Developments in symplectic topology

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Foreword

Symplectic topology is an area of mathematics that has seen rapid and continuous development in the past fifteen years. I hope to give here a view of some of the past and present progress.

As the field is very large, I shall focus on work and ideas of Donaldson, on symplectic submanifolds and symplectic Lefschetz pencils. This material also provides good reason to weave in a survey of some of what is known about the differences between the notions of symplectomorphism and diffeomorphism, both for a fixed pair of symplectic manifolds and for families.

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1. What is symplectic topology?

A symplectic manifold is a smooth manifold X , necessarily of even dimension $2n$, carrying a non-degenerate, closed 2-form ω . The non-degeneracy of ω as a bilinear form on the tangent spaces tells us for each point $p \in X$, we can find local coordinates $x_1, \dots, x_{2n}$ with the property that

$$\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$$

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at p . The important theorem of Darboux tells us that when ω is also closed, we can find such a coordinate system so that ω is given by the above formula everywhere in the coordinate patch. The existence of such *Darboux coordinates* means that there is no local structure which distinguishes one symplectic manifold from another in the way that curvature, for example, is a local invariant of Riemannian manifolds.

We can therefore equivalently define a symplectic manifold as a topological manifold X equipped with an atlas of charts $\{\phi_i\}_{i \in I}$ for which the transition functions are symplectomorphisms. That is, on their domains of definition in $\mathbb{R}^{2n}$, the transition functions $\phi_i \phi_j^{-1}$ are smooth and preserve the standard symplectic form (namely the 2-form that appears on the right-hand side above). This definition is in the same spirit as the usual definition of a smooth structure, a piecewise-linear structure, a complex structure or a foliation (to give the most familiar examples).

The list of questions that arises begins with the questions of existence and uniqueness for symplectic structures on a given smooth manifold. There are many other questions however, some of which are rather general and come rapidly to mind, but others of which are more special to the symplectic realm.

2. Questions and techniques

Symplectic capacities. The standard symplectic form on $\mathbb{R}^2$ is the standard area form. A linear transformation of $\mathbb{R}^2$ of the form $x_i \mapsto \lambda_i x_i$ preserves the symplectic form whenever $\lambda_1 \lambda_2 = 1$. We can therefore transform a standard rectangle into a narrower rectangle provided we also make it taller.

If we consider the same situation in dimension 4, the linear transformation $x_i \mapsto \lambda_i x_i$ is symplectic when *two* conditions are met: we need $\lambda_1 \lambda_2 = 1$ and $\lambda_3 \lambda_4 = 1$, so that area is preserved in both the (x_1, x_2) plane and the (x_3, x_4) plane. In order to find such a transformation that changes one standard 4-dimensional cuboid into another, it is no longer sufficient that they have the same 4-dimensional volume: we cannot stretch the first and second coordinates while shrinking the third and fourth.

A remarkable theorem of Gromov [11] says that what cannot be achieved here by a simple linear transformation cannot be achieved by any other means either:

THEOREM 2.1. *There is no symplectic embedding of the cube with side-lengths $(1, 1, 1, 1)$ in the cuboid with side-lengths $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ unless $\lambda_1 \lambda_2 \geq 1$, no matter how large λ_3 and λ_4 may be.*

This theorem was proved using pseudo-holomorphic curves, and it created the subject of symplectic capacities. (In this context, a symplectic capacity is a function that assigns to each open subset of $\mathbb{R}^{2n}$ a positive real number or infinity; it is to be invariant under symplectic transformations, monotonic under inclusions and should scale like 2-dimensional area. A non-triviality condition is needed to rule out, for example, the n th root of the volume.)

Examples of symplectic capacities that are apparently different from Gromov's were constructed by Ekeland and Hofer by a study of periodic orbits of Hamiltonian flows [8, 12, 13]. Any Hamiltonian function H on a domain $U \subset \mathbb{R}^{2n}$ gives rise to a Hamiltonian flow, which is defined for all time if H is constant outside a compact set. A capacity $c(U)$ can be defined by maximizing the variation of such a Hamiltonian subject to the constraint that the flow have no orbits of period less than 1. See [13] for an interesting survey of how things stood a few years ago.

Gromov's original 'squeezing' theorem has many extensions. For example, one can ask whether it is possible to symplectically embed the unit $2n$ -cube in $D^2(\lambda) \times X^{2n-2}$, where $D^2(\lambda)$ is a disk of area $\lambda^2 < 1$ and X^{2n-2} is an arbitrary symplectic manifold. See [15].

Pseudo-holomorphic curves. Gromov's 1985 paper [11] introduced pseudo-holomorphic curves as a tool in symplectic topology. This development opened up the area.

An almost complex structure J on a symplectic manifold (X^{2n}, ω) is *compatible* with ω if the bilinear form $g(u, v) = \omega(u, Jv)$ on the tangent spaces is symmetric and positive. In this case, each tangent space $T_p X$ can be identified with $\mathbb{R}^{2n}$ so that ω is standard, g is the euclidean metric, and J is the standard complex structure for the complex coordinates $z_1 = x_1 + ix_2$ etc. The compatible complex structures form a contractible space, for they are the sections of a fiber bundle over X with contractible fiber $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$.

A pseudo-holomorphic curve, or a J -curve, in an almost complex manifold (X, J) is a smooth map f of a Riemann surface (Σ, J_0) into X whose derivative

$df : T_s\Sigma \rightarrow T_{f(s)}X$ is complex linear: df intertwines the complex structure J_0 with J . If J is integrable, then we are talking about holomorphic curves in a complex manifold.

The technical tool in [11] is a compactness theorem for J -curves: a sequence of J -curves of bounded area either converges or degenerates in a manner that we can control. (The simplest example of such degeneration is the way in which a sequence of smooth conic curves in $\mathbb{CP}^2$ can degenerate to two projective lines meeting in a point.) The compactness theorem mirrors a similar theorem of Sacks and Uhlenbeck [17] concerning minimal surfaces. Indeed, in the integrable case, holomorphic curves have mean curvature zero (that is, they satisfy the variation equation for minimal surfaces), while in the general case, the mean curvature is uniformly bounded.

