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Recent Progress in Sphere Packing

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Most of this article will be about comparatively minor progress in the sphere packing problem in higher dimensions, but it is a pleasure to record that at last the original 3-dimensional density problem has been definitively solved. The attempted proof by Hsiang [**Hs93**] unfortunately turned out to be incomplete.

The Hales Proof of the Kepler Conjecture for 3-dimensional Sphere Packings

The problem of finding the greatest density of any packing of equal spheres in n-dimensional Euclidean space is an old and important one, which has applications in geometry, number theory, and information theory. It is also difficult, as is shown by the fact that although the 3-dimensional case was raised by Kepler in 1613, it has only recently been solved, by Thomas Hales in 1998, with an important contribution from S. Ferguson. I shall say only a few words about the argument, which is long and complicated in detail.

Hales starts from the familiar Delaunay tessellation, which has one cell for each point of space whose distance from the sphere centers is a local maximum, for which Sloane and I have introduced the snappy term "deep hole". Then the Delaunay cell that corresponds to a deep hole is the convex hull of the sphere centers nearest to that hole, and it is well known that these cells constitute a polyhedral decomposition of space.

In the standard face-centered cubic lattice packing, the Delaunay cells are alternately regular tetrahedra T and octahedra O, so that its density is

$$[vol(T') + vol(O')]/[vol(T) + vol(O)]$$

where T' and O' are the parts of T and O covered by the spheres. However, for a packing in which the sphere centers are in general position the Delaunay cells will be simplicial, and so it suffices to establish the density bound for packings with simplicial Delaunay cells.

C. A. Rogers long ago showed that $vol(T')/vol(T)$ was an upper bound for the density of any packing—this is now known as "the Rogers bound". It would be attained for a hypothetical packing whose Delaunay cells were all regular tetrahedra, so we may say that the difficulty in the 3-dimensional sphere packing problem is to show that the Rogers bound must inevitably be worsened to the extent that it is in the face-centered cubic packing by the presence of the octahedral cells.

In the first paper of his series, Hales defined a function called the "score" of a star of Delaunay cells, in terms of another function called the "compression" of a cell. The compression of the Delaunay simplex S is

$$vol(S') - vol(S).vol(O')/vol(O)$$

where S' is the part of S contained in the spheres, and O and O' are as above. In other words, the compression measures how much better S is covered than is the regular octahedral cell O of the face-centered cubic packing.

A tetrahedron S is naturally divided into four parts (the Voronoi cells of its vertices), the nth part S_n consisting of those points that are closer to the nth vertex than any other. In the case that S contains its circumcenter, its score at the nth vertex is defined to be its compression, if the circumradius is at most 1.41, and

$$4.solid(n)/3 - 4.vol(S_n).vol(O')/vol(O)$$

otherwise, where $solid(n)$ denotes the solid angle at that vertex. When S does not contain its circumcenter, the score is defined by continuing the analytic function corresponding to this expression. The score (measured in "points") of the star of Delaunay tetrahedra for a given vertex is the sum of these scores over all the tetrahedra at that vertex. These definitions ensure that the average of the score over all of space is the average of the compression, and reduce the problem to proving that the average score is at most 8 points.

Two sphere centers are called "close neighbors" if they are distant at most 2.51 from each other, and a Delaunay cell is called a quasi-regular tetrahedron if any two of its four vertices are close neighbors. In his first paper, Hales shows that the score of a star composed entirely of quasi-regular tetrahedra is indeed at most 8 points. The remaining papers of the sequence are addressed to the much harder task of extending this result to all the other configurations that might arise.

This involves a combinatorial classification of the possibilities, which is in itself very long, accompanied by an analytical proof of the appropriate inequality for each of thousands of cases. Both parts involved heavy use of machine computation, and careful selection of various parameters. Hales remarks for instance that

"The constant 2.51 was determined experimentally to have a number of desirable properties", and similar experimental determinations recur repeatedly throughout the paper. Several of the initial decisions had to be modified in the light of later calculations.

Of course a proof of this nature is not at all easy to read! But Hales and Ferguson have taken great care to make the entire proof accessible to readers who wish to check any detail. In particular they have started a practice that should serve as a model for future machine-dependent proofs, by keeping detailed logs of all their interactions with the machine, so that a potential auditor can discover that on such and such a day the following cases were handled, of which the last had to be split into two subcases for which the inequalities became ... . He or she can then rerun the programs which were used to verify the truth of these inequalities.

Great care was also taken with these proofs. Suppose, for instance, that the inequality $f(x) < g(x)$ has to be proved for all x in a certain interval. Then typically the machine will automatically find a dissection of this interval into a number of subintervals, and find upper bounds for the derivatives of f and g at the endpoints of these which yield linear bounds for them that still satisfy the inequality. [For higher-dimensional intervals this would involve linear programming.] All calculations of such bounds are done using interval arithmetic, which prevents errors that might otherwise arise from rounding the numbers.

