VOEVODSKY'S PROOF OF THE MILNOR CONJECTURE

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Introduction

Let F be a field. The Milnor's ring of F is defined as a factor-ring of the tensor algebra $T(F^*)$ of the multiplicative group of F modulo a homogenous ideal generated by tensors of the form $a \otimes (1-a)$ (with $a \in F^* \setminus 1$). Thus

$$K_*^M(F) = K_0^M(F) \oplus K_1^M(F) \oplus K_2^M(F) \oplus \dots$$

is a graded ring, whose n-th homogenous component $K_n^M(F)$ coincides with an abelian group generated by symbols $\{a_1, \dots, a_n\}$ $(a_i \in F^*)$ which are subject to two relations

(1) Multiplicativity in each variable
(2) $\{a_1, \dots, a_n\} = 0$ provided that $a_i + a_{i+1} = 1$ for some i.

Obviously $K_0^M(F) = \mathbb{Z}$, $K_1^M(F) = F^*$, furthermore the group $K_2^M(F)$ coincides with Quillen's $K_2(F)$ in view of the Matsumoto Theorem. One checks easily that $\{a, -a\} = 0$ for any $a \in F^*$ and furthermore $\{a, b\} + \{b, a\} = 0$ for any $a, b \in F^*$ - see [B-T]. The last relation shows that the ring $K_*^M(F)$ is (graded) commutative.

For any integer m prime to the characteristic of F Kummer Theory defines a natural isomorphism $\chi : F^*/(F^*)^m \xrightarrow{\sim} H^1(F, \mu_m)$. A well-known result (apparently due to John Tate) shows that $\chi(a) \cup \chi(1-a) = 0 \in H^2(F, \mu_m^{\otimes 2})$. Thus we get an induced homomorphism of graded rings

$$\chi : K_*^M(F)/m \to \coprod_{n=0}^{\infty} H^n(F, \mu_m^{\otimes n})$$

which is known as the norm residue Homomorphism. In degrees 0 and 1 the homomorphism χ is obviously an isomorphism. One of the most interesting and nontrivial conjectures in the Galois cohomology theory of fields states that the map

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χ is an isomorphism in all degrees (and for all m prime to *char* F). In this form this conjecture was apparently first formulated by Kazayo Kato [K], a similar (but slightly weaker) conjecture was proposed by S. Bloch [B-1]. We'll refer to the above conjecture as Bloch-Kato Conjecture. A special case of the Bloch-Kato Conjecture (for $m = 2$) was first considered by John Milnor [Mi-1] and so is often called the Milnor Conjecture. In a recent remarkable work [V 5] Vladimir Voevodsky proved the above conjecture of Milnor.

Theorem 1. [V 5] *For any field F of characteristic $\neq 2$ and any $n \geq 0$ the norm residue homomorphism*

$$K_n^M(F)/2K_n^M(F) \to H^n(F, \mathbb{Z}/2)$$

is an isomorphism.

The proof of this theorem uses essentially the motivic cohomology theory developed during the last years by V. Voevodsky in collaboration with A. Suslin and E. Friedlander - see [F-V], [S-V 1], [S-V 2], [S-V 3],[V-1], [V-2], [V-3], [V-4]. The proof also uses the Stable Homotopy Theory for schemes introduced by F. Morel and V. Voevodsky [M], [M-V]. Significant part of the latter theory is not published yet so that in a sence the proof is not quite complete.

Voevodsky works more generally with the Bloch-Kato conjecture for arbitraty m and reduces it to certain quite concrete questions concerning the universal splitting varieties. For $m = 2$ the corresponding question is known to have a positive answer due to the work of M. Rost [Ro 2]. Recently M. Rost proved that this question also has a positive answer when $m = 3$, $n = 3, 4$, thus proving that the norm residue homomorphism

$$K_n^M(F)/3K_n^M(F) \to H^n(F, \mu_3^{\otimes n})$$

is an isomorphism for $n = 3, 4$.

In preparing these notes I used heavily Bruno Kahn's report at Bourbaki seminar [K], which I found extremely helpful.

§1. Tne Bolch-Kato Conjecture.

In this section we make a few standard but useful general observations about the Bloch-Kato conjecture.

let F be a field of exponential characteristic p and let m be a positive integer prime to p. Consider the following statement

$BK_n(F, m)$. *The natural homomorphism*

$$\chi : K_n^M(F)/m \to H^n(F, \mu_m^{\otimes n})$$

is an isomorphism. In other words the Bloch-Kato conjecture modulo m holds for F in degree n.

The following remark is obvious from definitions.

Lemma 1.1. *Assume that $m = m_1 m_2$, where m_1 and m_2 are relatively prime. In this case the validity of $BK_n(F, m)$ is equivalent to the validity of $BK_n(F, m_1)$ and $BK_n(F, m_2)$. In particular the validity of $BK_n(F, m)$ for all m prime to p is equivalent to the validity of $BK_n(F, \ell^k)$ for all prime $\ell \neq p$ and all $k > 0$.*

¿From now on we fix a prime integer ℓ and consider only fields of characteristic $\neq \ell$. Using the transfer maps in Milnor K-theory and Galois cohomology one proves immediately the following result.

Lemma 1.2. *Let E/F be a finite field extension of degree prime to ℓ. Then the validity of $BK_n(E, \ell^k)$ implies the validity of $BK_n(F, \ell^k)$.*

The above Lemma allows us to consider only fields which have no extensions of degree prime to ℓ. In particular it suffices to consider only fields containing a primitive ℓ's root of unity. The following (well-known) fact is slightly less obvious.

