Global minimizers of the Mumford-Shah functional

Guy David
Université de Paris-Sud Orsay

The Mumford-Shah functional was introduced in the late eighties as a tool for (automatic) image segmentation [MuSh]. Let us give its definition in any dimension n, even though $n = 2$ in the initial setting. Let Ω be a simple bounded domain in $\mathbb{R}^n$, like a ball or a rectangle, and let $g \in L^\infty(\Omega)$ be given. In the context of image processing, $n = 2$, Ω is a screen, g is an image, and $g(x)$ represents the level of brightness (or colour if g is vector-valued) of the image at the point x. The functional is given by

$$(1) \qquad J(u,K) = \int\int_{\Omega\setminus K} |u-g|^2 + \int\int_{\Omega\setminus K} |\nabla u|^2 + H^{n-1}(K) \ ,$$

where the authorized competitors are pairs (u, K) such that K is a closed subset of Ω with finite Hausdorff measure $H^{n-1}(K)$ and u is a function on $\Omega\setminus K$ with one derivative ∇u in $L^2(\Omega\setminus K)$. One hopes that minimizers (u, K) of J will realize a good compromise between the three competing constraints that u be close to the original image g, u be fairly regular except on a singular set K where it may have jumps, and K itself be reasonably short. Of course the three terms of the functional should be given different weights that reflect the relative importance of the three constraints, but for theoretical mathematics the constants can be normalized out by multiplying g and u by the same constant and dilating the picture. The point here is not to have a very good accuracy (as in an image compression problem), but rather to get a fairly simple, cartoon-like description by (u, K) that would capture significant features of the image g.

Discrete variants of the functional J are used in practice for image segmentation; we refer to [MoSo] for an interesting description of some of them and further information on computer vision and image segmentation. When $n = 3$, the Mumford-Shah functional has been proposed as a model for the formation of cracks in a material, but I don't know how useful it is in this context.

Let us now go rapidly through some of the known properties of minimizers of J. The existence of minimizers (u, K) (for all choices of g) was not obvious, but is known ([DCL] and [Am]). (The trouble is the lack of semicontinuity of Hausdorff measure: it is easy to find sequences of compact sets K_n such that $H^1(K_n)$ tends to 0 but which converge to a line segment, say.) The main mathematical questions concern the regularity of K for minimizers (u, K) of J . Observe that it is a good thing for image processing if K is not too wild, because we think of it as edges in the image.

The focus is on K because u is determined by K (when K is fixed, minimizing J in u is a well-posed convex variational problem). In order to discuss regularity properties of K , we should restrict to "irreducible minimizers", i.e., minimizers (u, K) for which u does not extend to a function with a derivative in $L^2(\Omega \backslash K')$ for any K' strictly smaller than K. Otherwise we may artificially add any horrible compact set of H^1-measure 0 to K and ruin any nice description of K

The initial conjecture of Mumford and Shah [MuSh] is that when (u, K) is an irreducible minimizer, K is a finite union of C^1-arcs that may meet at their endpoints with 120° angles. This is still an open question, but significant progress has been made. Here are a few properties of K when (u, K) is an irreducible minimizer. First, K is rectifiable [DCL], which means that it is contained in a countable union of C^1-curves, plus possibly a set of H^1-measure zero. Also, it is Ahlfors-regular [DMS], which means that

$$(2) \quad C^{-1}r \ \leq \ H^1(K \cap B(x,r)) \ \leq \ Cr \text{ for all } x \in K \text{ and } 0 < r < 1.$$

Next, K is uniformly rectifiable: it is contained in an Ahlfors-regular curve (i.e., a compact curve that satisfies (2)) [DaSe1]. Finally, the best result of this type is that for every disk $B(x, r)$ centered on K and radius ≤ 1 there is another disk $B(y, t) \in B(x, r)$ centered on K, with $t \geq C^{-1}r$, and such that $K \cap B(y,t)$ is a C^1-curve. (See [Da] or [AFP]). These results have higher-dimensional analogues: see [CaLe1,2], [DaSe3], [AFP] respectively. The reader may consult [DaSe2] for a survey of some proofs, and [MoSo] for a more extensive study and other results that I did not quote here.

