Open Problems in Partial Differential Equations Arising from Fluid Dynamics

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Mathematical theory of fluid dynamics gives rise a galaxy of well-known nonlinear partial differential equations such as the celebrated Euler equations and Navier Stokes equations for the compressible and incompressible fluids ([1, 2]), nonlinear Boltzmann equations for rarefied gases ([3]) and their variations by taking into account of various additional physical effects. The fundamental importance in various applications in many different branches of sciences and the richness in the mathematical theory of these nonlinear systems have made them some of focuses of extensive researches in the last few decades by mathematicians and engineers, and tremendous progress has been made in solving these equations both theoretically and numerically, and in understand-

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ing the behavior of solutions to such nonlinear systems. Despite the substantial success and great effort by many mathematicians in the development of the mathematical theory for these nonlinear partial differential equations ([1, 3, & 4]), there remain many fundamental open problems which challenge the field now and in the future. In the following, I will list some of the open problems which, in my opinion, should be of importance in understanding these nonlinear partial differential equations.

§1. Compressible Euler Equations and the Mathematical Theory of Shock Waves

The system of Euler equations for compressible fluid ([1]) provides one of the most important examples of quasi-linear hyperbole conservation laws which is a notoriously difficult subject due to the richness and complexity caused by the fact that the wave speeds depend on the wave themselves. The past progress has been concentrated on one-dimensional theory with very few exceptions (see [4, 5, 6] and the reference therein). There remain many important problems to be solved. We start with one-dimensional problems.

1. Global (in time) well-posedness of entropy weak solutions to the Cauchy problems of the one-dimensional Euler equations for polytropic gas dynamics in the space of functions with bounded total variations (which is BV space). Even though the global existence of solutions in BV space has been obtained by the Glimm's scheme ([7, 4]), and their uniqueness and continuous dependence in the class of the viscosity solutions ([8]) has been shown ([8])

for initial data with small total variations, all the relevant questions remain open for solutions with large data in BV space. The significance of this problem lies in the facts that the BV space is natural space of basic waves such as shock waves, centered rarefaction waves, and contact discontinuities to develop and interact, and the physical relevance of strong linear and nonlinear waves ([1]). In fact, this problem has not been solved even for isentropic gases with general data in BV without the additional assumption of initial small oscillations ([13]), although in this case large amplitude weak solutions exist in the space of bounded measurable functions thanks to the theory of compensated compactness ([10, 11, 12]). However, little information is available on the structures of such solutions. Based on the insights provided by the studies of weakly nonlinear geometric optics approximations, there are recent interesting results which show finite time blow-up in BV norm for solutions to some systems of nonlinear hyperbolic conservation laws with initial data in BV space of small oscillation ([13, 14, 15]). However, such systems have different structures from those of Euler equations for polytropic fluids. Thus it is extremely interesting to know whether a weak solutions in BV space can blow-up in finite time for gas dynamics equations with large data in BV.

2. Uniqueness of the vanishing viscosity limits. Consider a weak solution to the one-dimensional Euler equations for polytropic fluids, which is a limit (in appropriate topology) of solutions of the corresponding Navier-Stokes equations as the dissipation parameters (viscosity and heat conduction coefficients) tend to zero.

Show that such a solution is the unique weak solution to the Cauchy problem for th Euler equations under appropriate entropy conditions. A less ambitious problem is to prove that such a solution is unique among the class of weak solutions which can be obtained as limit of solutions to the corresponding Navier-Stokes equations (or even the system of Euler equations with artificial viscosities) as dissipation goes to zero. This has been established only for very special cases ([16, 17]). One possible approach is to compare the vanishing viscosity limit solutions with the viscosity solutions defined in ([8]). This requires some deeper understanding of the asymptotic local behavior of solutions to the Navier-Stokes equations as dissipation becomes small. The recently developed theory of viscous shock waves ([18–24]) should yield considerable insight and provide some of the important analytical tools to settle this problem. A related problem is to show the uniqueness of weak solutions to Euler system for isentropic fluids for which the vanishing viscosity solutions have been obtained by the theory of the compensated compactness ([10, 12]).

