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O-minimal Structures and Real Analytic Geometry

Lou van den Dries

ABSTRACT. O-minimal structures originate in model theory. At the same time this subject generalizes topics like semialgebraic and subanalytic geometry, and provides an efficient framework for developing Grothendieck's *topologie modérée*. No previous knowledge of the topic is assumed, and we include proofs of some basic o-minimal results. Next we indicate applications in several areas, and discuss various ways of building o-minimal structures on the real field. These structures are displayed in an inclusion diagram. We conclude with a list of open problems.

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University of Illinois at Urbana-Champaign; Department of Mathematics; 1409 West Green Street; Urbana, IL 61801; vddries@math.uiuc.edu

Introduction

Semialgebraic and subanalytic geometry are characterized by many finiteness properties and other tameness features. These properties turn out to be consequences of a few simple axioms, namely the axioms for **o-minimal** (= order-minimal) structures. Several interesting o-minimal structures do not fit into the subanalytic universe, and it is this fact that makes a difference. I will discuss the subject of o-minimality starting from scratch, and consider some applications, especially those of a real algebraic and real analytic nature. Notions and results referred to in this introduction will be made explicit later in the paper.

The theory of o-minimal structures began in the early 1980's in [**12**], [**53**], [**54**], motivated by Tarski's famous problem on the decidability of the real exponential field [**68**], as well as by more general model-theoretic concerns. Khovanskii's suggestive work [**30**] on the topology of solution sets of exponential polynomial equations gave a strong impulse. It was noticed [**13**] that by a slight twist subanalytic geometry falls under the roof of o-minimality, which greatly clarified subanalytic matters for model theorists. These trends merged and were enriched by new ideas in Wilkie's solution [**72**] in 1991 of the *geometric* part of Tarski's original problem, which implies in particular the o-minimality of the real exponential field. (The remaining *arithmetic* part of Tarski's problem reduces to Schanuel's conjecture in transcendental number theory, as shown in [**42**].)

Another influence is Grothendieck's 1984 "Esquisse d'un Programme" [**24**], which contains an eloquent plea for developing tame topology (*topologie modérée*). Many suggestions in sections 5 and 6 of that program are strikingly similar to actual o-minimal results. This makes perfect sense: a lot of model theory is concerned with discovering and charting the "tame" regions of mathematics, where wild phenomena like space filling curves and Gödel incompleteness are absent, or at least under control. As Hrushovski put it recently:

Model Theory = Geography of Tame Mathematics.

O-minimality is only a small part of that enterprise. It should also be said that the subject owes much to semialgebraic and subanalytic geometry: many constructions and arguments go through with only minor changes, acquiring in this cheap way a much wider range of validity.

We stay mostly in the classical real setting, to take advantage of its special features and simplify the exposition. (The model-theoretic scope of o-minimality is larger and has its own special features which I will point out now and then.) Sections 1–3 contain very basic facts on o-minimality, with selected proofs, while after that the paper has more of a survey character. The book [**14**] treats many of the topics of sections 1–4 in more detail and greater generality.

Notation. We let m and n range over $\mathbf{N} = \{0, 1, 2, \dots\}$. Given $S \subseteq X \times Y$ and $x \in X$ we put $S_x := \{y \in Y : (x, y) \in S\}$, and given also a map $f : S \to Z$ we let $f_x : S_x \to Z$ be defined by $f_x(y) = f(x, y)$.

1. Generalities, and Examples

Structures on the real line and Definability.

DEFINITION 1.1. A **structure** $\mathcal{S}$ on the real line consists of a boolean algebra $\mathcal{S}_n$ of subsets of $\mathbf{R}^n$ for each $n = 0, 1, 2, \dots$, such that

1. the diagonals $\Delta_{ij} := \{x \in \mathbf{R}^n : x_i = x_j\}$, $1 \leq i < j \leq n$, belong to $\mathcal{S}_n$;
2. $A \in \mathcal{S}_m, B \in \mathcal{S}_n \Longrightarrow A \times B \in \mathcal{S}_{m+n}$;
3. $A \in \mathcal{S}_{n+1} \Longrightarrow \pi(A) \in \mathcal{S}_n$, where $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is the projection map defined by $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$;
4. the ordering $\{(x, y) \in \mathbf{R}^2 : x < y\}$ of the real line belongs to $\mathcal{S}_2$.

Such a structure is usually generated as follows. We are given a collection $\mathcal{A}$ of subsets of the cartesian spaces $\mathbf{R}^n$ for various n, such that the ordering $\{(x, y) : x < y\}$ belongs to $\mathcal{A}$. Then we add to $\mathcal{A}$ the diagonals, and close off under boolean operations, cartesian products and projections, to obtain $\mathrm{Def}(\mathcal{A})$, the smallest structure on the real line containing $\mathcal{A}$. Sets in $\mathrm{Def}(\mathcal{A})$ are said to be **definable from** $\mathcal{A}$, or just **definable** if $\mathcal{A}$ is clear from context. The definable sets may be much more complicated—topologically or otherwise—than the sets in $\mathcal{A}$. Given $\mathcal{A}$ one aims for a more or less effective characterization of the definable sets, but this can be difficult, or even impossible.

EXAMPLE 1.2. (Semialgebraic sets). Let $\mathcal{A} = \mathbf{alg}$ be the collection whose elements are $\{(x, y) : x < y\}$, the one-point sets $\{r\}$ with $r \in \mathbf{R}$, and (the graphs of) addition and multiplication (as subsets of $\mathbf{R}^3$).

A semialgebraic set in $\mathbf{R}^n$ is by definition a finite union of sets of the form

$$\{x \in \mathbf{R}^n : f(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0\}$$

where f and the g's are real polynomials in n variables. It is easy to see that semialgebraic sets are definable, but less obvious that they are exactly the definable sets. This last fact is essentially the content of the Tarski-Seidenberg theorem [**68**], [**63**], which says that the image of a semialgebraic set in $\mathbf{R}^{n+1}$ under the projection map $\mathbf{R}^{n+1} \to \mathbf{R}^n$ is semialgebraic in $\mathbf{R}^n$.

Thus the definable sets are generated from **alg** in a very simple way. They are very tame objects. For example, each semialgebraic set is a finite disjoint union of connected real analytic submanifolds of its ambient cartesian space, each of these manifolds being semialgebraic, and analytically diffeomorphic to some cartesian space $\mathbf{R}^k$.

EXAMPLE 1.3. Let $\mathcal{A} := \mathbf{alg} \cup \{\mathbf{Z}\}$, so $\mathcal{A}$ contains besides the sets of example 1.2 also the set of integers as an element. Then the situation changes drastically: all open subsets and even all Borel subsets of each $\mathbf{R}^n$ become definable. Generating $\mathrm{Def}(\mathcal{A})$ requires arbitrarily many alternations of projection and complementation operations; see [**29**], Exercise 37.6., and [**14**], p.16. These facts are closely related to the logical discoveries in the 1930's by Gödel et al. Since there are definable (even Borel) bijections between the line $\mathbf{R}$ and the plane $\mathbf{R}^2$, key geometric notions like dimension get obliterated.

It makes no difference if in the definition of $\mathcal{A}$ we replace the set $\mathbf{Z}$ by the (graph of the) sine function, since $\mathbf{Z} = \{x \in \mathbf{R} : \sin(\pi x) = 0\}$ is then definable.

Hereditary properties. Let us fix a structure $\mathcal{S}$ on the real line. It will be convenient to refer to the sets in $\mathcal{S}$ as the "definable sets", even though $\mathcal{S}$ may not be presented to us in the form $\mathrm{Def}(\mathcal{A})$. All intervals are definable; here and below, an *interval* is always an open interval $(a, b) \subseteq \mathbf{R}$, $-\infty \leq a < b \leq +\infty$. If $A \subseteq \mathbf{R}^n$ is definable, so is its closure $\mathrm{cl}(A)$. Given definable sets $A \subseteq \mathbf{R}^m$ and $B \subseteq \mathbf{R}^n$ we say that a map $f : A \to B$ is definable if its graph $\Gamma(f) \subseteq \mathbf{R}^{m+n}$ is definable. In that case, given any definable $A' \subseteq A$ and $B' \subseteq B$, the restriction $f|A' : A' \to B$, the image $f(A')$, and the inverse image $f^{-1}(B')$ are also definable. The definable sets with the maps between them as morphisms form a category under the usual composition of maps, and a morphism $f : A \to B$ is an isomorphism in this category if it is a bijection. This category has products: if $f : A \to B$ and $g : A \to C$ are morphisms, so are $(f, g) : A \to B \times C$, and the projection maps $B \times C \to B$ and $B \times C \to C$.

Suppose $\mathcal{S}$ contains addition and multiplication. Then, given a definable real valued function f on an open definable set $U \subseteq \mathbf{R}^n$ and a natural number k, the set $U^{(k)}$ of points in U in a neighborhood of which f is C^k is definable, and the partials of order at most k of f are definable functions on $U^{(k)}$.

There are many basic facts of this kind. They can be quickly verified by writing the condition for a point to belong to the set in question as a logical formula with variables ranging over $\mathbf{R}$, and observing that the logical connectives and quantifiers correspond to operations on sets under which $\mathcal{S}$ is closed, see [**18**], Appendix A for details. Thus familiar definitions in ϵ-δ style imply hereditary properties.

Definable manifolds. It is easy to go beyond the cartesian spaces $\mathbf{R}^n$ as ambient spaces: introduce definable manifolds. Let $k \in \mathbf{N} \cup \{\infty, \omega\}$. Then a **definable C^k-atlas** of dimension m on a set M is a finite collection $\{f_i : i \in I\}$ of bijections $f_i : U_i \to f_i(U_i)$ between sets $U_i \subseteq M$ and open definable sets $f_i(U_i) \subseteq \mathbf{R}^m$ such that

1. $M = \bigcup_i U_i$,
2. for all $i, j \in I$ the set $f_i(U_i \cap U_j) \subseteq \mathbf{R}^m$ is definable and open in $f_i(U_i)$, and the transition map

$$f_{ij} := f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_j \cap U_i)$$

is a definable C^k-diffeomorphism.

Two definable C^k-atlases of dimension m on a set M are equivalent if their union is also one. A **definable C^k-manifold** of dimension m is a set M equipped with an equivalence class of definable C^k-atlases of dimension m on M. Given such a definable manifold M with atlas $\{f_i : i \in I\}$ as above we equip M with the topology that makes the maps f_i homeomorphisms, and we declare $A \subseteq M$ to be definable if each set $f_i(A) \subseteq \mathbf{R}^m$ is definable. Given definable C^k-manifolds M and

N of dimensions m and n we make the cartesian product $M \times N$ into a definable C^k-manifold of dimension $m+n$ in the usual way. As in the cartesian setting we shall say that a map $f : A \to B$ between definable sets $A \subseteq M$ and $B \subseteq N$ is *definable* if its graph is a definable subset of $M \times N$. The category of definable subsets of cartesian spaces $\mathbf{R}^n$ and definable maps between them now appears as a full subcategory of the category of definable subsets of definable C^k-manifolds and the definable maps between them. Also the other hereditary properties of definable subsets of cartesian spaces and definable maps between them that we mentioned before go through. For $\mathcal{S} = \{\text{semialgebraic sets}\}$ the definable manifolds are known as Nash manifolds, see [**65**].

Suppose $\mathcal{S}$ contains addition and multiplication. Then, given $k > 0$ and a definable C^k-manifold M, we can construct in the obvious way the tangent and cotangent bundles TM and T^*M of M as definable C^{k-1}-manifolds. The various natural maps associated to these objects, such as the projection maps $TM \to M$ and $T^*M \to M$, and the addition of tangent vectors, are definable maps. We can talk about definable vector fields, definable differential forms, and so on ...

Enough has been said to make the point that $\mathcal{S}$ gives rise to a kind of self-contained universe in which we can freely carry out the usual finitary geometric constructions. But this universe may be so large that most of mathematics fits in it, as in example 1.3 above. When does a structure $\mathcal{S}$ on the real line produce a comparatively small world, and what parts of mathematics live there?

O-minimality. There is indeed a simple condition, namely "o-minimality", that guarantees a relatively tame mathematical world. Fortunately, this condition can be verified in cases of genuine interest.

DEFINITION 1.4. A structure $\mathcal{S}$ on the real line is said to be **o-minimal** if the sets in $\mathcal{S}_1$ are exactly the subsets of $\mathbf{R}$ that have only finitely many connected components, that is, the finite unions of intervals and points. (The "o-minimality axiom".)

The o-minimality axiom states the strongest possible compatibility of $\mathcal{S}$ with the ordering (and hence topology) of the real line. It has numerous consequences, see sections 2–4. The class of semialgebraic sets is clearly o-minimal. Here is a much simpler o-minimal structure. It is of interest because triangulation (see section 3) sometimes reduces problems to the semilinear setting:

EXAMPLE 1.5. (Semilinear sets). We let $\mathcal{A}$ consist of the ordering, the singletons $\{r\}$ ($r \in \mathbf{R}$), the scalar multiplications $x \mapsto \lambda x : \mathbf{R} \to \mathbf{R}$ ($\lambda \in \mathbf{R}$), and of addition. It is not hard to show that then $\mathrm{Def}(\mathcal{A})$ consists of the **semilinear sets**: the subsets of $\mathbf{R}^n$ (for each n) that are finite unions of sets of the form

$$\{x \in \mathbf{R}^n : f_1(x) = \cdots = f_k(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0\}$$

where the f_i and g_j are real polynomials in n variables of degree at most 1. Given any bounded semilinear sets $A_1, \ldots, A_k \subseteq \mathbf{R}^n$ there is a finite simplicial complex K in the affine space $\mathbf{R}^n$ such that each A_i is a union of open simplices of K.

Terminology. The o-minimality of $\mathrm{Def}(\mathcal{A})$ is also expressed by saying that $\mathbf{R}_{\mathcal{A}}$ is o-minimal. Here $\mathbf{R}_{\mathcal{A}}$ is thought of as $\mathbf{R}$ equipped with the sets in $\mathcal{A}$ as *basic* relations. Usually these relations other than the ordering are (graphs of) real valued functions, the *basic* or *primitive* functions of $\mathbf{R}_{\mathcal{A}}$.