I wish to insist more on a recent approach introduced by A. Bonnet [Bo1] in dimension $n = 2$. He first proves that the objects obtained by blowing up a reduced minimizer (u, K) of J are "global minimizers", as defined below. Blowing up here consists in taking a point $x_0 \in K$ and a sequence of numbers t_n tending to 0, and then considering the sets $K_n = x_0 + t_n^{-1}(K - x_0)$, and suitably normalized associated functions u_n. Limits exist (modulo taking subsequences), and are pairs (v, E) with the following properties.

First, E is a closed set in the plane, and v is a function on $\mathbb{R}^2 \setminus E$ with a derivative in $L^2_{loc}(\mathbb{R}^2 \setminus E)$ such that

(3)
$$H^1(E \cap B(0,R)) < \infty \text{ and } \int_{\mathbb{R}^2 \cap B(0,R)) \setminus E} |\nabla v|^2 < \infty \text{ for all } R > 0.$$

Competitors of the pair (v, E) are similar pairs (v^*, E^*), where E^* is also a closed set in the plane, v^* a function on $\mathbb{R}^2 \setminus E^*$, the analogue of (3) holds, and there is a radius $R > 0$ such that $v^* = v$ and $E^* = E$ out of $B(0, R)$. We also add the topological constraint that for R large enough, E^* separates all pairs of points $y, z \in \mathbb{R}^2 \setminus (E \cup B(0,R))$ that are separated by E (that is, such that y and z lie in different connected components of $\mathbb{R}^2 \setminus E$. The pair (v, E) is called a global (Mumford-Shah) minimizer if for all competitors (v^*, E^*) of (v, E) we have that

$$H^1(E \cap B(0,R)) + \int_{\mathbb{R}^2 \cap B(0,R)) \setminus E} |\nabla v|^2$$

$$\leq H^1(E^* \cap B(0,R)) + \int_{\mathbb{R}^2 \cap B(0,R)) \setminus E^*} |\nabla v^*|^2 \tag{4}$$

for R large enough.

Note that global minimizers can be thought of as solution of some Dirichlet problem at infinity (E and the values of v are known at infinity, and we only compare with compact perturbations). The following is a list of tentative global minimizers:

1. $E = \emptyset$ and v is a constant;

2. E is a line and v is constant on each of the two components of $\mathbb{R}^2 \backslash E$;

3. E is the union of three half lines with the same endpoint and that make 120° angles, and v is constant on each of the three components of $\mathbb{R}^2 \backslash E$;

4. Up to Euclidean motion, E is the half line $E = \{y = 0 \text{ and } x \leq 0\}$ and

$$v(x,y) = \pm\sqrt{\tfrac{2}{\pi}} \quad Im\{(x + iy)^{1/2}\} + C \quad .$$

The three first examples are easily seen to be global minimizers. Note that the topological constraint on competitors is useful here. Whether the fourth one is also a global minimizer is still an open question (of De Giorgi), although A. Bonnet may have a positive answer [Bo2]. If it were not, then it would not be possible to find isolated, compactly supported line segments or open C^1-curves in any minimizer of the (local) Mumford-Shah functional J above.

The main question, though, is whether all the global minimizers are in the list above. This would imply the Mumford-Shah conjecture above (and it is probably the best way to prove it).

A. Bonnet was only able to prove this for global minimizers such that E is connected, and because of this he was "only" able to show that if (u, K) is a reduced minimizer of the Mumford-Shah functional J, every **isolated** connected component of K is a finite union of C^1-curves that meet with 120° angles.