3. Spatial periodic solutions. In their fundamental work [25], Glimm and Lax obtained the existence and large time asymptotic behavior of spatial periodic solutions to the isentropic Euler equations. Their results display the amazing regularization effects of the genuined nonlinearity which damps out any initial oscillations, which lays the rigorous foundation for the later-on homogenization theory on the propagation of high frequency waves for this system. Despite its extreme importance and much effort in the past two decades, the theory of global (in time) existence and large time

asymptotic behavior of spatial periodic solutions to the full Euler equation for polytropic fluids remains completely open. Main difficulties include the lack of Riemann invariant coordinates and the resonant wave interactions through the entropy wave field which is linearly degenerate. Some of related but simpler questions are: Is there example of spatial periodic weak solution for the full Euler equation with piecewise constant periodic initial data which ceases to exist in finite time? Are there examples of special spatial periodic solutions which decay to their average values over each period with different rate of decay in time?

4. Local structures of solutions in BV space.

   Let $V(x, t) \in BV$ be a weak solution to the full Euler equations of polytropic fluids satisfying the physical entropy condition. What are the additional local structures of $V(x, t)$ in the physical space beyong those of a BV functions ([26])? Can $V(x, t)$ behave locally as an appropriate perturbation of a Riemann solutions as Glimm solutions do? In particular, can $V(x, t)$ be an viscosity solution in the sense of [8]? Some interesting results have been obtained by Dafermos in this directions for the isentropic Euler equations based on the generalized characteristic methods and entropy estimates [27].

We next turn to the multi-dimensional theory of shock waves which is an extremely important but difficult subject. There has been very little progress for the multi-dimensional shock wave theory for the compressible Euler system with a few exceptions (see [1] and [5] and the references therein). Some completely new ideas and techniques are needed to lay a foundation for the multi-dimensional shock wave theory. Some of the following open problems serve as steps to achieve better understanding of this great task. We also refer the readers to [5, 28] for some related perspective and open problems in this direction.

5. Structures of singularities and formation of shocks. The inevitable formations of shock waves in finite-time of solutions with generic smooth initial data is one of the most striking features of nonlinear hyperbolic conservation laws. This has been pretty well understood for one-dimensional Euler equations for polytropic fluids ([6, 29, 30]). In higher space dimensions, it is known that smooth solutions will blow-up in finite time for compressible Euler equations with generic smooth initial data ([31, 32]). However, the nature of singularities in the solutions is not clear except for special cases for which the singularities are shown to be shocks (see [33] and the reference therein). It is important to characterize the structures of singularities forming from smooth solutions in terms of initial data. In particular, it is of great interest to determine when shocks form first before the development of shell singularities.

6. The well-posedness of weak solutions. The well-posedness of weak solutions to the compressible Euler equations in higher space dimensions remains to be one of the most challenging open problems for mathematical theory of compressible fluid dynamics. It is not even clear that in which functional space the problem should be posed in contrast to the one-dimensional theory for which the spaces BV is natural (see [34]). Even the short time existence of piecewise smooth solutions are not proved except for the special cases that the initial data are suitable perturba-

tions of shock fronts or planary rarefaction waves (see [5] and the reference therein). Thus there are many basic questions to be answered. What are the geometric structures of generic discontinuities in a weak solution? How do the basic linear and nonlinear waves evolve and interact? What are the appropriate entropy conditions? Are the planary shock waves (satisfying the entropy conditions in [5]) and centered rarefactions waves asymptotically stable under non-planary perturbations? What is the appropriate space in which one can obtain either existence (even for short time) or uniqueness of general physically relevant weak solutions? When is the solution operator compact? Due to the complexity and the lack of understanding of the general problem, it might be very helpful to study some of the special physically relevant wave patterns for which a lot experimental data, numerical simulations, and formal asymptotics results are available. Some of the well-known important examples include: shock wave reflection phenomena ([35]), transonic flows and standing shocks in a nozzle [1], self similar flows, and other flows with various symmetry properties.

