Mirror Symmetry:
A Brief Review of the First 10 Years

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Through studies in superstring theory in the late 1980s, physicists began to suspect that there was an unexpected and powerful link between *a priori* unrelated topologically distinct Calabi–Yau manifolds. Using techniques of conformal field theory, these suspicions were subsequently put on firmer foundation, as explicit examples of *mirror manifolds* were found. These are pairs of topologically distinct Calabi-Yau manifolds which give rise to isomorphic string theories when either is chosen for the extra spatial dimensions required by the theory. Through their common physical universe, highly nontrivial relations between the members of a mirror pair can be found. Some of these lead, for example, to calculationally effective methods for determining the number of rational curves of arbitrary degree on various Calabi-Yau spaces. From a more physical standpoint, mirror symmetry has also played a pivotal role in understanding the first known examples of topology changing processes. Recently, aspects of mirror symmetry have been understood using more traditional mathematical methods.

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1. INTRODUCTION

Superstring theory is widely viewed as our best hope for constructing a consistent quantum theory of gravity and for realizing Einstein's dream of unified theory of the physical universe. Nothwithstanding such grandiose promise, some are justifiably skeptical about a theory whose natural energy scale may well be seventeen orders of magnitude higher than we can currently attain in the laboratory. To such doubters, the best one can do is give a pep talk on the importance of quantum gravity to a complete theory of physics, extoll the beauty and virtues of string theory's attempt to fill in this crucial gap in our understanding of nature, and recall that many features of the real world which we now take for granted (neutrinos, neutron stars, black holes, etc.) seemed like experimentally untestable hypothoses when *they* were first introduced. Some will find this covincing, others won't.

But there is another arena in which the utility and power of string theory (and its close cousin, supersymmetry) is unassailable: mathematics. Of course, this is little comfort to physicists trying to figure out how the universe works at its most fundamental level, but since these remarks are directed at a largely mathematical audience, we will exploit the success superstring theory has already had in this domain. And to keep the discussion managable, we will further focus on one particular area — mirror symmetry (for an in depth set of references, the reader can consult [28,29]).

Mirror symmetry is a property of Calabi–Yau spaces which relates topologically distinct manifolds in a surprising and powerful way. It was discovered by physicists working on string theory in the late 1980s and early 1990s. Although much insight into the mathematical underpinnings of mirror symmetry has been found over the last decade, its most potent expression is still in the language of string theory where it ensures that string propogation on the two manifolds of a mirror pair leads to identical physical theories. Among other things, this implies— from a purely mathematical point of view— surprising and unexpected equalities between geometrical quantities on the two members of a mirror pair. As years of studies have shown, these equalities are remarkably potent. For example, in some instances they provide a dictionary for translating difficult questions of mathematical interest on one manifold into far simpler questions on the other.

In the early days of mirror symmetry, although these mathematical results could be stated without reference to the underlying physical theory, there was no means of establishing their validity without recourse to physical methods. But over the years, some of the most famous mathematical implications of mirror symmetry have been proven with purely mathematical tools. A program which continues to be of central importance is to find the full mathematical framework which embodies mirror symmetry and hence ascertain its complete set of mathematical implications.

The organizers have asked me to give an introduction to and overview of some of the highlights of the first ten years of mirror symmetry, emphasizing the physical intuition and motivation which launched this subject—intuition which continues to be an extremely fruitful guide used by current research. As the anticipated audience is one whose interest in the subject originates in its mathematical aspects, we will try to avoid using detailed concepts from physics whenever possible, and emphasize those implications of the results for which the geometric interpretation is most manifest. However, now and then—especially in explaining physicist's motivation and the concrete results emerging from string theory— we will be forced to make use of some physical methods. These methods are standard and fully understood from the physics viewpoint but are likely to be unfamiliar to some readers. Happily, though, as time goes by, increasingly large number of mathematicians are learning and using these physical methods, facilitating yet more fruitful interactions between the two communities.

Before entering in upon some of the details of the discussion, we want to briefly describe mirror symmetry in heuristic physical terms. This informal description will facilitate our later discussion of mirror symmetry, as well as introduce the physical intuition which leads to the mirror symmetry conjecture; it is not required for an understanding of the more technical description which follows.

Studies in string theory in the mid 1980s established a correspondence between certain manifolds (and, in fact, more general spaces, as we shall discuss) and objects known as conformal field theories.[1]

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[1] In the appendix we briefly outline this Calabi–Yau / conformal field theory corresondence. In the early 1990s, Witten sharpened certain features of this correspondence through the use of so-called linear sigma models. The

Roughly, the correspondence involves considering the manifold as the spacetime in which strings propagate. Many of the basic mathematical properties of a given one of these manifolds (such as its cohomology ring) find direct expression in the associated conformal field theory. It turns out, however, that this association of manifolds and conformal field theories is not one–to–one: two and possibly more distinct manifolds may, in fact, correspond to the same conformal theory. It is thus worthwhile to partition the space of all manifolds which have conformal field theoretic realizations into classes of those which have the *same* realization. Such distinct manifolds, M and $\tilde{M}$, which correspond to the same conformal theory are called a *mirror pair* if they have Hodge diamonds satisfying the relation $h^{p,q}(M) = h^{d-p,q}(\tilde{M})$ where $h^{p,q}$ is the dimension of the (p,q) Dolbealt cohomology group of M, and d is the complex dimension of the manifold. We see that the common conformal theory associated to a mirror pair provides a previously unsuspected link between these topologically distinct manifolds. By passing through this link, intrinsic properties of one manifold of the pair are reflected in the other. Of much importance is the fact that some of these intrinsic properties are easier to understand and calculate when rephrased on the mirror. This is the origin of the famous success of [4] in calculating the number of rational curves of arbitrary degree on the quintic threefold – at the time, a seemingly impossible calculation which became thoroughly tractable when rephrased on the mirror manifold. We will discuss the essential ideas underlying these calculations and illustrate them with explicit examples of mirror manifolds.

In section two we will give a more formal discussion of the basic points covered in this introduction and state the original mirror conjecture. In section three we will review a somewhat less ambitious version of this mirror conjecture and present the evidence supporting it which physicists found in the early 1990s. In section four we will describe the mathematical implications of mirror symmetry found by physicists, and we will discuss some of the explicit calculations they provided to verify their results. In section five we will discuss some of the highlights as mathematicians have sought to put aspects of mirror symmetry on a firmer mathematical foundation, and in section 6

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interested reader can find a thorough review of such developments in my article in [31], for example.

we will discuss a few of the major unresolved issues currently under study.

Our purpose here is not to give detailed analyses and derivations as these can be found, for example, in the extensive article [31]. Rather, this article is an informal introduction and overview of mirror symmetry.

2. MIRROR SYMMETRY: A CONJECTURE

The manifolds of interest to our discussion are of the so–called *Calabi–Yau* type. These were introduced in the context of string theory in [7,8]. For our purposes we take the following definition:

Definition: A Calabi–Yau manifold is a compact, complex Kähler manifold with vanishing first Chern class and zero first Betti number.[2]

We note at the outset that it is insufficient for our purposes to limit attention to smooth Calabi–Yau manifolds. Rather, our discussion will also make use of *orbifolds* [8] of smooth Calabi–Yau manifolds by discrete group actions that generically have fixed points. We shall only need to consider, though, V-manifolds with at most abelian quotient singularities. It is known that these admit resolutions to smooth manifolds [9] and it is properties of the latter manifolds which will be of interest to us.