Thus $\mathbf{R}_{\mathbf{alg}}$, whose definable sets are exactly the semialgebraic sets, is o-minimal. We refer to $\mathbf{R}_{\mathbf{alg}}$ as **the real field**. A structure $\mathcal{S}$ on the real line that contains all semialgebraic sets is also called a **structure on the real field**, and if $\mathcal{A} \supseteq$ **alg**, then we call $\mathbf{R}_{\mathcal{A}}$ an **expansion of the real field.**

The next three examples of o-minimal expansions of the real field bring us roughly to the state of knowledge of 1994. (In section 5 we sketch the new o-minimal expansions of the real field that were constructed more recently.)

EXAMPLE 1.6. (Globally subanalytic sets). Let **an** be the collection whose elements are: the ordering, addition, multiplication, and the functions $f : \mathbf{R}^n \to \mathbf{R}$ (for all n) such that $f|I^n$ is analytic, $I = [-1, 1]$, and f is identically 0 outside I^n. We call these functions f *restricted analytic functions.* The choice of the cubes I^n is just a convenient normalization. The expansion $\mathbf{R}_{\mathbf{an}}$ of the real field is called **the real field with restricted analytic functions.** One can show that the definable sets of $\mathbf{R}_{\mathbf{an}}$ are exactly the so-called *globally subanalytic* sets.

These sets are obtained as follows. First, a subset A of an analytic manifold M is said to be **semianalytic in** M if M can be covered by open subsets U such that each $A \cap U$ is a finite union of sets of the form $\{x \in U : f(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0\}$ where f and the g's are analytic functions on U. Next, a subset B of an analytic manifold M is said to be **subanalytic in** M if M can be covered by open subsets U such that each $B \cap U$ is of the form $\pi(A)$ for some relatively compact $A \subseteq U \times \mathbf{R}^p$ which is semianalytic in $U \times \mathbf{R}^p$, where $\pi : U \times \mathbf{R}^p \to U$ is the projection map, and p may depend on B and U. Finally, a subset of $\mathbf{R}^n$ is said to be **globally subanalytic** if it is subanalytic in the larger compact analytic manifold $\mathbf{P}_1^n$, where $\mathbf{P}_1 = \mathbf{R} \cup \{\infty\}$ is the real projective line. One should be aware that the graph of the usual exponential function exp is subanalytic in $\mathbf{R}^2$, but not globally subanalytic. The twist of considering *globally* subanalytic sets is needed to make subanalyticity fall under o-minimality, see [**13**].

The theory of semianalytic sets is due to Łojasiewicz [**41**], who also showed that subanalytic sets in $\mathbf{R}^2$ are semianalytic in $\mathbf{R}^2$. Subanalytic sets were then introduced by Gabrielov [**22**] and Hironaka [**27**], who proved among other things the "theorem of the complement": if a subset of an analytic manifold M is subanalytic in M, then its complement is also subanalytic in M. This result easily implies that the definable sets of $\mathbf{R}_{\mathbf{an}}$ are exactly the globally subanalytic sets, as claimed. That $\mathbf{R}_{\mathbf{an}}$ is o-minimal follows then from Łojasiewicz's theorem that relatively compact semianalytic sets have only finitely many connected components. See Bierstone & Milman [**4**] for an efficient exposition of the Łojasiewicz-Gabrielov-Hironaka theory of semi- and subanalytic sets.

A different treatment of subanalyticity is in Denef-Van den Dries [**11**]. This approach has a p-adic analogue, with applications to Poincaré series of p-adic analytic sets. Another advantage is that the globally subanalytic sets in $\mathbf{R}^n$ get directly described in terms of equations and inequalities, in the style of semialgebraic sets. The polynomial functions in the defining equations and inequalities for semialgebraic subsets of $\mathbf{R}^n$ are replaced by the functions on $\mathbf{R}^n$ obtained via composition (substitution) from:

1. constant functions, coordinate functions $x_1, \ldots, x_n$, addition and multiplication,
2. the restricted analytic functions,
3. the reciprocal function $x \mapsto 1/x$, with the convention $1/0 := 0$.

EXAMPLE 1.7. (Subexponential sets). Let $\mathbf{alg}, \exp := \mathbf{alg} \cup \{\exp\}$, where exp is the usual exponential function on $\mathbf{R}$. Then the corresponding expansion $\mathbf{R}_{\mathbf{alg}, \exp}$ of the real field is called *the real exponential field.* Its definable sets are the subexponential sets, which are obtained as follows.

The **exponential sets** in $\mathbf{R}^n$ are by definition the sets of the form

$$\{x \in \mathbf{R}^n : P(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) = 0\},$$

where P is a real polynomial in $2n$ variables. A **subexponential set** in $\mathbf{R}^n$ is by definition the image of an exponential set in $\mathbf{R}^{n+k}$ (for some k) under the projection map $\mathbf{R}^{n+k} \to \mathbf{R}^n$. The notion of "exponential set" is ad hoc, but the class of subexponential sets is comparatively intrinsic and stable: Wilkie's theorem [**72**] says that the complement of a subexponential set in $\mathbf{R}^n$ is again subexponential in $\mathbf{R}^n$. This remarkable result implies immediately that the definable sets of $\mathbf{R}_{\mathbf{alg}, \exp}$ are exactly the subexponential sets, as claimed. By Khovanskii [**30**] the exponential sets, and hence the subexponential sets, have only finitely many connected components. Thus $\mathbf{R}_{\mathbf{alg}, \exp}$ is o-minimal.

Wilkie used model-theoretic techniques. A purely geometric proof has now become available, by combining recent works of Gabrielov [**23**] and of Lion & Rolin [**36**], as indicated in [**37**].

EXAMPLE 1.8. (Analytic-exponential sets). Let $\mathbf{an}, \exp := \mathbf{an} \cup \{\exp\}$. Then $\mathbf{R}_{\mathbf{an}, \exp}$ is o-minimal, as was first shown in [**17**] by adapting Wilkie's method. Subsequently, Van den Dries, Macintyre and Marker found a much shorter proof [**15**], which gave in addition an explicit description of the definable subsets of $\mathbf{R}^n$: just replace the polynomial functions in the defining equations and inequalities for semialgebraic subsets of $\mathbf{R}^n$ by the functions on $\mathbf{R}^n$ obtained via composition from the same functions as in the description above for globally subanalytic sets, *augmented* by the exponential function, and its inverse, the logarithm function (extended to all of $\mathbf{R}$ by setting $\log(x) = 0$ for $x \leq 0$).

The model-theoretic approach of [**15**] has suggested a completely geometric proof [**36**] of the above characterization of the definable sets, with useful constructive features.

Analytic-geometric categories. From $\mathbf{R}_{\mathbf{an}}$ and $\mathbf{R}_{\mathbf{an},\exp}$ we can pass to a category of manifolds different from that of definable manifolds. More generally, let $\mathbf{R}_{\mathcal{A}}$ be o-minimal, and suppose all restricted analytic functions are definable in $\mathbf{R}_{\mathcal{A}}$. Then, given any *analytic* manifold M of dimension m (definable or not) we introduce a collection $\mathscr{C}_{\mathcal{A}}(M)$ of distinguished subsets of M, by making $A \subseteq M$ belong to $\mathscr{C}_{\mathcal{A}}(M)$ if for each point $x \in M$ there is an open neighborhood U of x, an open $V \subseteq \mathbf{R}^m$ and an analytic isomorphism $h : U \to V$ such that $h(U \cap V)$ is definable in $\mathbf{R}_{\mathcal{A}}$. (One can always take $V = \mathbf{R}^m$.) For example, $\mathscr{C}_{\mathbf{an}}(M)$ turns out to be exactly the collection of subanalytic sets in M.

We obtain a so-called *analytic-geometric* category $\mathscr{C}_{\mathcal{A}}$: its objects are the pairs (A, M) as above with $A \in \mathscr{C}_{\mathcal{A}}(M)$, and its morphisms $(A, M) \to (B, N)$ are the continuous maps $A \to B$ with graph in $\mathscr{C}_{\mathcal{A}}(M \times N)$. The properties of these categories are routine consequences (using charts and partitions of unity) of those of the corresponding o-minimal structures. Typically, finiteness properties of $\mathbf{R}_{\mathcal{A}}$ become local finiteness properties in $\mathscr{C}_{\mathcal{A}}$. For details, see Van den Dries-Miller **[18]**, where these categories are also characterized axiomatically, independent of their o-minimal origin.

This categorical reorganization was useful to Schmid and Vilonen **[60]**, **[61]**, **[62]**. In their proof of the Barbasch-Vogan conjecture in the representation theory of Lie groups they had to go outside the subanalytic setting and work in the category $\mathscr{C}_{\mathbf{an},\exp}$.

General o-minimal structures. In the model theory of o-minimal structures we go beyond the real line as follows. Let $(R, <)$ be any nonempty dense linearly ordered set, without smallest or largest element, "dense" meaning that if $a < b$, then there is c with $a < c < b$. The ordering gives rise to a topology on R in the usual way, and each cartesian power R^n is equipped with the product topology. One defines the notion of a structure $\mathcal{S}$ on $(R, <)$ exactly as in the case of the real line, and o-minimality of $\mathcal{S}$ means that the sets in $\mathcal{S}_1$ are exactly the finite unions of intervals and points, where "intervals" are the sets (a, b) with $a < b$, $a, b \in R \cup \{-\infty, +\infty\}$. Thus, assuming $\mathcal{S}$ is o-minimal, definable subsets of R have a least upper bound in $R \cup \{-\infty, +\infty\}$. This suggests that much of basic real variable theory might go through for *definable* sets and maps. Indeed it does, with some notions suitably modified. (For example, the correct version of connectedness for a definable set in R^n is that it is not a union of two disjoint nonempty *definable* open subsets. A basic real variable result goes through in the following way: the image of a closed bounded *definable* set under a continuous *definable* map is again closed and bounded.)

Starting with Pillay & Steinhorn [**54**], the subject of o-minimality has been systematically pursued in this generality. This makes good sense even if our primary interest is in the real case: in the model-theoretic setting subsets of $\mathbf{R}^n$ acquire "generic" points outside the reals, with benefits similar to the use of generic points in algebraic geometry. Of greater weight perhaps is that model theory focuses attention on a novel kind of intrinsic property of mathematical structures, having to do with "classification up to bi-interpretability". In this connection new methods were developed to obtain coordinatization theorems that are reminiscent of geometric algebra and Hilbert's 5th problem. For o-minimal structures a definitive result of this nature was achieved by Peterzil & Starchenko in [**51**]. Roughly (and locally) speaking, these structures are shown to fall into three intrinsically different classes, called "trivial", "linear", and "nonlinear". It is hard to do justice to this direction in the subject without introducing elaborate terminology and machinery. Here is a consequence of the Trichotomy Theorem in [**51**] that pertains just to the nonlinear case and is easy to state.

COROLLARY 1.9. *Let $\mathcal{S}$ be an o-minimal structure on $(R, <)$. Suppose that there is a definable map $f : R^{n+1} \to R$ such that for any two intervals I and J there exists $a \in R^n$ for which f_a maps I onto J. Then there exist definable functions $+, \cdot : R^2 \to R$ such that $(R, <, +, \cdot)$ is an ordered real closed field. In particular, if $(R, <)$ is order isomorphic to the real line, then there is such an order isomorphism under which $\mathcal{S}$ corresponds to an o-minimal structure on the real field.*

In accordance with a general trend in model theory, the definable groups in an o-minimal structure (the group objects in its category of definable sets) have been studied extensively, see in particular the series of papers [**48**], [**49**], [**50**]. Loosely speaking, these objects behave like Lie groups and algebraic groups.

2. The Monotonicity Theorem and its Consequences

The monotonicity theorem is the fundamental one-variable result of the subject. We fix an o-minimal structure $\mathcal{S}$ on the real line, and give in this setting a quick proof of the monotonicity theorem and of its smooth version, following [**12**]. (The monotonicity theorem for arbitrary o-minimal structures is more difficult, and is due to Pillay & Steinhorn [**54**].)

Next we derive curve selection, introduce the polynomial/exponential dichotomy for o-minimal structures on the real field, indicate some Łojasiewicz type inequalities, and discuss a recent answer to an old question of Hardy.

MONOTONICITY THEOREM. *Let $f\colon I \to \mathbf{R}$ be a definable function on an interval $I = (a, b)$. Then there are $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ such that for each $i = 0, \ldots, k$ the restriction $f|(a_i, a_{i+1})$ is continuous, and either constant, or strictly increasing, or strictly decreasing.*

PROOF. We establish three claims.

1. f is continuous at some point of I.

If there is an interval $J \subseteq I$ such that $f(J)$ is finite, then for at least one point $c \in f(J)$, the inverse image $f^{-1}(c)$ is infinite and hence contains a subinterval of I, and f will be constant, hence continuous, on that particular subinterval. Thus we may assume that $f(J)$ is infinite for each interval $J \subseteq I$. We now construct inductively a descending sequence $[r_1, s_1] \supseteq [r_2, s_2] \supseteq \cdots$, with $[r_n, s_n] \subseteq I$, $0 < s_n - r_n < \frac{1}{n}$, $r_n < r_{n+1} < s_{n+1} < s_n$, such that $f([r_n, s_n]) \subseteq J_n$, an interval of length $< \frac{1}{n}$. (Then f will be continuous at the unique point in $\bigcap_{n=1}^{\infty}[r_n, s_n]$.) For J_1, take any interval of length < 1 contained in $f(I)$, and for $[r_1, s_1]$ take any segment with $0 < s_1 - r_1 < 1$ such that $[r_1, s_1] \subseteq f^{-1}(J)$. Having constructed $[r_n, s_n] \subseteq I$ with $0 < s_n - r_n < \frac{1}{n}$, we choose an interval $J_{n+1} \subseteq f([r_n, s_n])$ of length $< \frac{1}{n+1}$, and then choose $[r_{n+1}, s_{n+1}] \subseteq f^{-1}(J_{n+1}) \cap (r_n, s_n)$ such that $0 < s_{n+1} - r_{n+1} < \frac{1}{n+1}$. This proves Claim 1.

2. f is continuous at all but finitely many points.

To see this, note that the set $\{x \in I : f \text{ is not continuous at } x\}$ is definable. If it were infinite, it would contain an interval, contradicting Claim 1 for the restriction of f to that subinterval.

So we may assume for the rest of the proof that f is continuous.