Progress in other dimensions: The problem of "best" packings.

In other dimensions, our progress has been limited, and much of it only conjectural. The question we should like to answer is

"What are all the best sphere packings in a given dimension?"

Unfortunately, it is not even clear just what this question means! Despite this, the paper [**CS95**] of Conway and Sloane gives a conjectural answer to it in each dimension up to 10. We shall roughly follow the discussion in that paper, marching onward from dimension to dimension, discussing both the question and its answers as we go. As the reader may well suspect, packings of R^0 are not particularly intriguing, and so we begin with:

What are the best sphere packings in $\mathbf{R}^1$?

We can regard a sphere packing by spheres of radius r as a collection of points each pair of which is at least r apart. No matter what the best definition of "best" may be, the best packing of spheres of radius r in $\mathbf{R}^1$ is surely the lattice of points $2kr$:

And clearly, any packing that can be compacted further should not be called "best":

Note, perhaps a bit pedantically, that the spheres in a best packing in $\mathbf{R}^1$ are centered on the points of a lattice; we can rescale this to be the root lattice A_1, that is, the lattice generated by the vector $(1, -1)$ in $\mathbf{R}^2$. Root lattices will play a large role in our discussion of best packings; and since the root lattices are canonically scaled so that the minimum distance between points is $\sqrt{2}$, we will adopt the convention that all of our sphere packings are packings of spheres of radius $\sqrt{2}/2$.

What are the best sphere packings in $\mathbf{R}^2$?

Whatever "best" may mean, the best packing of the plane should of course be the hexagonal packing (Figure 1), with circles centered on the points of the root lattice A_2, the lattice generated by the vectors, say, $(1, -1, 0)$ and $(0, 1, -1)$:

Clearly "having the highest possible density" is a major component of any good meaning of "best", and Fejes Tóth has given an elegant proof that no packing in R^2 has density exceeding that of the hexagonal lattice A_2. However, if we allow this to be the entire definition we get some rather silly "best packings" since density is only defined by a limiting process.

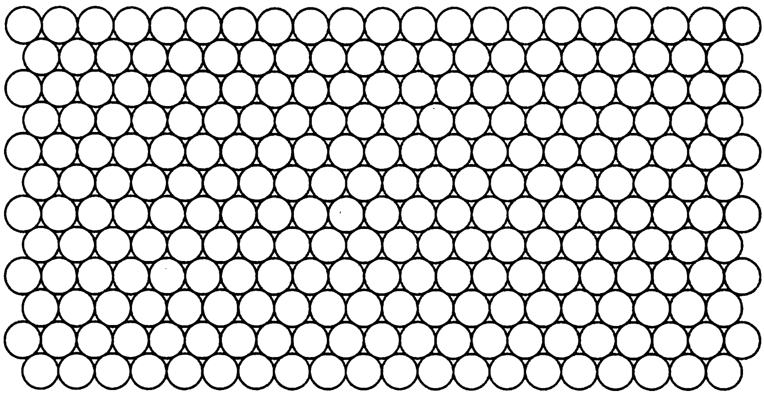


FIGURE 1. The A_2 lattice packing, Π_2

For example, beginning with this hexagonal packing, remove any finite number of circles. The depleted packing still has the same density, as does the "even worse" one in which we remove an entire infinite row of circles. In fact, we can even have a packing of the optimal density in which no circles are touching! Figure 2 below is derived from the hexagonal lattice by increasing all distances from the origin by a fixed multiple of their arctangents, an operation that does not change the overall density of the packing.

Perhaps we would do better with Tóth's notion of solidity: a packing is "solid" if there is only one way to replace any finite collection of spheres we have removed. Here is a little theorem:

Theorem 1. *If two packings in $\mathbf{R}^n$ have different densities, then the looser cannot be solid.*

The proof is quite simple. Consider the looser packing. If we carve out a vast but still finite region, we will be free to put all the spheres back in the denser configuration, with plenty of room around the sides. For example, in Figure 3, we have replaced 143 circles in the relatively loose square packing with 143 circles from a much denser hexagonal packing.

Solidity, then, implies having the highest density, and is a good candidate for the meaning of "best". But unfortunately there seem to be some not-so-good examples of solid packings. For example, consider the packing in Figure 4. The "upper half" of a hexagonal packing has been shifted slightly; the shaded circles show the fault line. Though this packing is clearly not "best", it is almost certainly solid.

Worse still it is extremely difficult to establish the solidity of any particular packing. The recent work of Hales, described above, does *not* suffice to establish the solidity of any Barlow packing.

But it is even difficult to show that there *is* a solid packing in a given space. It is possible, and quite plausible in higher dimensions, that solid packings simply don't exist.

