following. Let $f = P_c : z \mapsto z^2 + c$, with $c \in M$ but outside the main cardioid of the Mandelbrot set. Such a quadratic has a fixed point α which is a landing point of *more than one* external rays R_i cyclically permuted by f.)

So assume that the rays R_i divide D into the pieces $Y_i^{(0)}$, "puzzle pieces of depth 0". The puzzle pieces of depth n are defined as the closures of the components of $f^{-n}\,\mathrm{int}\,Y_i^{(0)}$. The pieces of depth n form a tiling $\mathcal{T}_n$ of the disk $D_n = f^{-n}D$. Moreover $\mathcal{T}_n$ is a refinement of $\mathcal{T}_{n-1}|D_n$. The pieces containing 0 are called *critical*. We will either label the critical pieces with subscript 0, or skip the subscript all together.

Thus any two puzzle pieces are either nested or have disjoint interiors, and moreover the image of any puzzle piece of depth $n > 1$ is a puzzle piece of depth $n - 1$. These two obvious facts express the extremely useful *Markov property* of the family of puzzle pieces. It prevents the intersection troubles mentioned above, and allows us to carry out a generalized renormalization procedure.

2.4 Construction of $T^n f$

Let $\mathcal{O} = \mathcal{O}(f)$ denote the *post-critical set*, that is, the closure of the orbit $\{f^n 0\}_{n=0}^{\infty}$ of the critical point.

Lemma 2.1 *Let $V = Y^{(n)}$ be a critical puzzle piece for a unimodal polynomial $f : z \mapsto z^d + c$ such that:*

- *V is compactly contained in the previous critical puzzle piece $Y^{(n-1)}$;*
- *The critical point returns infinitely many times to V.*

Then f admits a generalized unimodal renormalization $T_V f : \cup U_i \to V$ with range V.

Proof. Given a point $z \in V \cap \mathcal{O}$ which returns back to $\mathrm{int}\,V$, let $r(z)$ denote the first return time. Let us consider the pull-back $U(z)$ of V along the orbit $z, fz, \dots, f^{r(z)}z$ (that is, the puzzle piece containing z which is mapped under $f^{r(z)}$ onto V). It follows from the Markov property that all puzzle pieces $U(z)$ are contained in V, and any two of them either coincide or have disjoint interiors. Moreover, our first assumption implies that they are compactly contained in V. The map $g : \cup U(z) \to V$ defined as $g|U(z) = f^{r(z)}$ is the desired renormalization. □

Remarks. 1. Yoccoz showed that a non-renormalizable (in the sense of Douady & Hubbard) unimodal polynomial with all periodic points repelling always has a puzzle piece satisfying the first assumption. The second assumption is minor as polynomials with non-recurrent critical point can be easily treated.

2. In most interesting cases the domain of $T_V f$ consists only of finitely many pieces $U(z)$ (see, e.g., the Fibonacci maps below). This is the situation when the renormalization philosophy becomes really valuable.

By repeating the above construction we can now construct a sequence of generalized renormalizations

$$T^n f \equiv g_n : \bigcup_i V_i^n \to V_0^{n-1} \equiv V^{n-1}. \tag{2.1}$$

Namely, starting with a critical puzzle piece $V^0 \equiv V_0^0$, let us inductively define g_n as $T_{V^{n-1}}f$. The sequence of puzzle pieces

$$V^0 \supset V^1 \supset \dots \tag{2.2}$$

is called the ***principal nest.*** Understanding of this nest is the key to the full dynamical picture.

The maps $g_n : V^n \to V^{n-1}$ are quadratic-like in the sense of Douady & Hubbard. Note however that not all of them are different (remember that a polynomial-like map means a germ). Let $\{n(k)\}$ be the sequence of levels where new quadratic-like maps $g_n|V^n$ are created. (These levels are characterized by the property that the critical point returns to V^{n-1} later than to V^{n-2}). This sequence is finite if and only if the map f is renormalizable in the sense of Douady & Hubbard. (Indeed finiteness of this sequence means that one of the maps $g_n|V^n$ has a non-escaping critical point, so that it gives a renormalization of f.) Let us define the *height* $\chi(f)$ as the length of the sequence $\{n(k)\}$. This combinatorial parameter has a big impact on the geometry of the map.

2.5 Combinatorics of the Fibonacci maps

For every even degree d, there is a remarkable non-renormalizable map $F_d : z^d + c_d$ called Fibonacci, $c_d \in \mathbb{R}$. Such a map is combinatorially determined by the property that the closest returns of the critical point to itself occur at the Fibonacci moments. It is extremal in many respects which makes it a good candidate for different interesting properties and, on the other hand, a test example to work out general results.

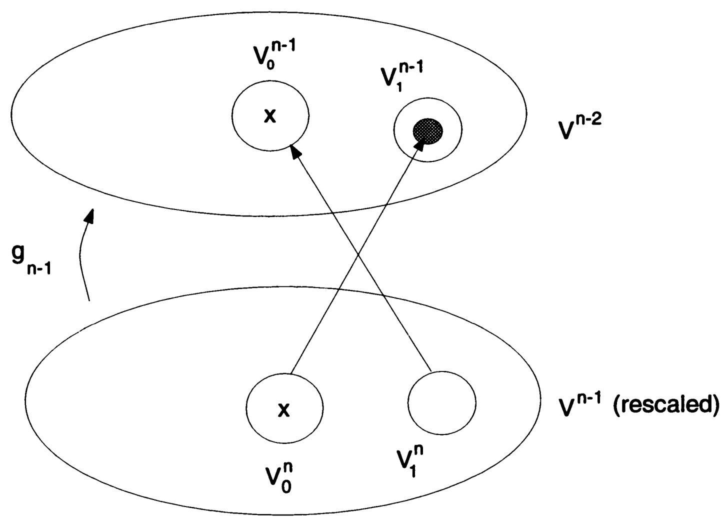

Figure 1. Fibonacci renormalization scheme.

The most efficient way to understand combinatorics of the Fibonacci maps is given by the generalized renormalization (2.1). It turns out that the domain of $g_n = T^n F_d$ consists of only *two* puzzle pieces, so that

$$g_n : V_0^n \cup V_1^n \to V_0^{n-1}, \tag{2.3}$$

where $g_n : V_0^n \to V_0^{n-1}$ is a d-to-1 covering, while $g_n : V_1^n \to V_0^{n-1}$ is univalent. Moreover, $g_n(0) \in V_1^n$, while $g_n(g_n 0) \in V_0^n$. This is in a sense the fastest possible recurrence of the critical point.

Figure 1 shows how to pass from one renormalization level of the Fibonacci map to the next. Even if you never heard about Fibonacci maps, the generalized renormalization analysis would inevitably lead you to this scheme.

Remark. Fibonacci maps appeared independently in several works: Branner & Hubbard [BH], Hofbauer & Keller [HK], Shibayama [Sh]... (see [LM] for more detailed account). The above renormalization scheme for these maps was suggested in [LM].

3 Parameter plane vs dynamical plane

3.1 Parameter rays vs dynamical rays

There is a remarkable similarity between the dynamical and parameter planes of conformal dynamical systems (actually it goes beyond conformal setting). Properties in the dynamical plane is usually reflected in the parameter plane. For example, little Julia sets in the dynamical plane reflect themselves as little copies of the Mandelbrot set in the parameter plane, see §2.1. As Douady put it: "You plow in the dynamical plane and then harvest in the parameter plane". There are different ways to pick this harvest. Historically the first one was based on a relation between the Riemann mapping

$$\phi_M : \mathbb{C} \setminus M \to \mathbb{C} \setminus \mathbb{D}$$

of the complement of the Mandelbrot set and the corresponding mappings in the dynamical plane [DH1].

Any quadratic polynomial $f = P_c : z \mapsto z^2 + c$ near infinity "looks like" z^2. The precise statement is that it is analytically conjugate to $z \mapsto z^2$, namely there is a conformal map $\phi = \phi_c$ near ∞, fixing ∞, tangent to id at ∞, and such that $\phi(fz) = (fz)^2$. There is a classical explicit formula, due to Böttcher, for this map, namely

$$\phi(z) = \lim_{n\to\infty} (f^n z)^{1/2^n}. \tag{3.4}$$

By means of this formula the map can be extended to larger domains until they hit the critical point 0. In the case of connected Julia set this gives the Riemann mapping $\phi : D(\infty) \to \mathbb{C} \setminus \mathbb{D}$ of the whole basin of infinity onto the complement of the unit disk. In the disconnected case it maps the complement of the "figure 8" onto the complement of some disk of radius $R > 1$.