3. There is a subinterval of I on which f is either constant or strictly monotone. For this, let $p, q \in I$, $p < q$, and suppose f is not constant on (p, q). Then $f([p, q])$ contains a segment $[c, d]$, $c < d$, and we define

$$g\colon [c, d] \to [p, q], \quad g(y) := \min\{x \in [p, q] : f(x) = y\}.$$

The function g is injective and definable, hence continuous on some interval $J \subseteq [c, d]$ by Claim 2. Therefore g is strictly monotone on J. Thus f is strictly monotone on the interval $g(J) \subseteq [p, q]$, which proves Claim 3.

Now let E be the set of all points $x \in I$ such that there is no subinterval of I around x on which f is constant or strictly monotone. Then E is definable, and cannot contain an interval, by Claim 3. So E has to be finite, say $E = \{a_1, \dots, a_k\}$. Then the theorem holds with these a_i's. □

For the rest of this section we shall assume that $\mathcal{S}$ contains addition and multiplication, that is, $\mathcal{S}$ is an o-minimal structure on the *field* of real numbers.

SMOOTH MONOTONICITY THEOREM. *Let p be a positive integer. Then in the Monotonicity Theorem we can take the a_i such that in addition f is of class C^p on each subinterval (a_i, a_{i+1}), $i = 0, \dots, k$.*

PROOF. First note that if the function f in the Monotonicity Theorem is differentiable, then f' is definable. Thus by induction on p it suffices to prove the desired result for $p = 1$. Next we establish a few claims.

1. Let $c \in I$. Then the right and left derivatives

$$f'(c^+) := \lim_{x \downarrow c} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad f'(c^-) := \lim_{x \uparrow c} \frac{f(x) - f(c)}{x - c}$$

of f at c exist in $\mathbf{R} \cup \{-\infty, +\infty\}$.

This follows from the Monotonicity Theorem: the difference quotients displayed are clearly definable functions of x.

2. Suppose f is continuous and $f'(c^+) > 0$ for all $c \in I$. Then f is strictly increasing, and its inverse f^{-1}, defined on the interval $f(I)$, satisfies

$$(f^{-1})'(d^+) = \frac{1}{f'(c^+)}, \text{ for } d := f(c), c \in I.$$

This is clear.

3. Suppose f is continuous and the functions $f'(x^+)$ and $f'(x^-)$ are real valued and continuous on I. Then f is of class C^1.

For this, it is enough to show that $f'(x^+) = f'(x^-)$ for all $x \in I$. Suppose that $f'(c^+) > f'(c^-)$ for some $c \in I$. Take some r such that $f'(c^+) > r > f'(c^-)$. Then by continuity there is an interval $J \subseteq I$ around c such that $f'(x^+) > r > f'(x^-)$ for all $x \in J$. Hence the continuous function $g\colon J \to \mathbf{R}$ defined by $g(x) = f(x) - rx$ satisfies $g'(x^+) > 0$ and $g'(x^-) < 0$ on J. By Claim 2, this means that g is both strictly increasing and strictly decreasing, which is impossible.

4. $\{x \in I : f'(x^+) = +\infty\}$ is finite.

Otherwise, there is an interval $J \subseteq I$ such that $f'(x^+) = +\infty$ for all $x \in J$. We may also assume that f is continuous on J, so f is strictly increasing on J, hence $f'(x^-) \geq 0$ for all $x \in J$. After shrinking J suitably, we may further assume that either

(i) $f'(x^-) = +\infty$ for all $x \in J$, or

(ii) $f'(x^-) \in \mathbf{R}$ for all $x \in J$.

In case (i) the inverse g of $f|J$ satisfies $g'(y^+) = g'(y^-) = 0$ for $y \in f(J)$. (Claim 2.) Therefore g is constant, contradicting injectiveness. In case (ii) we apply the same argument as in the proof of Claim 3. to reach a contradiction.

Next, by the Monotonicity Theorem we reduce to the case of a continuous f. By Claim 4 we may also assume that the functions $f'(x^+)$ and $f'(x^-)$ are real valued on I and (again by Monotonicity) continuous on I. Now apply Claim 3. □

REMARK. For all presently known o-minimal structures on the real field the definable functions are piecewise analytic, that is, the Smooth Monotonicity Theorem holds even with $p = \omega$.

Definable choice, and curve selection. We shall pick out from each non-empty definable set $A \subseteq \mathbf{R}^m$ an element $e(A) \in A$ in a "definable" way. The idea is to let $e(A)$ be the midpoint of A, if $m = 1$ and A is a bounded interval, and then use induction on m. In detail:

1. Let $A \subseteq \mathbf{R}$ be definable and non-empty. If A has a least element, take $e(A)$ as this least element. If not, let (a, b) be its leftmost interval, i.e.

$$a := \inf A, a \in \mathbf{R} \cup \{-\infty\}, \quad b := \sup\{x : (a, x) \subseteq A\}.$$

Then $a < b$, $(a,b) \subseteq A$. Put

$$e(A) := \begin{cases} 0, & \text{if } a = -\infty,\ b = +\infty, \\ b-1, & \text{if } a = -\infty,\ b \in \mathbf{R}, \\ a+1, & \text{if } a \in \mathbf{R},\ b = +\infty, \\ \frac{1}{2}(a+b), & \text{if } a, b \in \mathbf{R}. \end{cases}$$

2. Let $A \subseteq \mathbf{R}^m$ be definable, $A \neq \varnothing$, $m > 1$. Then $\pi(A) \subseteq \mathbf{R}^{m-1}$, and we assume $b := e(\pi(A)) \in \pi(A)$ has been chosen. Then we put $e(A) := (b, e(A_b))$.

Then we can conclude:

PROPOSITION 2.1. (Definable Choice)

(i) *If $S \subseteq \mathbf{R}^{m+n}$ is definable, then there is a definable map $f\colon \pi(S) \to \mathbf{R}^n$ such that $\Gamma(f) \subseteq S$, where $\pi\colon \mathbf{R}^{m+n} \to \mathbf{R}^m$ is the projection map given by $\pi(x_1, \ldots, x_{m+n}) := (x_1, \ldots, x_m)$.*

(ii) *For each definable equivalence relation on a definable set $X \subseteq \mathbf{R}^m$ there exists a definable subset of X that intersects each equivalence class in exactly one point.*

PROOF. For (i), define $f(x) := e(S_x)$ for $x \in \pi(S)$. For (ii), note that

$$\{e(A) : A \text{ is an equivalence class}\}$$

is a definable set of representatives as required. □

COROLLARY 2.2. (Curve Selection) *Let $X \subseteq \mathbf{R}^m$ be a definable set, and let $a \in \mathrm{cl}(X) \setminus X$. Then there is a definable continuous injective map*

$$\gamma : [0, \varepsilon) \to \mathbf{R}^m \qquad (\textit{for some } \varepsilon > 0)$$

such that $\gamma(0) = a$ and $\gamma(t) \in X$ for $0 < t < \varepsilon$.

PROOF. The definable set $\{|a - x| : x \in X\}$ contains arbitrarily small elements, hence contains an interval $(0, \varepsilon)$, $\varepsilon > 0$. That is, for each $t \in (0, \varepsilon)$ there is $x \in X$ such that $|a - x| = t$. By Definable Choice there is then a definable map $\gamma\colon [0, \varepsilon) \to X$ such that $|a - \gamma(t)| = t$ for $0 \leq t < \varepsilon$. By decreasing ε if necessary we may assume by the Monotonicity Theorem that γ is continuous. □

Given any positive integer p we can arrange that γ is of class C^p: use a definable reparametrization and the Smooth Monotonicity Theorem.

Hardy fields and the Growth Dichotomy. By the Monotonicity Theorem the germs at $+\infty$ of the definable real valued functions on intervals $(a, +\infty)$ form an *ordered field* $\mathrm{H}(\mathcal{S})$ with respect to the usual ring operations on germs; the germ of such a function f is positive if ultimately $f(x) > 0$. ("Ultimately" means "for all sufficiently large x".) We also write $\mathrm{H}(\mathbf{R}_{\mathcal{A}})$ for $\mathrm{H}(\mathcal{S})$ when $\mathcal{S} = \mathrm{Def}(\mathcal{A})$.

Examples. The field $\mathrm{H}(\mathbf{R}_{\mathbf{alg}})$ is isomorphic to the field of real Puiseux series in x^{-1} that are algebraic over the rational function field $\mathbf{R}(x)$. The field $\mathrm{H}(\mathbf{R}_{\mathbf{an}})$ is isomorphic to the field of real Puiseux series in x^{-1} that converge for all sufficiently large real values of x. The isomorphism is given in both cases by taking the sum of the series for sufficiently large real values of x.

By the Smooth Monotonicity Theorem $\mathrm{H}(\mathcal{S})$ is also a *differential* field, since each germ in $\mathrm{H}(\mathcal{S})$ has a differentiable representative whose derivative has its germ in $\mathrm{H}(\mathcal{S})$. Thus $\mathrm{H}(\mathcal{S})$ is a so-called *Hardy field* in the sense of Bourbaki [**6**], see also Rosenlicht [**57**]. The subject of Hardy fields is a modern incarnation of work on "Orders of infinity", see Hardy [**25**]. One aspect of o-minimality is that it can be seen as an extension of the theory of Hardy fields to higher dimensions.

Among many results by Rosenlicht on Hardy fields is Proposition 6 in [**58**]. It says that if f belongs to a Hardy field $K \supseteq \mathbf{R}(x)$ and $f > x^n$ for all n, then there is $g \in K$ such that $g(x)/\log f(x) \to 1$ as $x \to +\infty$ (letting x denote both the germ at $+\infty$ of the identity function on $\mathbf{R}$, and a real variable). Miller [**44**] noticed that this fact, in combination with special "closure under composition" properties of $\mathrm{H}(\mathcal{S})$, leads to the following surprising dichotomy among o-minimal structures on the real field.

THEOREM 2.3. *Either the exponential function is definable, or for each definable function $f : (a, +\infty) \to \mathbf{R}$ there exists a natural number n such that ultimately $|f(x)| < x^n$.*

Polynomial growth. We say that $\mathcal{S}$—and $\mathbf{R}_{\mathcal{A}}$ when $\mathcal{S} = \mathrm{Def}(\mathcal{A})$—is **polynomially bounded** if the second possibility in the last theorem is realized. For example, $\mathbf{R}_{\mathbf{alg}}$ and $\mathbf{R}_{\mathbf{an}}$ are polynomially bounded, by the Puiseux expansion results just mentioned. Suppose $\mathcal{S}$ is polynomially bounded. Then

$$\{r \in \mathbf{R} : \text{ the function } x \mapsto x^r : (0, +\infty) \to \mathbf{R} \text{ is definable }\}$$

is a subfield of $\mathbf{R}$, called the **field of exponents** of $\mathcal{S}$, and of $\mathbf{R}_{\mathcal{A}}$ when $\mathcal{S} = \mathrm{Def}(\mathcal{A})$. Thus the field of exponents of $\mathbf{R_{alg}}$ and of $\mathbf{R_{an}}$ is $\mathbf{Q}$. Any subfield E of $\mathbf{R}$ is the field of exponents of some polynomially bounded o-minimal expansion $\mathbf{R}_{\mathcal{A}}$ of the real field, in fact, one can take $\mathcal{A} = \mathbf{an} \cup \{x^r : r \in E\}$, where x denotes the identity function on $(0, +\infty)$, see [**46**]. The field of exponents turns out to be a significant invariant associated to a polynomially bounded $\mathcal{S}$, for partly model-theoretic reasons we won't go into. Here we just state a complement from [**44**] to the dichotomy theorem above.

PROPOSITION 2.4. *Suppose $\mathcal{S}$ is polynomially bounded. Then, given any definable function $f : \mathbf{R} \to \mathbf{R}$ such that ultimately $f(x) \neq 0$, there exist real constants c and r, with $c \neq 0$ and r in the field of exponents of $\mathcal{S}$, such that $f(x) = cx^r + o(x^r)$ as $x \to +\infty$.*

Hörmander-Łojasiewicz inequalities. There are several inequalities involving polynomials and analytic functions that go under that name, and they extend easily to the polynomially bounded o-minimal setting, see [**18**]. Here is an example, which is actually in the form of an equality (implying inequalities).

PROPOSITION 2.5. *Suppose $\mathcal{S}$ is polynomially bounded. Let $A \subseteq \mathbf{R}^n$ be closed, and let $f, g : A \to \mathbf{R}$ be continuous and definable with $f^{-1}(0) \subseteq g^{-1}(0)$. Then $g^N = hf$ for some positive integer N and some continuous definable function $h : A \to \mathbf{R}$. (Thus, if A is compact, there is a constant $C > 0$ such that $|g|^N \leq C|f|$.)*

In applications g is often the distance to $f^{-1}(0)$, or f the distance to $g^{-1}(0)$. There is a generalization to arbitrary $\mathcal{S}$: instead of raising g to the N^{th} power we apply a suitable one-variable function to g:

With f, g, A as in the last proposition and any integer $p > 0$ we have $\phi \circ g = hf$ for some continuous definable $h : A \to \mathbf{R}$ and some strictly increasing odd definable function $\phi : \mathbf{R} \to \mathbf{R}$ of class C^p, with $\phi^{(i)}(0) = 0$ for all $i < p$.

This result serves some of the same purposes as the standard Łojasiewicz inequalities, see [**18**].

Exponential growth. We say that $\mathcal{S}$—and $\mathbf{R}_{\mathcal{A}}$ when $\mathcal{S} = \mathrm{Def}(\mathcal{A})$—is **exponentially bounded** if for every definable real valued function f on an interval

$(a, +\infty)$ there is a natural number n such that ultimately $|f(x)| < \exp_n(x)$. Here $\exp_n(x)$ is the n^{th} iterate of the exponential function, e.g. $\exp_2(x) = \exp(\exp(x))$.

The expansion $\mathbf{R}_{\mathbf{alg},\exp}$ is exponentially bounded. This result from [**17**] is obtained by showing that each germ in the differential field $\mathrm{H}(\mathbf{R}_{\mathbf{alg},\exp})$ satisfies an algebraic differential equation, and then appealing to well-known theorems on Hardy fields, see [**58**]. At the end of [**17**] there is an incorrect claim that this route is also available for $\mathbf{R}_{\mathbf{an},\exp}$. Nevertheless, $\mathbf{R}_{\mathbf{an},\exp}$ is exponentially bounded, as shown in [**16**] by means of an explicit embedding of the ordered differential field $\mathrm{H}(\mathbf{R}_{\mathbf{an},\exp})$ into an ordered differential field of generalized formal series over $\mathbf{R}$, namely the field $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$ of so-called *logarithmic-exponential* series. This embedding extends the embedding of $\mathrm{H}(\mathbf{R}_{\mathbf{an}})$ into the field of real Puiseux series in x^{-1} discussed earlier, and can be seen as its analogue for $\mathrm{H}(\mathbf{R}_{\mathbf{an},\exp})$.