Lemma 1.3. *Assume that F contains a primitive ℓ's root of unity ξ. Assume further that $BK_n(F, \ell)$ and $BK_{n-1}(F, \ell)$ hold. Then $BK_n(F, \ell^k)$ holds for any $k > 0$.*

Proof. We proceed by induction on k. The induction step is made using the diagram chase in the commutative diagram

$$\begin{array}{ccccccc}
K^M_{n-1}(F)/\ell \otimes \mu_\ell & \longrightarrow & K^M_n(F)/\ell^{k-1} & \longrightarrow & K^M_n(F)/\ell^k & \longrightarrow & K^M_n(F)/\ell \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \cong\downarrow \\
H^{n-1}(F, \mu_\ell^{\otimes n}) & \longrightarrow & H^n(F, \mu_{\ell^{k-1}}^{\otimes n}) & \longrightarrow & H^n(F, \mu_{\ell^k}^{\otimes n}) & \longrightarrow & H^n(F, \mu_\ell^{\otimes n})
\end{array}$$

Here the bottom row is a part of the long exact cohomology sequence, corresponding to the short exact sequence of Galois modules

$$0 \to \mu_{\ell^{k-1}}^{\otimes n} \to \mu_{\ell^k}^{\otimes n} \to \mu_\ell^{\otimes n} \to 0,$$

the left horizontal arrow in the top row is given by the composition

$$K^M_{n-1}(F) \otimes \mu_\ell \to K^M_{n-1}(F) \otimes F^* \xrightarrow{\text{mult}} K^M_n(F) \to K^M_n(F)/\ell^{k-1}$$

and the left vertical arrow coincides with the isomorphism

$$K^M_{n-1}(F)/\ell \otimes \mu_\ell \xrightarrow{\cong} H^{n-1}(F, \mu_\ell^{\otimes n-1}) \otimes \mu_\ell = H^{n-1}(F, \mu_\ell^{\otimes n}).$$

Lemmas 1.2 and 1.3 show that the validity of $BK_{\leq n}(F, \ell)$ for all fields F (of characteristic $\neq \ell$) implies the validity of $BK_{\leq n}(F, \ell^k)$ for all F and all $k > 0$.

Lemma 1.4. *Let F be a complete discretely valued field with the valuation ring $\mathcal{O}$ and the residue field $\overline{F}$. Assume that $\ell \neq char\overline{F}$ and $BK_n(F,\ell)$ holds. Then $BK_n(\overline{F},\ell)$ holds as well.*

Proof. Using the fact that $\mathcal{O}$ is complete and *char* $\overline{F} \neq \ell$ one checks easily that the natural homomorphisms

$$K_n^M(\mathcal{O})/\ell \to K_n^M(\overline{F})/\ell, \quad H^n(\mathcal{O},\mu_\ell^{\otimes n}) \to H^n(\overline{F},\mu_\ell^{\otimes n})$$

are isomorphisms. This allows us to construct a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n^M(\overline{F})/\ell & \longrightarrow & K_n^M(F)/\ell & \xrightarrow{\partial} & K_{n-1}^M(\overline{F})/\ell & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^n(\overline{F},\mu_\ell^{\otimes n}) & \longrightarrow & H^n(F,\mu_\ell^{\otimes n}) & \xrightarrow{\partial} & H^{n-1}(\overline{F},\mu_\ell^{\otimes n}) & \longrightarrow & 0 \end{array}$$

The choice of the local parameter $\pi \in \mathcal{O}$ gives compatible splittings for the above short exact sequences and hence

$$BK_n(F,\ell) \equiv BK_n(\overline{F},\ell) + BK_{n-1}(\overline{F},\ell).$$

Lemma 1.4 allows us to reduce the general case of the Bloch-Kato conjecture modulo ℓ to the case of fields of characteristic zero.

§2. Motivic Complexes

In this section we fix a field F and consider the category Sm/F of smooth schemes of finite type over F. We make Sm/F into a site using one of the following three topologies: Zariski topology, Nisnevich topology or etale topology. For any $X \in Sm/F$ we denote by $L(X)$ the presheaf on the category Sm/F, given by the formula

$$L(X)(Y) = \text{The free abelian group generated by closed integral subschemes } Z \subset X \times Y \text{ finite and surjective over a component of } Y$$

One checks easily that the pesheaf $L(X)$ is actually a sheaf in the etale topology (and a fortiori in Zariski and Nisnevich topologies as well).

Consider the standard cosimplicial object $\Delta^\bullet$ in Sm/F. For any presheaf of abelian groups $\mathcal{F}$ on Sm/F we get a simplicial presheaf $C_*(\mathcal{F})$, by setting $C_n(\mathcal{F})(U) = \mathcal{F}(U \times \Delta^n)$. We'll use the same notation $C_*(\mathcal{F})$ for the corresponding complex (of degree -1) of abelian presheaves. Usually we'll be dealing with complexes of degree $+1$, in particular, we'll reindex the complex $C_*(\mathcal{F})$ (in the standard way), by setting

$$C^i(\mathcal{F}) = C_{-i}(\mathcal{F}).$$

Recall that a presheaf $\mathcal{F} : Sm/F \to Ab$ is said to be homotopy invariant, provided that for any $U \in Sm/F$ the natural homomorphism $\mathcal{F}(U) \to \mathcal{F}(U \times \mathbb{A}^1)$ is an isomorphism. One checks easily (cf. [S-V 1] Corollary 7.5) that homology presheaves of the complex $C^*(\mathcal{F})$ are homotopy invariant.