§2. Compressible Navier-Stokes Equation and Viscous Shock Wave Theory

Homogeneous Newtonian fluids are described by the compressible Navier-Stokes system when the viscosity and heat conduction are taken into account. This is a hyperbolic-parabolic composite type quasi-linear system with dissipative structure. Substantial progress has been achieved in the past several decades in understanding the solutions to the compressible Navier-Stokes equations. In particular, the well-posedness and large time asymptotic behavior towards constant states of small amplitudes smooth (or even BV) solutions in Sobolev spaces to the compressible Navier-Stokes equations for fixed viscosity and heat-conductivity have been well-established (see [36], [37]), the one-dimensional theory of nonlinear asymptotic stability toward linear and nonlinear waves has also been completed (see [18–24]), and even the existence of weak solution of large amplitudes to the 2D or 3D Navier-Stokes system for some isentropic fluids has been announced in [38]. However, there remain many more open problems which are of significance both mathematically and physically, such as the asymptotic equivalence in the limit of small dissipation between the compressible Navier-Stokes system and the Euler equations in the presence of discontinuities and boundaries; existence (or non-existence) of large amplitude general solutions; and large time asymptotic behavior toward basic waves, etc.

7. Global (in time) well-posedness of the Cauchy problem or initial-boundary value of problem for the compressible Navier-Stokes equations with large data. Consider the Navier Stokes equations for viscous polytropic flows with fixed positive viscosity and non-negative heat conductivity without exterior forcing. Is there formation of singularities in finite time (or even in infinite time) for the solutions with large smooth data ([39])? Can the vacuum state be formed in finite time even the data are given away from the vacuum? If there exist singularities forming from smooth solutions, can one characterize the structures of the singularity as in the case for harmonic maps? What are the appropriate spaces to study the weak solutions after formations of singularities? What

are the large time asymptotic behavior of the weak solutions?

8. Large time asymptotic states for the multi-dimensional compressible Navier-Stokes equations. The theory of large time asymptotic behavior of small solutions to the one-dimensional Navier-Stokes systems for viscous polytropic fluids is almost completely understood now in the sense that viscous shock waves ad rarefaction waves are nonlinearly stable and contact discontinuities are metastable with detailed pointwise asymptotic ansatz constructed and justified (see [17–24]). However, in higher space dimensions, only constant states are shown to be stable ([36]) except some scalar models ([40, 41, 42]). It is of great interest to know whether basic nonlinear wave patters such as planar viscous shock profiles and viscous rarefaction waves, and contact waves are nonlinearly stable under generic multi-dimensional perturbations. Can one determine the leading order (or even higher order) asymptotic ansatz of a solution to the compressible Navier-Stokes equations when its initial data are small perturbations of these basic wave patterns? These questions are not only important in the stability theory itself, but also should share light on the asymptotic equivalence between the compressible Navier-Stokes equations and Euler system in the limit of small dissipation which will be discussed in the next problem.

9. Asymptotic equivalence between Navier-Stokes system and Euler equations for compressible fluid in the limit of small dissipation. In the limit of small viscosity and heat conductivity, the solutions to the compressible Navier-Stokes equations display turbulent behavior whose understanding is still one of major challenging prob-

lems in fluid dynamics. This is particularly so if either the corresponding Euler flows contain discontinuities or there appear solid boundaries. Some of the main difficulties are due to the dramatic change of gradients of the flows across the discontinuities and near boundaries where the small scale effects are strong. In the case that the underlying Euler flows are piecewise smooth with finitely many entropy satisfying shock discontinuities, it is exported that solutions to the Navier-Stokes equation will converge uniformly to the corresponding solutions to the Euler equations except around thin layers near shock discontinuities in the zero dissipation limit, and furthermore, the precise asymptotic ansatz of the solutions to the Navier-Stokes system for small but nonzero dissipation can be determined by multi-scale matched asymptotic analysis. This has been proved only for one-dimensional problems ([23]). The case that the solutions to the Euler equation are piecewise smooth containing contact discontinuous surfaces is more subtle due to the instability of vortex sheets ([2]). The behavior of solutions around a contact discontinuity surface for multi-dimensional Navier-Stokes equations for small dissipation is open even formally. In the presence of physical boundaries, the well-known boundary layer phenomena occur due to discrepancy of the non-slip boundary conditions for the Navier-Stokes system and the prescribed normal component of the velocity field at the boundaries for the Euler equations ([2, 43]). Formal asymptotic analysis yields the well-known Prandtl's laminar boundary layer theory whose validity and rigorous justification are completely open in most physically relevant flows except for linearized Navier-Stokes equation ([44]). This is so even for short time before the Euler flows develop any singularities. There have been extensive studies on the steady Prandtl's boundary layer equations (see [45, 46]). Yet, very little is known for dynamical boundary layers. Even the short time well-posedness in Sobolev spaces of the unsteady Prandtl's boundary layer equations has not been achieved (but see [47, 48]).