Logarithmic-exponential series. This is a large topic related to asymptotic questions, Dulac's problem ([**21**], [**28**]), and o-minimality. It merits a survey by itself. Here we restrict ourselves to pointing out that this embedding $\mathrm{H}(\mathbf{R}_{\mathbf{an},\exp}) \longrightarrow \mathbf{R}((x^{-1}))^{\mathrm{LE}}$ is a natural byproduct of the model-theoretic treatment of $\mathbf{R}_{\mathbf{an},\exp}$ in [**15**], and that one can read off numerous properties of $\mathrm{H}(\mathbf{R}_{\mathbf{an},\exp})$ from this embedding.

For those familiar with model-theoretic lingo we can say more: given any o-minimal $\mathbf{R}_{\mathcal{A}}$ we can equip the ordered field $\mathrm{H}(\mathbf{R}_{\mathcal{A}})$ with natural extra operations so that it becomes an elementary extension of $\mathbf{R}_{\mathcal{A}}$, and equals the definable closure over $\mathbf{R}$ of (the germ of) x. Thus, by general properties of o-minimal structures from [**54**], given any other elementary extension $\mathcal{R}$ of $\mathbf{R}_{\mathcal{A}}$ and an element $y > \mathbf{R}$ from $\mathcal{R}$, we obtain a unique elementary embedding of $\mathrm{H}(\mathbf{R}_{\mathcal{A}})$ into $\mathcal{R}$ that is the identity on $\mathbf{R}$ and sends (the germ of) x to y. We now apply this to $\mathcal{A} = \mathbf{an}, \exp$. The ordered field $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$ is already equipped with a natural exponential operation extending the exponential function on $\mathbf{R}$. The restricted analytic functions also extend naturally to any field of formal series over $\mathbf{R}$ with divisible value group, and in particular to the field of logarithmic-exponential series. The quantifier elimination and complete axiomatization of the elementary theory of $\mathbf{R}_{\mathbf{an},\exp}$ in [**15**] implies that with these extended operations $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$ is an *elementary* extension of $\mathbf{R}_{\mathbf{an},\exp}$. Hence we obtain an elementary embedding $\mathrm{H}(\mathbf{R}_{\mathbf{an},\exp}) \longrightarrow \mathbf{R}((x^{-1}))^{\mathrm{LE}}$ sending the germ of

x to the element x of our formal series field. Exponential boundedness of $\mathbf{R}_{\mathbf{an},\exp}$ now follows immediately from the fact that by the construction of $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$ each logarithmic-exponential series is less than some iterate $\exp_n x$.

Logarithmic-exponential series were originally introduced by Dahn & Göring [**10**] in connection with Tarski's problem on real exponentiation. Another source is Ecalle's profound work [**21**] on Dulac's problem. One version of Ecalle's "trigèbre $\mathbf{R}[[[x]]]$ des transséries" is identical with $\mathbf{R}((x^{-1}))^{\mathrm{LE}}$, and one can show that the image of the embedding above is Ecalle's field of "transséries convergentes".

A question of Hardy. Here we consider an application of the above embedding. Hardy [**25**] views the *logarithmic-exponential* functions as providing a natural scale for asymptotic comparisons at $+\infty$. These functions are the (partially defined) one-variable functions that can be obtained by composition (or substitution) from semialgebraic functions, log and exp, where the semialgebraic functions may, like addition and multiplication, have several arguments. Logarithmic-exponential functions are clearly definable in $\mathbf{R}_{\mathbf{alg},\exp}$, and their germs at $+\infty$ form the smallest real closed Hardy field containing $\mathbf{R}(x)$ and closed under exp and log.

Hardy asked whether the compositional inverse of a logarithmic-exponential function $f(x)$ defined for all large x and increasing to $+\infty$ as $x \to +\infty$, is always asymptotic to some logarithmic-exponential function. He also suggested that the function $(\log x)(\log\log x)$ might be a counterexample.

Shackell [**64**] gave the first counterexample by showing that the compositional inverse of $(\log\log x)(\log\log\log x)$ is not asymptotic to any logarithmic-exponential function. His argument does not apply to $(\log x)(\log\log x)$. This last function is a counterexample as well, by [**16**], where properties of the embedding of $\mathrm{H}(\mathbf{R}_{\mathbf{an},\exp})$ into the field of logarithmic-exponential series are the key.

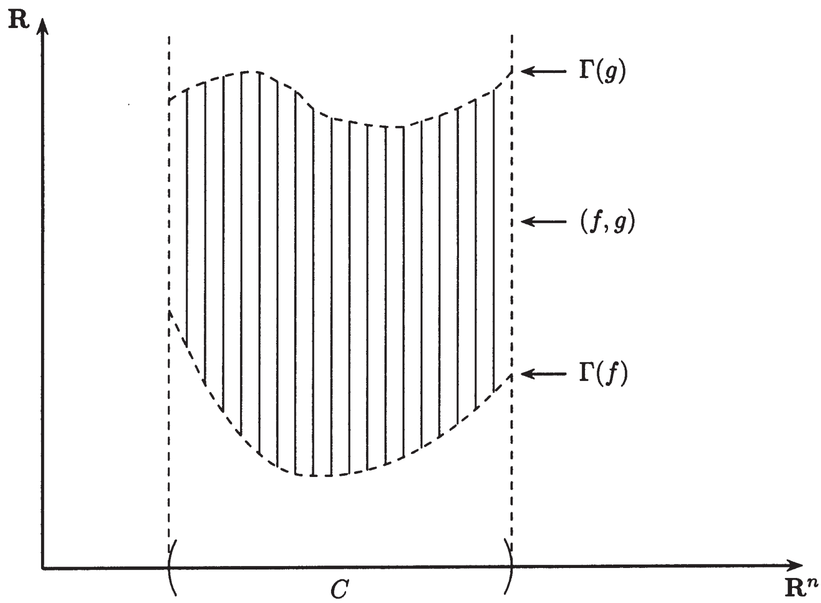
3. Cell Decomposition

In this section we fix again an o-minimal structure $\mathcal{S}$ on the real line. We then prove under a mild extra assumption the basic fact that each definable set in $\mathbf{R}^n$ can be partitioned into finitely many so-called *cells*. It is then easy to assign a dimension to definable sets, and to show this dimension function is well behaved. Next we establish a smooth version of the cell decomposition theorem, and indicate briefly how to improve further to get Whitney stratifications and triangulations.

Cells are nonempty definable sets of an especially simple nature. They are defined inductively as follows:

1. The cells in $\mathbf{R}^1 = \mathbf{R}$ are just the points $\{r\}$ and the intervals (a, b).
2. Let $C \subseteq \mathbf{R}^n$ be a cell. If $f, g : C \to \mathbf{R}$ are definable continous functions such that $f < g$ on C, then
$$(f, g) := \{(x, r) \in C \times \mathbf{R} : f(x) < r < g(x)\}$$
is a cell in $\mathbf{R}^{n+1}$. If $f : C \to \mathbf{R}$ is a definable continuous function, its graph $\Gamma(f) \subseteq C \times \mathbf{R}$, as well as the sets
$$(-\infty, f) := \{(x, r) \in C \times \mathbf{R} : r < f(x)\}$$
and
$$(f, +\infty) := \{(x, r) \in C \times \mathbf{R} : f(x) < r\}$$
are cells in $\mathbf{R}^{n+1}$. Finally, $C \times \mathbf{R} \subseteq \mathbf{R}^{n+1}$ is a cell.
3. By convention we also consider the one-point set $\mathbf{R}^0$ as a cell.

The picture on the next page might be useful to digest this definition. This notion of cell is from [**12**] and was inspired by Łojasiewicz's proof of Tarski's theorem on the real field, see [**41**]. Something like cells appears even in an old paper of Koopman and Brown [**34**].



It is very useful in inductive proofs to view the cells defined in clause 2 as fibered over their projection C. For each cell C in $\mathbf{R}^n$ there is a unique tuple $i = (i(1), \ldots, i(m))$ with $1 \leq i(1) < \cdots < i(m) \leq n$ such that the projection map $(x_1, \ldots, x_n) \mapsto (x_{i(1)}, \ldots, x_{i(m)}) : \mathbf{R}^n \to \mathbf{R}^m$ restricts to a homeomorphism π_C from C onto an open cell in $\mathbf{R}^m$. With this map π_C we reduce problems to the case of open cells. Let C be a cell in $\mathbf{R}^n$. It is easy to show:

1. if C is open, then C is homeomorphic to $\mathbf{R}^n$,
2. if C is not open, then its closure $\mathrm{cl}(C)$ has empty interior,
3. C is locally closed in $\mathbf{R}^n$.

DEFINITION 3.1. A **decomposition** of $\mathbf{R}^m$ is a special kind of partition of $\mathbf{R}^m$ into finitely many cells. We introduce decompositions inductively:

1. A decomposition of $\mathbf{R}^1 = \mathbf{R}$ is a collection

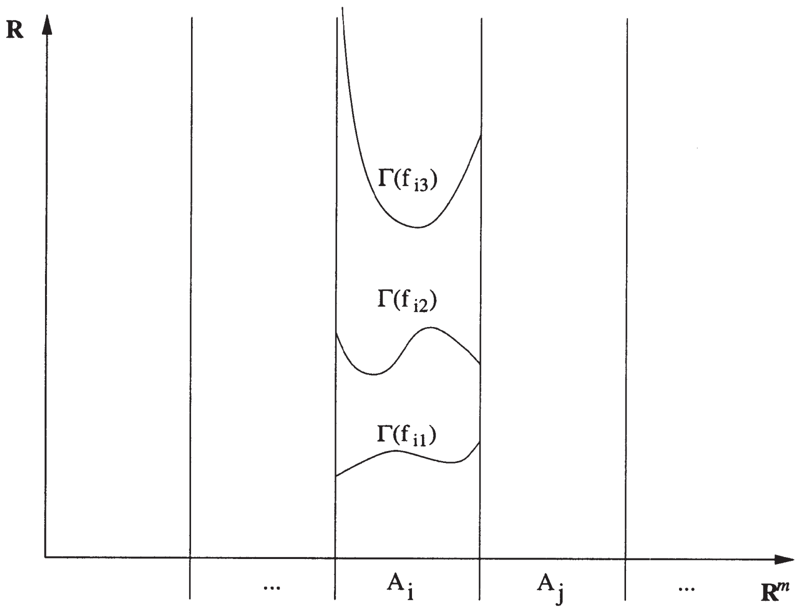
   $$\{(-\infty, a_1), \{a_1\}, (a_1, a_2), \ldots, \{a_k\}, (a_k, +\infty)\},$$

   where $a_1 < \cdots < a_k$ are points in $\mathbf{R}$.
2. Assuming that the class of decompositions of $\mathbf{R}^m$ has been defined, a decomposition of $\mathbf{R}^{m+1}$ is a finite partition $\mathcal{P}$ of $\mathbf{R}^{m+1}$ into cells such that $\pi(\mathcal{P}) := \{\pi(A) : A \in \mathcal{P}\}$ is a decomposition of $\mathbf{R}^m$. (Here and below $\pi : \mathbf{R}^{m+1} \to \mathbf{R}^m$ is the usual projection map.)

Let $\mathcal{D} = \{A_1, \ldots, A_k\}$ be a decomposition of $\mathbf{R}^m$, $A_i \neq A_j$ whenever $i \neq j$. Let $f_{i1} < \cdots < f_{in(i)} : A_i \to \mathbf{R}$ be continuous definable functions, $i = 1, \ldots, k$. Then

$$\mathcal{D}_i := \{(-\infty, f_{i1}), \Gamma(f_{i1}), (f_{i1}, f_{i2}), \ldots, \Gamma(f_{in(i)}), (f_{in(i)}, +\infty)\}$$

is a partition of $A_i \times \mathbf{R}$ into cells, and $\mathcal{D}^* := \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_k$ is a decomposition of $\mathbf{R}^{m+1}$. (See figure.) Each decomposition of $\mathbf{R}^{m+1}$ is obtained in this way from a decomposition $\mathcal{D}$ of $\mathbf{R}^m$. Note that $\mathcal{D} = \{\pi(C) : C \in \mathcal{D}^*\} = \pi(\mathcal{D}^*)$.



We say that a decomposition $\mathcal{D}$ of $\mathbf{R}^m$ *partitions a set* $S \subseteq \mathbf{R}^m$ if S is a union of cells in $\mathcal{D}$.

CELL DECOMPOSITION THEOREM.

(I_m) *Given any definable sets $A_1, \ldots, A_k \subseteq \mathbf{R}^m$, there is a decomposition of $\mathbf{R}^m$ partitioning each A_i.*

(II_m) *Given any definable function $f \colon A \to \mathbf{R}$, $A \subseteq \mathbf{R}^m$, there is a decomposition $\mathcal{D}$ of $\mathbf{R}^m$ such that $\mathcal{D}$ partitions A and each restriction $f|B \colon B \to \mathbf{R}$, with $B \in \mathcal{D}$, is continuous.*

As in [**12**] we prove this theorem here only under the extra assumption that $\mathcal{S}$ has the following *uniform finiteness property*: *Whenever* $S \subseteq \mathbf{R}^{m+1}$ *is definable and* $S_x \subseteq \mathbf{R}$ *is finite for each* $x \in \mathbf{R}^m$, *then there is* $N \in \mathbf{N}$ *such that*

$$|S_x| \leq N \quad \textit{for all } x \in \mathbf{R}^m.$$

Making this extra assumption is justifiable on pragmatic grounds: the proof that a particular kind of structure is o-minimal usually yields also the above uniform finiteness property. In fact, this uniform finiteness property is always satisfied, and is obtained along the way in the proof of the cell decomposition theorem for arbitrary o-minimal structures (not just those on the real line) by Knight, Pillay and Steinhorn [**54**], [**33**], see also [**14**].