Consider the presheaf $L((\mathbb{G}_m)^{\times n})$ (here $\mathbb{G}_m$ stands for the standard multiplicative group scheme $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$) and let $\mathcal{D}_n$ be the degenerate part of this presheaf, i.e. the sum of images of homomorphisms

$$L(\mathbb{G}_m^{\times n-1}) \to L(\mathbb{G}_m^{\times n})$$

given by the embeddings of the form

$$(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, 1, \dots, x_{n-1}).$$

One can verify easily that $\mathcal{D}_n$ is in fact a direct summand of $L(\mathbb{G}_m^{\times n})$. The corresponding projection $p : L(\mathbb{G}_m^{\times n}) \to \mathcal{D}_n$ is given by the formula

$$p = \sum_I (-1)^{card(I)+n-1} (p_I)_*,$$

where I runs through all proper subsets of $\{1, ..., n\}$ and $p_I : \mathbb{G}_m^{\times n} \to \mathbb{G}_m^{\times n}$ is the standard coordinate projection.

Definition 2.1. *The motivic complex $\mathbb{Z}(n)$ of weight n on Sm/F is the complex $C^*(L((\mathbb{G}_m)^{\times n})/\mathcal{D}_n)[-n]$. For a smooth scheme X over F we define its motivic cohomology groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ as hypercohomology*
$\mathbf{H}^i_{Zar}(X, \mathbb{Z}(n))$.

Note that $\mathbb{Z}(n)$ is a complex of sheaves with transfers in the Zariski topology (actually even in the etale topology) with homotopy invariant cohomology presheaves. The following lemma, which is a special case of results of [V 1], shows that motivic cohomology may be identified with Nisnevich hypercohomology:

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) = H^i_{Nis}(X, \mathbb{Z}(n)).$$

Lemma 2.2. *Let C^* be a complex (of degree +1) of Nisnevich sheaves with transfers with homotopy invariant cohomology presheaves.*

(1) *The cohomology sheaves (in the Zariski topology) $H^i(C^*)$ are homotopy invariant Nisnevich sheaves with transfers*
(2) *For any $X \in Sm/F$ $\quad H^*_{Zar}(X, C^*) = H^*_{Nis}(X, C^*)$.*

For any abelian group A we use the notation $A(n)$ for the tensor product complex $\mathbb{Z}(n) \otimes A$.

Proposition 2.3.

(1) *The complex $\mathbb{Z}(0)$ is naturally quasiisomorphic to $\mathbb{Z}$.*
(2) *The complex $\mathbb{Z}(1)$ is naturally quasiisomorphic to $\mathbb{G}_m[-1]$*

(3) *The complex $\mathbb{Z}(n)$ is acyclic in degrees $> n$.*
(4) *For any $n, m > 0$ there exist natural pairings (in the derived category of bounded above complexes of Nisnevich sheaves with transfers) $\mathbb{Z}(n) \otimes^L \mathbb{Z}(m) \to \mathbb{Z}(n+m)$ which are commutative and associative.*
(5) *For any m prime to char F the complex $\mathbb{Z}/m(n)$ being considered as a complex of sheaves on the etale site is naturally quasiisomorphic to $\mu_m^{\otimes n}$.*
(6) *The n-th cohomology presheaf of $\mathbb{Z}(n)$ coincides with the sheaf $\mathcal{K}_n^M$ of Milnor K-groups. In particular $H^n_{\mathcal{M}}(Spec\ F, \mathbb{Z}(n)) = K_n^M(F)$.*

Proof. The first and the third statements are obvious. The second follows easily from the computation of singular homology of relative curves - see [S-V 1]. Construction of the pairing $\mathbb{Z}(m) \otimes^L \mathbb{Z}(n) \to \mathbb{Z}(m+n)$ is given in [V 2]. The fifth statement follows immediately from results of [S-V 1]. The last result is proved (by an easy computation) in [S-V 3].

§3. Hilbert's Theorem 90

One of the main technical tools used in [M-S 1],[S 1] for the proof of the Bloch-Kato conjecture for K_2 was the following result, known as Hilbert's Theorem 90 for K_2.

Theorem 3.1 [M-S 1],[S 1]. *Let E/F be a cyclic Galois extension of prime degree ℓ. Let further σ denote a generator of $Gal(E/F)$. The following sequence is exact*

$$K_2^M(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N_{E/F}} K_2^M(F)$$

The same result was later proved (and used as the main technical tool in the proof of the Milnor conjecture in degree 3) for K_3^M and quadratic extensions in [M-S 2], [Ro 1]. Let's consider more generally the following statement.

$HT90_n(E/F)$. Let E/F be a cyclic Galois extension. Let further σ be a generator of the Galois group $Gal(E/F)$. Then the following sequence is exact

$$K_n^M(E) \xrightarrow{1-\sigma} K_n(E) \xrightarrow{N_{E/F}} K_n^M(F).$$

Hilbert's Theorem 90 for K_1 is due to Hilbert and is usually deduced from the following cohomological statement (also due to Hilbert): $H^1_{et}(F, \mathbb{G}_m) = 0$. Lichtenbaum proposed in [Li] the following cohomological version of Hilbert's Theorem 90 in degrees > 1.

$LiHT90_n(F, \ell)$. Let F be a field of characteristic $\neq \ell$, then $H^{n+1}_{et}(F, \mathbb{Z}_{(\ell)}(n)) = 0$.

The main goal of this section is to relate the above cohomological version of Hilbert's Theorem 90 with the Bloch-Kato Conjecture and with the mentioned above form of the Hilbert's Theorem 90 for K_n^M. We show also that $LiHT90$ holds for fields with divisible Milnor K-groups.

To simplify matters all fields in this section are assumed to be of characteristic zero.

We start with the following easy fact relating Zariski and etale cohomology with coefficients in $\mathbb{Q}(n)$.