We also note that when we speak of a Calabi–Yau manifold, we generally assume that a particular choice of polarization (i.e. Kähler structure) and complex structure has been made. Different choices of either the complex or the Kähler structure on the same underlying topological space are said to constitute different points in the moduli space of the Calabi–Yau space. Infinitesimal deformations of the complex structure of a Calabi–Yau manifold are parametrized by the cohomology group $H^{2,1}$ while $H^{1,1}$ parametrizes Kähler deformations. ¿From the point of view of string theory, as indicated in the appendix, the specification of a geometrical solution of the string consistency equations requires not only the topology of the Calabi–Yau manifold, but also a choice of Ricci–flat metric (the existence of which is guaranteed by Yau's theorem [10]; it is known that physical processes shift

[2] The requirement of vanishing first Betti number is a convenience for our discussion, but essentially all that we describe applies to tori as well.

the Ricci-flat metric to one that is not Ricci-flat, but a perturbative treatment must start with a metric that is Ricci-flat). The space of all Ricci–flat metrics is parametrized by the possible choices of complex and Kähler structures, thus matching the mathematical data specified above. Furthermore, although the cohomology group $H^{1,1}(M)$ is naturally real, string theory requires us to *complexify* the Kähler cone of M. Concretely, what we mean by this is as follows. If the Kähler metric on M is denoted by $g_{i\bar{j}}$, then in local coordinates the Kähler form can be written $J = ig_{i\bar{j}}dX^i \wedge dX^{\bar{j}}$. In considering string propagation it is important to include an additional structure on the manifold, which leads us to consider $J = (ig_{i\bar{j}} + b_{i\bar{j}})dX^i \wedge dX^{\bar{j}}$ where $b_{i\bar{j}}$ is some chosen element of $H^{1,1}(M)$.[3] It turns out that the physical model is insensitive to shifts in $b_{i\bar{j}}$, the so-called "antisymmetric tensor field," by integral elements of $H^2(M)$.

Having given the definition of Calabi–Yau manifolds (and their orbifolds) as well as indicating the additional data required to define a compactified string model, we would like to give a precise definition of mirror manifolds. The most precise definition, which we give below, relies upon conformal field theory. To motivate the definition we first spend a moment on a descriptive aspect of the Calabi–Yau/conformal field theory connection. Immediately after giving the definition of mirror manifolds we give two essential mathematical implications.

As mentioned, string theory leads us to associate with a chosen Calabi–Yau manifold an object known as a conformal field theory. Many of the geometrical properties of the manifold find a natural expression in its associated conformal field theory. In particular, let us specialize to the case of complex dimension three for the clarity of the discussion, though most of the results generalize to other dimensions. The Hodge diamond for a Calabi–Yau three fold takes the following

[3] Note that the usual positivity conditions on the imaginary part of J

are still expected to hold.

form:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{2,1} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

We see that the only Hodge numbers which are not uniquely determined are $h^{1,1} = h^{2,2}$ and $h^{2,1} = h^{1,2}$. Recall that deformations of M preserving the condition of Ricci-flatness are parametrized by precisely these cohomology groups: $H^{1,1}$ parametrizes infinitesimal deformations of the Kähler class while $H^{2,1}$ parametrizes infinitesimal deformations of the complex structure. The fact that $h^{3,0} = 1$ reflects the property that the canonical line bundle on M (more precisely, canonical sheaf) is trivial and hence admits a global holomorphic section Ω. The three–form Ω allows us to establish a canonical isomorphism between $H^{2,1}(M)$ and $H^1(M,T)$, where T is the holomorphic tangent bundle. Concretely, we have $\omega : H^1(T) \to H^{2,1}(M)$ given in local coordinates by

$$b^{j}_{\bar{i}} dX^{\bar{i}} \frac{\partial}{\partial X^j} \to \Omega_{ijl} b^{j}_{\bar{i}} dX^i dX^l dX^{\bar{i}} \ . \qquad (2.2)$$

As outlined in the appendix, these cohomology groups are related to a special class of fields in the conformal field theory, which generate deformations of the theory preserving its superconformal structure. These fields are labeled by their transformation properties under the symmetry algebra — the $N = 2$ supersymmetry algebra. Further, the difference between the two kinds of fields generating the two kinds of deformations lies in the *sign* of their charge (eigenvalue) under a $U(1)$ subalgebra, the choice of sign being a matter of convention. In other words, whereas elements of $H^{1,1}$ and $H^{2,1}$ are vastly different geometrical objects, their conformal field theory counterparts have no intrinsic differentiating characteristics. This is strange: How, for instance, should one determine *intrinsically* whether a given field in a conformal field theory arises from $H^{1,1}$ or from $H^{2,1}$? In the late 1980s, this led several physicists [5,1] to conjecture the existence of a second Calabi–Yau manifold, which would correspond to the *same* conformal field theory, but such that the identifications of fields with geometrical objects would follow the opposite convention for the sign of the $U(1)$ eigenvalue. That is, we *can't* tell *intrinsically* whether a given field in a conformal field theory arises from $H^{1,1}$ or from $H^{2,1}$ because, in actuality, it arises, say, from $H^{1,1}(M)$ *and* from $H^{2,1}(\tilde{M})$, with $\tilde{M}$ a different Calabi-Yau space.

We can now define what is meant by a mirror pair.

Definition: Two Calabi–Yau manifolds M and $\tilde{M}$ constitute a mirror pair if they correspond to the same conformal field theory, and the association of geometrical objects on the two manifolds to fields in the conformal field theory differs by a reversal of the charges under the left-moving $U(1)$ of all fields.

Technically, this is really the definition of *classical* or *perturbative* mirror symmetry because it turns out that conformal field theory provides the framework for analyzing quantum string theory in a perturbative expansion. The lowest order term in this expansion—classical string theory—is conformal field theory on the sphere; the next term is conformal field theory on the torus, and so forth. *Quantum* mirror symmetry does not only require an isomorphism between the conformal field theories associated to two Calabi-Yaus, but, in addition, an isomorphism between the full quantum string theories associated to each. As we briefly discuss later, this has some striking mathematical implications which, in fact, can be used to motivate the first purely mathematical notion of mirror symmetry.

For now, sticking to classical mirror symmetry (we will drop the word classical for most of the sequel), let's give two particular implications which follow from the definition. These embody the properties of mirror manifolds which have been used in most mathematical applications to date. As an aside, we note that some mathematicians have loosely refered to these implications as constituting the definition of mirror manifolds. However, it is crucial to emphasize that in the language of physics, mirror manifolds, as stated, correspond to the same conformal field theory. Thus, not only are their respective three point functions equal (which is the content of implication 2 below), but their n–point functions are also equal, for arbitrary n. The implications we give emphasize the three point functions as these have the most direct geometrical interpretation; it would be of interest to study the geometrical content of the other n–point functions.

Implication: If M and $\tilde{M}$ are a mirror pair then

1) $h^{p,q}(M) = h^{3-p,q}(\tilde{M})$

2) $H^{p,q}(M) \cong H^{3-p,q}(\tilde{M})$ under an isomorphism preserving the quantum triple products [4].

The first of these implies that the Hodge diamond of $\tilde{M}$ is related to that of M by a reflection about a diagonal axis; this is the origin of the term "mirror manifolds" [2]. In particular, except in very special cases, M and $\tilde{M}$ are topologically distinct. The quantum trilinear products are deformations of natural mathematical triple products based on the ordinary cup product of $H^*(M)$. Their motivation from the point of view of string theory is described in the appendix; they are an example of what has come to be known more generally as quantum cohomology. Here we give the explicit expressions for the nontrivial cases of interest to us. To do so, let us first define the natural mathematical triple products refered to above. These are [11,12]: $I^{1,1} : H^{1,1} \times H^{1,1} \times H^{1,1} \to \mathbb{C}$ given by

$$I^{1,1}(B_{(i)}, B_{(j)}, B_{(k)}) = \int_M B_{(i)} \wedge B_{(j)} \wedge B_{(k)} \tag{2.3}$$

and $I^{2,1} : H^{2,1} \times H^{2,1} \times H^{2,1} \to \mathbb{C}$ given by

$$I^{2,1}(A_{(i)}, A_{(j)}, A_{(k)}) = \int_M \Omega \wedge A^l_{(i)} \wedge A^m_{(j)} \wedge A^n_{(k)} \Omega_{lmn} \tag{2.4}$$

where Ω_{lmn} is the holomorphic three-form on M and $A_{(i)}$ are elements of $H^{2,1}$ (expressed as elements of $H^1(M,T)$ with their subscripts being tangent space indices). Notice that (2.3) is nothing but the cup product giving rise to the triple intersection matrix. Equation (2.4) is most

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[4] Language becomes a bit confusing here. Although we are working at the level of conformal field theory on the sphere— classical string theory— this field theory is affected by perturbative and nonperturbative physics that is often refered to as "quantum corrections." From the point of view of space-time physics, these corrections are "stringy" corrections (that is, they arise from the extended structure of the string) as opposed to quantum corrections, but the latter term is commonly used.

easily described as the natural map from $H^1(T)\times H^1(T)\times H^1(T)\to\mathbb{C}$ arising from the triviality of the canonical bundle. As written, (2.4) is this map after the isomorphism (2.2) has been employed.