Proof (Cell Decomposition Theorem). By induction on m. Note that (I_0), (II_0) are trivial and (I_1) is clear by the definition of o-minimality. Now let $m \geq 1$, and assume $(\mathrm{I}_1), \ldots, (\mathrm{I}_m), (\mathrm{II}_1), \ldots, (\mathrm{II}_{m-1})$; we shall derive (II_m) and then (I_{m+1}).

Let $f\colon A \to \mathbf{R}$ be definable. By (I_m), it suffices to show that then A is a union of definable sets $A_1, \ldots, A_k$ such that each $f|A_i$ is continuous. So by (I_m) we may as well assume that A is already a cell. We distinguish two cases:

1. *A is not an open cell.* Then consider the commuting diagram

$$\begin{tikzcd} A \arrow[d, "p"', "\cong"] \arrow[dr, "f"] & \\ p(A) \arrow[r, "f \circ p^{-1}"'] & \mathbf{R} \end{tikzcd}$$

   where $p := \pi_A$. Since $p(A)$ is a cell in $\mathbf{R}^n$ for some $n < m$, the inductive assumption gives us a partition of $p(A)$ into definable sets

$$p(A_1), \ldots, p(A_k), \qquad (A_1, \ldots, A_k \subseteq A)$$

   such that each $(f \circ p^{-1})|p(A_i)$ is continuous. Then each $f|A_i$ is continuous.

2. *A is an open cell.* Put

$$A' := \{x \in A : f \text{ is continuous at } x\}.$$

Note that A' is definable. We claim:

$$(*) \quad A' \text{ is dense in } A.$$

If we accept this claim, then we can use (I_m) to get a partition of A into cells $B_1, \dots, B_k$ such that $A \setminus A'$ and A' are unions of B_i's. If B_i is an open cell, then $A' \cap B_i \neq \varnothing$ by $(*)$, so $B_i \subseteq A'$, and thus $f|B_i$ is continuous. If B_i is not open, apply the first case to $f|B_i$.

So it remains to prove $(*)$. With "box" we mean a cartesian product of intervals. Let $B = (a_1, b_1) \times \cdots \times (a_m, b_m)$ be any box in A. It is enough to show that f is continuous at some point of B.

If f takes only finitely many values on some box $B' \subseteq B$, then by (I_m), B' is a union of cells $B_1, \dots, B_k$, on each of which f is constant. So at least one B_i has to be open; then f is continuous at each point of B_i.

From now on we assume f takes infinitely many values on each box $B' \subseteq B$. We now construct inductively a decreasing sequence $\{I_n\}_{n=1}^{\infty}$ of intervals and a sequence $\{B_n\}_{n=1}^{\infty}$ of bounded boxes in B such that for all $n \geq 1$:

$$\mathrm{length}(I_n) < \frac{1}{n}, \quad f(B_n) \subseteq I_n, \quad \text{and} \quad \mathrm{cl}(B_{n+1}) \subseteq B_n.$$

Then f will be continuous at each point of $\bigcap_{n=1}^{\infty} B_n$; by compactness of $\mathrm{cl}(B_n) \subseteq \mathbf{R}^m$, this intersection is not empty.

To find I_1, we write $f(B) = \left(\bigcup_{p=1}^{\infty} J_p\right) \cup F$, with each J_p an interval of length < 1 and F finite. Then

$$B = \bigcup_{p=1}^{\infty} (f^{-1}(J_p) \cap B) \cup \bigcup_{r \in F} (f^{-1}(r) \cap B).$$

By writing each of the sets $f^{-1}(J_p) \cap B$ and $f^{-1}(r) \cap B$ as a finite union of cells, we get B as a countable union of cells. By the Baire category theorem one of those cells has to be open, and such an open cell cannot be contained in $f^{-1}(r) \cap B$ for any $r \in F$.

Thus we obtain an interval I_1 of length < 1 and a bounded box $B_1 \subseteq B$ such that $f(B_1) \subseteq I_1$. Once we have constructed I_n and B_n with $\mathrm{length}(I_n) < \frac{1}{n}$, we obtain in the same way an interval $I_{n+1} \subseteq f(B_n)$ and a box $B_{n+1} \subseteq B$ such that $\mathrm{length}(I_{n+1}) < \frac{1}{n+1}$, $\mathrm{cl}(B_{n+1}) \subseteq B_n$, and $f(B_{n+1}) \subseteq I_{n+1}$.

This finishes the proof of $(*)$, and hence of (II_m). Now for the proof of (I_{m+1}). We first make two useful observations.

1. Given any two decompositions $\mathcal{D}_1$, $\mathcal{D}_2$ of $\mathbf{R}^{m+1}$, there is a decomposition $\mathcal{D}^*$ of $\mathbf{R}^{m+1}$ refining both, i.e. $\mathcal{D}^*$ partitions the cells in $\mathcal{D}_1 \cup \mathcal{D}_2$.

To see this, we reduce first (using (I_m)) to the case that there is a decomposition $\mathcal{D}$ of $\mathbf{R}^m$ such that

$$\mathcal{D} = \pi(\mathcal{D}_1) = \pi(\mathcal{D}_2).$$

Given $D \in \mathcal{D}$ let $f_1 < \cdots < f_p : D \to \mathbf{R}$ be definable and continuous such that the cells of $\mathcal{D}_1$ that project onto D under π are

$$(-\infty, f_1), \Gamma(f_1), (f_1, f_2), \ldots, \Gamma(f_p), (f_p, +\infty),$$

and let $g_1 < \cdots < g_q : D \to \mathbf{R}$ be definable and continuous such that the cells of $\mathcal{D}_2$ that project onto D under π are

$$(-\infty, g_1), \Gamma(g_1), (g_1, g_2), \ldots, \Gamma(g_q), (g_q, +\infty).$$

Now clearly this cell D can be partitioned further into definable sets D_λ ($\lambda \in \Lambda_D$, Λ_D a finite index set) such that for all $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, q\}$ either $f_i(x) < g_j(x)$ for all $x \in D_\lambda$, or $f_i(x) = g_j(x)$ for all $x \in D_\lambda$, or $f_i(x) > g_j(x)$ for all $x \in D_\lambda$. Now refine $\mathcal{D}$ to a decomposition $\mathcal{D}'$ of $\mathbf{R}^m$ that partitions for each $D \in \mathcal{D}$ all D_λ, $\lambda \in \Lambda_D$. Then it is clear that $\mathcal{D}_1$ and $\mathcal{D}_2$ can be refined to a common decomposition $\mathcal{D}^*$ of $\mathbf{R}^{m+1}$ such that $\pi_m(\mathcal{D}^*) = \mathcal{D}'$, which proves Observation 1.

Given a definable set $S \subseteq \mathbf{R}$ we define its *type* $\tau(S)$ as follows: Let $r_1 < \cdots < r_k$ be the points of $\mathrm{bd}(S)$; put $r_0 := -\infty$, $r_{k+1} := +\infty$; then $\tau(S) = (\tau_1, \ldots, \tau_{2k+1}) \in \{-1, 1\}^{2k+1}$ where

$$\tau_{2i+1} := \begin{cases} +1, & \text{if } (r_i, r_{i+1}) \subseteq S \\ -1, & \text{if } (r_i, r_{i+1}) \subseteq \mathbf{R} \setminus S \end{cases}$$

and

$$\tau_{2i} := \begin{cases} +1, & \text{if } r_i \in S \\ -1, & \text{if } r_i \in \mathbf{R} \setminus S. \end{cases}$$

Thus $\tau(\varnothing) = (-1)$, $\tau\left((1;2] \cup \{3\}\right) = (-1, -1, +1, +1, -1, +1, -1)$. With this notation, it follows from the uniform finiteness property:

2. Let $A \subseteq \mathbf{R}^{m+1}$ be definable. Then

$$\{\tau(A_x) : x \in \mathbf{R}^m\}$$

is finite, and for each $\tau = (\tau_1, \ldots, \tau_{2k+1}) \in \{-1, 1\}^{2k+1}$, $k \in \mathbf{N}$, the set

$$\{x \in \mathbf{R}^m : \tau(A_x) = \tau\}$$

is definable.

We can now finish the proof of (I_{m+1}). Using Observation 1, it suffices to consider a single definable set $A \subseteq \mathbf{R}^{m+1}$ and show there is a decomposition of $\mathbf{R}^{m+1}$ partitioning A. For this, we first use Observation 2 and (I_m) to find a decomposition $\mathcal{D}$ of $\mathbf{R}^m$ such that for each $D \in \mathcal{D}$ there is a tuple $\tau \in \{-1, 1\}^{2k+1}$ (k depending on $\mathcal{D}$) such that $\tau(A_x) = \tau$ for all $x \in D$. Let us fix our attention on one $D \in \mathcal{D}$ with τ as above. Then clearly there are definable functions $f_1 < \cdots < f_k \colon D \to \mathbf{R}$ such that for each $i = 0, \ldots, k$ ($f_0 := -\infty$, $f_{k+1} := +\infty$) either $(f_i, f_{i+1}) \subseteq A$ or $(f_i, f_{i+1}) \cap A = \varnothing$, and for each $i = 1, \ldots, k$, either $\Gamma(f_i) \subseteq A$ or $\Gamma(f_i) \cap A = \varnothing$. Now use (II_m) to partition D into finitely many cells such that the restriction of each f_i to each of those cells is continuous. Now apply (I_m) again. □

COROLLARY 3.2. *If $A \subseteq \mathbf{R}^n$ is definable, then A has only finitely many connected components (each definable and path connected).* □

Finiteness results like this often remain true with uniform finite bounds when there is a definable dependence on parameters:

COROLLARY 3.3. *If $S \subseteq \mathbf{R}^{m+n}$ is definable, then there is a natural number M such that for each point $a \in \mathbf{R}^m$ the fiber S_a has at most M connected components.*

For the proof, take a partition of S into M cells, and note that for a cell C in $\mathbf{R}^{m+n}$ and $a \in \mathbf{R}^m$ the fiber C_a is either empty or a cell in $\mathbf{R}^n$.

Dimension. For a cell C whose homeomorphic image under π_C is an *open* cell in $\mathbf{R}^d$, we put $\dim(C) = d$. We assign a dimension to *any* nonempty definable set A by taking a finite partition of A into cells, and letting $\dim(A)$ be the maximum of the dimensions of the cells in this partition (this maximum is independent of the partition). We also put $\dim(\varnothing) := -\infty$. This naive definition of dimension agrees with most other notions of dimension that make sense, but it is better behaved as the following facts indicate. Given definable sets $A \subseteq \mathbf{R}^m$, $B \subseteq \mathbf{R}^n$ and a definable (not necessarily continuous) map $f : A \to B$ we have:

1. $\dim(A) \geq \dim(f(A))$, with equality if f is injective,
2. for each natural number e the set

$$B(e) := \{b \in B : \dim f^{-1}(b) = e\}$$

is definable, and $\dim(f^{-1}(B(e)) = \dim(B(e)) + e$.

3. $\dim(\mathrm{cl}(A) \setminus A) < \dim(A)$ for *nonempty* A.

The last property is useful in inductive arguments and constructions. These facts on dimension are obtained using cell decomposition by simple direct arguments, with no need to appeal to deeper topological results like those of Brouwer.

For the rest of the section we assume that $\mathcal{S}$ contains addition and multiplication. Let $p \in \{1, 2, 3, \dots\} \cup \{\infty, \omega\}$. Then a *$C^p$-cell* is by definition just a cell that is also a C^p-submanifold of its ambient cartesian space. ("Submanifold" always means "embedded submanifold" in these notes.)

SMOOTH CELL DECOMPOSITION THEOREM. *Let $p \in \mathbf{N}$, $p > 0$. Then*

(I_m) *Given any definable sets $A_1, \dots, A_k \subseteq \mathbf{R}^m$, there is a decomposition of $\mathbf{R}^m$ into C^p-cells partitioning each A_i.*

(II_m) *Given any definable function $f\colon A \to \mathbf{R}$, $A \subseteq \mathbf{R}^m$, there is a decomposition $\mathcal{D}$ of $\mathbf{R}^m$ into C^p-cells such that $\mathcal{D}$ partitions A and each restriction $f|B\colon B \to \mathbf{R}$, with $B \in \mathcal{D}$ and $B \subseteq A$, is of class C^p.*

PROOF. We just do the case $p = 1$. (The general case then follows by induction on p.) If f and A are as in (II_m) and $x \in \mathrm{int}(A)$, we define

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right) \in \mathbf{R}^n,$$

provided these partials exist; if some $(\partial f/\partial x_i)(x)$ does not exist, then $\nabla f(x)$ is not defined. Take

$$A' := \{x \in A : x \in \mathrm{int}(A) \text{ and } \nabla f(x) \text{ is defined}\}.$$

Along with (I_m) and (II_m), we will prove

(III_m) $A \setminus A'$ has empty interior.

(Analogous to claim ($*$) in the proof of ordinary Cell Decomposition.)

We proceed by induction on m: (I_1) is trivial, (III_1) and (II_1) are easy consequences of the Smooth Monotonicity Theorem. Now we shall assume that (I_d), (II_d) and (III_d) hold for all $d \leq m$, and now derive successively (I_{m+1}), (III_{m+1}) and (II_{m+1}).

(I_{m+1}) Let $A_1, \ldots, A_k \subseteq \mathbf{R}^{m+1}$ be definable. We want to find a decomposition of $\mathbf{R}^{m+1}$ into C^1-cells partitioning each A_i. By ordinary cell decomposition, there is a decomposition $\mathcal{D}$ of $\mathbf{R}^{m+1}$ partitioning each A_i. Then $\pi(\mathcal{D}) = \{\pi(D) : D \in \mathcal{D}\}$ is a decomposition of $\mathbf{R}^m$. Let $\pi(\mathcal{D}) = \{C_1, \ldots, C_n\}$. For $i = 1, \ldots, n$, let the cells of $\mathcal{D}$ that project onto C_i be

$$(-\infty, f_{i1}), \Gamma(f_{i1}), (f_{i1}, f_{i2}), \ldots, \Gamma(f_{is}), (f_{is}, +\infty),$$

where $f_{ij}\colon C_i \to \mathbf{R}$ ($j = 1, \ldots, s = s(i)$) is definable and continuous, $f_{i1} < \cdots < f_{is}$. By (I_m) and (II_m) we may assume, after suitably refining $\pi(\mathcal{D})$, and $\mathcal{D}$ accordingly, that all C_i are C^1-cells and all f_{ij} are C^1. Then $\mathcal{D}$ is a decomposition as required in (I_{m+1}).