In many situations, the compactness theorem allows one to conclude that a smooth pseudo-holomorphic curve that exists for one almost complex structure, J , persists as J is deformed. For example, in $\mathbb{CP}^2$ with its standard symplectic form and complex structure J_0 , there is a complex line joining any two points; as a J -curve, the line is unique up to reparametrization of the domain by $\mathrm{SL}(2, \mathbb{C})$. From the compactness theorem and a little geometry, one can deduce that the same is true for any other almost complex structure J compatible with ω . Gromov used this idea to prove:

THEOREM 2.2 (Gromov, [11]). *The group of symplectomorphisms of $\mathbb{CP}^2$ with its standard symplectic structure retracts onto the isometry group, $\mathrm{PU}(3)$. The same is true of $S^2 \times S^2$, equipped with a symplectic structure as the product of two spheres of equal area.*

In the case of $S^2 \times S^2$, the proof rests on the existence, for any J , of two families of J -curves, modelled on the two rulings of $S^2 \times S^2$ with the product complex structure. A closely related result of Gromov's is that there is no non-standard symplectic structure on $\mathbb{R}^4$ that is standard at infinity.

The idea of 'filling' by pseudo-holomorphic disks, with the boundaries constrained to lie on two-dimensional surface or the boundary of a symplectic 4-manifold, is an idea which has been applied in many directions by Eliashberg; see for example the survey [9].

Diffeomorphisms and symplectomorphisms. In Gromov's theorem concerning $S^2 \times S^2$ quoted above, the condition that the two spheres have equal area is important. An interesting story unfolds when this condition is dropped, as already noted by Gromov and developed further by McDuff and Abreu in [16], [1] and [2].

To explain why the condition is important, let us recall that Gromov's proof rests on the existence, for any J , of two rulings by J -curves. Let x and y denote the homology classes of the spheres in the two standard rulings. The J -curves in these rulings have areas $\int_x \omega$ and $\int_y \omega$. The positivity of the integral of ω is a necessary condition for the existence of a J -curve, so the equality of these two areas does not allow a J -curve Σ in the homology class of $x - y$, for example. However, if we change ω so that the area of x is larger, then such a curve Σ may exist for some J . In this case, the compactness theorem may allow a family of J -curves in the homology class of x to degenerate as a union of two curves, representing the homology classes y and $x - y$. (Exactly this sort of degeneration is seen when a smooth quadric surface degenerates into a cone. In this case, since x and y have the same area, the geometry degenerates. But if one blows up the cone at the singular point, the exceptional set is a curve in the homology class $x - y$.) This breakdown of the proof is what has to be picked up and turned around: with the construction of an explicit family of symplectomorphisms realizing the resulting obstruction, one establishes the existence of some topology in the quotient of the symplectomorphism group by the isometry group.

A different slant on the topology of the group of symplectomorphisms has been given by Seidel, [18]. In particular, Seidel showed that for a symplectic 4-manifold (X, ω) , the inclusion of the symplectomorphisms in the diffeomorphisms is very often not injective at the level of the component groups: there are symplectomorphisms that can be deformed to the identity through diffeomorphisms but not through symplectomorphisms. One consequence of this is a construction of 6-manifolds with more than one isotopy class of symplectic form in a given homology class: one can form a fibration over T^2 whose monodromy on one generator is one of Seidel's distinguished symplectomorphisms of the 4-dimensional fiber.

Earlier examples of 'exotic' symplectic structures in dimension 6 were exhibited using Gromov's idea of defining invariants by counting J -curves. Ruan gave a particularly striking example. No such example is known in dimension four: it

may be that two symplectic 4-manifolds are symplectomorphic whenever they are diffeomorphic by a diffeomorphism that respects the cohomology class of the form and the first Chern class of a compatible almost-complex structure.

Taubes' applications of gauge theory. The subjects of gauge theory and 4-dimensional symplectic topology came together with a result of Taubes in 1994 [19]. Among other results, Taubes proved that in any symplectic 4-manifold (X, ω) with $b^+(X) > 1$, with an generic compatible almost-complex structure J , there is always a J -curve representing the homology class Poincaré dual to the canonical class. (The condition on b^+ means that there is at least a 2-dimensional subspace of H^2 on which the cup-square is positive. The *canonical class* is minus the first Chern class of the almost complex structure.) This is the only general existence theorem of its kind for pseudo-holomorphic curves: previously, it was quite possible that the whole theory of J -curves was empty for the most general symplectic 4-manifold.

In the special case that (X, ω) is a complex projective surface, one has the relationship $b^+ = 1 + 2p_g$, where the geometric genus p_g is the number of linearly independent holomorphic sections of the canonical line bundle. The condition $b^+ > 1$ is precisely the condition that the canonical line bundle has sections. The zero set of a section is a curve dual to the canonical class. In this non-generic situation of an integrable complex structure J , the canonical divisor might not be cut out transversely.

A related result from [19] is the existence of 'projective lines' in the manifold $\mathbb{CP}^2$ equipped with *any* symplectic structure: given any symplectic form and any compatible complex structure J , one can find a pseudo-holomorphic $\mathbb{CP}^1$ in $\mathbb{CP}^2$ representing the generator of homology. (In the much more elementary existence theorem of Gromov, only J varies, while ω is the standard structure.) In combination with Gromov's filling construction, this existence theorem implies:

THEOREM 2.3. *If ω_0 be the standard symplectic form on $\mathbb{CP}^2$ and ω is any other, then there is a diffeomorphism of $\mathbb{CP}^2$ that pulls back ω to some multiple of ω_0 . In this sense, every symplectic form on $\mathbb{CP}^2$ is standard.*

This theorem has been extended by various authors to the case of other rational surfaces.