In the parameter plane one can write down the similar formula, only instead of iterates of a single polynomial one should consider the sequence of polynomials $Q_n(c) = Q_{n-1}(c)^2 + c$, $Q_0(c) \equiv 0$, which keeps track of the orbit of the critical point

$$0 \mapsto c \mapsto c + c^2 \mapsto (c + c^2)^2 + c \mapsto \dots$$

Namely

$$\phi_M(c) = \lim_{n\to\infty} (Q_n c)^{1/2^n} \tag{3.5}$$

This yields connectivity of the Mandelbrot set together with the following remarkable formula for the corresponding Riemann mapping (Douady-Hubbard, Sibony [DH1]):

$$\phi_M(c) = \phi_c(c), \quad c \in \mathbb{C} \setminus M. \tag{3.6}$$

So a point $c \in \mathbb{C} \setminus M$ has a double personality: as a parameter value and as the critical value for the corresponding polynomial P_c. Both personalities can be identified by their uniformizing coordinates: the external angles and equipotential levels. Formula (3.6) says us that these identifications coincide!

3.2 Combinatorial classes

In [McM1] dynamical rational laminations $\lambda_{\mathbb{Q}}(P_c)$ are defined. We can similarly define a parameter rational lamination $\lambda_{\mathbb{Q}}(M)$ which describes how the rational rays in $\mathbb{C} \setminus M$ land. Douady & Hubbard [DH1] gave a full combinatorial description of this lamination. In particular, they proved using (3.6) that rational rays with odd denominators land at parabolic points, while rational rays with even denominators land at post-critically finite points (also called Misiurewicz).

Moreover formula (3.6) shows that the dynamical lamination bifurcates exactly at the moments when c crosses the parameter rational rays. Thus combinatorial classes in the quadratic family can be defined as the pieces on which the parameter rational rays partition the Mandelbrot set. Notice that this definition gives an extension of the notion of a combinatorial class to the maps with indifferent cycles as well (compare [McM1], Theorem 6.1). This also explains the relation between combinatorial rigidity and local connectivity of M (the last property is usually abbreviated as MLC).

Indeed, let us show that combinatorial rigidity implies MLC. Assume that the combinatorial class of $c \in M$ is a single point $\{c\}$. Then there is a nest of topological disks $D_1 \supset D_2 \supset \dots$ bounded by rational parameter external rays and shrinking to c. The complement $M \setminus D_k$ splits into finitely many disjoint parts. It follows that the intersection $M \cap D_k$ is connected (for otherwise the whole Mandelbrot set would be disconnected), and local connectivity at c follows.

The reverse property requires a finer combinatorial analysis similar to the proof that MLC implies density of expanding maps (see [DH1, Sch]).

3.3 Parapuzzle

The dynamical puzzle constructed in the previous section can also be transferred to the parameter plane. First splitting is given by the combinatorial rotation number of the fixed point α. As we mentioned before, there are finitely many external rays landing at α which are cyclically permuted by dynamics. The rotation number q/p of this permutation is called the *combinatorial rotation number of α.*

To determine this number looking at the parameter plane, you should do the following. Take a bifurcation point b on the main cardioid of the Mandelbrot set. The polynomial P_b has a parabolic point α_b with rotation number $e^{2\pi i q/p}$. The component of $M \setminus \{b\}$ which does not intersect the main cardioid is called *q/p-limb* $L_{q/p}$ of M. It turns out that the combinatorial rotation number of α_c is equal to q/p if and only if $c \in L_{q/p}$.

To construct further the nest of parapuzzle pieces $Z^{(n)}(c)$ about a parameter value $c \in M$ we should consider the corresponding dynamical nest $Y^{(n)}(c)$ about the critical value c. By definition, the parameter piece $Z^{(n)}(c)$ is bounded by the external rays and equipotentials of the same arguments and levels as the dynamical pieces $Y^{(n)}(c)$. By (3.6), all quadratics within the parameter piece have the same combinatorics up to depth n. By the discussion of §3.2, combinatorial rigidity of of a non-renormalizable polynomial P_c amounts to shrinking of these parameter pieces to c.

In particular, we have a *principal parapuzzle nest* $\{W^n(c)\}$ corresponding to the principal nest $\{V^n(c)\}$. The generalized renormalizations $T^n P_b$ have the domains with the same combinatorics when b ranges over $W^{(n)}(c)$.

4 Geometric bounds

4.1 Compactness.

Convergence = pre-compactness + uniqueness of a limit point. This triviality often helps to understand better a nature of a specific deep problem we deal with.

If we are after convergence of the sequence of renormalizations $R^n f$, we should first try to prove its pre-compactness or, at least, boundedness in some metric. Both approaches turn out to be fruitful, and both amount to the same analytical issue, namely complex *a priori* bounds.

Let $\mathcal{PL}_d$ denote the space of complex unimodal maps of degree d up to conformal equivalence (normalized so that 0 is the critical point). This space can be supplied with a *Carathéodory topology.* Convergence of a sequence $f_n : U_n \to V_n$ to $f : U \to V$ in this topology means Carathéodory convergence of pointed domains $(U_n, 0)$ and ranges $(V_n, 0)$ to $(U, 0)$ and $(V, 0)$ respectively, and compact-open convergence of the corresponding maps (all after appropriate choice of representatives of conformal classes).

Let $\mathcal{PL}_d(\mu)$ denote the subspace of $\mathcal{PL}_d$ consisting of maps which have a fundamental annulus of modulus at least $\mu > 0$ (we will also express this by saying that $\operatorname{mod} f \geq \mu$). A normality argument yields the following fact (see [McM2]):

Lemma 4.1 *For any $\mu > 0$, the space $\mathcal{PL}_d(\mu)$ is compact.*

Thus compactness of the sequence of renormalized maps $R^n f$ amounts to complex *a priori bounds* for these maps: $\operatorname{mod}(R^n f) \geq \mu$, $n = 0, 1, \dots$, for some $\mu > 0$.

On the other hand, in §2.1 we introduced the Teichmüller metric on the hybrid classes $\mathcal{H}(f)$ of polynomial-like maps. Let $\mathcal{H}(f, \mu)$ denote the subspace of the hybrid class consisting of maps with $\operatorname{mod}(f) \geq \mu$. Since the dilatation of the straightening map depends only on the modulus of the fundamental annulus, we have the following fact:

Lemma 4.2 *The set $\mathcal{H}(f, \mu)$ has a bounded diameter in the Teichmüller metric.*

So both approaches lead us to the problem of *a priori* bounds.

4.2 Real bounds

Before passing to the complex plane let us analyze the situation on the real line. For an infinitely renormalizable real unimodal map f the postcritical set $\mathcal{O}$ is a Cantor set with the following structure:

$$\mathcal{O} = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{q_n - 1} I_k^n,$$

where $I_0^1 \supset I_0^2 \supset \dots \ni 0$, and for any n the intervals I_i^n are cyclically permuted by f. The *real* n-fold renormalization $R^n f$ is just $f^{q_n} | I_0^n$. (Like in the complex situation, there is some flexibility in the choice of intervals I_0^n). The ratios $p_n = q_{n-1}/q_n$ are called *the relative periods.* Every interval of level $n-1$ contains exactly p_n intervals of the next level n. The Feigenbaum-Coullet-Tresser situation discussed in the introduction corresponds to the doubling on all levels: $p_n = 2$, $n = 1, 2, \dots$.

One says that f has a *bounded combinatorics* if the relative periods are uniformly bounded. By the *gaps* of level n we mean the connected components of $I_k^{n-1} \setminus I_j^n$. One says that the Cantor set $\mathcal{O}$ has bounded geometry if for any I_k^{n-1}, all intervals and all gaps of level n belonging to I_k^{n-1} are commensurable (with a constant independent of level n and the interval I_k^{n-1}). Cantor subsets of $\mathbb{R}$ with bounded geometry have Hausdorff dimension strictly in between 0 and 1.