The quantum triple products are deformations of these natural mathemtical constructs in a manner dictated by string theory as outlined in the appendix. In particular, these are $I_Q^{1,1}: H^{1,1}\times H^{1,1}\times H^{1,1}\to\mathbb{C}$ given by [13]

$$
\begin{aligned}
I_Q^{1,1}(B_{(i)},B_{(j)},B_{(k)}) &= I^{1,1}(B_{(i)},B_{(j)},B_{(k)}) \\
&+\sum_{I_{n,m}} m^{-3}\, e^{\left(-\int_{I_{n,m}}\iota^*(J)\right)} \int_{I_{n,m}}\iota^*(B^{(i)})\int_{I_{n,m}}\iota^*(B^{(j)})\int_{I_{n,m}}\iota^*(B^{(k)})
\end{aligned}
\tag{2.5}
$$

where $I_{n,m}$ is an m-fold cover of a rational curve on M of degree n, $\iota: I_{n,m}\hookrightarrow M$ the inclusion, and J is the Kähler form of M. The definition of the quantum deformation of (2.4) is much simpler: $I_Q^{2,1}: H^{2,1}\times H^{2,1}\times H^{2,1}\to\mathbb{C}$ is given by

$$
I_Q^{2,1}(A_{(i)},A_{(j)},A_{(k)}) = \int_M \Omega\wedge A^l_{(i)}\wedge A^m_{(j)}\wedge A^n_{(k)}\Omega_{lmn}\,. \tag{2.6}
$$

In other words, the relevant triple product arising in string theory is precisely the natural mathematical construct in this case [15].

Having defined what we mean by the quantum triple products, lets briefly explain why the implications given above follow from the definition of mirror manifolds. To do so, let M and $\tilde{M}$ be a mirror pair of Calabi–Yau manifolds which correspond to the conformal field theory K. This implies that every construct in K has *two geometrical interpretations* – one on M and one on $\tilde{M}$. In particular, this is true of the correlation functions in K. As discussed in the appendix, there are two special classes of correlation functions in K. One such class has the geometrical interpretation on M as $I_Q^{1,1}$ while the other is given geometrically by $I_Q^{2,1}$. Now, the crucial point of mirror symmetry is that fields in K which correspond to $(1,1)$ forms ($(2,1)$ forms) on M correspond to $(2,1)$ forms ($(1,1)$ forms) on $\tilde{M}$. Thus, correlation functions in K which are geometrically interpretable as $I_Q^{1,1}$ on M necessarily are geometrically interpretable as $I_Q^{2,1}$ on $\tilde{M}$. Similarly, correlation functions in K which are geometrically interpretable as $I_Q^{2,1}$ on M are geometrically interpretable as $I_Q^{1,1}$ on $\tilde{M}$. Hence, we arrive at the equality stated in implication 2. In summary then, the identities which arise from mirror symmetry have their origin in the fact that a given object in the conformal field theory has two vastly different geometrical interpretations on the manifolds of a mirror pair.

We now state the *mirror conjecture* and return shortly to a less robust version that can be elevated to the level of a theorem.

Conjecture: For every[5] Calabi–Yau manifold M there exists a *mirror* Calabi–Yau manifold $\tilde{M}$. In particular, for $d = 3$

$$h_M^{p,q} = h_{\tilde{M}}^{3-p,q} \tag{2.7a}$$

and

$$I_Q^{1,1}(M) = I_Q^{2,1}(\tilde{M}) \tag{2.7b}$$

$$I_Q^{2,1}(M) = I_Q^{1,1}(\tilde{M}) \ . \tag{2.7c}$$

We note that by the latter equalities of triple products we mean that there exists a suitable choice of bases of the relevant cohomology groups such that the equalities hold. In the next section we review that for a certain subclass of Calabi–Yau spaces we can establish the mirror conjecture and explicitly construct the mirror manifold $\tilde{M}$.

3. A CONSTRUCTION OF MIRROR MANIFOLDS

In this section we focus on a subclass of Calabi–Yau spaces for which we can establish the mirror conjecture described previously.

[5] The precise meaning of 'every' in this context has never been made precise. The last decade has shown that the conjecture appears to be true for Calabi-Yau spaces constructed as complete intersections in toric varieties, but beyond this class no general statements have been shown. In the last section we will note some relatively recent insight on this issue. Relatedly, in the early days of mirror symmetry, a natural question that was asked was: what happens when one member of a supposed mirror pair is rigid, i.e. $h^{d-1,1} = 0$? Then the supposed mirror would appear to be non–Kähler. This question has a natural solution, once one realizes the fact, discussed later, that the moduli space of Calabi-Yau manifolds has a cell structure in which certain cells have a natural physical interpretation but no geometrical interpretation. Moreover, there are degenerate moduli spaces where only the latter cells appear—these turn out to be the mirror pairs of rigid Calabi-Yau manifolds.

The proof involves an explicit construction of the mirror manifold pair; we will see in section four that this allows the extraction of nontrivial information about both members of the pair. We again specialize the discussion to the case of complex dimension three for definiteness, although most of what we say is independent of this choice. We begin with the main result of [2]:

Theorem Let M be a Calabi–Yau hypersurface of degree n in weighted projective four space of Fermat type. Let G be the group of coordinate scaling symmetries which preserve the holomorphic three form on M. Then, the mirror $\tilde{M}$ of M exists and is given by the quotient M/G.

We note that M/G is a singular space since G generally acts with fixed points. However, due to [9] we know that M/G admits a desingularization to a smooth Calabi–Yau manifold; it is the cohomology of the latter which we refer to. (The role of singularity resolution in mirror symmetry will be discussed shortly.)

A related point and one that must be emphasized is that although we perform the explicit construction of the mirror for a specific manifold representing a very special point in the parameter space (the Fermat point) the theorem in fact allows us to construct the mirror manifold in (at least) a neighborhood of this point. This is because mirror symmetry yields an isomorphism of the tangents to the paired moduli spaces, allowing us to relate deformations of one manifold away from the Fermat point to deformations of the mirror manifold from the Fermat mirror. The isomorphism relates deformations of the complex structure of M to deformations of the (complexified) Kähler structure of $\tilde{M}$ and vice versa. Note that in general the deformed version of M and the correspondingly deformed version of $\tilde{M}$ are not related by an orbifolding operation (as are M and $\tilde{M}$) and are not of Fermat type – nonetheless they are a mirror pair. The result that these corresponding deformations preserve the mirror symmetry property is best understood from the conformal field theory point of view. It reflects the fact that we have *one* conformal theory with two geometric interpretations. Deforming the Kähler structure on $\tilde{M}$ corresponds to a particular deformation of the conformal field theory (by turning on a vacuum expectation value for some truly marginal operator). This deformation of the conformal theory (by this particular marginal operator) is interpretable on M as a complex structure deformation (by the corresponding complex structure modulus). After deformation, then, we again have one conformal theory (a deformation of the original) with two geometrical interpretations (each a deformation of one of the orginal spaces).