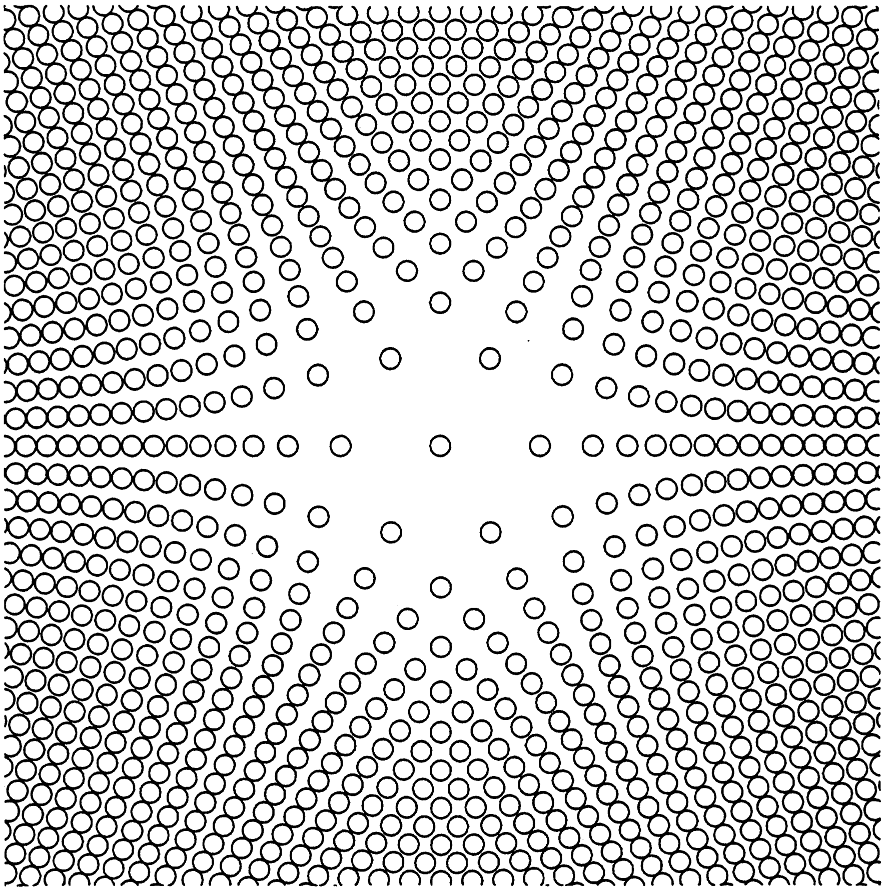


FIGURE 2. A bad but maximally dense packing in $\mathbf{R}^2$

We might define "best" as something like "having maximum density over finite regions". But this definition seems to fail. Beyond the completely compacted packing in $\mathbf{R}^1$, every sphere packing in $\mathbf{R}^n$ has plenty of empty space; over just which finite regions should we measure this density?

So just how should we define "best"? Conway and Sloane were unable to find a satisfactory definition and eventually gave up on this aspect of the problem. This does not force us to give up on the real and apparently meaningful problem of finding out just what the "best" packings actually are. Since in any case, we can't reasonably expect to find more than a conjectural answer to the question, the fact that we can't assign a precise meaning to it is not as distressing as it might have been!