Let me finish this description with a formula that should be useful. It is a (surprisingly simple) Euler-Lagrange formula that was discovered by J.-C. Léger. Identify $\mathbb{R}^2$ with the complex plane, and set

$$F(x+iy) = \frac{\partial v}{\partial z}(x+iy) = \frac{1}{2}[\frac{\partial v}{\partial x}(x+iy) - i\frac{\partial v}{\partial y}(x+iy)]$$

$$\text{for } (x,y) \in \mathbb{R}^2 \backslash E \ . \tag{5}$$

Note that v is harmonic because it minimizes the integral of $|\nabla v^2|$ locally, and so F is holomorphic on $\mathbb{C} \backslash E$. The formula is

$$F(z)^2 = \frac{-1}{2\pi} \int_E \frac{dH^1(w)}{(z-w)^2} \quad \text{for } z \in \mathbb{C} \backslash E \ . \tag{6}$$

It has a few applications, like the fact that v is determined by E modulo an additive constant and a change of sign on each component (which was not clear a priori), or the fact that all global minimizers for which E is contained in a line (or a sufficiently flat chord-arc curve) are in the list above, but so far it does not have as striking applications as I think it should.

One can also start to dream about higher dimensions. It would be nice to have a list of tentative global minimizers in dimension 3, even if the chances of proving that it is the right list are quite small at this stage. Maybe interesting new phenomena will show up.

REFERENCES

[Am] L. Ambrosio, Existence theory for a new class of variational problems, Arch. Rational Mech. Anal. 111 (1990), 291-322.

[AFP] L. Ambrosio, N. Fusco and D. Pallara, Partial regularity of free discontinuity sets II., preprint, Univ. de Pisa 1994.

[Bo1] A. Bonnet, On the regularity of edges in image segmentation, Ann. Inst. H. Poincaré, Analyse non linéaire, Vol 13, 4 (1996), 485-528.

[Bo2] A. Bonnet, Personal communication, 1996, to be retrieved.

[CaLe1] M. Carriero and A. Leaci, Existence theorem for a minimum problem with free discontinuity set , Nonlinear Analysis, Theory, Methods and applications, Vol 15, 1990, 661-677.

[CaLe2] M. Carriero and A. Leaci, S^k-valued maps minimizing the L^p norm of the gradient with free discontinuities, Ann. Sc. Norm. Sup. Pisa, Vol. 18, 1991, 321-352.

[DMS] G. Dal Maso, J.-M. Morel, and S. Solimini, A variational method in image segmentation: Existence and approximation results, Acta Math. 168 (1992), 89-151.

[Da] G. David, C^1-arcs for minimizers of the Mumford-Shah functional, SIAM Journal of Appl. Math., Vol. 56, 3, 1996, 783-888.

[DaSe1] G. David and S. Semmes, On the singular sets of minimizers of the Mumford-Shah functional, Journal de Math. Pures et Appl. 75 (1996), 299-342.

[DaSe2] G. David and S. Semmes, On a variational problem from image processing, proceedings of the conference in honor of J.-P. Kahane, special issue of the Journal of Fourier Analysis and Applications, 1995, 161-187.

[DaSe3] G. David and S. Semmes, Uniform rectifiability and Singular sets, Annales de l'Inst. Henri Poincaré, Analyse non linéaire, Vol 13, 4 (1996), p. 383-443.

[DCL] E. De Giorgi, M. Carriero, and A. Leaci, Existence theorem for a minimum problem with free discontinuity set, Arch. Rational Mech. Anal. 108 (1989), 195-218.

[DiKo] F. Dibos and G. Koepfler, Propriété de régularité des contours d'une image segmentée, Comptes Rendus Acad. Sc. Paris 313 (1991), 573-578.

[MoSo] J.-M. Morel and S. Solimini, Variational methods in image segmentation, Progress in nonlinear differential equations and their applications 14, Birkhauser 1995.

[MuS] D. Mumford and J. Shah, Optimal approximations by piecewise smooth functions and associated variational problems, Comm. Pure Appl. Math. 42 (1989), 577-685.

Guy David,
Université de Paris-Sud Orsay,
Institut Universitaire de France,
and MSRI
guy.david@math.u-psud.fr
gdavid@msri.org