In the early days of mirror symmetry, physicists emphasized their belief that there is no obstruction to extending this local deformation argument to the whole of moduli space thus yielding the result: *if a Calabi–Yau space lies in the same moduli space as a Fermat hypersurface then it has a mirror.* This statement made some mathematicians uncomfortable, especially in light of the fact the global structure of the (complexified) Kähler moduli space of one Calabi-Yau and the complex structure moduli space of its mirror appear to be quite different: the former is a bounded domain while the latter is a quasi-projective variety. Resolving this issue took a number of years and uncovered a substantial amount of interesting physics and mathematics, as we we briefly indicate shortly.

An immediate corollary of the theorem which underlies its mathematical applications thus far is:

Corollary:

$$I_Q^{1,1}(M) = I_Q^{2,1}(\tilde{M}) \qquad (3.1a)$$
$$I_Q^{2,1}(M) = I_Q^{1,1}(\tilde{M}) \ . \qquad (3.1b)$$

As noted in [2], there is an interesting limit of this corollary – namely, if the 'radius' of M, say, should tend to "infinity" [6] then $I_Q^{1,1}(M)$ tends to the intersection matrix on M, i.e. $I^{1,1}(M)$. Then, (3.1) relates this topological quantity on M to a quasi–topological (depends on the complex structure as well) quantity on the topologically distinct space $\tilde{M}$. We will return to these ideas shortly.

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[6] The meaning of "infinity" in this context proves to be both subtle and crucial. As we shall see, the Kähler moduli space of a Calabi-Yau should be enlarged by attaching the Kähler cones of particular Calabi-Yaus that are birational to the original, among others, along common walls. Then, within each of the Kähler cones for the birational Calabi-Yaus there is a notion of going to "infinity" — i.e. approaching the deep interior point of the complexified Kähler cell, as we will discuss.

Before moving on to a description of the proof of the theorem, we illustrate the result with one well known example; we will return to this example in the final section, as the object of study in the first applications of mirror symmetry

Consider the quintic Fermat hypersurface in $\mathbb{CP}^4$. The coordinate scaling symmetries constitute a $(\mathbb{Z}_5)^5$. One of these (the diagonal) is trivial because we work in projective space. Another does not preserve the holomorphic three form. Thus, G is the group $(\mathbb{Z}_5)^3$. Hence,

$$Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 + Z_5^5 = 0 \tag{3.2a}$$

and

$$\frac{Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 + Z_5^5 = 0}{(\mathbb{Z}_5)^3} \tag{3.2b}$$

constitute a mirror pair. We note that the former has $h^{1,1} = 1$ and $h^{2,1} = 101$ while the latter has $h^{1,1} = 101$ and $h^{2,1} = 1$. In particular, $\chi(M) = -200$ and $\chi(\tilde{M}) = 200$, where χ denotes the Euler characteristic. In table 1 we list some other explicit examples of the construction.

We now turn to a brief summary of how we establish the mirror symmetry theorem stated above. It is at this point in the discussion that conformal field theory ideas play a crucial role. Almost a decade after this result was established with such physical methods, we are still not aware of a means of establishing the results without recourse to these techniques. Recent insights, discussed in the last section, have made headway toward a proof using more conventional mathematical tools, but a finding complete argument has proved to be a difficult challenge. Our goal here is to communicate the most skeletal description of the arguments, referring the interested reader to [2] and [29] for details.

The strategy for constructing mirror manifolds based on Fermat hypersurfaces is as follows: We seek a nontrivial *automorphism* $\mathcal{O}$ of the conformal field theory K which realizes an operation on the corresponding geometrical construction which is *not* an automorphism of the corresponding Calabi–Yau hypersurface M. Rather, we seek an $\mathcal{O}$ such that $\mathcal{O} : M \to \tilde{M}$ with $\tilde{M}$ being Calabi–Yau and not isomorphic to M. In this way, M and $\tilde{M}$ would be intimately linked by their common conformal field theory. In particular, as mentioned above, if $\mathcal{O}$ acts on the conformal field theory by reversing the sign of all charges under the left-moving $U(1)$ symmetry, we have the following situation: Both M and $\mathcal{O}(M)$ correspond to the same conformal field theory since K and $\mathcal{O}(K)$ are isomorphic. Furthermore, since the explicit isomorphism between K and $\mathcal{O}(K)$ is the reversal of the sign of the left moving $U(1)$ charges, the identification of conformal fields with differential forms is *reversed* on $\mathcal{O}(M)$ relative to M. Thus M and $\mathcal{O}(M)$ would constitute a mirror pair, if such an operation $\mathcal{O}$ could be found. In particular, we would have

$$h^{2,1}_M = h^{1,1}_{\mathcal{O}(M)} \tag{3.3a}$$

$$h^{1,1}_M = h^{2,1}_{\mathcal{O}(M)} \,. \tag{3.3b}$$

In [2] such an operation $\mathcal{O}$ was constructed. Its existence relies on the notion of orbifolding. If a manifold M is invariant under a group of symmetries G (in our case these will generally be holomorphic automorphisms) we can consider the quotient space $\tilde{M} = M/G$. We restrict attention to groups G which ensure that $\tilde{M}$ is also Calabi-Yau. Such G are called 'allowable'.[7]. Similarly, if a conformal theory K respects a symmetry group G, we can consider the quotient theory K/G. The crucial property of these operations for our argument, is that the quotient conformal theory describes propagation on the quotient of the underlying Calabi-Yau manifold by the same allowable group actions G[8] The relationship between group actions in the two pictures is furnished by the identification of homogeneous coordinates on the manifold with primary fields in the conformal field theory given by the arguments of [16,17] and more precisely in the description of [27].

The correspondence between conformal field theories and Calabi–Yau manifolds is most explicitly understood for what are known as 'minimal model' conformal field theories and represent Calabi–Yau hypersurfaces in weighted projective space [18,16,17]. The proof of the theorem given in [2] relies on the algebraic construction of these

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[7] We note that G is allowable if it preserves the holomorphic d–form present on the initial Calabi-Yau d–fold.

[8] To be more precise, the quotient will have singularities at the fixed points of the G action. The associated conformal field theory, though, is perfectly well defined.

'minimal model' conformal field theories (derived from simple representations of the superconformal algebra). Using this formulation, one can show that the operation $\mathcal{O}$ defined by taking the quotient by the group of 'allowable' scaling symmetries G is an automorphism of the conformal field theory satsfying the conditions mentioned. Namely, the explicit isomorphism involves reversing the $U(1)$ eignevalues of all fields. Let us note here the points in the argument at which the conformal field theory enters in a crucial way. Initially, arguments from conformal field theory motivated the construction of the particular quotient described above; in a sense this is inessential, since in principle the construction itself can be carried out with no reference to the physical interpretation. One simply considers a given Fermat Calabi–Yau hypersurface together with the appropriate orbifolds thereof. One then finds two manifolds with Hodge diamonds related in the requisite way. The crucial contribution of the conformal field theory argument is in showing that the relation goes quite a bit deeper, giving the isomorphism between the relevant cohomology classes and the equality of the quantum triple products (and more generally quantum n–products) as well as the extension of these equalities through deformation arguments.

Before discussing the implications of mirror symmetry, we pause here to emphasize three important points. First, the construction above works equally well on any orbifold of the theories under discussion. That is, the space of all orbifolds of a given theory can be partitioned into mirror pairs. We illustrate this with the quintic hypersurface in table 2. We note that the mirror of a theory M/H with $H \subset G$ is given by M/H^* where H^* is the complement of H in G (that is, the smallest group containing H and H^* is G). Second, the arguments are not specific to complex dimension three and immediately generalize to other dimensions. Third, as discussed earlier, although our discussion to this point has been tied to very special points in the respective Calabi-Yau moduli spaces by deformation arguments we can immediately extend our results to more general points in moduli space.