Theorem 4.3 (see [G, BL, S2]) *Let f be an infinitely renormalizable real unimodal map. Then:*

- *There is an absolute $\delta > 0$ and intervals $0 \in S^n \subset T^n$ such that $f^{q_n} : (S^n, \partial S^n) \to (T^n, \partial T^n)$ is a unimodal map, and $|T^n| \geq (1+\delta)|S^n|$;*
- *The real renormalizations $R^n f$ form a pre-compact family in C^1 topology;*
- *If f has bounded combinatorics then the post-critical set $\mathcal{O}$ has bounded geometry.*

4.3 Sullivan's bounds

Sullivan's idea is to complexify the real bounds using the following hyperbolic disks. Let $I \subset \mathbb{R}$ be an interval. Let us consider the complex plane with two slits, $D_0(I) = \mathbb{C} \setminus (\mathbb{R} \setminus I)$. It is conformally equivalent to the unit disk $\mathbb{D}$ and thus can be supplied with the hyperbolic metric. By symmetry, I is a hyperbolic geodesic in this metric. It is easy to check that the hyperbolic r-neighborhood of this geodesic is the union $D_\theta(I)$ of two symmetric segments of Euclidean disks which meet the real line at angle $\theta = \theta(r)$ (see Figure 2). As analytic maps contract the Poincaré metric, we have:

Schwarz lemma. *Let $\phi : D_0(J) \to D_0(J')$ be an analytic map which transforms the interval J into J'. Then for any $\theta \in (0, \pi)$, $\phi(D_\theta(J)) \subset D_\theta(J')$.*

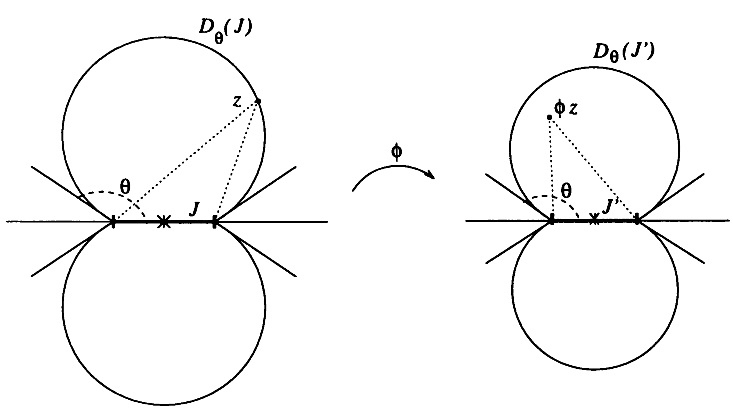

Figure 2. Poincarè disks.

Let us now have a renormalizable map $f \in \mathcal{PL}_d$ preserving the real line, with renormalization $Rf = f^p$. By Theorem 4.3, there is an orbit of intervals

$$J_0 \to J_1 \to \dots \to J_p \supset J_0 \tag{4.7}$$

with the following properties: $f : J_0 \to J_1$ is unimodal with $f(\partial J_0) \subset \partial J_1$ (a real version of a double covering), while all $f : J_k \to J_{k+1}$ are diffeomorphisms for $k = 1, \dots, p-1$. Moreover, $J_p \supset (1+\delta)J_0$ with an absolute $\delta > 0$.

Let us now take a Poincaré disk $V_p = D_\theta(I_p)$, and pull it back along the orbit (4.7): let V_k be the component of $f^{-1}V_{k+1}$ containing J_k. By the Schwarz lemma $V_k \subset D_\theta(I_k)$ for all $k = 1, 2 \dots, p$, but not necessarily for $k = 0$.

This little phenomenon is a source of big troubles. The way Sullivan settles it is the following: For the maps of bounded type, he first proves the so called Sector Lemma asserting that the pull-back of the *whole slit complex* plane $D_0(I_p)$ (think of it as the Poincare disk of infinite radius) is contained in the union of two symmetric ϵ-sectors based on I_0, with some $\epsilon > 0$ dependent only on the combinatorial bounds on f. It follows that for sufficiently small θ, the pull-back of $D_\theta(I_p)$ under f^p is contained well inside itself. This gives complex *a priori* bounds for infinitely renormalizable real maps of bounded type. As the Sector Lemma fails for unbounded combinatorics, this case requires a different treatment.

4.4 Divergence property

A totally different methods to estimate geometric moduli have been developed in the framework of the puzzle. The first results of this kind appeared in the works of Branner & Hubbard [BH] and Yoccoz (see [H, M2]) on dynamics of polynomials with one non-escaping critical point of quadratic type. We will state the results in the quadratic case. Recall that $Y^{(n)}(z)$ stands for the Yoccoz puzzle pieces of depth n containing z, while $Z_n(c)$ stands for the nest of parapuzzle pieces about c (see §2.3).

Theorem 4.4 *Let f be a non-renormalizable quadratic polynomial with all periodic points repelling. Then for any $z \in J(f)$,*

$$\sum \operatorname{mod}(Y^{(n)}(z) \setminus Y^{(n+1)}(z)) = \infty.$$

Hence $\operatorname{diam}(Y^{(n)}(z)) \to 0$ *as $n \to \infty$.*

The last conclusion follows from the Grötzcsh inequality.

Quantifying in ingenious way the ideas outlined in §3, Yoccoz has transferred the last result to the parameter plane:

Theorem 4.5 *Let $c \in M$ be a non-renormalizable parameter value. Then*

$$\sum \operatorname{mod}(Z_n(c) \setminus Z_{n+1}(c)) = \infty.$$

Hence the Mandelbrot set is locally connected at c and the quadratic P_c is combinatorially rigid.

These results are easily extended to at most finitely renormalizable quadratic polynomials (with an appropriate choice of the puzzle), but they are not enough for infinitely renormalizable maps. Also, even for non-renormalizable maps, many geometric issues need better bounds, which will be discussed next.

4.5 Growth of the principal moduli

Given a point $c \in \mathbb{C}$, let $d(c)$ stand for the distance from c to the union of the expanding domain of the Mandelbrot set bounded by the main cardioid and all expanding components attached to it. With the notations of §3.3 for the principal nest, we have the following estimate:

Theorem 4.6 ([L5, L6]) *Let P_c be a quadratic polynomial with $c \in M$ and $d(c) \geq \epsilon > 0$. Then*

$$\mathrm{mod}(V^{n(k)-1} \setminus V^{n(k)}) \geq C(\epsilon)k.$$

Remark. For real maps a related result was independently obtained by Graczyk & Swiatek [GS1].

Theorem 4.6 implies that infinitely renormalizable maps with sufficiently big height on all levels have big moduli:

Corollary 4.7 *Let P_c be an infinitely renormalizable quadratic polynomial. Let $c_n \in M$ label the hybrid class of the renormalization $R^n f$, while χ_n stand for its height. Assume that*

- $d(c_n) \geq \epsilon > 0,\ n = 0, 1, \dots ;$
- $\chi_n \geq \chi$.
  Then $\mathrm{mod}(R^n f) \geq \mu(\epsilon, \chi),\ n = 0, 1 \dots,$ *where for any given* $\epsilon > 0$, $\mu(\chi, \epsilon) \to \infty$ *as* $\chi \to \infty$.

Thus the renormalizations $R^n f$ are becoming purely quadratic as the height χ grows. Let us now state the parameter counterpart of these results (with notations of §3.3 for the principal parapuzzle nest):

Theorem 4.8 ([L9, W]) *Let M' be a maximal copy of the Mandelbrot set with $d(c) \geq \epsilon > 0$ for all $c \in M'$. Then for $c \in M'$,*

$$\mathrm{mod}(W^{n(k)-1}(c) \setminus W^{n(k)}(c)) \geq C(\epsilon)k.$$

Corollary 4.9 *Let the assumptions of Corollary 4.7 be satisfied throughout an infinitely renormalizable combinatorial class $Com(c) = \cap M_n$, where M_n is the corresponding nest of the little Mandelbrot copies. Then the M_n exponentially shrink. In particular $Com(c) = \{c\}$, so that P_c is combinatorially rigid.*

The way we derive these parameter results from the dynamical ones is different from the Yoccoz's method. It is based on the theory of holomorphic motions (see §5.2): the transversal quasi-conformal structure is the key which allows us to compare dynamical and parameter moduli.

4.6 Geometry of the quadratic Fibonacci puzzle.

Let us illustrate the above geometric results in the Fibonacci case. Let R_i^n denote the maximal annulus in $V^{n-1} \setminus (V_0^n \cup V_1^n)$ which goes around V_i^n but does not go around V_{1-i}^n (see Figure 1). Let us consider the following asymmetric combination:

$$\sigma_n = \operatorname{mod} V_0^n + \frac{1}{2} \operatorname{mod} V_1^n.$$

Pulling these annuli back according to the Fibonacci scheme, one can see that $\sigma_{n+1} \geq \sigma_n$. This yields the bounds $\operatorname{mod}(V^{n-1} \setminus V^n) \geq \epsilon > 0$, $n = 0, 1, \dots,$ but not yet the growth of the moduli.