3. Donaldson's theorem on symplectic submanifolds

Before 1994, next to nothing was known about what is perhaps the most primitive question one can ask about compact symplectic manifolds, namely the question of which smooth, compact manifolds X carry symplectic forms. There are two rather trivial necessary conditions. First, the structure group of the tangent bundle must reduce from $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{Sp}(2n, \mathbb{R})$. This is the condition for the existence of a non-degenerate 2-form. Because both $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ retract onto $\mathrm{U}(n)$, the existence of such a 2-form is the same constraint as the existence of an almost-complex structure on X . The second constraint is that there must be a class o in $H^2(X, \mathbb{R})$ (represented by the closed form ω) with the property that o^n is non-zero in $H^{2n}(X, \mathbb{R})$. In fact, the bundle reduction or almost complex structure determines an orientation of X , and we should ask that $\langle o^n, [X] \rangle$ is positive (though this is no further restriction if n is odd). In any event, $b_2(X)$ must be non-zero. Refining the existence question slightly, one can ask:

QUESTION 3.1. Given an almost complex manifold (X, J) and a class $o \in H^2(X)$ with $o^n > 0$, does there exist a symplectic form ω representing o in de Rham cohomology and determining the same bundle reduction as J up to homotopy?

Donaldson answered this question with a 'no' for 4-dimensional manifolds X , as a consequence of an existence theorem for symplectic submanifolds [7].

The existence theorem, and corollaries. An even-dimensional submanifold Y^{2k} of a symplectic manifold (X^{2n}, ω) is a *symplectic submanifold* if the restriction of ω to Y as a 2-form is non-degenerate. A symplectic submanifold therefore inherits a symplectic structure from X . The existence of any non-trivial, closed symplectic submanifolds in a general symplectic manifold is not obvious. Indeed, before Donaldson's paper [7], it remained open whether one should expect to find them. The main theorem of [7] is the following:

THEOREM 3.2. *Let (X, ω) be a compact symplectic manifold, and suppose that the de Rham cohomology class of $[\omega]$ lifts to an integer class h . Then for sufficiently large integers $k > 0$, there exists a symplectic submanifold Y of real codimension 2, representing the homology class Poincaré dual to kh . More generally, without the*

hypothesis that ω lifts to an integer class, symplectic submanifolds of codimension 2 exist.

REMARK. It is the first part of the theorem that has most of the content. Note also that, once one has submanifolds of codimension 2, one can repeat the construction to obtain symplectic submanifolds of every even codimension.

Before saying something about the proof, it is worth pointing out that the theorem has non-trivial consequences for the existence question posed above, at least in dimension 4. In the 4-dimensional case, the theorem asserts the existence of an embedded surface $\Sigma \subset X$ on which ω is everywhere positive. It is not difficult to see that there is then an almost complex structure J on X with respect to which Σ is a pseudo-holomorphic curve. In particular then, there is a C^∞ , complex-linear splitting of the tangent bundle $TX|_\Sigma$ into two complex line bundles, $T\Sigma \oplus N\Sigma$ (the tangent bundle plus the normal). The degrees of these two line bundles are respectively $\chi(\Sigma)$ (the euler number, $-2g+2$) and $\Sigma \cdot \Sigma$ (the self-intersection number of the homology class, the dual of the cup-square). From this one obtains the *adjunction formula* which expresses the genus of a symplectic submanifold in terms of its homology class and the first Chern class of the almost complex structure:

$$(1) \quad 2g - 2 = \Sigma \cdot \Sigma - \langle c_1(X, J), \Sigma \rangle.$$

Donaldson's existence theorem therefore gives:

COROLLARY 3.3. *If (X, ω) is a compact symplectic 4-manifold and ω represents an integer class h , then the class dual to kh is represented by a smoothly embedded surface satisfying the adjunction inequality (1) for all sufficiently large k .*

The existence of a smooth surface Σ having a particular genus and representing a particular homology class is a matter of differential topology on X . For example, it was shown in [14] that in a large class of 4-manifolds, *any* smoothly embedded surface of positive genus satisfies $2g - 2 \geq \Sigma \cdot \Sigma$. (The proof used Donaldson's polynomial invariants from Yang-Mills theory, and the class of 4-manifolds to which the theorem applies are the 4-manifolds X with $b^+(X) > 1$ for which at least one polynomial invariant is non-zero.) Combining this fact with the above corollary, one can deduce:

COROLLARY 3.4. *Let X be a 4-manifold to which the theorem of [14] applies, let h be an integer class in $H^2(X)$ and J an almost complex structure. A necessary condition for the existence of a symplectic form representing the class h and determining the same bundle reduction as J is that*

$$\langle c_1(X, J), h \rangle \leq 0.$$

As a particular example, consider a smooth quintic surface in $\mathbb{CP}^3$. This is a complex manifold of real dimension 4, and its first Chern class is $-H$, where H is the restriction of the positive generator on $\mathbb{CP}^3$. It carries a symplectic form, namely the restriction of the Fubini-Study form, whose class is a positive multiple of H . This is a 4-manifold to which the theorem applies, and we can deduce:

COROLLARY 3.5. *There is no symplectic form ω on the quintic surface satisfying:*

1. *the compatible almost complex structures have the same first Chern class as the standard complex structure; and*
2. *the cohomology class $[\omega]$ is a negative multiple of H .*

One can also find examples of such negative results in which the class $[\omega]$ is unconstrained. For example, if X is a regular elliptic surface with $p_g > 1$, there is no symplectic structure with $c_1 = 0$, despite the existence of almost complex structures with this property.

Taubes results, discussed in the previous section, followed hot on the heels of Donaldson's theorem, and gave stronger constraints. For example, we now know that the constraint $\langle c_1(X, J), h \rangle \leq 0$ must hold on any symplectic 4-manifold X with $b^+(X) > 1$. (Note that $\mathbb{CP}^2$ is a symplectic manifold with $b^+ = 1$ and $\langle c_1(X, J), h \rangle > 0$.)