Moreover, as mentioned in the introduction, many believe that the phenomenon of mirror symmetry extends well beyond this class of examples. Concrete evidence for this comes from a variety of sources. First, simultaneous with the above construction of a class of mirror manifolds, in [3] Candelas, Lynker and Schimmrigk completed a thorough computer search of Calabi-Yau manifolds in weighted projective four space (not restricted to Fermat type). They found that the set of all such constructions is almost invariant under the interchange of $h^{1,1}$ and $h^{2,1}$. While finding two manifolds differing by the interchange of these Hodge numbers by no means assures that the manifolds are a mirror pair (to be a mirror pair they must correspond to the same conformal field theory) their results was an early indicator that mirror duality extends beyond the particular class of models treated in [2].

Second, a few years after the papers of [2] and [3], Batyrev [32](inspired in part by earlier work of Roan [6]) used methods of toric geometry to extend significantly the class of purported mirror pairs of Calabi-Yau spaces. In particular, Batyrev noticed that a connected family of Calabi-Yaus realized as hypersurfaces in toric varieties is defined by combinatorial data involving a pair of reflexive polyhedra. This naturally led Batyrev to conjecture that mirror symmetry amounts to the *interchange* of these two reflexive polyhedra—a conjecture which he supported by much circumstantial evidence such as the correct interchange of Hodge numbers of the purported Calabi-Yau mirrors, as well as the fact that this conjecture reduces to the mirror construction via orbifolding discussed earlier, when applied to that case. Since Batyrev's paper (as well as a follow-up generalization to complete intersections in toric varieties done in collaboration with Lev Borisov [33]) physicists have subjected his conjecture to a great deal of detailed tests and it has survived all in tact. Most string theorists believe that Batyrev's conjecture does in fact construct pair of mirror Calabi-Yau manifolds, although no one has been able to establish this fully.

Having outlined the arguments by which mirror symmetry was shown to exist and the various generalizations discovered over the last decade, let us briefly review some consequences. In the initial paper of [2] describing the construction of mirror manifolds, a number of important implications of mirror symmetry were given. In the next section we review the arguments that lead to these claims, as well as subsequent work by other groups which lead to a spectacular verification of their veracity as well as to some new applications.

4. APPLICATIONS OF MIRROR SYMMETRY

In the previous section we have briefly summarized the arguments which establish the existence of a class of mirror manifolds as well as a larger class of conjectured mirror manifolds. In this section we will review two applications of this construction. The first centers on the consequences of (2.7*b*), which we reiterate here explicitly:

$$\int_{\tilde{M}} \tilde{\Omega} \wedge \tilde{B}^l_{(i)} \wedge \tilde{B}^m_{(j)} \wedge \tilde{B}^n_{(k)} \tilde{\Omega}_{lmn} = \int_M B_{(i)} \wedge B_{(j)} \wedge B_{(k)}$$

$$+ \sum_{I_{n,m}} m^{-3} e^{\left(-\int_{I_{n,m}} \iota^*(J)\right)} \int_{I_{n,m}} \iota^*(B^{(i)}) \int_{I_{n,m}} \iota^*(B^{(j)}) \int_{I_{n,m}} \iota^*(B^{(k)}) \tag{4.1}$$

where we remind the reader that the $\tilde{A}^{(i)}_a$ are $(2,1)$ forms (expressed as elements of $H^1(\tilde{M}, T)$ with their subscripts being tangent space indices) $\tilde{\Omega}$ is the holomorphic three form, the B^i are $(1,1)$ forms correspondingto the $\tilde{B}^{(i)}_a$, $I_{n,m}$ is an m-fold cover of a rational curve on M of degree n, $\iota : I_{n,m} \hookrightarrow M$ the inclusion, and J is the Kähler form of M. In the physics literature, the terms in the second line of (4.1) are called instanton corrections (and constitute an example of the "quantum" corrections referred to earlier). We recall the crucial role played by the underlying conformal field theory in deriving this equation. If we simply had two manifolds whose Hodge numbers were interchanged we could not make any such statement. This result [2] is rather surprising and clearly very powerful. We have related expressions on *a priori* unrelated manifolds which probe rather intimately the structure of each. Furthermore, the left hand side of (4.1) is directly calculable while the right hand side requires, among other things, knowledge of the rational curves of every degree on the space.[9]

In perhaps the most famous of the early papers on mirror symmetry, Candelas, de La Ossa, Green and Parkes analyzed (4.1) for the case of the quintic–mirror-quintic pair (3.2). In particular, they considered a one parameter family of mirror manifolds given by deforming along the single Kähler modulus (complex structure modulus)

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[9] We recall a remark made earlier – (4.1) is but one of an infinite series of equalites implied by having a pair of mirror manifolds. As yet, the full geometrical content of any of the other equalities has not been studied in much detail.

on the quintic (mirror quintic), as we discussed in the general context in section 2. Through a careful analysis of the map between Kähler and complex structure deformations, the authors were able to extract information about the right hand side of (4.1) from direct calculation of the left hand side using standard techniques of variation of Hodge structure. In practice, this amounted to solving certain Picard-Fuchs differential equations which govern the dependence of the periods of the holomorphic three-form Ω on the complex structure of $\tilde{M}$. A comparison to the right hand side then yielded a relation between the numbers of rational curves of various degrees on M and the coefficients in an asymptotic expansion of solutions to these Picard-Fuchs equations of $\tilde{M}$. This enabled these authors to find the *number* of rational curves (holomorphic instantons) of any desired degree on the quintic hypersurface simply by carrying out this asymptotic expansion to any desired order.

This result took mathematicians by surprise, and led to a good natured showdown at the Mathematical Sciences Research Institute meeting on Mirror Symmetry in 1991. For degree 1 and 2 curves, the well known number of rational curves (d=1: 2875, d=2: 609250) were reproduced by this mirror symmetry calculation. The degree 3 case, though, proved puzzling. Candelas and his collaborators found, using mirror symmetry, that the number of curves of degree 3 was 317206375. But Ellingsrud and Stromme found, using more conventional mathematical methods, that the number of curves of degree 3 on the quintic was 2682549425. Beyond sorting out this one particular result, there was a great deal of motivation to determine who was right because the mirror symmetry calculation could easily be extended to curves of arbitrary degree while, at that time, it seemed nearly impossible to similarly extend the mathematical calculation. Although the question of who was right was not settled at the MSRI meeting, shortly thereafter Ellingsrud and Stromme found an error in their computer code, which when corrected, confirmed the results emerging from mirror symmetry.

Since then, a great deal of effort has been expended on both verifying the formulae for rational curves predicted by mirror symmetry and on finding a means of performing justified calculations of these numbers using more rigorous mathematical tools. Many mathematicians have played a pivotal role in these developments and the talk of Kefung Liu at this meeting will update the impressive progress that has been made to date.

The second application of mirror symmetry has both mathematical and physical implications. As mentioned earlier, from the point of view of physics, in the early 1990s it was conjectured that once a mirror pair of Calabi-Yau spaces is found, this ensures that the mirror relationship can be extended to all points in the connected component of moduli space to which either Calabi-Yau space belongs. The basic reasoning behind this conjecture relied on deformation theory: every continuous deformation to one member of a pair has a mirror deformation defined on the mirror Calabi-Yau which modifies the underlying physical model in precisely the same way (thereby preserving the mirror relationship). Taken at face value, this physics reasoning leads to the mathematical statement that the complex structure moduli space of one Calabi-Yau manifold M is isormorphic to the (complexified) Kähler moduli space of its mirror $\tilde{M}$. But is this true? As alluded to earlier, the mathematical structure of these moduli spaces is plain different and hence the answer is clearly no. The resolution to this apparent conflict is simple, but quite interesting.