To prove the growth one needs to analyze the positions of the V_i^n in V^{n-1}, and the shapes of these puzzle pieces. What, after all, makes the moduli grow is *pinching* of the puzzle pieces. Carring this renormalization analysis further, we can find exactly what is the asymptotic shape of the quadratic Fibonacci puzzle pieces: It is just the filled Julia set of $z \mapsto z^2 - 1$! (see Figure 3)

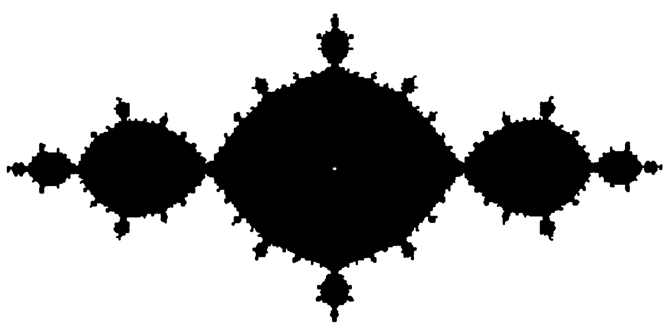

Figure 3. Degree two Fibonacci puzzle piece (made by B. Yarrington).

To explain this phenomenon, let us consider the triples of points $t_n = \{0, g_n 0, a_n\}$, where a_n is an appropriately chosen point of $\partial V^n \cap \mathbb{R}$. Then t_{n+1} is the pull-back of t_n by the map $g_n : V^n \to V^{n-1}$, which is exponentially close to a quadratic map (according to Theorem 4.6). This pull-back coincides (up to an exponentially small error) with the Thurston transformation τ in the Teichmüller space of thrice punctured planes (see [McM1] and the discussion in the next section). As τ is contracting, t_n converge to its fixed point, which corresponds to the superattracting period two cycle $0 \mapsto -1 \mapsto 0$ of $z \mapsto z^2 - 1$.

Let us finally mention the following consequence of the above discussion: the post-critical set $\mathcal{O}$ of the quadratic Fibonacci map is a Cantor set with exponentially decaying geometry [LM] (that is, the intervals of the next level are exponentially small as compared with the intervals of the previous level). In particular, this set has zero Hausdorff dimension. This is quite different from the bounded geometry of the Feigenbaum attractors (see §4.2). What is more surprising that this is also different from the geometry of the post-critical sets for higher degree Fibonacci maps. This curious phenomenon will be discussed in §4.8.

4.7 Complex bounds for real quadratics

Notice that among Poincaré disks introduced in §4.3 there is one especially nice, namely the Euclidean disk $D(I) \equiv D_{\pi/2}(I)$. What if to try to create a (generalized) polynomial-like map by pulling it back?

Let us have an orbit of intervals (4.7) (not-necessarily corresponding to the renormalization level). Take the Euclidean disk $D(J_p)$ and pull it back along this orbit. We will obtain a sequence of pull-backs $V_k \subset D(I_k)$, $k = p, p-1, \dots, 1$. To settle the trouble with the last square root pull-back indicated in §4.3, we need some control of the position of the critical value $f0$ in J_1. If this position is sufficiently "high" (that is, $|fJ_0|/|J_1|$ is sufficiently big), we are fine; otherwise $V_0 \not\subset D(I_0)$.

The appropriate estimate was made first for the Fibonacci maps [LM], and then extended to much wider range of combinatorial patterns. Let f be a renormalization $R^m P_c$ of a quadratic polynomial. Recall that $\chi(f)$ stands for the height of f (see §2.4).

Lemma 4.10 ([L4]) *There are absolute $\epsilon_* > 0$ and χ_* with the following property. If $\chi(f) \geq \chi_*$, then f admits a generalized renormalization $g_n : \cup U_i \to D(I)$ with a definite modulus:* $\mathrm{mod}(D(I) \setminus U_0) \geq \epsilon_* > 0$.

After a generalized polynomial-like map is created, according to Theorem 4.6 the moduli start to grow, so that we have:

Corollary 4.11 *If f is renormalizable in the sense of Douady and Hubbard then* $\mathrm{mod}(Rf) \geq \mu(\chi(f))$ *where $\mu(\chi) \to \infty$ as $\chi \to \infty$.*

Remark. The "height" in the above results can be replaced by a finer combinatorial parameter called "essential period" (see [LY] for the definition). If the essential period is bounded, the true period can be big only because some of the maps $g_n : V^n \to V^{n-1}$ are combinatorially close to $z \mapsto z^2 + 1/4$. This parameter is responsible for the dichotomy between "decaying" and "essentially bounded" geometry.

The combinatorial condition of Corollary 4.11 is a kind of complementary to Sullivan's bounded type. A gap between these two results has been recently filled in [LY], by an appropriate extension of Sullivan's sector lemma.

Theorem 4.12 *Any infinitely renormalizable real quadratic polynomial has complex a priori bounds.*

Different proofs of this result have been independently given by Graczyk & Swiatek [GS2] and Levin & van Strien [LS]. The latter work also treats the higher degree unimodal case $x \mapsto x^d + c$.

4.8 Higher degree Fibonacci maps

The geometry of higher degree Fibonacci maps F_d is quite different from the quadratic case, and more similar to the geometry of infinitely renormalizable maps of bounded type:

Lemma 4.13 *The post-critical set $\mathcal{O}$ for higher degree Fibonacci maps ($d > 2$) has bounded geometry. There are a priori bounds for generalized polynomial-like renormalizations $T^n f$.*

The latter property means that there is an $\epsilon > 0$, a choice of the domains of the generalized renormalizations $T^n F_d : V_0^n \cup V_1^n \to V^{n-1}$ (see §2.5) and annuli $R_i^n \subset V^{n-1} \setminus (V_0^n \cup V_1^n)$ defined in §4.6 such that $\mathrm{mod}(R_i^n) \geq \epsilon > 0$. (The former property shows that the $\mathrm{mod}(R_i^n)$ are also bounded from above). These *a priori* bounds for Fibonacci maps of higher degree are obtained by pulling back Euclidean disks in the same way as in the quadratic case [LM].

The reason why higher degrees differ degree 2 can be roughly seen in the following way. Let $I^n = V^n \cap \mathbb{R}$ stand for the real traces of the principal puzzle pieces. Let $\lambda_n = |I^n|/|I^{n-1}|$ be the corresponding *scaling factors.* Then one can write a recurrent relation between these scaling factors, which looks (up to bounded factors) like this

$$\lambda_{n+1}^d \approx \frac{1}{d}\lambda_n \lambda_{n-1}.$$

Pretending that this relation is precise, we see that its solutions decay to 0 for $d \leq 2$, and stay bounded away from 0 for $d > 2$.

Remark. This difference between Fibonacci maps of degree two and higher degrees was first pointed out in [LM]. Curiously there is a similar phenomenon for quite a different class of maps (circle maps with flat spot) which had been earlier studied by Tangerman & Veerman [TV].

5 Rigidity

In this section we will continue McMullen's discussion of the rigidity problem, see [McM1], §§2,5.

5.1 Deformation spaces

Action of a rational function on the Fatou set produces a Riemann surface $S(f)$ with affine foliation on some components. As described in [McM1], §2, there is a way to deform a rational function f by deforming its Riemann surface $S(f)$ (respecting the affine foliation). Namely, a conformal structure on $S(f)$ compatible with the affine structure on the leaves can be lifted to an f-invariant measurable structure μ on the Riemann sphere with bounded dilatation (on the Julia set μ coincides with the standard structure σ). By the Measurable Riemann Mapping Theorem (see the Appendix), there a quasi-conformal map $h_\mu : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $(h_\mu)_*(\mu) = \sigma$. Then $f_\mu = h_\mu \circ f \circ h_\mu^{-1}$ is a new rational function (defined up to conformal equivalence). Let $\mathcal{D}ef(f)$ stand for the space of functions (modulo conformal equivalence) which can be obtained in such a way. This space is parametrized by the Teichmüller space of $S(f)$.