The complex case. The model is the case of a Kähler manifold, with ω being the Kähler form. Kodaira's embedding theorem states that, in the case that ω lifts to an integer class h , a Kähler manifold is a projective variety: it admits a holomorphic embedding in $\mathbb{CP}^N$ in such a way that the restriction of the generator of $H \in H^2(\mathbb{CP}^N, \mathbb{Z})$ is a large multiple of h . A holomorphic (and hence symplectic) submanifold of X then arises as the intersection of X with a generic hyperplane.

The proof of Kodaira's theorem starts with the elementary topological fact that, when a 2-form ω lifts to an integer class h , there exists a complex line bundle $L \rightarrow X$ having first Chern class h and carrying a unitary connection ∇ with curvature 2-form $F_\nabla = (2\pi/i)\omega$. When ω is also compatible with a complex structure, the line bundle with connection ∇ is holomorphic: the equation for holomorphic sections s of L is locally integrable. From the line bundle L , one can form for each k the tensor power L^k , which is a line bundle with first Chern class kh and curvature kF_∇ . The heart of the proof of the theorem is in the existence of many global holomorphic sections of L^k when k is large: there are sufficiently many sections to separate points and distinguish tangent vectors on X . A collection of global sections $s_0, s_1, \dots, s_N$ defines a map to $\mathbb{CP}^N$ by a standard construction: on local patches, we can trivialize L^k by a non-vanishing holomorphic section s and so obtain $N + 1$ holomorphic functions $f_i = s_i/s$. In homogeneous coordinates, the map to projective space is defined by $[f_0, \dots, f_N]$; it is independent of s and therefore globally defined, as long as the s_i have no common zero.

Once X is holomorphically embedded in $\mathbb{CP}^N$, it is quite easy to see that the general complex hyperplane meets X transversely, and so cuts out a complex submanifold. On the other hand, the hyperplane sections are simply the zero sets of the sections s of L^k obtained as linear combinations of the chosen s_i . The argument can therefore be summarized by saying first that L^k has a large collection of global holomorphic sections s_i once k is large; and second that some linear combination s of these sections has a transverse zero set.

Almost holomorphic sections. Donaldson's idea was to imitate some aspects of this proof in the symplectic case. First, we can construct again the line bundle $L \rightarrow X$ with unitary connection ∇ and curvature $F_\nabla = (2\pi/i)\omega$. Next, we need to choose a complex structure J on X compatible with ω . This also determines a Riemannian metric $g(u, v) = \omega(u, Jv)$. Because we can no longer expect J to be integrable, there will usually be no holomorphic sections of L or of L^k , even locally. However, in order for the zero set to be a symplectic submanifold, it is only necessary a section to be transverse to zero and *approximately* holomorphic, in the following sense.

Let us focus on the case on that the dimension of X is four. Consider a map

$$a : \mathbb{C}^2 \rightarrow \mathbb{C}$$

which is linear over the *reals*, but not necessarily complex linear. We can write a uniquely as a sum of two maps,

$$a = a' + a'',$$

where $a' : \mathbb{C}^2 \rightarrow \mathbb{C}$ is complex linear and a'' is conjugate linear: we simply define $a'(v) = \frac{1}{2}(a(v) - ia(iv))$. Note that if a takes values in $\mathbb{R} \subset \mathbb{C}$, then a'' is the complex conjugate of a' , and we therefore see more generally that a fails to have full rank over the reals when $a'' = e^{i\theta}\bar{a}'$ for some θ .

When a has full rank, its kernel is a real 2-plane in $\mathbb{C}^2$ (or in $\mathbb{R}^4$), and it is naturally oriented because both the domain and range are complex and so naturally oriented. We can ask whether this oriented 2-plane is symplectic; that is, whether the standard 2-form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ is positive on the 2-plane. The answer is that the 2-plane is symplectic precisely when $|a''| < |a'|$. In all, we have the following elementary lemma:

LEMMA 3.6. *Let $a : \mathbb{C}^2 \rightarrow \mathbb{C}$ be linear over $\mathbb{R}$ and let $a = a' + a''$ be its decomposition into complex linear and conjugate linear parts. Then*

1. *The map a fails to have real rank 2 when $a'' = e^{i\theta}\bar{a}'$ for some θ .*
2. *The kernel of a is a symplectic as an oriented 2-plane precisely when $|a'| > |a''|$.*
3. *The kernel is symplectic with the opposite orientation when $|a'| < |a''|$.*
4. *When $|a'| = |a''|$ but condition (1) does not hold, then the kernel is an isotropic 2-plane: the restriction of ω is zero.*
5. *Finally, the kernel is a complex line in $\mathbb{C}^2$ when $a'' = 0$.*

Note that $|a|^2 = |a'|^2 + |a''|^2$, so that the condition in (2) can also be written $|a| > \sqrt{2}|a''|^2$.

Now let us return to a section s of L (or L^k) on X , still of dimension 4. Recall that L has a unitary connection ∇ ; let $|\cdot|$ be a compatible hermitian metric. The covariant derivative ∇s is a linear bundle map $TX \rightarrow L$. At each point p , we have a real linear map $a = (\nabla s)_p$ from the 2-dimensional complex vector space $(T_p X, J)$

to L_p . The decomposition of a into its two parts is the decomposition of ∇s as

$$\nabla s = \partial s + \bar{\partial} s.$$

At a point where s vanishes transversely, the tangent space to the zero set is the kernel of ∇s , and we can apply the lemma to this situation. (Note that the complex conjugate of ∂s does not make sense unless we trivialize the fiber of L , but the condition in part (1) of the lemma is nevertheless unambiguous.) To conclude, we have the still elementary fact:

PROPOSITION 3.7. *The zero set of a section s of L^k is a symplectic submanifold representing the class kh provided that, at all points of the zero set, we have $|\partial s| > |\bar{\partial} s|$, or equivalently $|\nabla s| > \sqrt{2}|\bar{\partial} s|$.*

A section s is holomorphic when $\bar{\partial} s = 0$, but as we have said, such sections will not exist in the general almost-complex case. What we need is a section with $\bar{\partial} s$ small and ∇s large on the zero set. This is what Donaldson's theorem provides.