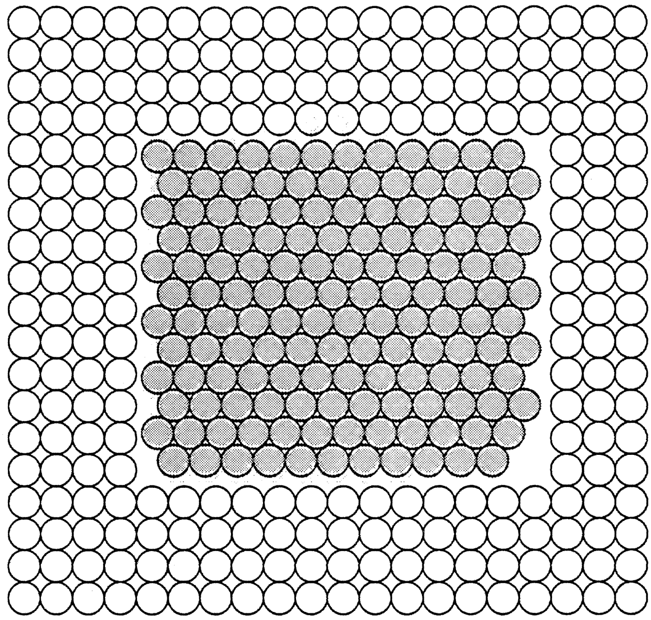


FIGURE 3. Theorem 1

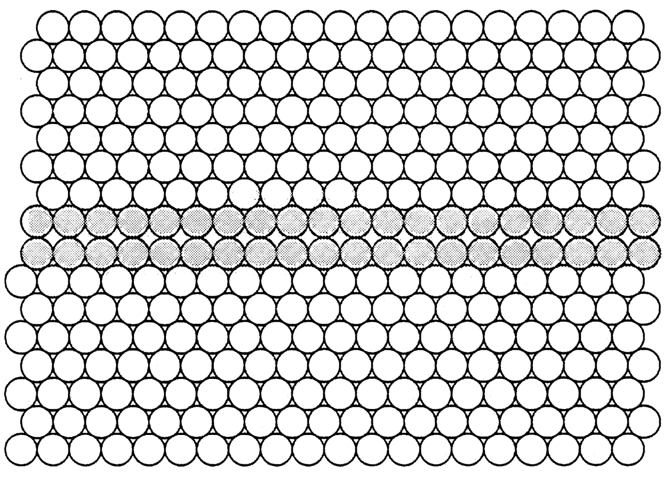


FIGURE 4. "Solid" ≠ "best"

What are the "best" sphere packings in $\mathbf{R}^3$?

We have already commented on one of the most spectacular developments in recent years, Thomas Hales' settling of the so-called Kepler Conjecture, that no packing in $\mathbf{R}^3$ has greater density than that based on the face-centered cubic lattice $A_3 \cong D_3$, *viz.*, the points of the form (i, j, k) with $i + j + k$ even.

As it happens, the face-centered cubic lattice packing is far from being uniquely the densest; it is only the most symmetrical of an uncountable family of equally good "Barlow packings", named for the great English crystallographer who studied some of them in 1883 [**Ba83**].

It is conceivable that there are other packings in $\mathbf{R}^3$ that deserve the name "best", for Hales showed only that the density of the Barlow packings cannot be surpassed. Nonetheless, despite the lack of a sound definition of "best", it is quite reasonable to suggest that the Barlow packings are indeed all the best packings in $\mathbf{R}^3$.

To make a Barlow packing of your own, begin with a sheet of spheres arranged into a two-dimensional hexagonal packing. There are precisely two ways to stack another such sheet neatly on top of the first: we can fit the next layer of spheres above either of two sets of holes, as shown in Figure 5. Repeating the process, we face two choices at each layer and altogether can produce an uncountably infinite family of packings as promised, all with exactly the same density.

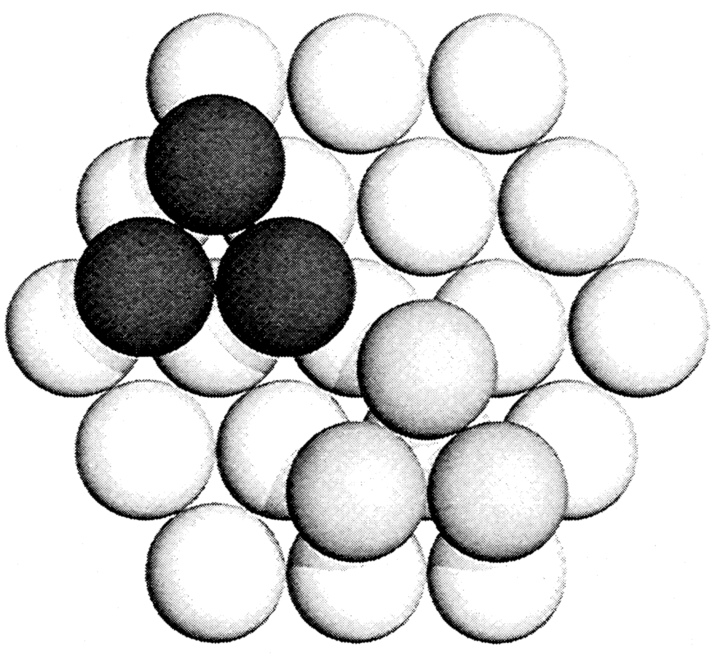


FIGURE 5. Stacking hexagonal lattice packings

Only two of these are "uniform", however, in the sense that there is a symmetry of the packing taking any one sphere to any other. To see this, look down upon a

Barlow packing from above. All the centers of the spheres in any given layer lie in one of three sets of points, colored white [0], gray [1] and black [2] in figure 6. In fact, these points form the unique three-coloring of the hexagonal lattice A_2. To put it another way, the three sets of points represent three cosets of the unique sub-lattice of index 3 in A_2; this sub-lattice is of course isomorphic to A_2 itself.