§3. Incompressible Euler and Navier-Stokes equations

In the limit of small Mach numbers, one derives from the compressible Euler and Navier-Stokes systems the incompressible Euler and Navier-Stokes equations respectively ([2, 49]), which govern various flows such as waters in the oceans and air in the atmosphere. The mathematical theory for the incompressible fluid is more mature than that for the compressible fluid (see [50, 51, 52]). In particular, a rather complete well-posed theory exists in two space dimensions for both Navier-Stokes and Euler equations except the important non-smooth flow—vortex sheets; and under minimal assumptions on the data and exterior forcing, the Leray-Hopf weak solutions to the 3-dimensional Navier-Stokes exist globally in time ([50]), and the partial regularity of the weak solutions for 3-D Navier-Stokes equation ([53]), large time decay of the solutions ([54]), and existence of large strong solutions under the assumption of certain spatial symmetries (see [55] and reference therein), have been established. Yet many of the central problems in incompressible fluid mechanics are still challenging the field. We will mention only a few well-known problems here.

10. Well-posedness of the Cauchy problem for the incompressible Euler equations. Do smooth solutions to the 3-dimensional incompressible Euler equations break down in finite time? A positive answer to this question has a profound physical implication that the singularities signify the onset of turbulence, and their structures should share light on the internal cascade in turbulent flows at high Reynolds numbers. This problem has been studied extensively by both analytical and numerical methods in the past several decades which strongly suggest the possible breakdown in finite time of the smooth solution (see [52], and the reference therein). Yet the formation of singularities in finite time has not been rigorously confirmed so far. The lack of understanding of the structures of the possible singularities also prevents one from obtaining the well-posedness of weak solutions to the 3-D Euler equations even though the existence of measure-valued solutions has been proved [57]. The principal mechanism, responsible for the possible formations of singularities, is the stretching and accumulation of the vorticity (see [52]). In the cases that there is no stretching of vorticity, such as 2-dimensional flows and axisymmetric flows without swirls, smooth global (in time) solutions to the Euler equations are obtained [52, 58].

11. Vortex sheets problem and concentration of energy for 2-dimensional Euler equations. Although 2-dimensional smooth inviscid incompressible flows are very well understood to some extent, yet many important questions involving 2-dimensional singular flows remain to be answered. This is particularly so in the case of the vortex sheets problem, in which the vorticity is a finite Radon

measure concentrated or a curve which is initially smooth, and the corresponding velocity field has locally finite energy. This gives a classical example of ill-posed initial value problem in the sense of Hadamard. At later time, a singularity in the sheet may develop and the nature of solutions past the singularity formation is unknown [59]. The existence of classical weak solutions to the 2-D Euler equations with vortex sheets initial data has been achieved by the concentration-cancellations technique ([56, 57]) under the additional assumptions that the initial vorticity has one sign [50]. It is of fundamental importance both physically and mathematically to know whether such a concentration-cancellation occurs to yield the existence of classical weak solution (in contrast to measure-valued solutions) to 2-D Euler equations with general vortex sheets initial data. Another related problem is to understand the structures of the approximate solutions to the vortex sheets problem generated by either viscous approximation or vortex methods [56], which is of considerable significance both physically and numerically. It has been shown ([61, 62]) that concentration-cancellation always occurs for such approximate solutions to vortex sheets problem provided that the initial vorticity has one sign. Does the same phenomena occur for general vortex sheets initial data? Does energy-concentration occur at all dynamically for such approximate solutions in the sense that the energy-concentration occurs later on in time even though the initial data for the approximate solutions converges strongly to that of the initial vortex sheets data in energy norms. The later question is open even in the case that the initial vorticity has one sign.. Finally, the uniqueness of weak solutions to the 2-D Euler equations are wildly open except the case where initial vorticity are bounded in the super-norm [52].