Let $\mathcal{M}$ be an analytic family of rational functions modulo conformal equivalence (examples to keep in mind: a family of rational functions of degree d, a family of polynomials of degree d, a family of complex unimodal maps $z \mapsto z^d + c$ modulo the cyclic group Γ_{d-1}) . Let

$$\mathcal{C}om_{\mathcal{M}}(f) \supset \mathcal{T}op_{\mathcal{M}}(f) \supset \mathcal{QC}_{\mathcal{M}}(f)$$

denote respectively the combinatorial, topological and quasi-conformal classes of f in this family modulo conformal equivalence (actually combinatorial classes are so far defined for polynomials only). We skip the label $\mathcal{M}$ unless it may lead to confusion.

Deformation Conjecture. *For any f,*

- $\mathcal{C}om(f) = \mathcal{T}op(f)$ *whenever* $\mathcal{C}om(f)$ *is defined;*
- $\mathcal{T}op(f) = \mathcal{QC}(f)$*;*
- $\mathcal{QC}(f) = \mathcal{D}ef(f)$*, except for the Lattès examples.*

The last statement is equivalent to the absence of invariant line fields on the Julia set. By [MSS], this would imply density of expanding maps among rational maps. In the case when $\mathcal{D}ef(f) = \{f\}$ (that is, the Riemann surface $S(f)$ is rigid), the Deformation Conjecture turns into the *Rigidity Conjecture.* It can be refined as the *combinatorial/topological/quasi-conformal rigidity* conjecture which would assert that the corresponding class is a single map.

5.2 Structural stability

The maps belonging to open topological classes are called *structurally stable.* In early 80th the following advance towards the topological classification of rational maps was achieved (see [MSS, ST], and also [L1] for a part of this result):

Theorem 5.1 (Structural stability) *Structurally stable maps are dense in any holomorphic family. Connected components of structurally stable maps represent open quasi-conformal classes.*

The proof of this result based on the notion of a "holomorphic motion" is at least as important as the result itself. A holomorphic motion is a family $h_\lambda : X \to \mathbb{C}$ of injections of a set $X \subset \mathbb{C}$ holomorphically depending on λ (ranging withing some analytic manifold with a reference point λ_0) and such that $h_{\lambda_0} = \mathrm{id}$. A great property of a holomorphic motion (called "the λ-lemma") is that it automatically admits an extension to a holomorphic motion $h_\lambda : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ of the whole Riemann sphere, and that the maps of this motion are automatically quasi-conformal.

To prove Theorem 5.1 one constructs a holomorphic motion conjugating a function f_0 and a nearby function f. One can begin the construction of the motion with repelling periodic points, then extend it by the λ-lemma to the Julia set, then go to little neighborhoods of attracting cycles and spread the motion by dynamics onto the whole attracting basin, etc. There are obstructions for this construction, like parabolic cycles or coincidence of the grand orbits of two critical points, but one can show that they don't occur on a dense set of maps.

5.3 Unimodal families

Let us refine the above discussion in the unimodal case $P_c : z \mapsto z^d + c$. First note that in this case the set of expanding components (also called "hyperbolic") is the complement $\mathbb{C} \setminus M_d$ of the connectedness locus union the *expanding* components of $\operatorname{int} M_d$. For c belonging to an expanding component H, the polynomial P_c has an attracting cycle α_c. The *center* of the expanding component is the parameter value c_H for which the attracting cycle becomes superattracting (by a theorem of Douady and Hubbard [DH1], the center is indeed unique in any expanding component). Non-expanding components of $\operatorname{int} M_d$ are called *queer*. The maps in queer components must have invariant line fields on the Julia set. The set of structurally stable maps is the union of $\mathbb{C} \setminus M_d$, punctured expanding components $H \setminus \{c_H\}$ and queer components.

Now, we have the following quasi-conformal classification of the unimodal families

Theorem 5.2 (Quasi-conformal classification) *Quasi-conformal classes of the complex unimodal maps $z \mapsto z^d + c$ (modulo the cyclic group Γ_{d-1}) are the following:*

- $(\mathbb{C} \setminus M_d)/\Gamma_{d-1}$*;*
- *punctured hyperbolic components $H \setminus \{c_H\}$ of* $\operatorname{int} M_d/\Gamma_{d-1}$*;*
- *queer components;*
- *single points.*

Thus all maps P_c with $c \in \partial M_d$ are quasi-conformally rigid. To complete the proof of the Quasi-conformal Rigidity Conjecture, one should prove that there are no queer components.

Proof of Theorem 5.2. Note that the Measurable Riemann Mapping Theorem implies that quasi-conformal classes are always connected, and in the unimodal family they are either open, or single points. Indeed, let f and $\tilde{f}$ be two quasi-conformally equivalent maps, and h be the corresponding conjugacy. Then h induces an f-invariant conformal structure $h^*\sigma$ represented by the Beltrami differential $\mu = \bar{\partial}hd\bar{z}/\partial hdz$ with $\|\mu\|_\infty < 1$. But then we actually have a complex one-parameter family of f-invariant conformal structures corresponding to Beltrami differentials $\lambda\mu$ with $|\lambda| < 1/\|\mu\|_\infty$. By the Measurable Riemann Mapping Theorem, there is a family of quasi-conformal maps h_λ such that $(h_\lambda)_*(\lambda\mu) = \sigma$. The family of maps $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$ is an analytic disk in the quasi-conformal class $\mathcal{QC}(f)$ containing $\tilde{f}$. Moreover, in the unimodal case, this disk gives a neighborhood of f contained in $\mathcal{QC}(f)$. □

The following remark shows that in the unimodal families the first two parts of the Deformation Conjecture yield the last one:

Lemma 5.3 *Assume that for some $c \in M_d$, $Com(c) = \mathcal{QC}(c)$. Then c is combinatorially rigid.*

Proof. Indeed the combinatorial classes are closed as the intersections of parapuzzle pieces, while the quasi-conformal classes are either open, or single points. Thus if two classes coincide, they must be a single point.

5.4 Bounds and line fields

Queer components are always associated with the invariant line fields on the Julia set. Thus to complete quasi-conformal classification of the unimodal families (and to prove density of expanding components) we need to show that line fields don't exist: a nice relation between the rigidity problem and ergodic theory. In turn, the latter problem has been reduced by McMullen to the problem of ***a priori*** bounds:

Theorem 5.4 ([McM2]) *Let f be an infinitely renormalizable complex unimodal map with a priori bounds: $R^n f \geq \epsilon > 0$. Then there are no invariant line fields on the Julia set $J(f)$.*

The idea going back to the works of Mostow and Sullivan in Kleinian groups is to dynamically blow up the invariant line field near a density point where it is almost constant. This will show that the field is a.e. compatible with a real analytic one, which easily leads to a contradiction.

To carry out the "blow up" procedure, one needs to know that dynamics is fairly expanding. McMullen managed to exploit a quite modest amount of expansion following from *a priori* bounds: In the fundamental annuli of the $R^n f$, the map is uniformly expanding with respect to the hyperbolic metric in $\mathbb{C} \setminus \mathcal{O}$.

5.5 Pull-back argument for bounded geometry

Lemma 5.3, or Theorem 5.4 reduce the rigidity problem (under appropriate circumstances) to a construction of a quasi-conformal conjugacy between two combinatorially equivalent maps f and $\tilde{f}$. The main method to carry this out is called "the pull-back argument". It was originated (at least in the dynamical setting) in the work of Thurston on post-critically finite maps (see [Th, DH3] and discussion in [McM1],§5). The idea is to start with a quasi-conformal map of a right homotopy type which preserves some dynamical data, to lift it up by iterates of f and $\tilde{f}$, and to obtain a quasi-conformal (or even conformal) conjugacy in the limit. However you need some luck to carry this procedure out: the respected dynamical data you start with should allow you to go through an infinite lifting procedure. In the simplest cases this data is just dynamics on the post-critical set.

Assume, for instance, you wish to show that two topologically equivalent post-critically finite maps are quasi-conformally equivalent. Start with any K-quasi-conformal map h_0 which conjugates f and $\tilde{f}$ on their post-critical sets $\mathcal{O}$ and $\tilde{\mathcal{O}}$, and homotopic to a topological conjugacy rel $\mathcal{O}$. Then h_0 can be lifted to a map h_1 homotopic to h_0 rel Ω. Moreover this map is also K-quasi-conformal since f and $\tilde{f}$ are analytic. Hence you can lift it again, etc.

By interpreting this procedure as iterates of a contracting transformation in the Teichmüller space of punctured spheres, Thurston proved that h_i converge, unless f is a Lattès example. The limit map is a quasi-conformal conjugacy between f and $\tilde{f}$.