To accommodate the parameter k in the statement of the theorem it is convenient to measure the size of derivatives on X using not the metric g but the metric defined on vectors by $g_k(u, v) = kg(u, v)$, so that 1-forms are smaller by a factor of $k^{1/2}$. (The hermitian metric on sections of L^k is unaffected.) We write $|\nabla s|_k$, for example, when this metric is involved. The theorem then states:

THEOREM 3.8 (Donaldson, [7]). *There are constants C and $\eta > 0$ such that for all sufficiently large k , there is a section σ_k of L^k satisfying*

$$|\bar{\partial} \sigma_k|_k \leq Ck^{-1/2},$$

everywhere on X , and

$$|\nabla \sigma_k|_k > \eta$$

at all points where $|\sigma_k| < \eta$ (and in particular on the zero set).

The zero set of σ_k is thus the submanifold whose existence is asserted by the first theorem. The submanifold is better than just symplectic: it is approximately holomorphic for the almost complex structure J , in that its tangent space is everywhere close to a complex subspace of TX . There is a uniqueness result for these symplectic submanifolds, suggested by Donaldson and proved by Auroux [4]. Call

a sequence of sections $\{\sigma_k\}_{k \geq k_0}$ of the line bundles L^k *asymptotically holomorphic* if they satisfy the first condition of the theorem (we actually need to insist that $|\nabla \bar{\partial} \sigma_k|_k$ also decays like $k^{-1/2}$), and say that the sequence is *η -transverse* if the second condition holds. Auroux's uniqueness theorem then says that if we have two asymptotically holomorphic, η -transverse families of sections, then the zero sets Σ_k and Σ'_k are isotopic for sufficiently large k . Indeed, there is even a symplectic isotopy of X carrying Σ_k to Σ'_k (that is, a 1-parameter family of symplectomorphisms starting at the identity).

Ideas in the proof. Constructing sections s of L^k with $|\bar{\partial}s|$ small is not the hard part of the proof. Let L be a line bundle on $\mathbb{C}^n$ with hermitian metric and unitary connection ∇ . Let the curvature of ∇ be $(2\pi/i)\omega$, where ω is the standard 2-form on $\mathbb{C}^n = \mathbb{R}^{2n}$. This model bundle has a global holomorphic section s on $\mathbb{C}^n$ with Gaussian decay: we have

$$|s(x)| = e^{-\pi|x|^2},$$

with similar decay of the derivatives of s . Equipped with the metric g_k , the manifold X has diameter $O(k^{1/2})$; and if we look at the g_k -unit ball about a fixed point p , the geometry of this ball (with metric g_k , symplectic form $\omega_k = k\omega$ and almost complex structure J) approaches the geometry of the unit ball in $\mathbb{C}^n$ ever more closely as k increases. Indeed, this is true also for the ball about p of g_k -radius, say, k^λ when compared to the corresponding ball in $\mathbb{C}^n$, as long as the power k^λ is growing more slowly than $k^{1/2}$, so that the radius in the original metric g is going to zero.

In suitable coordinates, therefore, it is possible to transfer the model section s from $\mathbb{C}^n$ to a ball in X centered on any point p . After using a cut-off function to obtain a section that is everywhere defined, we have a section s_p of L^k on X whose length is given approximately by the Gaussian formula above (where $|x|$ is now replaced by the g_k -distance to p), and whose support is in a ball of g_k -radius $O(k^\lambda)$. If λ is not too large ($\lambda = 1/6$ will do), then the section s_p , while no longer holomorphic, is approximately holomorphic, in that $|\bar{\partial}s_p| \leq Ck^{-1/2}$ for some constant C depending on the geometry of X .

In this way, a large collection of approximately holomorphic sections is obtained, as finite linear combinations of these local Gaussian sections:

$$(2) \quad s = \sum_i c_i s_{p_i}.$$

More precisely, for each k , we choose a finite collection of points p_i such that the g_k -unit balls about these points cover X , and we consider linear combinations with complex coefficients c_i of norm at most 1. Note that because the g_k -diameter of X is increasing as $k^{1/2}$, the number of points p_i will grow as $O(k^n)$; but as long as these points are distributed over X with reasonable economy, the Gaussian decay of each term will ensure that the sum s still has $|\bar{\partial}s|_k = O(k^{-1/2})$, while $|s| = O(1)$.

Transversality. The difficult part of the argument is to find the coefficients c_i in (2) so as to achieve transversality. The plan is to select each c_i so as to achieve η -transversality on the g_k -unit ball B_i around p_i . Note first that η -transversality is an open condition, in that if s is η -transverse on some open set and s' is C^1 close to s , then s' is η' -transverse for some η' a little smaller than η .

With this idea in mind, suppose we have successfully selected coefficients c_j for $j < i$ such that the linear combination

$$s' = \sum_{j < i} c_j s_{p_j}$$

is η_1 -transverse on the union unit balls about the points p_j . We seek a suitable coefficient c_i so as to achieve η_2 -transversality on the unit ball B_i about p_i , and at the same time we want to ensure that the addition of the extra term $c_i s_i$ to the sum does not much affect the η_1 -transversality that has already been achieved. Looking at the second matter first, note that if p_j is not close to p_i (say their distance apart is at least d , where d will depend on η_1 but not on k), then the additional term will not substantially effect the η_1 -transversality on the ball B_j , because of the Gaussian decay of s_i and its derivatives. The difficulty arises from nearby balls. If p_i is close to p_j for some $j < i$, then we must choose c_i small in comparison to η_1 in order not to disturb the existing transversality on B_j .