12. Regularity and asymptotic behavior of solutions to the incompressible Navier-Stokes equations. Although there is no evidence either numerically or physically to support the breaking down of smooth solutions to the 3-D incompressible Navier-Stokes equations, yet the global (in time) existence of smooth to the 3-D incompressible Navier-Stokes system is one of the most challenging mathematical problems to be solved. This is related to the regularity of Leray-Hopf weak solution [50], for which only partial regularity, results are available (see [53, 64, 65]). There are recent interesting results which shows self-similar blow-up (Leray's self-similar solutions) are impossible [66]. These problems are open even for general viscous axisymmetric flows. Other important open problems include: asymptotic behavior of solutions to the Navier-Stokes equations either as time approaches infinitely or as Reynold number goes to infinity: do inertial manifolds exist for 2-D or 3-D Navier-Stokes equations [50]? Is Prandtl's boundary layer theory valid for the viscous incompressible fluids in the presence of boundary [43, 46]? What are the asymptotic forms of the solutions to the Navier-Stokes system as the Reynold number goes to infinity in the case of corresponding Euler solutions are singular?

§4. Nonlinear Boltzmann Equation

The general Bolzmann equations of kinetic theory gives a statistical description of a gas of interacting particles. An important property of this equations is its asymptotic equivalence to the Euler and Navier-Stokes systems of compressible fluid dynamics in the limit of mean free path ([3, 67]). It is expected physically that away from initial layers, shock layers, and boundary layers, the solution to the Boltzmann equation should relax to its equilibrium states (local Maxwellian state) in the limit of small mean-free path so that the gases are governed the macroscopic equations such as Euler or Navier-Stokes equations ([3]). This is predicted by the method of normal solutions (or normal region) based on the Hilbert expansion and the Chapman-Enskeg expansions. The rigorous mathematical justification of this fluid-dynamic approximations of Boltzmann solutions poses a challenging open problem in most important cases, in particular, in the case that there are shock discontinuities in the fluid flow and/or in the presence of boundaries. There has been extensive studies on this hydrodynamic limit problem. However, most work deal with smooth flows and without boundaries, or some model linearized Boltzmann equation (see [67, 3, 68, 69, 70]) with exceptions [71–74]. A qualitative theory for boundary layer problem exists for some models for steady Boltzmann equations [75], very little is know for unsteady problems. The hydro-dynamic limit problem in the presence of discontinuities in the fluid flows is completely open for general nonlinear Boltzmann equation except for Broadwell model [72, 76]. Even a formal asymptotic theory for the shock layer and boundary layer problems has not been achieved. One of the main difficulties in analyzing these problems is due to the complexity of the nonlocal collision operator in the Boltzmann equation, which makes it difficult to study the structures of the layer problems associated with the formal matched asymptotic analysis. Even when the formal asymptotic solutions can be constructed such as for the Broadwell model [72], the convergence analysis is still highly nontrivial due to the fact that the fluid-dynamic limits are strongly singular. However, since shock waves are essential for compressible fluids and boundary layer theory describes the interactions of the gas molecules with the molecules of the solid body, to which one can trace the origin of the drag exerted by the gas on the body and the heat transfer between the gas and the solid boundaries, it is of fundamental importance to understand the fluid dynamic approximation to the nonlinear Boltzmann equation in the presence of shocks and boundaries. Other related questions, such as global existence of large amplitude smooth solution to the general nonlinear Boltzmann equation [3], and the regularity of the renormalized weak solutions to Boltzmann equations due to Diperna–Lions ([77]) and its hydro-dynamic limit, are waiting to be attacked too.

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