For more complicated combinatorics, a problem arises at the very beginning of the procedure: Why is there a quasi-conformal map which conjugates f and $\tilde{f}$ on their post-critical sets? Such a fact depends on the geometry of the post-critical set, which thus becomes crucial for the rigidity problem. Real bounds of §4.2 allow one to handle the problem in the real infinitely renormalizable case of bounded type. Indeed, by Theorem 4.3 the post-critical sets have bounded geometry in this case. Then their complements can be constructed by gluing standard pairs of pants (a standard pair of pants is a round disk with several round disks removed) with bounded geometry. Then there is a K-quasi-conformal map between the respective pairs of pants which is affine on the boundary circles and orientation preserving on the real line (with a uniform K). By the Gluing Lemma from the Appendix, the complements of the post-critical sets are K-quasi-conformally equivalent, with the same K. Applying the Gluing Lemma again (remember that our Cantor sets lie on the real line), we obtain the desired quasi-conformal map h_0 to start with.

Now, the pull-back argument allows us to turn h_0 into a quasi-conformal conjugacy. Indeed, let us extend h_0 to a quasi-conformal map on the whole complex plane in such a way that it conjugates f to $\tilde{f}$ outside some equipotentials E and $\tilde{E}$ (remember that both maps are conformally equivalent to $z \mapsto z^d$ outside the Julia set). This map can be lifted to a map h_1 homotopic to h_0 rel the post-critical sets. Moreover, h_1 is K-quasi-conformal with the same dilatation K as h_0. Similarly h_1 can be lifted to a K-quasi-conformal map h_2, etc. (as in the post-critically finite case). By the Compactness Lemma from the Appendix, we can select a subsequence $h_{n(i)}$ uniformly converging to a K-quasi-conformal map h. Outside the Julia sets this map conformally conjugates f and $\tilde{f}$. As the Julia set of a polynomial with all periodic points repelling is nowhere dense, h conjugates f to $\tilde{f}$ on the whole plane, and the construction is completed.

Let us summarize the above discussion in the following rigidity result:

Proposition 5.5 (see [MvS, S2]) *For any infinitely renormalizable bounded combinatorial type τ, there is at most one real quadratic polynomial of type τ.*

5.6 Pull-back argument for decaying geometry

To fix the idea, let us consider a quadratic Fibonacci map $f \equiv F_2$ (we pretend that we yet don't know that such a map is unique). For this map we have a sequence of generalized renormalizations $g_n : V_0^n \cup V_1^n \to V_0^{n-1}$ with linearly increasing moduli $\mathrm{mod}(V^{n-1} \setminus V^n)$ (see Theorem 4.6). So the pairs of pants $V_0^{n-1} \setminus (V_0^n \cup V_1^n)$ don't have bounded geometry. However we will check that the corresponding pairs of pants stay bounded "Teichmüller distance away", that is, they are K-quasi-conformal equivalent with a uniform K.

We will mark the objects corresponding to $\tilde{f}$ with tilde. Note that all puzzle pieces come together with the boundary parametrization, induced e.g., by the Böttcher coordinate in the complement of the Julia set. Let us have a K-quasi-conformal map

$$h_n : (V^{n-1}, V_0^n, V_1^n) \to (\tilde{V}^{n-1}, \tilde{V}_0^n, \tilde{V}_1^n),$$

respecting the boundary parametrization of the pieces. We would like to lift this map to a quasi-conformal map

$$h_{n+1} : (V^n, V_0^{n+1}, V_1^{n+1}) \to (\tilde{V}^n, \tilde{V}_0^{n+1}, \tilde{V}_1^{n+1})$$

with the same property. What causes a problem is that h_n does not carry the critical values $v_n = g_n(0)$ to $\tilde{v}_n = \tilde{g}_n(0)$. However, as $\mathrm{mod}(V_0^{n-1} \setminus V_1^n)$ is linearly big, $h_n(v_n)$ is exponentially close to $\tilde{v}_n$ in the hyperbolic metric of $\tilde{V}^{n-1}$.

By lifting h_n to the non-central puzzle pieces $V_1^n \to \tilde{V}_1^n$ via the univalent maps $g_n : V_1^n \to V^{n-1}$ and $g_n : \tilde{V}_1^n \to \tilde{V}^{n-1}$, we obtain a K-quasi-conformal map $\hat{h}_n : V^{n-1} \to \tilde{V}^{n-1}$ matching with h_n on $V^{n-1} \setminus V_1^n$, with even better property: $\hat{h}_n(v_n)$ is exponentially close to $\tilde{v}_n$ in the hyperbolic metric of $\tilde{V}_1^n$.

Now we can replace $\hat{h}_n$ by another map H_n matching with it on $V^{n-1} \setminus V_1^n$, respecting the critical values and having dilatation K(1+exp small term). This map can be already lifted to V^n. It needs not yet respect boundary parametrization of V_i^{n+1} but one more repetition of the pull-back procedure will do the job.

Repeating this procedure we will construct a quasi-conformal equivalence between the pairs of pants of all levels with uniformly bounded dilatation (as the dilatation increases by exponentially small amount on every step, it stays bounded). Spreading it around the post-critical set, we conclude that the post-critical sets of two Fibonacci quadratics are quasi-conformally equivalent in the right homotopy class (respecting dynamics on the sets). Now the pull-back argument described in §5.5 turns this quasi-conformal map to a quasi-conformal conjugacy on the whole plane.

This argument can be carried out for all non-renormalizable quadratics which gives a different proof of Theorem 4.5. In the infinitely renormalizable case one needs complex *a priori* bounds in order to start this argument from scratch on every renormalization level. This leads to the following Rigidity Theorem (compare Corollary 4.9).

Theorem 5.6 ([L5, L7]) *Let P_c be an infinitely renormalizable quadratic polynomial. Let $c_n \in M$ label the hybrid classes of its renormalizations $R^n f$. Let us consider the following two properties.*

- *$d(c_n) \geq \epsilon > 0$, $n = 0, 1, \dots$ (where $d(c)$ is defined in §4.5);*
- *P_c has a priori bounds: $\operatorname{mod}(R^n P_c) \geq \epsilon$, $n = 0, 1, \dots$*
  Every combinatorial class contains at most one quadratic polynomial with these two properties.

5.7 Rigidity of real maps

Theorem 5.4 and Theorem 4.12 yield:

Theorem 5.7 ([McM2]) *Non-expanding real quadratic polynomials are quasi-conformally rigid.*

(Actually McMullen derived this result from the real bounds, as the complex bounds were not available at that time).

Theorem 4.12 and Theorem 5.6 yield a stronger conclusion:

Theorem 5.8 *Combinatorial classes of non-expanding real quadratic polynomial intersect the real line in single points.*

Corollary 5.9 *Expanding real quadratics are dense in the family $z^2 + c$, $c \in \mathbb{R}$.*

The last two results were first announced by Swiatek [Sw] who approached them from the point of view of real dynamics. The above proof follows [L5, L7].

6 Universality

Let us now consider a combinatorial class $\mathcal{C}^{\mathbb{R}} \equiv Com^{\mathbb{R}}(M')$ of infinitely renormalizable real unimodal maps f which admit polynomial-like extensions to the complex plane and have a stationary combinatorial type $\tau(f) = (M', M', \dots)$. The Universality Law asserts that in this class there is a unique R-invariant map f_*, and the renormalizations of all other maps $f \in \mathcal{C}^{\mathbb{R}}$ converge to f_*. Moreover, the renormalization operator is hyperbolic at f_*, with one dimensional unstable manifold. We will sketch two approaches to the construction of the fixed point f_* and the stable manifold (due to Sullivan and McMullen), and then the author's approach to the unstable direction.

6.1 Sullivan's Contraction Lemma

By Proposition 5.5, all maps of $\mathcal{C}^{\mathbb{R}}$ are hybrid equivalent, $\mathcal{C}^{\mathbb{R}} \subset \mathcal{H}(f)$, so that this space can be supplied with the Teichmüller metric (see §2.1). If two maps f and $\tilde{f}$ are hybrid conjugate by a quasi-conformal map h then their renormalizations Rf and $R\tilde{f}$ are conjugate by a restriction of this map. It follows that the renormalization transformation R is contracting with respect to the Teichmüller metric.