The work of [27] and [26] showed that string theory requires that we enlarge the Kähler moduli space of a Calabi-Yau manifold by adjoining a number of related moduli spaces along the various walls of a given Kähler cell. Some of these cells correspond to the complexified Kähler moduli spaces of Calabi-Yau manifolds which are birational to the original Calabi-Yau $\tilde{M}$ but differ from it by flops of rational curves. Physically, such cells contain the Kähler moduli of a conformal field theory built on the flopped Calabi-Yau model. Other cells only admit a more abstract physical interpretation which we will not discuss here. The essential point, though, is that the totality of these Kähler cells — the so-called fully enlarged Kähler moduli space — *is* isomorphic to the complex structure moduli space of M, clearing up the the previous conflict. Mirror symmetry, therefore, maps points in the complex structure moduli space of M to points in the fully enlarged Kähler moduli space of $\tilde{M}$—*not* simply to the Kähler moduli space of M as was initially thought.

From the point of view of physics this has an important implication: Singularities in the complex structure moduli space of M are at most complex codimension one. Since the Kähler parameters of M can be held fixed at a "large" value (a value ensuring the validity of perturbative methods), we can conclude that physical singularties are also at most complex codimension one. (In fact, subsequent research has shown that even this statement can be significantly relaxed.) By mirror symmetry, the same statement must hold true for the fully enlarged Kähler moduli space of $\tilde{M}$. But this implies that a generic path in the fully enlarged Kähler moduli space of $\tilde{M}$ which penetrates through cell wall(s), passes through *smooth* physical models. In other words, paths along which the topology of a Calabi-Yau changes via a flop transition are perfectly smooth from the point of view of physics, establishing the first concrete example of smooth topology changing processes in physics.

5. IMPORTANT RECENT DEVELOPMENTS

There have been many important developments in mirror symmetry over the past few years, but there are three general directions which stand out most notably.

The first surrounds attempts to establish the mirror symmetry results in enumerative geometry by using more traditional mathematical methods. A long list of impressive results have been as we will hear about from Liu. These works appear to establish the counting of rational curves originally found using mirror symmetry.

The second development, found by Strominger, Yau, and Zaslow [30] involves a careful application of notions emerging from the second superstring revolution—techniques which give access to nonperturbative effects in string theory— to mirror symmetry. Their work focuses on the stronger notion of *quantum* mirror symmetry which requires that M and $\tilde{M}$ satisfy the constraints of mirror symmetry as used to this point, but, additionally, M and $\tilde{M}$ must give rise to isomorphic compactified *quantum* string theories. Although it would take us too far afield to review all the physics underlying their results, let's summarize the mathematical content. According to Strominger, Yau, and Zaslow, for M and $\tilde{M}$ to constitute a quantum mirror pair, $\tilde{M}$ must be the moduli space (suitably compactified) of special Lagrangian (real three dimensional) submanifolds of M. (Just by way of physics motivation, these Lagrangian submanifolds make an appearance because they are associated with so-called Dirichlet 3-branes which wrap around "supersymmetric" 3-cycles in a Calabi-Yau space. Realizing the Calabi-Yau, locally, as a supersymmetric T^3 fibration, T-duality on the fibers takes us to an isomorphic theory in which 3-branes turn into 0-branes. As the moduli space of 0-branes on a Calabi-Yau space $\tilde{M}$ is the Calabi-Yau itself, we learn that $\tilde{M}$ is also isomorphic (by T-duality) to the moduli space of supersymmetric 3-cycles on M.) Putting the physics motivation aside, this characterization of mirror symmetry is intrinsic in the sense that it does not require reference to the underlying string theory. Therefore, it is amenable to direct mathematical analysis without the mathematical imprecision that accompanies a quantum mechanical theory. A number of mathematicians have and continue to pursue this direction which holds the hope of a complete translation of the physics of mirror symmetry into rigorous mathematics.

Third, a great deal of progress has been made on establishing the connectivity of the full space of Calabi-Yau manifolds—a notion contemplated by both mathematicians and physicists over many years. Specifically, through the conifold transition results of [34] and numerous generalizations, we now have a means of performing physically smooth deformations which take us between Calabi-Yau manifolds with different Hodge numbers. Mathematically, these transitions involve degenerating real 3-cycles in an original Calabi-Yau space (satisfying nontrivial homology relations) and then resolving the resulting singular space by performing small resolutions. Since we now know that such manipulations are perfectly smooth from the point of physics, we can extend the deformation arguments given earlier and claim that mirror symmetry is established for all Calabi-Yau spaces that are part of the huge web of Calabi-Yaus joined by conifold transitions and there generalizations. Some believe that this web may well include all Calabi-Yau spaces, thereby ensuring that mirror symmetry holds for all. This has yet to be established.

6. CONCLUSIONS

Mirror symmetry is almost a decade old, and these first ten years have established it as a powerful, unexpected, and as yet incompletely understood structure in string theory and in algebraic geometry. Relative to the earliest days of the subject, there is now a substantial amount of mathematical insight into mirror symmetry, but there is certainly a way to go before complete analytic understanding is achieved. Moreover, it is almost a certainty that mirror symmetry, as currently formulated, is but a small piece of a larger physical and mathematical structure. Namely, in heterotic string theory, the data defining a string model includes all that was described above (a Calabi-Yau manifold M, an element of $H^2(M)$, as well as additional data required to define the full quantum string theory) but, additionally, it includes the specification of a stable, holomorphic vector bundle V whose first Chern class vanishes and whose second Chern class equals that of the tangent bundle of M. To this point, we have implicitly chosen V to be identically T_M and have therefore not called attention to this data. This choice gives rise to a so-called $(2,2)$ model, where these numbers refer to the number of left and right moving supersymmetries. More general choices for V give rise to $(0,2)$ models.

At the level of heterotic string theory, mirror symmetry as so far described establishes a physical isomorphism between string propagation on (M, T_M) and string propagation on $(\tilde{M}, T_{\tilde{M}})$. But, although less well understood, one can make different choices for the bundle V. Then, one expects there to be a more general version of mirror symmetry establishing an isomorphism between string propagation on (M, V) and $(\tilde{M}, \tilde{V})$. A little more precisely, the $(2,2)$ mirror relationship between $(2,1)$ forms on M and $(1,1)$ forms on $\tilde{M}$ and vice versa, arises from an isomorphism between $H^1(M, T_M)$ and $H^1(\tilde{M}, T_{\tilde{M}})$ (and vice versa) at the level of quantum cohomology. The $(0,2)$ version of this statement is likely to relate $H^1(M, V)$ and $H^1(\tilde{M}, \tilde{V})$, again, at the level of an isomorphism between the quantum cohomologies of $H^*(M, \wedge^* V)$ $H^*(\tilde{M}, \wedge^* \tilde{V})$. Recent work heading toward this goal has been carried out by C. Vafa, E. Sharpe, R. Blumenhagen, and others but there is much further to go before $(0,2)$ mirror symmetry is understood at the level of $(2,2)$ mirror symmetry. This is an important problem for the future.

Appendix A. Conformal Field Theory and Algebraic Geometry

This appendix attempts to briefly describe the correspondence between conformal field theories and Calabi–Yau manifolds given by string theory. The intent is more to convey a glossary of terms and the spirit of the relationship than to present a full derivation of the results mentioned.