Lemma 3.1 [V 1]. *Let $C^\bullet$ be a bounded above complex of etale sheaves of $\mathbb{Q}$-vector spaces with transfers. Assume that homology presheaves of $C^\bullet$ are homotopy invariant. Then for any $X \in Sm/F$ the natural homomorphisms*

$$H^*_{Zar}(X, C^\bullet) = H^*_{Nis}(X, C^\bullet) \to H^*_{et}(X, C^\bullet)$$

are isomorphisms. In particular $H^*_{Zar}(X, \mathbb{Q}(n)) \xrightarrow{\sim} H^*_{et}(X, \mathbb{Q}(n))$.

Denote by $\pi : (Sm/F)_{et} \to (Sm/F)_{Zar}$ the obvious morphism of sites.

Corollary 3.2. *The natural homomorphism of complexes of sheaves with transfers*

$$\mathbb{Q}(n) \to R\pi_*\mathbb{Q}(n)$$

is a quasiisomorphism.

The following theorem (which is the main result of [S-V 3]) is much more difficult.

Theorem 3.3.

(1) *Assume that for any field F of characteristic zero, any $i \leq n$ and any $k > 0$ the Bockstein homomorphism*

$$H^i(F, \mu_{\ell^k}^{\otimes i}) \to H^{i+1}(F, \mu_\ell^{\otimes i})$$

is trivial. Then $BK_{\leq n}(F, \ell)$ holds for any field F of characteristic zero.

(2) *Assume that $BK_{\leq n}(F, \ell)$ holds for any field of characteristic zero. Then the natural homomorphism of complexes of Zariski sheaves with transfer*

$$\mathbb{Z}/\ell^k(n) \to \tau_{\leq n} R\pi_*(\mathbb{Z}/\ell^k(n))$$

is a quasiisomorphism for all $k > 0$.

Proposition 3.4. *Assume that $LiHT90_{\leq n}(F, \ell)$ holds for all fields F of characteristic zero. Then $BK_{\leq n}(F, \ell)$ holds for any field of characteristic zero.*

Proof. Consider the following commutative diagram of complexes of etale sheaves with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_{(\ell)}(n) & \xrightarrow{\ell^k} & \mathbb{Z}_{(\ell)}(n) & \longrightarrow & \mathbb{Z}/\ell^k(n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & \mu_\ell^{\otimes n} & \longrightarrow & \mu_{\ell^{k+1}}^{\otimes n} & \longrightarrow & \mu_{\ell^k}^{\otimes n} & \longrightarrow & 0 \end{array}$$

Vanishing of $H^{n+1}_{et}(F, \mathbb{Z}_{(\ell)}(n))$ implies that the Bockstein homomorphism

$$H^n_{et}(F, \mu_{\ell^k}^{\otimes n}) \to H^{n+1}_{et}(F, \mu_\ell^{\otimes n})$$

corresponding to the bottom row is trivial. Thus it suffices to use Theorem 3.3.

Corollary 3.5. *Assume that $LiHT90_{\leq n}(F,\ell)$ holds for all fields F of characteristic zero. Then the natural homomorphisms of complexes of Zariski sheaves with transfer*

$$\mathbb{Z}_{(\ell)}(n) \to \tau_{\leq n} R\pi_*(\mathbb{Z}_{(\ell)}(n)) \to \tau_{\leq n+1} R\pi_*(\mathbb{Z}_{(\ell)}(n))$$

are quasiisomorphisms.

Proof. The cohomology sheaves of the complex $R\pi_*(\mathbb{Z}_{(\ell)}(n))$ are homotopy invariant Zariski sheaves with transfer. Since the $(n+1)$-st homology sheaf vanishes on fields it is equal to zero -see [V 1]. This shows that the second map is a quasi-isomorphism. The first map is a quasiisomorphism after inverting ℓ - according to Corollary 3.2 and after factoring out ℓ - according to Theorem 3.3 and Proposition 3.4 and hence is a quasiisomorphism itself.

The following result is essentially due to S. Lichtenbaum [Li].

Proposition 3.6. *Assume that $LiHT90_{\leq n}(F,\ell)$ holds for all fields F of characteristic zero. Then $HT90_{\leq n}(E/F)$ holds for any cyclic ℓ-primary extension of fields of characteristic zero.*

Proof. Set $G = Gal(E/F)$ and consider the following short exact sequence of Galois modules (=sheaves on the small etale site of *Spec* F)

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \xrightarrow{1-\sigma} \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

Taking Ext to $\mathbb{Z}_{(\ell)}(n)$ we get a spectral sequence converging to zero, with the following E_1-term:

$$H^*_{et}(F,\mathbb{Z}_{(\ell)}(n)) \xleftarrow{N_{E/F}} H^*_{et}(E,\mathbb{Z}_{(\ell)}(n)) \xleftarrow{1-\sigma} H^*_{et}(E,\mathbb{Z}_{(\ell)}(n)) \leftarrow H^*_{et}(F,\mathbb{Z}_{(\ell)}(n))$$

Corollary 3.5 shows that $H^n_{et}(F,\mathbb{Z}_{(\ell)}(n)) = K^M_n(F)\otimes\mathbb{Z}_{(\ell)}$. Since $H^{n+1}_{et}(F,\mathbb{Z}_{(\ell)}(n)) = 0$ we conclude that the sequence

$$K^M_n(E) \xrightarrow{1-\sigma} K^M_n(E) \xrightarrow{N_{E/F}} K^M_n(F)$$

becomes exact after tensoring with $\mathbb{Z}_{(\ell)}$. On the other hand it's trivial to see that $Ker\ N_{E/F}/Im\ (1-\sigma)$ is killed by $[E:F] = \ell^k$.

Lemma 3.7. *Let E/F be a cyclic extension of degree ℓ of fields of characteristic zero. Assume that F has no extensions of degree prime to ℓ. Assume further that the norm map $N_{E/F} : K^M_{n-1}(E) \to K^M_{n-1}(F)$ is surjective and $HT_{n-1}(E/F)$ holds. Then $HT90_n(E/F)$ holds as well.*

Proof. The proof is identical to the proof of the corresponding statement for K_2 given in [S 1].