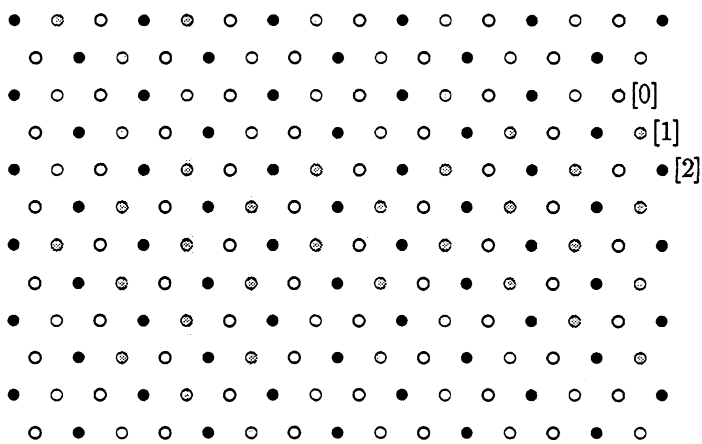
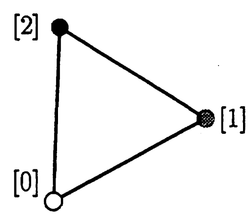


FIGURE 6. Three cosets in A_2.

We call the possible projections of the layers *placements*, and label them by the "colors" [0], [1] and [2]. Now there is only one restriction on the possible sequence of placements of the layers in the Barlow packing, namely that we never use the same color for two adjacent layers. So the Barlow packings are exactly in correspondence with doubly infinite paths in this *placement graph*, 3_2 (we will use v_k to denote a graph with v vertices each of valence k):



Alternatively, the Barlow packings are in precise correspondence with the 3-colorings of A_1.

Now any *uniform* packing exactly corresponds to a path whose vertices are equivalent under any symmetries of this graph. Clearly, up to labeling, there are only two such paths: ... abababababa ... and ... abcabcabcabc ... where a, b, c is some permutation of the colors [0], [1], [2]. Small neighborhoods of these two packings are shown in Figure 7.

The packing in which the layers cyclically permute the colors is the more special; this is the face-centered lattice packing with which we began, and it is the only *lattice* packing among the Barlow packings.

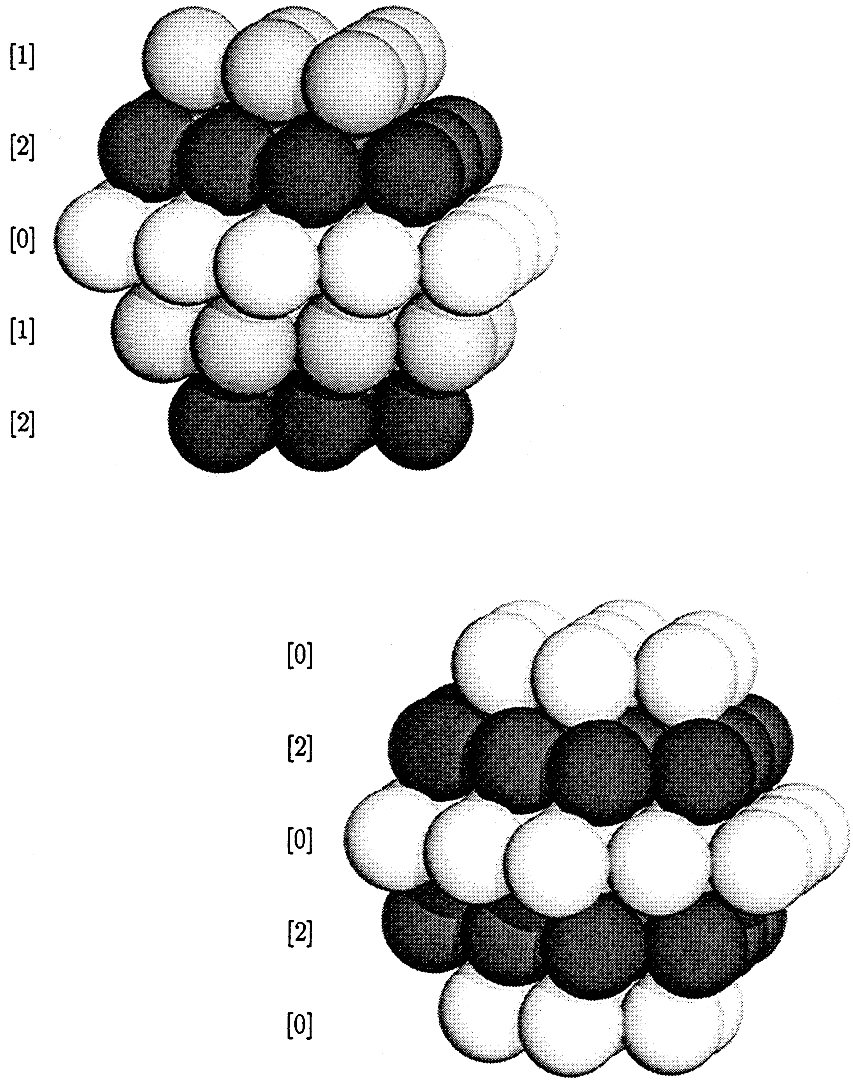


FIGURE 7. Portions of the two uniform Barlow packings: the face-centered cubic lattice packing and the hexagonal close packing