The heart of the matter then, is that we must choose c_i to be small so as not to disturb the transversality on the previous nearby balls, and yet we must obtain a useful amount of transversality for the section $s' + c_i s_i$ on B_i from our choice. Since s_i is non-vanishing on the unit ball B_i , we can write $s' = f s_i$ for

some function f (which is approximately holomorphic, in that $\bar{\partial}f$ is $O(k^{-1/2})$). In terms of f , our aim is to find a small c_i so that $f + c_i$ is η_2 -transverse to zero, for an η_2 that is usefully large. Of course, Sard's theorem assures us that we can find c_i arbitrarily close to zero so that the zero set of $f + c_i$ is transverse. We need a quantitative version of Sard's theorem for approximately holomorphic functions f . This is Donaldson's result, which is based on a theorem of Yomdin [20].

PROPOSITION 3.9. *Suppose $f : B^n \rightarrow \mathbb{C}$ is a function on the unit polydisk in $\mathbb{C}^n$, satisfying $|f| \leq 1$. Suppose f is approximately holomorphic, in that*

$$|\bar{\partial}f| + |\nabla \bar{\partial}f| \leq \eta' = \eta/(\log |\eta|)^p,$$

for some $\eta < 1/2$. Then there is a $c \in \mathbb{C}$ with $|c| \leq \eta$ such that $f + c$ is η' -transverse to zero on a ball $B' \subset B^n$ of radius $1/2$. Here p is a positive integer depending only on the dimension.

The proof uses the approximately holomorphic nature of f to approximate f by a polynomial function of controlled degree. The book-keeping that is needed to deduce the main theorem from this local proposition is more delicate than might be suggested by our account, but it is in the end elementary.

4. Symplectic Lefschetz pencils

Holomorphic Lefschetz pencils. If s_1 and s_2 are two sections of a complex line bundle $L \rightarrow X$, their ratio $f = s_1/s_2$ gives a map to $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, which is defined everywhere except on the locus Z where both s_1 and s_2 vanish:

$$f : X \setminus Z \rightarrow \mathbb{CP}^1.$$

If $L \rightarrow X$ is a holomorphic line bundle over a complex manifold and the sections s_i are holomorphic, then f is a holomorphic map. If the zero sets of the two sections intersect transversely, then Z is a smooth complex submanifold of complex codimension 2 (real codimension 4). The fibers of f in $X \setminus Z$ are the zero sets of the sections $as_1 + bs_2$. In X , each of these zero sets also contains Z .

Still in the holomorphic case, we can ask a little more of f . In its domain of definition, f will have critical points, where the differential $\nabla f = \partial f : T_p X \rightarrow T_{f(p)} \mathbb{CP}^1$ vanishes. The fiber of f through such a point p will be singular. Consider

the model case of a holomorphic map

$$f : B^n \rightarrow \mathbb{C}$$

defined on a ball in $\mathbb{C}^n$. In coordinates, the derivative is a map $\partial f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, and we suppose ∂f vanishes at the origin, so that the fiber $f^{-1}(c)$ is singular, with $c = f(0)$. If ∂f *vanishes transversely* at zero, then holomorphic Hessian $\partial\partial f|_0$ is non-singular, and in suitable coordinates the power series expansion of f begins

$$f = c + \sum_{i=1}^n z_i^2 + O(z^3).$$

The first result of singularity theory is the Morse lemma, which in its holomorphic version tells us we can find local coordinates in which f is given by its quadratic terms.

The transversality of ∂f to zero therefore assures us that the singularities of f are standard. In real dimension 4, the model for a singular fiber is the singular conic: a union of two lines obtained as the degeneration of the general fiber, a smooth conic. In local complex coordinates, we are simply looking at the map $f(z_1, z_2) = z_1 z_2$ near 0.

When the simultaneous zero set Z of the sections s_i is cut out transversely, and the zeros of ∂f are transverse (so that f has only finitely many critical points and each is a standard quadratic singularity), the map f is called a Lefschetz fibration. The family of complex submanifolds defined as the zero set of the linear combination of curves $as_1 + bs_2$ is a *Lefschetz pencil*. (One can also ask that the values of f at the critical points are distinct, so that each curve in the pencil has at most one singular point).

If there are sufficiently many global holomorphic sections of L so that, for each $p \in X$, the derivatives of the holomorphic sections which vanish at p span $\text{Hom}(T_p X, L_p) \cong \mathbb{C}^n$, then an application of Sard's theorem tells us that a Lefschetz pencil exists. As in the case of Kodaira's theorem, if the curvature of L is $(2\pi/i)\omega$, where ω is a Kähler form, then this condition on global sections is met by the line bundle L^k for large enough k .

The map $f : X \setminus Z \rightarrow \mathbb{CP}^1$ can be extended to a map $f : \tilde{X} \rightarrow \mathbb{CP}^1$ by blowing up X along the smooth locus Z . Topologically, this means replacing each point of

z of Z by a copy of $\mathbb{CP}^1$ (the projectivization of the normal bundle). Each $\mathbb{CP}^1$ in $\tilde{X}$ is mapped isomorphically to the $\mathbb{CP}^1$ which is the range of f .

To give the simplest examples in complex dimension 2, a Lefschetz pencil of lines in $X = \mathbb{CP}^2$ consists of all the projective lines through a point z . None of the curves in the pencil are singular, the blow up of X at z is topologically $\mathbb{CP}^2 \# \bar{\mathbb{CP}}^2$ and the extension of f is a smooth fibration of $\mathbb{CP}^2 \# \mathbb{CP}^2$, exhibiting it as a 2-sphere bundle over S^2 . A Lefschetz pencil of conics consists of all the conics through four points. After blowing up at these four points, we have a map from $\mathbb{CP}^2 \# 4\bar{\mathbb{CP}}^2$ to S^2 . The map has three critical points: three of the fibers are reducible conics, corresponding to the three ways to draw a pair of lines through four points. A Lefschetz pencil of cubics displays the rational elliptic surface: $\mathbb{CP}^2$ blown up at nine points maps to $\mathbb{CP}^1$ with general fiber an elliptic curve; there are twelve critical points, where the fibers are nodal cubics.