Lemma 3.8 [V-5]. *Assume that $LiHT90_{\leq n-1}(F,\ell)$ holds for any field F of characteristic zero. Let further F be a field of characteristic zero which has no extensions of degree prime to ℓ and let E/F be a cyclic extension of degree ℓ. Then the following sequence (in which ϕ denotes the character corresponding to the extension E/F) is exact*

$$H^{n-1}(E,\mathbb{Z}/\ell) \xrightarrow{N_{E/F}} H^{n-1}(F,\mathbb{Z}/\ell) \xrightarrow{\cup\phi} H^n(F,\mathbb{Z}/\ell) \to H^n(E,\mathbb{Z}/\ell)$$

We skip the proof since in the most interesting case $\ell = 2$ this statement is true (and well-known) without any assumptions on F.

Corollary 3.9. *Assume that $LiHT90_{\leq n-1}(F,\ell)$ holds for any field F of characteristic zero. Let further F be a field of characteristic zero without extensions of degree prime to ℓ and such that $K_n^M(F) = \ell K_n^M(F)$. Then the formula $K_n^M(E) = \ell K_n^M(E)$ holds also for any finite extension E of F.*

Proof. This follows immediately from Lemma 3.7, using the same argument as in the case of K_2 - see [S 1].

Theorem 3.10. *Assume that $LiHT90_{\leq n-1}(F,\ell)$ holds for any field F of characteristic zero. Let further F be a field of characteristic zero without extensions of degree prime to ℓ and such that $K_n^M(F) = \ell K_n^M(F)$. Then $LiHT90_n(F,\ell)$ holds.*

Proof. The group $H_{et}^{n+1}(F,\mathbb{Z}_{(\ell)}(n))$ is ℓ-torsion in view of Lemma 3.1. The exact sequence

$$H_{et}^n(F,\mu_{ell}^{\otimes n}) \to H_{et}^{n+1}(F,\mathbb{Z}_{(\ell)}(n)) \xrightarrow{\ell} H_{et}^{n+1}(F,\mathbb{Z}_{(\ell)}(n))$$

shows that it would suffice to prove that $H_{et}^n(F,\mu_{ell}^{\otimes n}) = 0$.

§4. Splitting Varieties.

As we have seen in the priveous sections to prove the general case of the Bloch-Kato conjecture modulo ℓ it suffices to consider fields of characteristic zero only. Moreover it suffices to establish for such fields the cohomological version of the Hilbert's Theorem 90. Voevodsky proves that $LiHT90_n(F,\ell)$ holds by induction on n. We assume that $LiHT90_{\leq n-1}(F,\ell)$ holds for any field F of characteristic zero and try to prove that $LiHT90_n(F,\ell)$ holds for any such F as well. One important step is already done in Theorem 3.10. There we saw that $LiHT90_n(F,\ell)$ holds provided that F has no extensions of degree prime to ℓ and $K_n^M(F)/\ell = 0$. Now the general strategy is to show that for any field F of characteristic zero there exists a field extension F'/F such that

(1) F' has no extensions of degree prime to ℓ.
(2) $K_n^M(F')/\ell = 0$
(3) The natural map $H_{et}^{n+1}(F,\mathbb{Z}_{(\ell)}(n)) \to H_{et}^{n+1}(F',\mathbb{Z}_{(\ell)}(n))$ is injective.

The main step in the construction of such an extension is to construct an extension which splits a given symbol $\{a_1,...,a_n\} \in K_n^M(F)$ and to verify the injectivity of the associated map on $H_{et}^{n+1}(-,\mathbb{Z}_{(\ell)}(n))$.

Definition 4.1. *Let $\underline{a} = (a_1, ..., a_n)$ be a n-tuple of elements of F^*. We say that an extension K/F is the universal splitting field for the symbol $\{\underline{a}\} = \{a_1, ..., a_n\} \in K_n^M(F)$ (modulo ℓ) provided that*

1. *$\{a_1, ..., a_n\} \in \ell \cdot K_n^M(K)$.*
2. *If L/F is a field extension such that $\{a_1, ..., a_n\} \in \ell \cdot K_n^M(L)$ then there exists an F-point of K with values in L.*

We say that a smooth projective variety X/F is a universal splitting variety for the symbol $\{\underline{a}\} = \{a_1, ..., a_n\} \in K_n^M(F)$ provided that the field $F(X)$ is a universal splitting field for $\{a_1, ..., a_n\}$. In other words iff the field $F(X)$ splits the given symbol and moreover for any splitting field L/F there exists a point $x \in X$ and a field embedding (over F) $F(x) \hookrightarrow L$.

Example 4.2. *In case $\ell = 2$ the construction of universal splitting varieties is quite well-known. Consider the n-fold Pfister form $<< a_1, ..., a_n >>$ and let ϕ be a subform of $<< a_1, ..., a_n >>$ of dimension $> 2^{n-1}$ (what is called in the theory of quadratic forms a Pfister neighbour) then the corresponding quadric Q_ϕ is the universal splitting field of the symbol $\{a_1, ..., a_n\} \in K_n^M(F)$ modulo 2.*

The following statements explaine the further strategy.

Lemma 4.3.