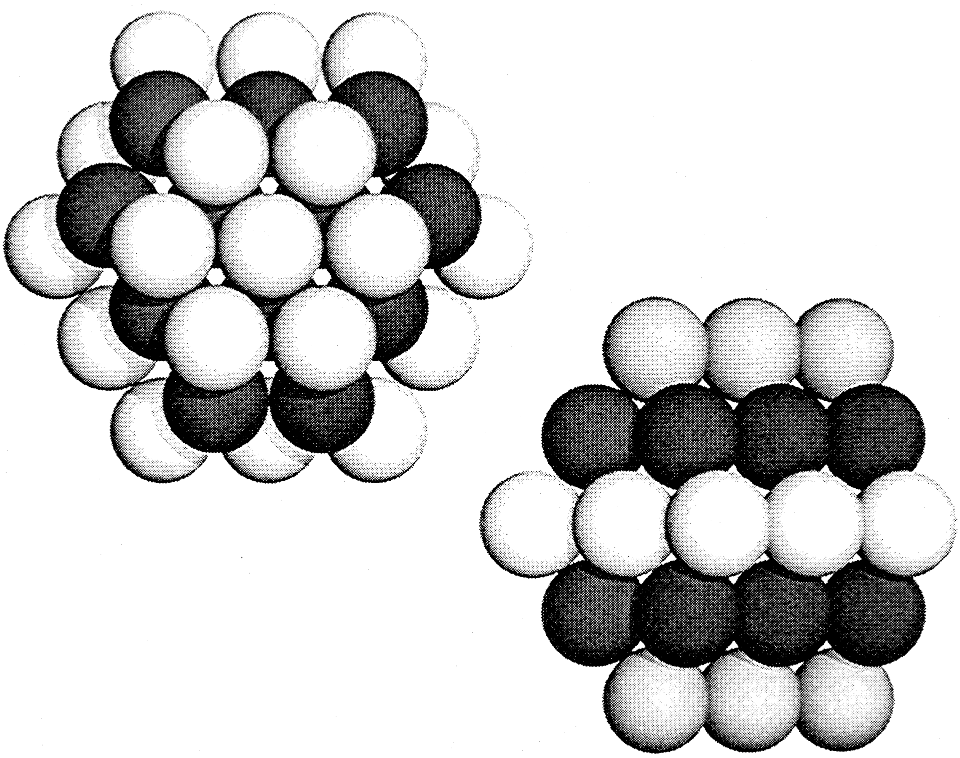


FIGURE 8. No Barlow packing completely occludes all light ...

It is quite amusing to note that the other uniform packing, the so-called hexagonal close-packing, in which the layers occur in just two alternating colors, is the "least occluding" of all the Barlow packings.

As Figure 8 suggests, no Barlow packing completely occludes all light (part of a generic packing, is on the lower right; the close-packed hexagonal packing, seen from a different vantage is on the upper left.) Equivalently, one can always drive rather thin, infinitely long stakes in one of three "horizontal" directions, without touching any of the spheres in the packing. In the face-centered cubic lattice, there is enough additional symmetry that one can drive thin stakes in three more directions. But the hexagonal close packing is the only Barlow packing in which there is enough room for really thick stakes, more than 1/7th the width of the spheres themselves. These stakes are "vertical" and run through the holes above the third, unused coset in A_2, as seen in Figure 9.