This is not, though, enough to conclude that R has a globally attracting fixed point: To this end one needs a definite contraction. A result of this kind proved by Sullivan [MvS, S2, S3] is the following:

Lemma 6.1 (Contraction) *There exists a $\lambda \in (0,1)$ with the following property. For any two maps $f, g \in \mathcal{C}^{\mathbb{R}}$ there is an n such that*

$$\mathrm{dist}_T(R^n f, R^n g) \le \lambda\, \mathrm{dist}_T(f,g).$$

It follows that there is at most one limit map f_* for any orbit $\{R^m f\}$, which thus must be R-invariant. Moreover, this point is independent of f. On the other hand, due to *a priori bounds* and Compactness Lemma 4.1, any orbit has at least one accumulation point. It follows that there is a unique fixed point $f_* \in \mathcal{C}^{\mathbb{R}}$ which attracts all $f \in \mathcal{C}^{\mathbb{R}}$.

Sullivan's proof of the above Contraction Lemma uses the full scale machinery of the Teichmüller theory extended to objects called "Riemann surface laminations" [MvS, S2, S3].

6.2 McMullen's towers

Let f be an infinitely renormalizable unimodal map with complex *a priori bounds*. By Lemma 4.1, the orbit $\{R^n f\}$ is Carathéodory compact. Let Λ be the set of limit points of this orbit. Then the restriction $f|\Lambda$ is invertible, so that for every $g \in \Lambda$, there is a two sided R-orbit $\hat{g}$

$$\ldots \mapsto g_{-1} \mapsto g \mapsto g_1 \mapsto \ldots \tag{6.8}$$

With appropriate normalization, such an orbit can be realized as a kind of multi-valued conformal dynamical system called "tower": given a point $z \in \mathbb{C}$, you can apply to it infinitely many maps of (6.8) and all their admissible compositions. The Julia set $J(\hat{g})$ of the tower is defined as $\mathrm{cl}(\bigcup J(g_n))$. With these concepts in hands, McMullen globalized his Rigidity Theorem 5.4:

Theorem 6.2 (Towers rigidity [McM3]) *Let f be an infinitely renormalizable map with a priori bounds, and $\hat{g}$ an associated tower. Then $J(\hat{g}) = \mathbb{C}$, and there are no invariant line fields on $J(\hat{g})$.*

So, ***a priori*** bounds imply pre-compactness of the orbit $\{R^n f\}$, combinatorial rigidity of the straightened maps, and rigidity of limit towers. Altogether these yield convergence: $R^n f \to f_*$. Indeed, let $g, g' \in \Lambda$ be two limit maps, and $\hat{g}, \hat{g}'$ be the corresponding towers. By the combinatorial rigidity (Proposition 5.5), $\hat{g}$ and $\hat{g}'$ are quasi-conformally equivalent. Hence by Theorem 6.2, they are conformally equivalent. Thus g and g' represent the same point in the hybrid class $\mathcal{H}(f)$, so that $\Lambda = \{g\}$, and convergence of $\{R^n f\}$ follows.

Carrying further these ideas McMullen has proven exponential convergence:

Theorem 6.3 (Exponential convergence [McM3]) *For any $f \in \mathcal{H}(f_*)$, the orbit $R^n f$ converges to the fixed point f_* exponentially fast in the Carathéodory topology.*

Thus the hybrid class $\mathcal{H}(f)$ is contained in the strong stable manifold $W^s(f_*)$ of the renormalization fixed point.

The last theorem has been independently announced by Jeremy Kahn.

6.3 Unstable direction

To complete the renormalization picture, we need to analyze the unstable direction of the renormalization operator (see Figure 4). Our analysis is based upon the Rigidity Theorem 5.6 and the following general lemma:

Lemma 6.4 (Small orbits) *Let B be a complex Banach space, $R : (U, 0) \to (U', 0)$ be an analytic map in a neighborhood of 0, $L = DR(0)$, and U' is compact. Let* $\mathrm{spec}(L) = \mathrm{spec}^s(L) \cup \mathrm{spec}^n(L)$ *where the* $\mathrm{spec}^s(L)$ *is contained in a disk of radius $r < 1$, while* $\mathrm{spec}^n(L)$ *lies in the unit circle. Let W^s be the strong stable manifold of R (corresponding to* $\mathrm{spec}^s(L)$*). Then R has small orbits outside W^s, that is, for any neighborhood $V \ni 0$, there is a point $f \notin W^s$ whose forward orbit $\{R^m f\}_{m=0}^{\infty}$ is contained in V.*

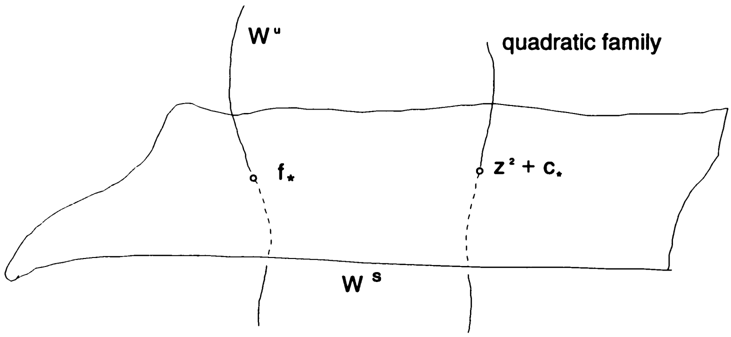

Figure 4. Renormalization picture.

Theorem 6.5 (Hyperbolicity [L8]) *The renormalization operator R is hyperbolic at the fixed point f_* with one dimensional unstable manifold.*

Proof. Clearly $W^s(f_*)$ cannot be open in the space of quadratic-like maps, since it intersects the quadratic family $z \mapsto z^2 + c$ at a single point on the real line (by Proposition 5.5).

If $\operatorname{codim} W^s(f_*) > 1$ then there would be a codimension 2 complex analytic submanifold X (in the space of quadratic-like maps) transversal to $W^s(f_*)$. Then the straightening map χ on $X \setminus \{f_*\}$ would omit the parameter value c_* representing the hybrid class of f_*. But one can see (using an index argument) that every nearby complex one dimensional submanifold $Y \subset X$ contains maps f with $\chi(f) = c_*$.

To prove hyperbolicity of R at f_*, let us assume by contradiction that $\operatorname{spec}(DR(f_*))$ is contained in the closed unit disk. Then by the Small Orbits Lemma, there is a quadratic-like map $f \notin W^s(f_*)$ such that the whole orbit $R^n f$ stays near f_*. But this means that f is an infinitely renormalizable map with *a priori* bounds. By Theorem 5.6, $f \in \mathcal{H}(f_*) \subset W^s(f_*)$, which is a contradiction. □

Remarks. 1. Note that even for real combinatorics the proof of Theorem 6.5 is complex is nature as Lemma 6.4 fails over reals. Moreover, the proof works equally well for any complex situation, provided R has a fixed point f_*.

2. For bounded type combinatorics, the renormalization operator has a hyperbolic invariant set instead of a single point.

6.4 Milnor's Hairiness Conjecture

In [M1] Milnor conjectured that the Mandelbrot set is becoming dense in small scales near the Feigenbaum-Coullet-Tresser point c_*. In other words, the rescalings of M by $T_\lambda : c \mapsto \lambda(c - c_*) + c_*$ tend (in the Hausdorff metric) to the whole complex plane $\mathbb{C}$ as $\lambda \to \infty$.

The dynamical counterpart of this conjecture asserts that the rescalings of the corresponding Julia set $J(P_{c_*})$ about 0 are becoming dense. This follows from the first part of McMullen's Theorem 6.2.

Now we can transfer this result to the parameter plane in the following way. The unstable manifold $W^u \equiv W^u(f_*)$ is a complex one-parameter family of quadratic-like maps, so that we can consider the Mandelbrot set $M^u \subset W^u$. For any $f \in W^u$, the inverse orbit $f_{-n} = R^{-n} f \to f_*$ can be viewed as a one-sided tower $\hat{f}$ with *a priori* bounds. If $\bigcup R^n M^u$ were not dense in W^u then the tower $\hat{f}$ would admit a holomorphic motion over the region $W^u \setminus \bigcup R^n M^u$. But this would contradict to the quasi-conformal rigidity of towers (an extended version of Theorem 6.2).

Thus $\bigcup R^n M^u$ is dense in W^u, so that M^u is hairy. Now one can transfer hairiness of M^u into the hairiness of the genuine Mandelbrot set M by means of holonomy from W^u to the quadratic family along the hybrid classes. This holonomy is transversally quasi-conformal since it can be viewed as a holomorphic motion.