1. *Assume that for any field F (of characteristic zero) and any n-symbol $\{a_1, ..., a_n\} \in K_n^M(F)$ there exists a splitting field L/F such that $H_{et}^{n+1}(F, \mathbb{Z}_{(\ell)}(n)) \to H_{et}^{n+1}(L, \mathbb{Z}_{(\ell)}(n))$ is injective. Then $LiHT90_n(F, \ell)$ holds for any F.*
2. *Assume that there exists a splitting field L/F for the symbol $\{a_1, ..., a_n\} \in K_n^M(F)$ such that $H_{et}^{n+1}(F, \mathbb{Z}_{(\ell)}(n)) \to H_{et}^{n+1}(L, \mathbb{Z}_{(\ell)}(n))$ is injective. Let further X/F be a universal splitting variety for $\{a_1, ..., a_n\}$, then the natural map $H_{et}^{n+1}(F, \mathbb{Z}_{(\ell)}(n)) \to H_{et}^{n+1}(F(X), \mathbb{Z}_{(\ell)}(n))$ is also injective.*

The main step in Voevodsky's proof of the Milnor Conjecture is the following result.

Theorem 4.4. *Let $\underline{a} = (a_1, ..., a_n)$ be a n-tuple of elements of F^*. Denote by $Q_{\underline{a}}$ the projective quadrique defined by the Pfister neighbour $<< a_1, ..., a_{n-1} >> \perp < -a_n >$. Then the natural map $H_{et}^{n+1}(F, \mathbb{Z}_{(\ell)}(n)) \to H_{et}^{n+1}(F(Q_{\underline{a}}), \mathbb{Z}_{(\ell)}(n))$ is injective.*

Injectivity of the map

$$H_{et}^{n+1}(F, \mathbb{Z}_{(\ell)}(n)) \to H_{et}^{n+1}(F(Q_{\underline{a}}), \mathbb{Z}_{(\ell)}(n))$$

is deduced from vanishing of an appropriate motivic cohomology group of a certain simplicial scheme. We procede to develop the necessary technique.

For any scheme of finite type X/F denote by $\check{C}(X)$ the simplicial scheme equal in degree n to X^{n+1} and face and degeneracy operators of which are given by partial projections and partial diagonal maps. Consider the following chain of morphisms of (simplicial) schemes

$$Spec\ F(X) \to X \to \check{C}(X) \to Spec\ F$$

Lemma 4.5.

(1) *Assume that X is a smooth irreducible scheme with a rational point. Then the natural homomorphisms*

$$H^*_{Zar}(F, \mathbb{Z}(n)) \to H^*_{Zar}(\check{C}(X), \mathbb{Z}(n))$$

are isomorphisms for all n.

(2) *For any smooth irreducible scheme X the natural homomorphisms*

$$H^*_{et}(F, \mathbb{Z}(n)) \to H^*_{et}(\check{C}(X), \mathbb{Z}(n))$$

are isomorphisms.

Denote by $Co(n)$ the cone of the canonical map $\mathbb{Z}_{(\ell)}(n) \to \tau_{\leq n+1} R\pi_*(\mathbb{Z}_{(\ell)}(n))$. The Lichtenbaum conjecture predicts that this complex has to be quasiisomorphic to zero. Assuming that $LiHT90_{\leq n-1}(\ell)$ holds we conclude easily from cohomological purity for motivic cohomology the following result

Lemma 4.6. *For any smooth irreducible scheme X/F the natural restriction map $H^*_{Zar}(X, Co(n)) \to H^*_{Zar}(F(X), Co(n))$ is an isomorphism*

We say, following Saltman [Sa] that an integral scheme X/F is a rational retract iff there is a non-empty open subset $U \subset X$ which is isomorphic to a retract in an open subscheme of the affine space $\mathbb{A}^N_F$ (for some $N > 0$).

Corollary 4.7. *Let $f : Y \to X$ be a dominant morphism of smooth irreducible schemes over F. Assume that the generic fiber of f is a rational retract. Then the induced map*

$$H^*_{Zar}(X, Co(n)) \to H^*_{Zar}(Y, Co(n))$$

is an isomorphism.

Proposition 4.8. *Let X/F be a smooth irreducible scheme of finite type. Assume further that $X_{F(X)}$ is a rational retract (over $F(X)$).*

(1) *The natural homomorphisms*

$$H^*_{Zar}(\check{C}(X), Co(n)) \to H^*_{Zar}(X, Co(n)) \to H^*_{Zar}(F(X), Co(n))$$

are isomorphisms.

(2) *We have an exact sequence*

$$H^{n+1}_{Zar}(\check{C}(X), \mathbb{Z}_{(\ell)}(n)) \to H^{n+1}_{et}(F, \mathbb{Z}_{(\ell)}(n)) \to H^{n+1}_{et}(F(X), \mathbb{Z}_{(\ell)}(n))$$

Since the quadric $Q_{\underline{a}}$ from Theorem 4.4 is rational over its function field we see from Proposition 4.8 that to prove Theorem 4.4 it suffices to establish the following result

Theorem 4.9. *The motivic cohomology group*

$$H^{n+1}_{Zar}(\check{C}(Q_{\underline{a}}), \mathbb{Z}_{(2)}(n))$$

is trivial.

Voevodsky deduces the latter theorem from the following two results (the first of which is of topological nature whereas the second one is more arithmetical).

Theorem 4.10. *Assume that $LiHT90_{\leq n-1}(\ell)$ holds for all fields of characteristic zero. Let further F be a subfield of $\mathbb{C}$ and let X/F be a smooth projective variety of dimension $d = \ell^{n-1} - 1$ such that $s_d(X(\mathbb{C})) \not\equiv 0 \mod \ell^2$, where s_d is the Chern number associated to the d-th Newton polynom. Then there is a natural injective map*

$$H^{n+1}_{Zar}(\check{C}(X), \mathbb{Z}_{(\ell)}(n)) \xrightarrow{\alpha} H^{2\frac{\ell^{n-1}-1}{\ell-1}+1}(\check{C}(X), \mathbb{Z}_{(\ell)}(\frac{\ell^{n-1}-1}{\ell-1}+1)).$$

We sketch the proof of this theorem in the next sections.