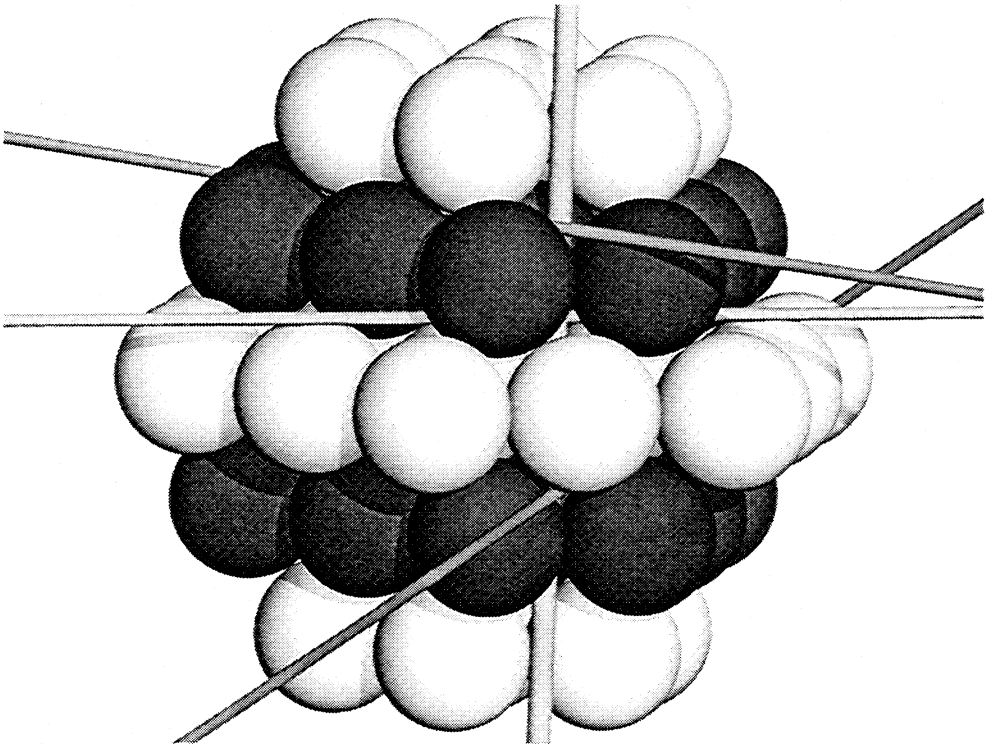


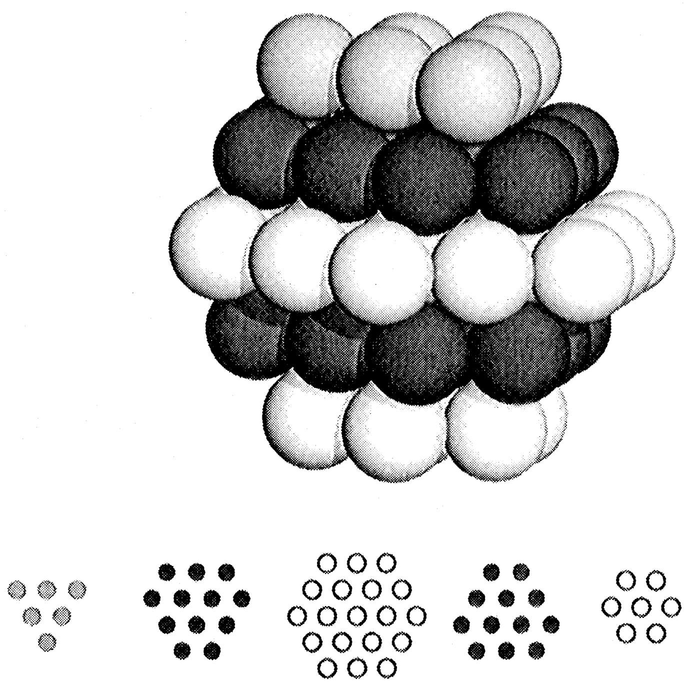
FIGURE 9. ... but the hexagonal close packing is the least occluding of all

Crystal Balls. The *first crystal ball* of a sphere in a packing consists of that sphere and those touching it, and inductively, the $(n+1)$*st crystal ball* consists of the nth crystal ball and all the spheres that touch it.

In all of the Barlow packings, every sphere has exactly 12 neighbors, so the first crystal ball of any sphere has size 13. However, the *second* crystal ball can have size 55, 56, or 57. Consider five adjacent layers labeled a, b, c, d, e, where each color is one of [0], [1] or [2]. Then the second crystal ball has size 55 plus the number of "Yes" answers to the two questions:

$$\text{Is } a = c? \qquad \text{Is } c = e?$$

The case when $a \neq c$ and $c = e$, so that there are 56 spheres in the second crystal ball, is shown in Figure 10.



$$7 \quad + \quad 12 \quad + \quad 19 \quad + \quad 12 \quad + \quad 6 \quad = \quad 56$$

FIGURE 10. A second crystal ball has 56 spheres when $a \neq c$ and $c = e$

There are just four packings that are determined by the set of sizes of their second crystal balls:

If no crystal ball has size 56, then the answers to alternate questions from the sequence

$$\ldots \qquad \text{Is } a = c? \qquad \text{Is } b = d? \qquad \text{Is } c = e? \qquad \ldots$$

must always agree and we obtain one of three packings:

Packing	Answers	Coloring	Sizes
Λ_3^1	... NNNN ...		55
Λ_3^2	... YYYY ...		57
Λ_3^3	... YNYN ...		55,57

We have already seen Λ_3^1 and Λ_3^2; these are the two uniform packings, *viz.,* the face centered cubic lattice packing and the hexagonal close packing.

If all the crystal balls have size 56, then alternate answers must always be different and we have a fourth packing

Λ_3^4 ... YYNNYYNNYY ...  56

There are uncountably many Barlow packings with some second crystal balls of size 56 and some with size 55 or 57 or both, so these four are the only Barlow packings determined by the sizes of their second crystal balls.