The symplectic case. Using the same set of tools as for the previous existence theorem for symplectic submanifolds, Donaldson has shown that a symplectic manifold also admits ‘Lefschetz fibrations’ [5]. We shall explain what this means, and indicate some of what is involved.

First, we set up the same framework: we have the hermitian line bundle L with connection ∇ and curvature $(2\pi/i)\omega$, and on X an almost complex structure J compatible with ω . From the previous section, we know that L^k has a large number of almost holomorphic sections s : in the dilated metric g_k , the length of $\bar{\partial}s$ is of order $k^{-1/2}$ and we can achieve the same for higher derivatives of $\bar{\partial}s$. We can also find such sections s that are η -transverse, so that for large k their zero sets are symplectic.

The first new demand is to find two such sections s_1 and s_2 whose zero sets meet transversely. One way to set this up is to regard a pair of sections s_1 and s_2 as a single section of the bundle $\mathbb{C}^2 \otimes L^k$. What we need is an approximately holomorphic section of this rank 2 bundle which is η -transverse to zero for some η . This will ensure that the complex codimension 2 zero set Z is cut out transversely and is also symplectic, or even approximately holomorphic in the sense already discussed. One way to go about this is to extend the local transversality result, Proposition 3.9 to $\mathbb{C}^2$ -valued maps. (In the higher rank case, η -transversality can

be defined in terms of the reciprocal of the norm of a 1-sided inverse to ∂f ; the norm of ∂f alone no longer tells us whether the derivative is invertible.)

At this stage we have an approximately holomorphic map $f : X \setminus Z \rightarrow \mathbb{CP}^1$, whose behavior near Z is close to the holomorphic model. However, we do not yet have any assurance that the general fiber of f is a symplectic submanifold, or that the structure of the singularities of f resembles the model case.

The main extra step is to achieve η -transversality of the complex part of the derivative, ∂f . Note that ∂f is a section of rank n complex vector bundle on a manifold of complex dimension n . We need two tools to achieve this. First, and quite easily, we need a supply of approximately holomorphic sections of L^k whose derivatives $(\partial s)_p$ span $\mathbb{C}^n$ at each point p (see the discussion of the holomorphic case above). These are quite easily constructed starting with our model Gaussian section s of the line bundle on $\mathbb{C}^n$: we simply take as our new model the sections $z_i s$, where z_i are the coordinate functions, and we graft these models onto X as before. The more difficult step is to extend the quantitative transversality result (3.9) to the case of $\mathbb{C}^n$ -valued maps.

The η -transversality of ∂f assures us of two things. First, provided we have kept a uniform bound on the second derivatives of f , we now know that ∂f is small only near to points where ∂f vanishes. (Small might mean less than $\eta' = \eta/C$, where C is independent of k). And second, ∂f vanishes only at finitely many points.

We are not quite done. The singular points of f are the points where its derivative, as a linear map $\mathbb{R}^{2n} \rightarrow \mathbb{R}^2$, fails to have full rank. In the case $n = 2$, for example, we see from Lemma 3.6 that the singular points are where ∂f and $\bar{\partial} f$ are complex conjugates, up to phase. A necessary condition in this case is that ∂f and $\bar{\partial} f$ have the same length. Since $\bar{\partial} f$ is everywhere small when k is large, and since ∂f is small only near the zeros of ∂f , we see that the critical points are all close to the finitely many zeros of ∂f . But unlike the holomorphic case, the general situation is still that the critical points are locus of (real) dimension 1 in the domain.

It is interesting to look at the model case for $n = 2$. The standard model can be described in complex coordinates as the map $z_1^2 + z_2^2$. As a real polynomial map, it can be deformed by altering just the imaginary part, so as to give a map $\mathbb{C}^2 \rightarrow \mathbb{C}$

whose real part is a before, but whose imaginary part is, say

$$\operatorname{Im}(z_1^2 + (z_2 - i\epsilon)^2)$$

for some real ϵ . After this deformation, the singular locus is a small, smooth closed curve in the domain. Its image in the codomain is a curve with three cusps. The three cusps correspond to fibers of the map where there is a real singularity of type A_2 .

Thus the complex model is not generic for maps which are not holomorphic. To ensure that the singularities of f are isolated points and that the map resembles the complex model, we need to arrange that $\bar{\partial}f$ vanishes at the points where ∂f vanishes, and that $\nabla\bar{\partial}f$ is small. Once this is achieved, we are assured that $|\partial f| > |\bar{\partial}f|$ except at the finitely many zeros, and that the fibers other fibers are therefore regular and symplectic. Finally, these conditions ensure that there are local coordinates which identify the singularity with the complex model.

Further results and an example. Auroux [3] has extended the ideas above in the direction of considering linear systems of higher rank. On a symplectic 4-manifold X for example, three generic sections of a line bundle L define a map $X \rightarrow \mathbb{CP}^2$, and Auroux shows that one can imitate the holomorphic case in obtaining a map whose locus of singularities is a smooth symplectic submanifold in X , which projects to a singular symplectic submanifold with cusps in the codomain. (This is the generic A_2 cusp singularity for a holomorphic map $\mathbb{C}^2 \rightarrow \mathbb{C}^2$.) Auroux's paper also contains a proof of Donaldson's theorem, assuming the basic quantitative transversality result.

In dimension 4, the existence of symplectic Lefschetz fibration has non-trivial topological implications. The existence of a smooth map $f : X \# r\mathbb{CP}^2 \rightarrow S^2$, for some r , having singularities modelled on the complex case (with correct orientation), is a statement about the differential topology of X . Indeed, as Gompf has observed, it does imply the existence of a symplectic structure on $X \# r\mathbb{CP}^2$, and must constrain X .