(III_{m+1}) Let $f\colon A \to \mathbf{R}$ be definable, $A \subseteq \mathbf{R}^{m+1}$, and define A' as above. It suffices to show that, given a box $U \times (a,b) \subseteq A$, where U is a box in $\mathbf{R}^m$, we have $\big(U \times (a,b)\big) \cap A' \neq \varnothing$. By Smooth Monotonicity, there is, for each $u \in U$, an interval $\big(\alpha(u), \beta(u)\big) \subseteq (a,b)$ such that $\big(\partial f/\partial x_{m+1}\big)(u,t)$ exists for all $t \in \big(\alpha(u), \beta(u)\big)$. By Definable Choice, we may assume that $\alpha, \beta\colon U \to \mathbf{R}$ are definable. Using Cell Decomposition we can shrink U such that α, β are continuous on U. Shrinking U further and changing a and b we may assume that $\partial f/\partial x_{m+1}$ is defined on $U \times (a,b)$.

Take any $t \in (a,b)$. By applying (III_m) to the function

$$x \mapsto f(x,t) : U \to \mathbf{R},$$

we see that there must exist $x_0 \in U$ such that $\big(\partial f/\partial x_i\big)(x_0,t)$ exist for $i = 1, \ldots, m$. Also $\big(\partial f/\partial x_{m+1}\big)(x_0,t)$ exists, since $(x_0,t) \in U \times (a,b)$. Hence $(x_0,t) \in A'$.

(II_{m+1}) Let $f\colon A \to \mathbf{R}$ be definable, $A \subseteq \mathbf{R}^{m+1}$. Put

$$A' := \big\{x \in \operatorname{int}(A) : \nabla f(x) \text{ exists}\big\}.$$

Take a decomposition $\mathcal{D}$ of $\mathbf{R}^{m+1}$ partitioning A and A' such that ∇f is continuous on each *open* cell of $\mathcal{D}$ contained in A'. (Such a $\mathcal{D}$ exists, by ordinary cell decomposition.) Similar to an argument in the proof of ordinary Cell Decomposition, and using the inductive assumption, we obtain, for each non-open cell $C \in \mathcal{D}$ with $C \subseteq A$, a finite partition $\mathcal{P}_C$ of C into definable sets such that $f|D$ is C^1 for each $D \in \mathcal{P}_C$.

By (I_{m+1}), there is a decomposition $\mathcal{D}'$ of $\mathbf{R}^{m+1}$ that partitions each cell in $\mathcal{D}$ and each set in $\mathcal{P}_C$, for all $C \in \mathcal{D}$, $C \subseteq A$, and consists entirely of C^1-cells. It is easy to check that if $B \in \mathcal{D}'$, $B \subseteq A$, then $f|B\colon B \to \mathbf{R}$ is C^1. This finishes the inductive step, and hence the proof of the theorem. □

REMARK. For all known o-minimal structures on the real field the Smooth Cell Decomposition Theorem holds even with $p = \omega$.

Stratification. In a cell decomposition the cells C need not fit nicely together at their frontiers $\partial C := \mathrm{cl}(C) \setminus C$. Since $\dim(\partial A) < \dim(A)$ for nonempty definable A, we can use standard inductive arguments to improve matters:

Given definable sets $A_1, \ldots, A_k$ in $\mathbf{R}^m$ and an integer $p > 0$ there exists a finite Whitney stratification of $\mathbf{R}^m$ that partitions each A_i, and all of whose strata are C^p-cells.

Definable maps can be similarly stratified. All this extends easily to the various categories of definable manifolds, and to the analytic-geometric category associated to $\mathcal{S}$ if **an** $\subseteq \mathcal{S}$. Also, if the Smooth Cell Decomposition Theorem holds for $\mathcal{S}$ with $p = \infty$, respectively, $p = \omega$, then the above goes through with $p = \infty$, respectively $p = \omega$. See [**18**] for precise statements, definitions, proofs and references.

Triangulation. Another way of improving on cell decomposition is by triangulation. The following is the triangulation theorem from [**14**].

THEOREM 3.4. *For any definable sets $A_1, \ldots, A_k \subseteq A \subseteq \mathbf{R}^m$ there is a definable homeomorphism h from A onto a finite union of open simplices of a finite simplicial complex K in $\mathbf{R}^m$ such that each $h(A_i)$ is a finite union of open simplices of K.*

Here is a brief sketch of the proof, which proceeds by induction on m as in the semialgebraic case [**5**]. Let $m > 0$ and assume triangulation holds for lower values of m. We may assume A is compact. Let B be the union of the boundaries of A and of the A_i's, so B is definable with $\dim(B) < m$. In the semialgebraic case this means that B is contained in the zero set of a nonconstant polynomial, so that by Noether normalization we can assume that each "vertical" line $p + \mathbf{R}.e_m$ (where $e_m = (0, \ldots, 0, 1)$ is the vertical basis vector of $\mathbf{R}^m$) intersects B in a uniformly bounded finite number of points. Next we triangulate the projection of A in $\mathbf{R}^{m-1}$ compatibly with the projections of the A_i's, and show that this triangulation can be "lifted" to a triangulation as desired. In the general case we proceed roughly in the same way, using as a substitute for Noether normalization the following "good directions" lemma [**14**].

LEMMA 3.5. *Let $B \subseteq \mathbf{R}^m$ be definable, $m > 0$, with* $\dim(B) < m$. *Then there is a unit vector $e \in \mathbf{R}^m$ such that all lines $p + \mathbf{R}.e$ intersect B in only finitely many points.*

4. Further Properties of Definable Sets

In this section we discuss the following items:

- Euler characteristic.
- Generic triviality of definable continuous maps.
- Shiota's o-minimal "Hauptvermutung".
- Results on (Hausdorff) limits of definable families of sets.
- The Vapnik-Chervonenkis property of definable families of sets.

This is just a selection among possible topics. Throughout this section we fix an o-minimal structure $\mathcal{S}$ on the *real field*, and "definable" is with respect to $\mathcal{S}$. (Most results discussed actually go through for o-minimal structures on the *real line.*)

Euler characteristic. (See [**14**], Chapter 4, and also [**59**] for the semilinear case.) The "naive" Euler characteristic $E(A)$ of a definable set $A \subseteq \mathbf{R}^m$ is given by $E(A) := \sum_{i \in I}(-1)^{\dim(C_i)}$, where A is the disjoint union of cells C_i, with i ranging over the finite index set I. With some effort one shows this is independent of the choice of C_i's, and that $E(A) = E(B)$ whenever there is a definable (not necessarily continuous) bijection from A onto $B \subseteq \mathbf{R}^n$. This naive Euler characteristic provides a finitely additive integer valued measure on the boolean algebra of definable subsets of any given definable set, and is also multiplicative under cartesian products. Using the triangulation theorem one can show that two definable sets A and B are definably equivalent (that is, there is a definable bijection from A onto B) if and only if $\dim(A) = \dim(B)$ and $E(A) = E(B)$.

Generic Triviality. Let $A \subseteq \mathbf{R}^m$ and $B \subseteq \mathbf{R}^n$ and $f : A \to B$ be definable, with f continuous. The map f gives rise to a family $\{f^{-1}(b)\}_{b \in B}$ of definable subspaces of A, and it is natural to ask how the topology of $f^{-1}(b)$ varies with b. On this level of generality the following theorem is an answer.

THEOREM 4.1. *We can partition B into finitely many definable subsets $B_1, \ldots, B_k$ such that each restriction $f_i := f|f^{-1}(B_i) : f^{-1}(B_i) \to B_i$ is $\mathcal{S}$-trivial, that is, given any $b_i \in B_i$ there is a definable homeomorphism*

$$h_i : f^{-1}(B_i) \cong B_i \times f^{-1}(b_i)$$

such that $f_i = \pi_i \circ h_i$ where $\pi_i : B_i \times f^{-1}(b_i) \to B_i$ is the projection map.

Hence all fibers over B_i are (definably) homeomorphic, and thus there are only finitely many topological types among the fibers $f^{-1}(b)$ as b ranges over B. In the semialgebraic case the theorem is due to Hardt [**26**]. That case implies for example the well-known result that for each pair $(n, d) \in \mathbf{N}^2$ there are only finitely many topological types among the zero-sets in $\mathbf{R}^n$ of real polynomials in n variables of total degree at most d.

See [**14**], Chapter 9 for arbitrary $\mathcal{S}$, and refinements, and also for details concerning the applications to be discussed now.

Applying the generic triviality theorem to $\mathcal{S} = \{\text{subexponential sets}\}$ we obtain a finiteness result for polynomials similar to the above, with "degree" replaced by "number of monomials" :

COROLLARY 4.2. *For each pair $(n, k) \in \mathbf{N}^2$ there are only finitely many topological types among the zero-sets in $\mathbf{R}^n$ of real polynomials in n variables having at most k monomials.*

This result is in the spirit of earlier theorems by Khovanskii [**30**], [**31**], [**32**], bounding the Betti numbers of such sets in terms of n and k. Indeed, the idea is to use Khovanskii's change of variables $x_i = e^{y_i}$, $i = 1, \ldots, n$ to construct finitely many continuous subexponential maps such that each zero-set as above occurs as a fiber of one of those maps, and then to apply the generic triviality theorem to $\mathbf{R}_{\mathbf{alg},\exp}$. A minor problem that this change of variables only works in the region where all $x_i > 0$. In fact, slightly different changes of variables are used.

An elaboration along these lines proves the following finiteness result conjectured by Benedetti and Risler [**3**]:

COROLLARY 4.3. *Given any $n, p, k \in \mathbf{N}$ there are only finitely many different embedded homeomorphism types of semialgebraic subsets of $\mathbf{R}^n$ defined by at most p polynomial equations and (strict) inequalities where each polynomial has additive complexity at most k.*

Here the *embedded homeomorphism type* of a set $S \subseteq \mathbf{R}^n$ refers to the homeomorphism type of the pair $(\mathbf{R}^n, S)$. (The generic triviality theorem extends to *pairs* of definable sets.) The *additive complexity* of a rational function $f(X_1, \ldots, X_n) \in \mathbf{R}(X_1, \ldots, X_n)$ is said to be $\leq k$ if f can be obtained from the variables $X_1, \ldots, X_n$ and real constants using at most k additions and an unlimited number of multiplications and divisions. (The additive complexity of a polynomial is more natural than its number of monomials, since the latter can jump drastically under a simple change of variables like substituting $X + 1$ for a variable X.)

One can also generalize by considering the topological types of *maps* rather than sets, where maps f and g are equivalent if there is a homeomorphism between their domains and a homeomorphism between their codomains transforming f to g. Coste [**9**] has positive results for $\mathbf{R}$-valued maps in the o-minimal context similar to the above results for sets. (These results do not seem to extend to $\mathbf{R}^2$-valued maps, but see [**66**].)

Shiota's work. This is closely connected to the topics above. Shiota [**66**] obtains some basic results of differential topology in an o-minimal setting, for example the transversality theorem and the first and second isotopy lemmas of Thom. The difficulty here is that the objects to be constructed from the definable data must remain definable. Thus the method of integrating vector fields cannot be used, and new proofs have to be found, some of which are long and complicated. Shiota's book also develops some o-minimal singularity theory where various kinds of definable sets and maps and their germs are classified under suitable equivalence relations. A very remarkable theorem ([**66**], Chapter III) is the following.

THEOREM 4.4. *Any two definably homeomorphic compact semilinear sets are semilinearly homeomorphic.*

We remark that the compact semilinear sets in $\mathbf{R}^n$ are exactly the polyhedra $|K|$ spanned by finite simplicial complexes K in $\mathbf{R}^n$, and that a semilinear homeomorphism between such spaces is the same as a PL-homeomorphism. In other words, the theorem says that the "Hauptvermutung" holds in the o-minimal setting. It is not clear (to me) whether Shiota's proof provides a "definable" way of passing from a given definable homeomorphism to a semilinear one.

Limit Sets. Let $A \subseteq \mathbf{R}^{m+n}$ be definable. Then A gives rise to a collection $\{A_x : x \in \mathbf{R}^m\}$ of definable subsets of $\mathbf{R}^n$. Identifying each subset of $\mathbf{R}^n$ with its characteristic function $\mathbf{R}^n \to \{0,1\}$ produces an inclusion

$$\{A_x : x \in \mathbf{R}^m\} \subseteq \{0,1\}^{\mathbf{R}^n}.$$

The right hand product has the product topology with the discrete topology on each factor $\{0,1\}$, making the product space a totally disconnected compact Hausdorff space. Thus a set $Y \subseteq \mathbf{R}^n$ is in the closure of $\{A_x : x \in \mathbf{R}^m\}$ in this product space if and only if Y agrees on each finite set $F \subseteq \mathbf{R}^n$ with some A_x, that is, $Y \cap F = A_x \cap F$ for some $x \in \mathbf{R}^m$ depending on F. Let us call Y in that case a **limit set** of $\{A_x : x \in \mathbf{R}^m\}$. (For example, the union of an increasing sequence of A_x's is such a limit set, as well as the intersection of a decreasing sequence of A_x's.)