A.1. From Manifolds to Conformal Field Theories

Perturbative string theory describes the physics of one-dimensional extended objects. As they propagate, strings sweep out a two dimensional "worldsheet", and calculations in string theory lead us to consider a quantum field theory on this two-dimensional surface. A quantum field theory on a surface Σ is composed of two essential ingrediants: *fields* and an *action.* The former are sections of chosen bundles on Σ while the latter is a real valued functional of the sections. For the case at hand, the fields are simply coordinates parametrizing the embedding of Σ in the chosen Calabi–Yau manifold M, and the action is basically the area of the image of Σ in the induced metric (more preceisly, this is true after solving the Euler–Lagrange equations for the metric on Σ). [10] Explicitly, the action S is defined by

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma\sqrt{|h|}\left(h^{\alpha\beta}G_{ij}(X)\partial_\alpha X^i\partial_\beta X^j + \epsilon^{\alpha\beta}B_{ij}(X)\partial_\alpha X^i\partial_\beta X^j\right) . \tag{A.1}$$

In (A.1) the 'fields' X^i comprise the map $X : \Sigma \to M$ embedding the worldsheet in a (spacetime) manifold M, $h_{\alpha\beta}$ is a metric on Σ, $d^2\sigma\sqrt{|h|}$ an invariant integration measure on Σ, and G_{ij} a metric on M; the antisymmetric tensor B_{ij} is an additional term, of crucial importance in string theory but absent in discussions of particle propagation; α' is a dimensionful parameter related to the 'tension' of the string. The fundamental objects in a quantum field theory are the *correlation functions* of fields ϕ_j (constructed for example from X and its derivatives) inserted at points z_j on Σ which are formally calculated using the action S as

$$\langle\phi_1(z_1)\cdots\phi_N(z_N)\rangle = \int DXe^{-S(X)}\phi_1(z_1)\cdots\phi_N(z_N) . \tag{A.2}$$

A rigorous definition of the integral (A.2) over maps X is a very difficult and in general unsolved problem, though in the situations we discuss it is unusually well-behaved. For our purposes, to motivate the construction presented later, it will suffice to consider it merely as a formal device. For example, for small α' we can evaluate the

[10] For all of our discussion, the reader can safely take Σ to have the topology of a sphere.

integral as an asymptotic series by a saddle-point calculation.[11] In the theories of interest to us, there are additional (fermionic) degrees of freedom, and the action (A.1) contains other terms, but for our heuristic purposes the explicit form of these will not be necessary.

The consistency conditions of string theory lead us to study a specific class of these "sigma models", namely those which lead to a field theory with $N = 2$ superconformal symmetry. We note that superconformal symmetry embodies conformal symmetry (invariance under conformal transformations on Σ) and supersymmetry. The latter symmetry is not evident in our action since the fermionic degrees of freedom crucial to this symmetry have been suppressed for clarity of exposition. Applying the requirements that (A.1) give rise to a theory respecting superconformal invariance we find a number of conditions.[12] To lowest order in α' these are as follows. The manifold M is required to be a three dimensional complex Kähler manifold, and the metric G to have vanishing Ricci tensor. This is the origin of the interest in Calabi–Yau manifolds on the part of string theorists. In other words, the quantum field theory action (A.1) is conformally invariant if the manifold M is chosen to be Calabi–Yau.[13] The anti-symmetric tensor B_{ij} is required to be such that $B_{ij}dX^i dX^j$ is a harmonic $(1,1)$ form on M. The space of solutions to these consistency equations is the space of consistent string theories – loosely referred to in this context as the "moduli space" of the physical problem. Locally, then, this space is parametrized by the inequivalent Ricci-flat metrics on M and by an element of $H^{1,1}$ representing B. The former space is well known to decompose locally as a product of deformations of the complex structure of M parametrized by harmonic $(2,1)$ forms

[11] Note that α' can be absorbed into G by a rescaling so the true (dimensionless) expansion parameter is $\frac{\sqrt{\alpha'}}{R}$. This limit thus corresponds to large (nearly flat) manifolds.

[12] We are also imposing an additional requirement that the central charge of the representation have a prescribed value – this is required for a consistent string theory.

[13] We note that maintaining conformal invariance in the presence of quantum corrections forces adjustments to be made to the Ricci–flat metric on M but they do not change the topological property of M having vanishing first Chern class.

and deformations of the Kähler class of G parametrized by harmonic $(1,1)$ forms. The way the two terms appear in the action leads us to consider the variation of B_{ij} as *complexifying* the Kähler cone.

From this discussion, we see that starting from a Calabi–Yau manifold with the additional structure of the harmonic $(1,1)$ form B, we can build a unique conformal field theory. The point of our discussion in this paper is that going from a conformal field theory to a corresponding Calabi–Yau manifold is *not* unique. Rather, more than one Calabi–Yau manifold can yield the same conformal field theory.

The fields in our theory can be described by their transformation properties under the superconformal symmetry. In the models of interest, they transform as irreducible highest weight representations of this algebra. The detailed structure of the algebra will not be needed here but we note that the highest weight vectors are labeled by their conformal weight h and their charge q under a $U(1)$ symmetry. In fact, in a conformal field theory we find two copies of this algebra which act on holomorphic (resp. antiholomorphic) degrees of freedom, in terms of complex coordinates on Σ. There is a sector of the theory which is of special interest in string theory. It is composed of those highest weight fields with conformal dimension and $U(1)$ charge $(h,q) = (1,\pm 1)$. These "exactly marginal" fields parametrize deformations of the theory which preserve the consistency conditions mentioned above, thus representing tangents to the space of consistent theories. These deformations may be related to deformations of the action, and thus to the geometrically constructed moduli space discussed above. In particular, the space of fields with $q=1$ under both the holomorphic and the antiholomorphic algebras is related to $H^{2,1}$, and the space of fields with $(q=-1)$ $q=1$ under the (anti-)holomorphic algebras to $H^{1,1}$. This identification can be made quite explicit using the construction of the fields in terms of the coordinates X^i (the form structure is encoded in the fermionic degrees of freedom we have omitted here), and one may thus translate the evaluation of correlation functions of these fields to geometrical calculations on M. This is not a trivial calculation in general but for a certain set of correlators, the results are especially simple. These are the so-called "three-point functions" involving three operators of either type.[14] Because of an exact $SL(2,\mathbb{C})$ symmetry of the theory,

[14] We are not being quite precise here; the correlation functions of the

these are in fact independent of the points z_i on Σ at which the fields are inserted (see (A.2)) and thus lead to a triple product structure on the relevant cohomology groups which appears quite natural: We find $I^{2,1} : H^{2,1} \times H^{2,1} \times H^{2,1} \to \mathbb{C}$ given by

$$I^{2,1}(A_{(i)}, A_{(j)}, A_{(k)}) = \int_M \Omega \wedge A^l_{(i)} \wedge A^m_{(j)} \wedge A^n_{(k)} \Omega_{lmn} \qquad \text{(A.3)}$$

where Ω_{lmn} is the holomorphic three-form on M and $A_{(i)}$ are elements of $H^{2,1}$ (expressed as elements of $H^1(M,T)$ with their subscripts being tangent space indices), and $I^{1,1} : H^{1,1} \times H^{1,1} \times H^{1,1} \to \mathbb{C}$ given by

$$I^{1,1}(B_{(i)}, B_{(j)}, B_{(k)}) = \int_M B_{(i)} \wedge B_{(j)} \wedge B_{(k)}. \qquad \text{(A.4)}$$

We thus see that both the toplogy of M and its complex structure are very naturally embodied in properties of the conformal field theory it defines.

The considerations above were limited to the first nontrivial order in an expansion in small α'. In string theory, the correlation functions (A.2) are understood to be well-defined as functions of α', and the first term in an asymptotic series contains information only about some limit of these functions. However, in the case we are discussing, the existence of the superconformal symmetry places severe restrictions on the dependence of correlation functions on α' . These are embodied in "nonrenormalization theorems", which guarantee that the statements made above at lowest order in α' are in fact true to *all* orders in the expansion [24].[15] Since the series is asymptotic, "nonperturbative" contributions to the exact answers can still appear. In fact, they do appear, and one may obtain an explicit form for them [13,15]

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fields we discussed in fact vanish (for example by conservation of the $U(1)$ charge). What we are really referring to are the correlation functions of other fields in the theory which are canonically related to the marginal fields by the action of elements of the superconformal algebra.