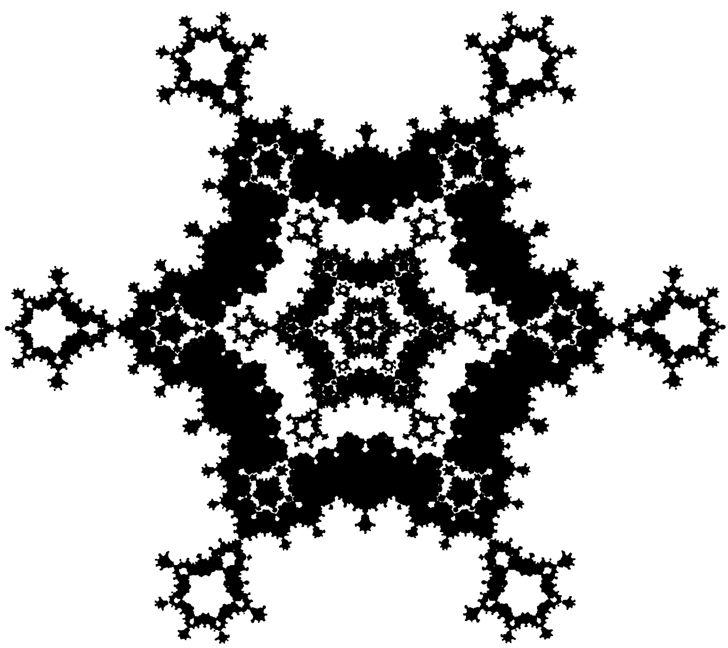

Figure 5. Degree six Fibonacci puzzle piece (made by Scott Sutherland).

6.5 Universality law for higher degree Fibonacci maps

Generalized renormalization and *a priori* bounds of Lemma 4.13 allow us to carry the above discussion for the Fibonacci maps of even degree $d > 2$. For such a d, let $\mathcal{F}_d$ stand for the combinatorial class of polynomial-like Fibonacci maps f preserving the real line.

Theorem 6.6 *For any Fibonacci map $f \in \mathcal{F}_d$ of even degree $d > 2$, the generalized renormalizations $T^n f$ converge to a cycle $\{f_1, f_2\}$ of order two depending only on d.*

The combinatorial difference between f_1 and f_2 is that the restrictions of these maps on the corresponding non-critical puzzle pieces V_1^n have opposite orientation on the real line. The picture of the principal nest for degree 6 Fibonacci map below in Figure 5 shows that all puzzle pieces have approximately the same shape.

As in the quadratic case, these puzzle pieces have asymptotical shapes of the Julia set of an appropriate polynomial-like map, which explains all the pinchings visible at the picture. Unlike the quadratic case though, this polynomial-like map is not a genuine polynomial.

7 Measure of Julia sets

In this last section we will refine a bit McMullen's discussion of the measure problem (see Theorems 5.8 and 6.5 in [McM1]). It is intimately related to renormalization and rigidity.

7.1 Measure-theoretic attractors

The global dynamics of a rational map depends first of all on the structure of the post-critical set. One of the results of such a kind is that the post-critical set is a "measure-theoretic attractor" for the dynamics on the Julia set:

Lemma 7.1 ([L2]) *For Lebesgue almost all $z \in J(f)$, either* $\mathrm{orb}(z)$ *is dense in $\mathbb{C}$, or $\omega(z) \subset \mathcal{O}$.*

In the second case, conjecturally there are at most finitely many measure-theoretic attractors A_i such that the orbit of almost every z converge to one of these attractors: $\omega(z) = A_i$. The basins of these attractors are going to be ergodic components $f|J(f)$. Such results also depend on the geometric bounds, and have been resolved in the cases when the geometry is under a good control.

7.2 Quadratic case

The divergence property of Branner and Hubbard yields that the Julia set of the corresponding cubic polynomial is a Cantor set of zero measure. (The latter consequence was pointed out by McMullen, see [BH]). The passage from this result to the measure zero result for quadratic maps (Lyubich-Shishikura [L2]) can be made by means of the generalized renormalization

Tf. Indeed, by McMullen's argument, the Julia set of Tf has zero measure. Then by Lemma 7.1 the big Julia set $J(f)$ must have zero measure as well.

7.3 Higher degree case

In particular, the Julia set of the quadratic Fibonacci map F_2 has zero measure. What about higher degree Fibonacci maps $f = F_d$? There is an approach to this problem based on the renormalization theory and a random walk construction. Given the principal nest $V^0 \supset V^1 \supset \dots$, let us think of the annuli $A^n = V^{n-1} \setminus V^n$ as the states of the random walk with the transitions induced by $g_n|A^n$. Then drift of the random walk orbits to the higher levels means that the f-orbits converge with positive probability to the post-critical set: In this case the Julia set $J(f)$ has positive measure. The computer experiment carried out by the author jointly with Scott Sutherland suggested that this is indeed the case for $d = 32$.

G. Keller stated a rigorous lemma about random walks which is applicable to such kind of situations, while T.Nowicki & S. van Strien [SN] gave necessary geometric estimates to show that for sufficiently big d, there is a drift to higher levels. Altogether this gives the first example of a polynomial with the Julia set of positive measure.

8 Appendix: Quasi-conformal maps

Quasi-conformal maps play an outstanding role in conformal dynamics (see §2 of [McM1]). They are sufficiently regular to be a subject of analysis, and, on the other hand, so irregular that produce fractal geometric objects (e.g., Jordan curves with Hausdorff dimension greater than 1). By definition, a homeomorphism $h : U \to V$, where $U, V \subset \mathbb{C}$, is called quasi-conformal if it has locally integrable distributional derivatives ∂h, $\bar\partial h$, and $|\bar\partial h/\partial h| \le k < 1$ almost everywhere. As this local definition is conformally invariant, one can define quasi-conformal homeomorphisms between Riemann surfaces.

One can associate to a quasi-conformal map an analytic object called Beltrami differential, namely

$$\mu = \frac{\bar\partial h}{\partial h}\frac{d\bar z}{dz},$$

with $\|\mu\|_\infty < 1$. The corresponding geometric object is a measurable family of infinitesimal ellipses (defined up to dilation), pull-backs by h_* of the field of infinitesimal circles. The eccentricities of these ellipses are ruled by $|\mu|$, and are uniformly bounded almost everywhere, while the orientation of the ellipses is ruled by the $\arg \mu$. The dilatation $K_h = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$ of h is the essential supremum of the eccentricities of these ellipses. A quasi-conformal map is called K-quasi-conformal if $K_h \le K$.

One of the most remarkable facts of analysis is that the above statements can be reversed: Any Beltrami differential with $\|\mu_\infty\| < 1$ (a measurable field of ellipses with essentially bounded eccentricities) is locally generated by a quasi-conformal map, unique up to post-composition with an analytic map. Thus such a Beltrami differential on a Riemann surface S induces a conformal structure quasi-conformally equivalent to the original structure of S. Together with the Riemann mapping theorem this leads to the following statement:

Measurable Riemann Mapping Theorem. *Let μ be a Beltrami differential on $\bar{\mathbb{C}}$ with $\|\mu_\infty\| < 1$. Then there is a quasi-conformal map $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ which solves the Beltrami equation: $\bar{\partial}h/\partial h = \mu$ a.e.*

In what follows by a conformal structure we will mean a structure associated to measurable Beltrami differentials μ with $\|\mu\|_\infty < 1$. We will denote by σ the standard structure corresponding to zero Beltrami differential.

Another fundamental property of the space of quasi-conformal maps is compactness:

Compactness Lemma. *The space of K-quasi-conformal maps $h : \mathbb{C} \to \mathbb{C}$ normalized by $h(0) = 0$ and $h(1) = 1$ is compact in the uniform topology on the Riemann sphere.*

The following gluing property is also important:

Gluing Lemma. *Let us have two disjoint domains D_1 and D_2 with a smooth piece γ of their common boundary. Let $D = D_1 \cup D_2 \cup \gamma$. If $h : D \to \mathbb{C}$ is a homeomorphism such that $h|D_i$ is K-quasi-conformal, then h is K-quasi-conformal.*

One of Sullivan's leading ideas was the idea of the Teichmüller metric on the space of deformations of a conformal dynamical systems. The prototype for this metric is the classical Teichmüller metric on the space of marked Riemann surfaces. The distance $\mathrm{dist}(S_1, S_2)$ between two marked Riemann surfaces is defined as the infimum of the dilatations K_h, where $h : S_1 \to S_2$ ranges over quasi-conformal homeomorphisms in the homotopy class respecting the marking.

A basic references on quasi-conformal maps is [A].

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