THEOREM 4.5. *Each limit set of $\{A_x : x \in \mathbf{R}^m\}$ is definable. In fact, there is a definable set $B \subseteq \mathbf{R}^{m'+n}$* (for some m') *such that*

$$\{B_{x'} : x' \in \mathbf{R}^{m'}\} = \textit{ the collection of limit sets of } \{A_x : x \in \mathbf{R}^m\}.$$

Hausdorff Limits. With definable $A \subseteq \mathbf{R}^{m+n}$ as before, we now focus on

$$\mathcal{K}_A := \{A_x : x \in \mathbf{R}^m, A_x \text{ is compact}\},$$

the collection of compact fibers, which is a subset of the space $\mathcal{K}(\mathbf{R}^n)$ of compact subsets of $\mathbf{R}^n$. We equip $\mathcal{K}(\mathbf{R}^n)$ with the Hausdorff metric induced by the usual euclidean metric on $\mathbf{R}^n$. It is well-known that then $\mathcal{K}(\mathbf{R}^n)$ is locally compact. A **Hausdorff limit** of $\mathcal{K}_A$ is by definition a compact set $Y \subseteq \mathbf{R}^n$ (i.e. a point of $\mathcal{K}(\mathbf{R}^n)$) that is in the closure of $\mathcal{K}_A$ in $\mathcal{K}(\mathbf{R}^n)$. We have the following analogue of the theorem on limit sets:

THEOREM 4.6. *Each Hausdorff limit of $\mathcal{K}_A$ is definable. More precisely, if $\mathcal{K}_A \neq \varnothing$, then there is a definable set $B \subseteq \mathbf{R}^{m'+n}$ such that $\{B_{x'} : x' \in \mathbf{R}^{m'}\}$ is exactly the collection of Hausdorff limits of $\mathcal{K}_A$.*

The only known proofs of these theorems on limit sets use model theory, see Marker and Steinhorn [**43**], Pillay [**52**], and also Bröcker [**7**] for the semialgebraic case. The connection with model theory arises in an entirely natural way as follows. The definable set $A \subseteq \mathbf{R}^{m+n}$ extends naturally to a definable set $A(R) \subseteq R^{m+n}$, where R is an elementary extension of the expansion $\mathbf{R}_{\mathcal{S}}$. For sufficiently saturated R the limit sets of the collection $\{A_x : x \in \mathbf{R}^m\}$ are exactly the sets $A(R)_x \cap \mathbf{R}^n$ with $x \in R^m$. Hausdorff limits can be characterized in a similar way.

The Vapnik-Chervonenkis Property. Let $A \subseteq \mathbf{R}^{m+n}$ be definable. The collection of sets $\{A_x : x \in \mathbf{R}^m\}$ also has a remarkable combinatorial property:

THEOREM 4.7. *There are numbers $C = C(A) > 0$ and $d = d(A) \in \mathbf{N}$ such that each finite set $F \subseteq \mathbf{R}^n$ has at most $C \cdot |F|^d$ subsets of the form $F \cap A_x$ with $x \in \mathbf{R}^m$, where $|F|$ is the cardinality of F.*

This is a purely combinatorial manifestation of the fact that the sets A_x vary in a highly restricted way as x ranges over $\mathbf{R}^m$: the total number of subsets of a finite set F is $2^{|F|}$, which grows much faster than $C \cdot |F|^d$ as $|F| \to \infty$.

In probabilistic terms the theorem says that $\{A_x : x \in \mathbf{R}^m\}$ is a Vapnik-Chervonenkis class [**70**]. This means in particular that if the sets A_x are events in a probability space, then the convergence of their relative frequencies to their probabilities is *uniform* in x in certain ways. This plays a role in the mathematics of pattern recognition, neural networks, and learning theory, see Vapnik [**71**].

In model-theoretic terms the theorem is equivalent to saying that $\mathbf{R}_{\mathcal{S}}$ does not have Shelah's independence property, and in this form it already appears in [**54**]. Laskowski [**35**] observed this equivalence and gives a more combinatorial proof of the theorem. (With some preliminaries added this last proof is essentially reproduced in [**14**], Chapter 5.)

There is also more recent work by Wilkie, Macintyre and Sontag on these matters.

5. Building o-minimal Structures on the Real Field

In this section we discuss how one actually goes about proving that some given expansion $\mathbf{R}_{\mathcal{A}}$ of the real field is o-minimal. We also give a brief account of recent work in constructing such expansions, and finish with a diagram displaying the main known o-minimal expansions of the real field.

Needless to say, a "natural" class of real functions—on some interval, say—is only a candidate for generating an o-minimal structure, if these functions behave "tamely"; in particular, they should not oscillate infinitely often, or have infinitely many isolated zeros.

Elimination theory. This is a general idea that makes sense whenever a set is structured by endowing it with functions and relations. We give a precise definition in the relevant case. Let the collection $\mathcal{A}$ consist of the ordering on the real line, and of real valued functions on $\mathbf{R}^n$ for various n (among which may be addition and multiplication). We then define the class of $\mathcal{A}$-functions to be the smallest collection of real valued functions on the spaces $\mathbf{R}^n$ (for all n) such that the coordinate functions $(x_1, \dots, x_n) \mapsto x_i : \mathbf{R}^n \to \mathbf{R}$ $(1 \leq i \leq n)$ are $\mathcal{A}$-functions and such that we can substitute in functions of $\mathcal{A}$, that is, if $f : \mathbf{R}^n \to \mathbf{R}$ belongs to $\mathcal{A}$, and $g_1, \dots, g_n : \mathbf{R}^m \to \mathbf{R}$ are $\mathcal{A}$-functions, then $f(g_1, \dots, g_n) : \mathbf{R}^m \to \mathbf{R}$ is an $\mathcal{A}$-function. (Hence the $\mathcal{A}$-functions are definable in $\mathbf{R}_{\mathcal{A}}$.)

Next, an $\mathcal{A}$-set in $\mathbf{R}^n$ is by definition a finite union of sets of the form

$$\{x \in \mathbf{R}^n : f_1(x) = \cdots = f_k(x) = 0, g_1(x) > 0, \dots, g_l(x) > 0\}$$

where the f's and g's are $\mathcal{A}$-functions on $\mathbf{R}^n$. The $\mathcal{A}$-sets are clearly definable in $\mathbf{R}_{\mathcal{A}}$. (In logical terms they are exactly the "quantifier-free definable" sets, the sets that can be defined in $\mathbf{R}_{\mathcal{A}}$ by formulas without quantifiers.)

DEFINITION 5.1. We say that $\mathbf{R}_{\mathcal{A}}$ **eliminates quantifiers** if for each n the image of an $\mathcal{A}$-set in $\mathbf{R}^{n+1}$ under the projection map $\mathbf{R}^{n+1} \to \mathbf{R}^n$ is an $\mathcal{A}$-set in $\mathbf{R}^n$.

Thus the definable sets are exactly the $\mathcal{A}$-sets in the (a priori unlikely) case that $\mathbf{R}_{\mathcal{A}}$ eliminates quantifiers. To see why this resembles the old idea of "elimination", note that given a system of $\mathcal{A}$-equations and $\mathcal{A}$-inequalities with n *parameters* and one *unknown*, the definition above expresses that solvability of the system in $\mathbf{R}$

is equivalent to the parameters satisfying certain $\mathcal{A}$-equations and $\mathcal{A}$-inequalities. In logical terms it implies that any formula with quantifiers (in a certain formal language associated to $\mathbf{R}_{\mathcal{A}}$) is equivalent in $\mathbf{R}_{\mathcal{A}}$ to a quantifier-free formula.

Suppose $\mathcal{A}$ contains all $\{r\}$, $r \in \mathbf{R}$, and we manage to show that the $\mathcal{A}$-subsets of $\mathbf{R}$ are finite unions of intervals and points, and that $\mathbf{R}_{\mathcal{A}}$ eliminates quantifiers. Then $\mathbf{R}_{\mathcal{A}}$ is obviously o-minimal.

For the field of real numbers $\mathbf{R}_{\mathbf{alg}}$ the classical Sturm theorem on the number of real zeros of a real polynomial is the key fact underlying such an elimination theory, as Tarski [**68**] realized. There is a variety of other proofs ([**63**], [**41**]) that $\mathbf{R}_{\mathbf{alg}}$ eliminates quantifiers. Among these is a beautiful model-theoretic argument by A. Robinson [**56**], which makes it appear as a natural corollary to the Artin-Schreier theory [**2**] of real closed fields. This model-theoretic approach has been suggestive and influential among model-theorists in finding elimination theories in other situations. The case $\mathbf{R}_{\mathbf{an,exp}}$ was treated this way in [**15**], using also ideas of Ressayre [**55**]. It is worth noting that in this case one has to add the logarithm function (extended to all of $\mathbf{R}$ by some convention) to eliminate quantifiers. This is typical: a correct choice of "basic" functions and relations is essential for elimination to be possible. (Another example is $\mathbf{R}_{\mathbf{an}}$, where we add $1/x$ to the basic functions to eliminate quantifiers.)

Model-completeness. This notion is a weaker variant of quantifier elimination, and is much less sensitive to the choice of basic relations and functions generating our structure. It applies again very generally, but we just introduce it in the case of interest. With $\mathcal{A}$ as above, a **sub-$\mathcal{A}$-set** in $\mathbf{R}^n$ is by definition the image of an $\mathcal{A}$-set in $\mathbf{R}^{n+k}$ (for some k) under the projection map $\mathbf{R}^{n+k} \to \mathbf{R}^n$. (In logical terms these sets are exactly the "existentially definable" sets, the sets that can be defined in $\mathbf{R}_{\mathcal{A}}$ by so-called existential formulas.)

DEFINITION 5.2. We say that $\mathbf{R}_{\mathcal{A}}$ is **model-complete** if the complement of each sub-$\mathcal{A}$-set in $\mathbf{R}^n$ is again a sub-$\mathcal{A}$-set in $\mathbf{R}^n$.

Thus if $\mathbf{R}_{\mathcal{A}}$ is model-complete, then the definable sets are exactly the sub-$\mathcal{A}$-sets.

Suppose $\mathcal{A}$ contains all $\{r\}$, $r \in \mathbf{R}$, all $\mathcal{A}$-sets have only finitely many connected components, and $\mathbf{R}_{\mathcal{A}}$ is model-complete. Then $\mathbf{R}_{\mathcal{A}}$ is o-minimal.

As the name suggests, model-completeness has a model-theoretic significance. This was established by A. Robinson. As with quantifier elimination, there are useful model-theoretic criteria that can help in proving model-completeness. In the terminology above, Gabrielov's "Theorem of the Complement" **[22]** means model-completeness for $\mathbf{R}_{\mathbf{an}}$. Gabrielov also proved an interesting strengthening, where not *all* analytic functions on cubes, but only those belonging to a collection closed under partials are considered, see **[23]**. While the arguments in **[22]** and **[23]** are very geometric, Wilkie's proof of the model-completeness of $\mathbf{R}_{\mathbf{alg},\mathbf{exp}}$ did take place in Robinson's model-theoretic setting.

We now describe some new (post '94) expansions whose o-minimality was obtained via quantifier elimination or model-completeness. Later in this section we turn to recent work by Wilkie and others that have led to o-minimality in a quite different way.

Generalized convergent power series. We consider an expansion $\mathbf{R}_{\mathbf{an}^*}$ of the real field, in which we can define for example all functions of the form

$$x \mapsto \sum_{n=0}^{\infty} c_n x^{\alpha_n} : [0,a] \to \mathbf{R}, \quad a > 0$$

with strictly increasing exponents $\alpha_n \geq 0$ such that $\sum |c_n|(a+\epsilon)^{\alpha_n} < \infty$ for some $\epsilon > 0$ (so the displayed sum is safely convergent). In particular, $\mathbf{R}_{\mathbf{an}^*}$ defines the function $\zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log n}$ on $[0, e^{-2}]$.

More generally, let $X = (X_1, \ldots, X_n)$ be a tuple of indeterminates, and $r = (r_1, \ldots, r_n)$ be a tuple of positive reals. Then we define $\mathbf{R}\{X^*\}_r$ to be the ring of all generalized formal power series

$$f(X) = \sum_{\alpha \in [0,\infty)^n} c_\alpha X^\alpha, \text{ with } c_\alpha \in \mathbf{R},\ X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

such that $\{\alpha : c_\alpha \neq 0\} \subseteq S_1 \times \cdots \times S_n$ for suitable well-ordered subsets S_i of $[0, \infty)$, and such that $\sum_\alpha |c_\alpha| r^\alpha < +\infty$. A series $f(X)$ of this form defines a real valued function

$$x \mapsto f(x) := \sum c_\alpha x^\alpha$$

on the cube $[0, r_1] \times \cdots \times [0, r_n]$ which is analytic on its interior. Let $\mathbf{an}^*$ consist of the ordering, addition, multiplication, and all functions on $\mathbf{R}^n$ (for all n) that are given on $[0,1]^n$ by a series in $\mathbf{R}\{X^*\}_r$ as above with $r_1 > 1, \ldots, r_n > 1$, and that vanish identically outside $[0,1]^n$. Then the main result from [**19**] by Van den Dries and Speissegger is:

THEOREM 5.3. *The expansion* $\mathbf{R}_{\mathbf{an}^*}$ *of the real field is model-complete, o-minimal, and polynomially bounded.*

The proof uses inductive arguments involving Weierstrass preparation and blow-up maps (much as in Tougeron [**69**]) to reduce to a situation where a judicious application of Gabrielov's technique of "fiber cutting" is possible. Unlike $\mathbf{R}_{\mathbf{an}}$ it does not seem that $\mathbf{R}_{\mathbf{an}^*}$ can be made to eliminate quantifiers by just adding $1/x$ to the basic functions.

Multisummable power series. The paper [**19**] was a kind of warm-up for dealing with $\mathbf{R}_{\mathcal{G}}$ in [**20**]. The geometric arguments are very similar, but the analytic preliminaries are more demanding. As before $\mathcal{G}$ extends **alg**, but to list exactly the basic functions of $\mathcal{G}$ in detail would lead too far. The main point is that we get into the territory of functions with divergent asymptotic expansions. Among the functions definable in $\mathbf{R}_{\mathcal{G}}$ (but not in $\mathbf{R}_{\mathbf{an}^*}$ or $\mathbf{R}_{\mathbf{an},\exp}$) are:

1. The function f on $[0,1]$ given by
$$f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt.$$
It is analytic on $(0,1]$ but only C^∞ at 0, Its Taylor expansion at 0 is the divergent series $\sum_{n=0}^\infty (-1)^n n! X^n$ as
2. The function ψ on $[0,1]$ given by $\psi(0)=0$ and
$$\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log\pi + \psi\left(\frac{1}{x}\right)$$
for $x \geq 1$ (Stirling expansion).

For the complicated definition of the functions in $\mathcal{G}$ one has to go to the complex plane and use the theory of multisummability, as extended in a rather subtle way to several variables by Tougeron [**69**]. We refer to [**20**] for details, and the proof of the following.

THEOREM 5.4. *The expansion* $\mathbf{R}_{\mathcal{G}}$ *of the real field is model-complete, o-minimal, and polynomially bounded.*

Both $\mathbf{R}_{\mathbf{an}^*}$ and $\mathbf{R}_{\mathcal{G}}$ admit analytic cell decomposition.