[15] An exception are the conditions on G. These receive corrections at every order; the convergence of this series has not been proved.

$$I^{1,1} \to I_Q^{1,1} = I^{1,1} \tag{A.5a}$$
$$+ \sum_{I_{n,m}} m^{-3}\, e^{(-\int_\Sigma X^*(J))} \int_\Sigma X^*(B^{(i)}) \int_\Sigma X^*(B^{(j)}) \int_\Sigma X^*(B^{(k)})$$

$$I^{2,1} \to I_Q^{2,1} = I^{2,1} \tag{A.5b}$$

where $I_{n,m}$ is an m-fold cover of a rational curve on M of degree n, $X : \Sigma \to I_{n,m}$, and J is the Kähler form of M. The factor m^{-3} in (A.5a) was discovered phenomenologically in [4] by an application of mirror symmetry and was put on a firm footing in [14]. The fact that $I^{2,1}$ is not corrected was shown in [15]. Heuristically, the nonrenormalization theorems tell us that the saddle-point approximation to (A.2) is exact. The equations (A.5) express the contribution (or its absence) from nontrivial extrema.

A.2. From Conformal Field Theories to Manifolds

We have described how string theory associates a conformal field theory to a geometrical space of an appropriate type. The construction, however, is quite complicated and does not seem like a very promising approach to studying the geometric situation itself. Additionally, various questions of convergence and rigour are left unsatisfactorily addressed. For a particular class of Calabi–Yau spaces, this situation was dramatically improved in [18]. In this work, Gepner constructed superconformal field theories which are exactly solvable in the sense that the correlation functions (A.2) can be explicitly evaluated. This is achieved by constructing a twisted product of simple, well-studied representations of the $N = 2$ superconformal algebra. What is remarkable is that this purely algebraic construction leads to exactly the same field theory as the geometrical construction described above. More precisely, one can obtain in this way the conformal field theory corresponding to any Calabi–Yau manifold given by the vanishing locus of a polynomial of Fermat type in a weighted projective space, with a prescribed Kähler structure corresponding to a "radius"

To avoid any confusion we emphasize that only a limited subclass of conformal theories have a geometrical interpretation.

of order 1 (in units of $\sqrt{\alpha'}$). This correspondence was conjectured by Gepner on the basis of the representation content of the theories under large discrete symmetry groups, and further strengthened by the wotk of [15] in which the values of $I^{2,1}$ were shown to agree exactly between the two approaches (to within an undetermined normalisation). A formal argument proving the equivalence based upon path integrals was later given in [16,17] and a more rigorous derivation of some of the results using more traditional methods in [25]. For these specified points in certain moduli spaces we thus have exact formulas for the correlation functions of interest. As mentioned above, we also have an explicit identification of deformations of the conformal field theory with deformations of the geometrical structure. This allows us to extend the solution to a neighborhood of the exactly solved point.

Globally, the validity of the identification of stringy moduli space with the geometric construction is not guaranteed, and one may expect some additional identifications between inequivalent points in the geometrical parameter space when considered as string theories. For example, one readily sees that (A.2) is unchanged by adding to B_{ij} an element of $H^{1,1}(M, \mathbb{Z})$ so this variable is to be thought of as periodic, and there are other such identifications. Mirror symmetry may be considered a highly nontrivial generalization of these, relating two manifolds which are in fact topologically distinct.

A.3. A Simple Example

An example may serve to clarify some of the preceding discussion. The simplest example of a compact Ricci-flat manifold is a torus (this is not three-dimensional but we can imagine a tensor product of this with a "trivial" theory). Since M is in fact flat, the discussion simplifies considerably and the conformal field theory is exactly solvable throughout the moduli space. In the discussion above, there are no higher-order corrections to G in this case. The infinitesimal deformations of the conformal field theory are generated by two (complex) fields – one corresponding to a deformation of the complex structure and one to a Kähler deformation. Indeed, in this example the global moduli space is known: the flat metrics G are parametrized by the complex structure labelled by a point τ in the fundamental domain of $SL(2, \mathbb{Z})$, and the volume $|G|$. When we incorporate the B field we are led to the complex combination $\rho = B_{12} + i\sqrt{|G|}$ parametrizing the upper half plane. In fact, the conformal field theory is invariant under $B \to B + 1$ as mentioned above, but also under $\rho \to -1/\rho$ so that inequivalent conformal field theories on a torus are parametrized by two copies of the fundamental domain. The "modular group" [4] is thus $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. Finally, there is an additional identification, representing mirror symmetry, which interchanges the roles of complex structure and Kähler structure deformations. In this example the symmetry is simply $\tau \leftrightarrow \rho$.

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M	$h^{2,1}$	$h^{1,1}$	S	M'
$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$	101	1	$\mathbb{Z}_5^5$	$M/(\mathbb{Z}_5^3)$
$z_1^5 + z_2^{10} + z_3^{10} + z_4^{10} + z_5^2 = 0$	145	1	$\mathbb{Z}_5 \times \mathbb{Z}_{10}^3 \times \mathbb{Z}_2$	$M/(\mathbb{Z}_{10}^2)$
$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$ $z_1 x_1^3 + z_2 x_2^3 + z_3 x_3^3 = 0$	35	8	$\mathbb{Z}_3 \times \mathbb{Z}_9^3$	$M/(\mathbb{Z}_3 \times \mathbb{Z}_9)$
$z_1^3 + z_2^4 + z_3^5 + z_4^6 + z_5^{20} = 0$	47	23	$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{20}$	$M/(\mathbb{Z}_2)$
$z_1^3 + z_2^4 + z_3^4 + z_4^{12} + z_5^{12} = 0$	89	5	$\mathbb{Z}_3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{12}^2$	$M/(\mathbb{Z}_3 \times \mathbb{Z}_4^2)$
$z_1^4 + z_2^6 + z_3^{21} + z_4^{28} + z_5^2 = 0$	52	28	$\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{21} \times \mathbb{Z}_{28}$	$M/(\mathbb{Z}_2)$
$z_1^5 + z_2^6 + z_3^{10} + z_4^{30} + z_5^2 = 0$	91	7	$\mathbb{Z}_5 \times \mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{30}$	$M/(\mathbb{Z}_2 \times \mathbb{Z}_{10})$

Table 1
Mirrors M' of Some Minimal Model Compactifications M

The equations listed define M as a hypersurface in an appropriate weighted projective 4-space (such that the equation is homogeneous). The last column gives the mirror manifold M' as a quotient of M. The third entry defines a submanifold of $\mathbb{CP}^3 \times \mathbb{CP}^2$, demonstrating that the restriction to minimal models sometimes extends beyond hypersurfaces.

Symmetries	$h^{2,1}$	$h^{1,1}$	χ
	101	1	-200
[0,0,0,1,4]	49	5	-88
[0,1,2,3,4]	21	1	-40
$[0,1,1,4,4]$ $[0,1,2,3,4]$	21	17	-8
[0,1,1,4,4]	17	21	8
$[0,1,3,1,0]$ $[0,1,1,0,3]$	1	21	40
$[0,1,4,0,0]$ $[0,3,0,1,1]$	5	49	88
[0,1,0,0,4] [0,0,1,0,4] [0,0,0,1,4]	1	101	200

Table 2
Orbifolds of $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$ in $\mathbb{CP}^4$

The generators of each symmetry group are represented by the powers of a fundamental fifth root of unity by which they multiply the homogeneous coordinates of $\mathbb{CP}^4$. Thus $[0,0,0,1,4]$, for example, represents

$$(z_1, z_2, z_3, z_4, z_5) \to (z_1, z_2, z_3, \alpha z_4, \alpha^4 z_5)$$

with $\alpha^5 = 1$.