Theorem 4.11. *Assume that $LiHT90_{\leq n-1}(2)$ holds for all fields of characteristic zero. Let $Q_{\underline{a}}$ be the quadric of Theorem 4.4. Then $s_d(Q_{\underline{a}}(\mathbb{C})) \not\equiv 0 \mod 4$ and $H^{2^n-1}_{Zar}(\check{C}(Q_{\underline{a}}), \mathbb{Z}_{(2)}(2^{n-1})) = 0$.*

Computation of the Chern numbers of quadrics over $\mathbb{C}$ is well-known and easy. In particular we have $s_d(Q_{\underline{a}}(\mathbb{C})) = 2(2^{2^{n-1}-1} - 2^{n-1} - 1)$ - see [Mi-St]. The second statement of the Theorem is essentially equivalent to the Theorem of Rost [Ro 2], concerning injectivity of the norm map $A_0(X, K_1) \to F^*$ in case of Pfister quadrics. Computation of the group $A_0(X, K_1)$ for universal splitting varieties is the only part still missing in the proof of the Bloch-Kato conjecture for primes $\ell \neq 2$.

§5. Stable Homotopy Category of Schemes.

Denote by $Shv_{Nis}(Sm/F)$ the category of Nisnevich sheaves on Sm/F. Consider further the category $\Delta^{op}Shv_{Nis}(Sm/F)$ of simplicial sheaves. For any simplicial smooth scheme X_* we'll use the same notation X_* for the corresponding simplicial representable sheaf. Every scheme $X \in Sm/F$ may be considered as a constant simplicial scheme and hence also as an object of $\Delta^{op}Shv_{Nis}(Sm/F)$. A morphism of simplicial sheaves $\phi : \mathcal{X} \to \mathcal{Y}$ is said to be a weak equivalence provided that for any $u \in U \in Sm/F$ the associated map of simplicial sets $\mathcal{X}(U^h_u) \to \mathcal{Y}(U^h_u)$ is a weak equivalence. Denote by $\mathcal{H}_s(\Delta^{op}Shv_{Nis}(Sm/F))$ the localization of the category $\Delta^{op}Shv_{Nis}(Sm/F))$ with respect to the weak equivalences. An object $\mathcal{X} \in \Delta^{op}Shv_{Nis}(Sm/F))$ is said to be $\mathbb{A}^1$-local provided that for any $\mathcal{Y} \in \Delta^{op}Shv_{Nis}(Sm/F))$ the natural map $Hom_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X}) \to Hom_{\mathcal{H}_s}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$ is bijective . A morphism $f : \mathcal{Y} \to \mathcal{Y}'$ in $\Delta^{op}Shv_{Nis}(Sm/F))$ is said to be a weak $\mathbb{A}^1$-equivalence provided that for any $\mathbb{A}^1$-local $\mathcal{X}$ the corresponding map

$$Hom_{\mathcal{H}_s}(\mathcal{Y}', \mathcal{X}) \xrightarrow{f^*} Hom_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X})$$

is bijective. A morphism ϕ of simplicial sheaves is said to be a cofibration provided that it is injective. One checks easily (see [M-V]) that $\Delta^{op} Shv_{Nis}(Sm/F)$ with weak $\mathbb{A}^1$-equivalences and cofibrations is a closed model category in the sence of Quillen [Q]. The corresponding homotopy category $\mathcal{H}(F)$ is called the homotopy category of schemes over F. We'll use the notation $\mathcal{H}_\bullet(F)$ for the corresponding category of pointed simplicial sheaves. For any pointed simplicial sheaves $\mathcal{X}, \mathcal{Y}$ we define their smash producr as the sheafification of the presheaf $U \mapsto \mathcal{X}(U) \wedge \mathcal{Y}(U)$. This gives $\mathcal{H}_\bullet$ a structure of a simplicial monoid category.

There are two fundamental circles in the category $\mathcal{H}_\bullet$

(1) The usual simplicial circle S^1_s considered as the constant simplicial sheaf.
(2) S^1_t is the pointed scheme $(\mathbb{G}_m, 1)$ considered as a representable sheaf (constant in the simplicial direction).

We denote by T the pointed simplicial sheaf given by the cocartesian square

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ Spec\ F & \longrightarrow & T. \end{array}$$

There are canonical isomorphisms in $\mathcal{H}_\bullet$:

$$S^1_s \wedge S^1_t \cong T \cong (\mathbb{P}^1, 0).$$

For any morphism $f : \mathcal{X} \to \mathcal{Y}$ of pointed simplicial sheaves one defines it's cone as the sheafification of the presheaf $U \mapsto cone(\mathcal{X}(U) \to \mathcal{Y}(U))$. Repeating this construction one gets in the usual way the cofibration sequence

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \to cone(f) \to S^1_s \wedge \mathcal{X} \to \ldots$$

Definition 5.1. *A T-spectrum over F is a family of pointed simplicial sheaves and their morphisms* $\mathbf{E} = (E_i, e_i : T \wedge E_i \to E_{i+1})_{i \in \mathbb{Z}}$

A morphism of T-spectra is defined in an obvious way. Using the above $\mathbb{A}^1$-equivalences and cofibrations one defines in the usual way weak equivalences and fibrations of spectra and finally defines the associated stable homotopy category $\mathcal{SH}(F)$. For any pointed simplicial sheaf $\mathcal{X}$ one defines its suspension spectrum in the usual way, by setting

$$\Sigma^\infty_T(\mathcal{X}) = (T^{\wedge i} \wedge \mathcal{X}, Id).$$

Abusing the language we'll use sometimes the notation $\mathcal{X}$ instead of $\Sigma^\infty_T(\mathcal{X})$.