In the existence theorem (3.2) for symplectic submanifolds Y_k , Poincaré dual to $k[\omega]$ on a symplectic manifold X , Donaldson also notes that the particular submanifolds that arise are rather special. The sections s_k that cut out Y_k are asymptotically holomorphic and η -transverse, while remain bounded along with their derivatives

in the g_k metric. Even in the complex case, when the s_k are holomorphic, the η -transversality (with η independent of k) is a non-trivial condition. In general, Donaldson shows, amongst other things, that the Y_k converge to ω , after rescaling, as currents: for any test form $\phi \in \Omega^{2n-2}(X)$, we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{Y_k} \phi = \int_X \phi \wedge \omega.$$

This means in particular that the Y_k ‘fill out’ all of X . Some other interesting observations about the topology of Y_k are made in [3].

The local structure of the Y_k is very intricate. If we take a ball of radius, say, $k^{\epsilon-1/2}$ about a point p in X , then in the g_k metric we have an approximately standard ball of radius k^ϵ , for small positive ϵ . As k goes increase, we obtain from the Y_k a sequence of asymptotically holomorphic submanifolds in an increasing sequence of balls in $\mathbb{C}^n$, and we can ask what these look like. While we should not expect too precise an answer perhaps, we can exhibit submanifolds in $\mathbb{C}^n$ which exhibit the properties that Y_k exhibits. We can take an abelian variety $A = \mathbb{C}^n / \Lambda$, where Λ is a suitable lattice of rank $2n$. If $Y \subset A$ is a smooth ample divisor, then its inverse image $\tilde{Y}$ in the universal cover $\mathbb{C}^n$ is an example of the sort of curve we should be thinking about. Indeed, an η -transverse family of submanifolds Y_{j^2} of A , for integers $k = j^2$ that are perfect squares, can be obtained by taking the inverse image of Y by the covering map $A \rightarrow A$ given by multiplication by j . The simplest non-trivial case arises when $n = 2$ and A is a suitable small deformation of a product $A_0 = E_1 \times E_2$ of two elliptic curves. For the right deformation, there is an ample curve Y of genus 2 which is close to the singular curve $E_1 \cup E_2$ in A_0 . Its covering is an interesting submanifold of $\mathbb{R}^4$. It is an elementary observation that, for any n , the submanifold of $\mathbb{C}^n$ which we obtain in this way is actually independent of the choices of A and the ample divisor: any two choices give us submanifolds that are diffeomorphic by a diffeomorphism that is the composite of a real-linear transformation and a diffeomorphism that commutes with the translations of some lattice.

In a similar vein, one can take a Lefschetz pencil on an abelian variety, and consider the resulting Lefschetz fibration $\mathbb{C}^n \setminus Z \rightarrow \mathbb{CP}^1$. In the case $n = 2$, the set of points Z at which $\mathbb{C}^2$ must be blown up is discrete and 4-fold periodic. This

reflects the fact that in Donaldson's symplectic fibrations, the blow-up locus $Z = Z_k$ also fills out X as k increases.

Questions. It is hard to look at these results without asking questions about uniqueness of symplectic submanifolds up to isotopy. In dimension 4, for example, it seems just about possible that two homologous symplectic submanifolds are always isotopic. A suggestion was put forward by Donaldson [6] for an evolution equation, as a possible first step to a proof. In contrast, without the symplectic condition, there are many exotic embeddings of surfaces in 4-manifolds: there are knotted spheres in S^4 , and much more subtle examples in [10].

It is also tempting to look for other links between symplectic structures and fibrations. For example, it is not known whether a compact 4-manifold of the form $M^3 \times S^1$ can be symplectic if the 3-manifold M^3 is not fibered by surfaces.

References

1. M. Abreu, *Topology of symplectomorphism groups of $S^2 \times S^2$* , *Inventiones Mathematicae* **131** (1998), 1–23.
2. M. Abreu and D. McDuff, *Topology of symplectomorphism groups of rational ruled surfaces*, in preparation.
3. D. Auroux, *Symplectic 4-manifolds as branched coverings of $\mathbb{CP}^2$* , preprint.
4. ———, *Asymptotically holomorphic families of symplectic submanifolds*, *Geom. Funct. Anal.* **7** (1997), no. 6, 971–995.
5. S. K. Donaldson, in preparation.
6. ———, unpublished.
7. ———, *Symplectic submanifolds and almost-complex geometry*, *J. Differential Geom.* **44** (1996), no. 4, 666–705.
8. I. Ekeland and H. Hofer, *Symplectic topology and Hamiltonian dynamics, I*, *Math. Zeit.* **200** (1990), 355–378.
9. Yakov Eliashberg, *Filling by holomorphic discs and its applications*, *Geometry of low-dimensional manifolds, 2* (Durham, 1989), Cambridge Univ. Press, Cambridge, 1990, pp. 45–67.
10. Ronald Fintushel and Ronald J. Stern, *Surfaces in 4-manifolds*, *Math. Res. Lett.* **4** (1997), no. 6, 907–914.

11. M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, *Inventiones Mathematicae* **82** (1985), 307–347.
12. H. Hofer and E. Zehnder, *A new capacity for symplectic manifolds*, *Analysis et cetera* (P. Rabinowitz and E. Zehnder, eds.), Academic press, 1990, pp. 405–428.
13. ———, *Symplectic invariants and Hamiltonian dynamics*, The Floer memorial volume (H. Hofer et al., ed.), Birkhäuser, 1990, pp. 525–544.
14. P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces, I*, *Topology* **32** (1993), no. 4, 773–826.
15. F. Lalonde and D. McDuff, *The geometry of symplectic energy*, *Ann. Math.* **141** (1995), 349–371.
16. D. McDuff, *Examples of symplectic structures*, *Inventiones Mathematicae* **89** (1987), 13–36.
17. J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of two-spheres*, *Ann. Math.* **113** (1981), 1–24.
18. P. Seidel, *Floer homology and the symplectic isotopy problem*, Ph.D. thesis, Oxford University, 1997.
19. Clifford Henry Taubes, *The Seiberg-Witten and Gromov invariants*, *Math. Res. Lett.* **2** (1995), no. 2, 221–238.
20. Y. Yomdin, *The geometry of critical and near-critical values of differentiable mappings*, *Math. Ann.* **264** (1983), no. 4, 495–515.