Adding exponentiation. Using a combination of model theory and valuation theory [**20**] also proves:

THEOREM 5.5. *Suppose* $\mathbf{R}_{\mathcal{A}}$ *is a polynomially bounded o-minimal expansion of the real field in which the restriction of the exponential function to* $[0,1]$ *is definable. Then* $\mathbf{R}_{\mathcal{A},\exp}$ *is o-minimal and exponentially bounded, where* $\mathcal{A},\exp := \mathcal{A} \cup \{\exp\}$.

In fact, we have a quantifier elimination for $\mathbf{R}_{\mathcal{A},\exp}$ "relative to $\mathbf{R}_{\mathcal{A}}$", by taking as basic functions all functions on $\mathbf{R}^n$ (for all n) that are definable in $\mathbf{R}_{\mathcal{A}}$, as well as the functions exp and log.

In particular we obtain some new exponentially bounded o-minimal expansions of the real field:

1. $\mathbf{R}_{\mathbf{an}^*,\exp}$, in which for example functions given on a suitable interval $(a, +\infty)$ by a convergent Dirichlet series are definable,
2. $\mathbf{R}_{\mathcal{G},\exp}$, in which the Gamma function on the positive real line is definable.

The embedding of $\mathrm{H}(\mathbf{R}_{\mathbf{an},\exp})$ into the field of logarithmic-exponential series has a natural extension to an embedding of $\mathrm{H}(\mathbf{R}_{\mathcal{G},\exp})$ into this field.

Adding Pfaffian functions. Charbonnel [**8**] proposes a method completely different from those discussed so far to obtain the o-minimality of the exponential field of reals. While the proof claimed in [**8**] has serious flaws, the new direction suggested by it has turned out to be viable.

Throughout this subsection $\mathbf{R}_{\mathcal{A}}$ denotes an expansion of the real field, such that, except for the ordering, $\mathcal{A}$ consists of C^∞ functions on $\mathbf{R}^n$ for various n. Wilkie [**73**] shows:

THEOREM 5.6. *Suppose that for each* $\mathcal{A}$*-set* S *in* $\mathbf{R}^{m+n}$ *there is a bound* $B = B(S) \in \mathbf{N}$ *such that for all* $a \in \mathbf{R}^m$ *the fiber* S_a *has at most* B *connected components. Then* $\mathbf{R}_{\mathcal{A}}$ *is o-minimal.*

Take the smallest collection $\mathcal{A}^*$ of subsets of the spaces $\mathbf{R}^n$ such that $\mathcal{A}^*$ contains all $\mathcal{A}$-sets and is closed under taking finite unions and intersections, topological closures (in an ambient cartesian space), and projections (images under projection maps $\mathbf{R}^{n+1} \to \mathbf{R}^n$). A salvagable part of [**8**] says that the tameness hypothesis of the theorem is inherited by the sets in $\mathcal{A}^*$. Wilkie shows that $\mathcal{A}^*$ is also closed under taking complements. Thus in the theorem the definable sets of $\mathbf{R}_{\mathcal{A}}$ are exactly the sets in $\mathcal{A}^*$. This result produces new o-minimal expansions of the real field as follows. Let $f : \mathbf{R}^n \to \mathbf{R}$ be **Pfaffian** over $\mathbf{R}_{\mathcal{A}}$, that is, f is C^∞ and

$$\frac{\partial f}{\partial x_i}(x) = F_i(x, f(x)) \text{ throughout } \mathbf{R}^n \text{ , for } i = 1, \dots, n$$

for suitable C^∞ functions F_i on $\mathbf{R}^{n+1}$ that are definable in $\mathbf{R}_{\mathcal{A}}$.

COROLLARY 5.7. *If $\mathbf{R}_{\mathcal{A}}$ is o-minimal, and f as above is Pfaffian over $\mathbf{R}_{\mathcal{A}}$, then $\mathbf{R}_{\mathcal{A},f}$ is also o-minimal, where $\mathcal{A}, f := \mathcal{A} \cup \{f\}$.*

This follows from the theorem above applied to $\mathcal{A}, f$ instead of $\mathcal{A}$. The hypothesis of the theorem is satisfied for $\mathcal{A}, f$ by a variant of Khovanskii's theorem [**30**]. Note that the process of adjoining Pfaffian functions can be repeated, since the new basic function f is C^∞.

For the concrete o-minimal structures on the real field mentioned so far, one can indeed always take a generating set $\mathcal{A}$ that satisfies the C^∞ assumption made in the beginning of this subsection.

Adding Rolle leaves. Geometrically it is more natural to consider 1-forms and integral manifolds that are not necessarily graphs of functions. Moussu and Roche [**47**] have an extension of Khovanskii's finiteness theorem to intersections of so-called Rolle leaves of analytic 1-forms with semianalytic sets. Wilkie's work inspired Lion & Rolin [**38**] to obtain o-minimality in that setting. This has been further extended and simplified by Speissegger [**67**] so that no analyticity or C^∞ assumptions are needed any longer, and one can work over an arbitrary o-minimal expansion of the real field.

Here are more details. Let $\mathbf{R}_{\mathcal{A}}$ be an o-minimal expansion of the real field, and let $\omega = a_1 dx_1 + \cdots + a_n dx_n$ be a definable 1-form of class C^1 on some definable open set $U \subseteq \mathbf{R}^n$, where the definability of ω means that the coefficient functions a_i are definable in $\mathbf{R}_{\mathcal{A}}$. Let

$$\mathrm{Sing}(\omega) := \{x \in U : a_1(x) = \cdots = a_n(x) = 0\}$$

be the singular set of ω. We view the equation $\omega = 0$ as defining a hyperplane field on the open set $U \setminus \mathrm{Sing}(\omega)$. A **Rolle leaf** of the equation $\omega = 0$ is a connected (embedded) integral manifold L of this hyperplane field, such that L is closed in

$U \setminus \mathrm{Sing}(\omega)$ and any C^1-curve $\gamma : [0,1] \to U \setminus \mathrm{Sing}(\omega)$ with $\gamma(0), \gamma(1) \in L$ is tangent to the hyperplane field at some point, that is, $\omega(\gamma(t), \gamma'(t)) = 0$ for some $t \in (0,1)$. (The Rolle condition on the leaf prevents non o-minimal behaviour such as spiraling. The graph of a Pfaffian function f as in the last subsection is a Rolle leaf of $F_1 dx_1 + \cdots + F_n dx_n - dx_{n+1} = 0$.) With this terminology we have **[67]**:

THEOREM 5.8. *Let $\mathbf{R}_{\mathcal{A}}$ be an o-minimal expansion of the real field. Let $\mathcal{A}^+$ be $\mathcal{A}$ augmented by all Rolle leafs of all equations $\omega = 0$, where ω ranges over the definable 1-forms of class C^1 on definable open sets. Then $\mathbf{R}_{\mathcal{A}^+}$ is o-minimal.*

This process can now be repeated with $\mathcal{A}^+$ to produce $\mathcal{A}^{++}$, and continuing this way, and taking the union of $\mathcal{A}$, $\mathcal{A}^+$, $\mathcal{A}^{++}$, ..., we obtain a collection $\mathrm{Pf}(\mathcal{A})$ such that $\mathbf{R}_{\mathrm{Pf}(\mathcal{A})}$ is an o-minimal expansion of the real field and $\mathrm{Pf}(\mathcal{A})$ contains every Rolle leaf of any equation $\omega = 0$ with ω a definable 1-form of class C^1 on a definable open set, where "definable" refers to definability in $\mathbf{R}_{\mathrm{Pf}(\mathcal{A})}$. We call $\mathrm{Pf}(\mathcal{A})$ the *Pfaffian closure* of $\mathcal{A}$. It is not known if it gives rise to the same definable sets as the "Pfaffian closure" obtained by iterating Wilkie's method when we start with an $\mathcal{A}$ satisfying the C^∞-assumption in Wilkie's theorem.

The way the theorem is proved gives considerable control over the definable sets, and allows Lion & Speissegger **[40]** to show that if $\mathbf{R}_{\mathcal{A}}$ admits "analytic cell decomposition", so does $\mathbf{R}_{\mathrm{Pf}(\mathcal{A})}$.

This concludes our survey of the various constructions. The inclusion diagram on the next page summarizes which o-minimal expansions of the real field can be obtained by these methods. The idea of such a display came from Macintyre.

$$\begin{tikzcd}
 & & & \mathbf{R}_{\mathrm{Pf}(\mathcal{G})} \\
\mathbf{R}_{\mathrm{Pf}(\mathbf{alg})} \arrow[r] & \mathbf{R}_{\mathrm{Pf}(\mathbf{an})} \arrow[r] \arrow[urr] & \mathbf{R}_{\mathrm{Pf}(\mathbf{an}^*)} & \\
 & & & \mathbf{R}_{\mathcal{G},\exp} \arrow[uu] \\
\mathbf{R}_{\mathbf{alg},\exp} \arrow[r] \arrow[uu] & \mathbf{R}_{\mathbf{an},\exp} \arrow[r] \arrow[uu] \arrow[urr] & \mathbf{R}_{\mathbf{an}^*,\exp} \arrow[uu] & \\
 & & & \mathbf{R}_{\mathcal{G}} \arrow[uu] \\
\mathbf{R}_{\mathbf{alg}} \arrow[r] \arrow[uu] & \mathbf{R}_{\mathbf{an}} \arrow[r] \arrow[uu] \arrow[urr] & \mathbf{R}_{\mathbf{an}^*} \arrow[uu] &
\end{tikzcd}$$

The known o-minimal expansions of the real field.

In this diagram an arrow $\mathbf{R}_{\mathcal{A}} \longrightarrow \mathbf{R}_{\mathcal{B}}$ means that $\mathrm{Def}(\mathcal{A}) \subseteq \mathrm{Def}(\mathcal{B})$. The bottom arrows connect the polynomially bounded expansions, the upward pointing ones go to the expansions that can be built on top of the polynomially bounded ones by adding exp and taking the Pfaffian closure. It seems to be unknown if $\mathbf{R}_{\mathrm{Pf}(\mathbf{alg})}$ is exponentially bounded.

Added in proof: Recently C. Miller and P. Speissegger have shown that all expansions in the diagram above are exponentially bounded.

6. Open Problems

We start with some clear-cut and well-known questions.

1. Is each o-minimal structure on the real field exponentially bounded?
2. Do all o-minimal structures on the real field have analytic cell decomposition? Same question with "analytic" replaced by "C^∞".
3. Is there a largest o-minimal structure on the real field?

A positive answer to any of these questions would be too good to be true.

Here are some more open-ended questions.

4. Let the function $f : [0,1]^2 \to \mathbf{R}$ be continuous and definable in the o-minimal expansion $\mathbf{R}_{\mathcal{A}}$ of the real field, and let $g : [0,1] \to \mathbf{R}$ be given by $g(x) = \int_0^1 f(x,y)dy$. Is then $\mathbf{R}_{\mathcal{A},g}$ o-minimal, where $\mathcal{A}, g := \mathcal{A} \cup \{g\}$?
5. Does quasianalyticity imply o-minimality?
6. Do the "analysable functions" in the sense of Ecalle [**21**] generate an o-minimal structure on the real field?

Question 4 was asked some years ago by L. Bröcker for $\mathbf{R}_{\mathbf{alg}}$, and has a positive answer in that case. More generally, let $f : \mathbf{R}^{m+n} \to \mathbf{R}$ be globally subanalytic such that for each $x \in \mathbf{R}^m$ the function f_x on $\mathbf{R}^n$ is (Lebesgue) integrable. (For example, f could be the characteristic function of a bounded subanalytic set in $\mathbf{R}^{m+n}$.) Then the function

$$x \mapsto \int_{\mathbf{R}^n} f(x,y)dy$$

on $\mathbf{R}^m$ is definable in $\mathbf{R}_{\mathbf{an,exp}}$. More precise results of this kind have been obtained by Lion & Rolin [**39**]. What happens is that integration introduces logarithms, as in $\int_1^x y^{-1}dy = \log x$. Here is a rough idea of [**39**]. First, reduce to considering functions on cubes $[0,1]^{m+n}$. As shown in [**36**], globally subanalytic functions on such cubes can be expanded into "Puiseux-Laurent" type series with respect to the last variable, piecewise uniformly with respect to the other variables. Integrating these series termwise with respect to the last variable may introduce logarithms of globally subanalytic functions, but these enter only in a polynomial way, which allows the process to be repeated for other variables.

As to question 5, let Q be a ring of real valued C^∞ functions on a connected open set $U \subseteq \mathbf{R}^n$, and suppose Q is closed under taking partial derivatives. Call Q *quasianalytic* if for each point $a \in U$ the Taylor homomorphism

$$f \mapsto \sum_{\alpha \in \mathbf{N}^n} \frac{f^{(\alpha)}(a)}{\alpha!} X^\alpha : Q \longrightarrow \mathbf{R}[[X]], \text{ where } X = (X_1, \ldots, X_n),$$

is injective. For a polynomially bounded o-minimal expansion $\mathbf{R}_{\mathcal{A}}$ of the real field, the definable C^∞ functions on (definable) U form such a quasianalytic ring, see [**45**]. Question 5 asks for a kind of converse to this. For example, by the Denjoy-Carleman theorem, given an interval I, the C^∞ functions f on I such that for some positive integer k,

$$\frac{|f^{(n)}(x)|}{n!} \leq \left(\prod_{i=1}^{k} \log_i n \right)^n \quad \text{for all } x \in I \text{ and } n > \exp_k(1),$$

form such a quasianalytic ring (where $\log_i$ denotes the i^{th} iterate of log). Do the restrictions of these functions to $[0,1]$ (with $I \supset [0,1]$) generate a polynomially bounded o-minimal expansion of the real field? All signs point to a positive answer—in which case the first part of question 2 would have a negative answer, since these Denjoy-Carleman functions on $[0,1]$ are in general not analytic (although all analytic functions on $[0,1]$ are among them).

A positive answer to question 6 might be of interest in connection with the part of Hilbert's 16th problem that concerns the number of limit cycles of polynomial vector fields on the real plane. Ecalle [**21**] and Il'yashenko [**28**] have both written formidable books proving that this number is always finite for each particular polynomial vector field on the plane. It would be very desirable to have this finiteness property as part of the o-minimality of a suitable expansion of the real field. It is this consideration that leads to question 6.

We should also mention Arnold's paper [**1**]. Some of the problems listed there are very much in the spirit of o-minimality.

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