Theorem 5.2 [V-M]. *The category $\mathcal{SH}(F)$ has a natural structure of the tensor triangulated category such that*

(1) *the shift functor $\mathbf{E} \mapsto \mathbf{E}[1]$ coincides with the smash product by S^1_s.*
(2) *The functor Σ^∞_T takes cofibration sequences to distinguished triangles.*
(3) *The functor Σ^∞_T takes smash products of pointed simplicial sheaves to tensor products of spectra.*
(4) *The object T of $\mathcal{SH}(F)$ is invertible.*

In the standard way each spectrum defines a cohomology theory on the category Sm/F. This theory is now bigraded (since we have two circles). More precisely to each T-spectrum $\mathbf{E}$ we associate a cohomology theory which associates to each (simplicial) smooth scheme $\mathcal{X}$ the bigraded abelian group

$$\mathbf{E}^{p,q}(\mathcal{X}) = Hom_{\mathcal{SH}(F)}(\Sigma_T^\infty(\mathcal{X}_+)), \mathbf{E}(q)[p]),$$

where $\mathbf{E}(q)[p]$ is the spectrum $\mathbf{E} \wedge S_t^q \wedge S_s^{p-q})$. We need the Eilenberg-MacLane spectra which represent the motivic cohomology theory.

For any $n > 0$ denote by $K(\mathbb{Z}(n), 2n)$ the sheaf of abelian groups $L(\mathbb{A}^n)/L(\mathbb{A}^n \backslash 0)$, considered as a pointed (by zero) simplicial sheaf. For all n there are canonical morphisms of pointed simplicial sheaves

$$e_n : T \wedge K(\mathbb{Z}(n), 2n) \to K(\mathbb{Z}(n+1), 2n+2)$$

Thus we get a spectrum $\mathbf{H}_\mathbb{Z}$ called the Eilenberg-MacLane spectrum. In the same way one defines the Eilenberg-MacLane spectra $\mathbf{H}_{\mathbb{Z}_{(\ell)}}$, $\mathbf{H}_{\mathbb{Z}/\ell}$.

Theorem 5.3 [V-M]. *For any smooth simplicial scheme $\mathcal{X}$ we have natural isomorphisms*

$$\mathbf{H}_\mathbb{Z}^{p,q}(\mathcal{X}) = H^p_{Zar}(\mathcal{X}, \mathbb{Z}(q)).$$

§6. The Steenrod Operations

Denote by $\mathcal{A}^{*,*}(F, \mathbb{Z}/\ell)$ the motivic Steenrod algebra modulo ℓ. Thus

$$\mathcal{A}^{p,q}(F, \mathbb{Z}/\ell) = Hom_{\mathcal{SH}(F)}(\mathbf{H}_{\mathbb{Z}/\ell}, \mathbf{H}_{\mathbb{Z}/\ell}(q)[p])$$

The structure of the motivic Steenrod algebra is studied in [V 4].

Theorem 6.1 [V 4].

(1) $\mathcal{A}^{p,q}(F, \mathbb{Z}/\ell) = 0$ *for* $q < 0$.
(2) $\mathcal{A}^{0,0}(F, \mathbb{Z}/\ell) = \mathbb{Z}/\ell$.

Theorem 6.2 [V 4]. *There exists a unique series of cohomology operations $P^i \in \mathcal{A}^{2i(\ell-1),\ell-1}$, $i \geq 0$, with the following properties*

(1) $P^0 = Id$
(2) *For any simplicial smooth scheme $\mathcal{X}$ and any $u \in H^n(\mathcal{X}, \mathbb{Z}/\ell(i))$ one has $P^i(u) = 0$ for $n < 2i$ and $P^i(u) = u^\ell$ for $n = 2i$.*
(3)

$$\Delta(P^i) = \sum_{a+b=i} P^a \otimes P^b + \tau \sum_{a+b=i-2} \beta P^a \otimes \beta P^b$$

where β is the Bockstein operation and τ is multiplication by $-1 \in H^0(F, \mathbb{Z}/2(1)$ if $\ell = 2$ and $\tau = 0$ otherwise.

Using the Steenrod operations P^i one defines in the standard way the Milnor operations $Q_i \in \mathcal{A}^{2\ell^i-1,\ell^i-1}$ by setting $Q_0 = \beta, Q_{i+1} = [Q_i, P^{\ell^i}]$. The same as in the usual topological situation the operations Q_i are square-zero and admit a lifting to operations $\tilde{Q}_i$ in motivic cohomology with coefficients in $\mathbb{Z}_{(\ell)}$. Define the operation

$$H^{n+1}_{Zar}(\check{C}(X), \mathbb{Z}_{(\ell)}(n)) \xrightarrow{\alpha} H^{2\frac{\ell^{n-1}-1}{\ell-1}+1}(\check{C}(X), \mathbb{Z}_{(\ell)}(\frac{\ell^{n-1}-1}{\ell-1}+1)).$$

as the composition $\tilde{Q}_{n-2}...\tilde{Q}_1$. The proof of injectivity of α (in the situation of Theorem 4.11) is based on the following result

Theorem 6.3. *In the assumptions and notations of Theorem 4.11 let $\mathcal{X} \in \mathcal{SH}(F)$ denote the fiber of $\Sigma^\infty_T(\check{C}(X)_+) \to S^0$. Then for any $i < n$ the complex*

$$... \to H^{p-2(\ell^i-1),q-\ell^i+1}(\mathcal{X}, \mathbb{Z}/\ell) \xrightarrow{Q_i} H^{p,q}(\mathcal{X}, \mathbb{Z}/\ell) \xrightarrow{Q_i} ...$$

is acyclic.

Voevodsky gives two proofs of this theorem. One of them uses essentially the algebraic cobordism spectrum and the topological realization functor. The other is more elementary and amounts to the construction of the explicit contracting homotopy operator for the above complex .

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