Since all of the second crystal balls in the face-centered cubic lattice packing contain 55 balls, and all of those in the hexagonal close packing contain 57 balls, the average number of spheres in the second crystal ball in any large region of a Barlow packing is therefore between that of the two uniform Barlow packings. And so it is natural to suspect that the number of spheres in the nth crystal ball of any sphere in any Barlow packing is bounded above by the size of an nth crystal ball in the hexagonal close packing, and below by the size of the nth crystal ball in the face-centered cubic lattice packing.

Conway and Sloane verefied this by showing that the number of spheres in any nth crystal ball in the face centered-cubic lattice packing is $(5\Delta n^4 + \Delta n^2)/6$ where $\Delta n^k = (n+1)^k - n^k$; similarly the number of spheres in any nth crystal ball in the hexagonal close packing is $[7\Delta n^4/8]$. And indeed the number of spheres in any nth crystal ball in any Barlow packing is bounded below by the former and above by the latter (a conjecture that was only later made by O'Keefe).

The Hypothetical Answers

The Barlow packings generalize quite nicely to a conjectured description of *all* the best packings in low dimensions:

A n-dimensional packing P_n *fibers* over an m-dimensional packing P_m if P_n can be decomposed into sets (e.g. layers) of points lying in parallel m-dimensional spaces, each one of which is a packing of type P_m. In particular, then, the Barlow packings all fiber over the hexagonal packing in R^2.

Hypothesis n : *The best packings in $\mathbf{R}^n$ fiber over one of the best packings in $\mathbf{R}^m$, where m is the largest power of 2 less than n.*

Conway and Sloane believe (but cannot prove) that Hypothesis n is true for $2 \leq n \leq 8$. Hypothesis 9 requires some adjustments, and the Hypotheses are irredeemably false for $n \geq 10$.

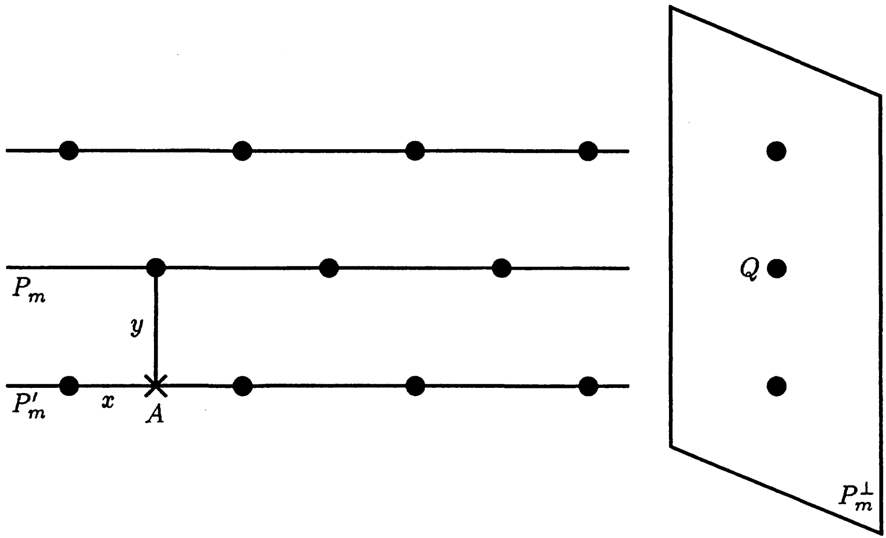


FIGURE 11

Let us fix some ideas: Suppose P_n is an n-dimensional packing that fibers over an m-dimensional packing P_m. Let A be the projection onto the space of a layer P_m of a point in another layer P'_m. Let x be the distance of A from the closest point of P_m and let y be the separation between the layers P_m and P'_m as illustrated in Figure 11. Then x is at most the covering radius R of P_m and $x^2 + y^2 \geq 2$ (because our spheres are of radius $1/\sqrt{2}$, so y is at least $\sqrt{2 - R^2}$).

Let Q be the projection of P_n onto the space $P_m^\perp$. Surrounding each point of Q by a sphere of diameter $\sqrt{2 - R^2}$, we obtain a sphere packing in $P_m^\perp$. If P_n is to be "best", plainly the packing in $P_m^\perp$ must be "best" (The hypothesis asserts that all best packings can be obtained this way taking m to be the greatest power of 2 less than n.) This can only occur if for adjacent layers P_m and P'_m, every point of P_m lies above a deep hole of P'_m.

Before marching off at the rapid rate the Hypotheses will allow us, let us turn back and see Hypotheses 2 and 3 applied to produce the best packings in R^2 and R^3.

Proposition 2 (using Hypothesis 2) *The unique "best" packing in* $\mathbf{R}^2$ *is the triangular packing* A_2.

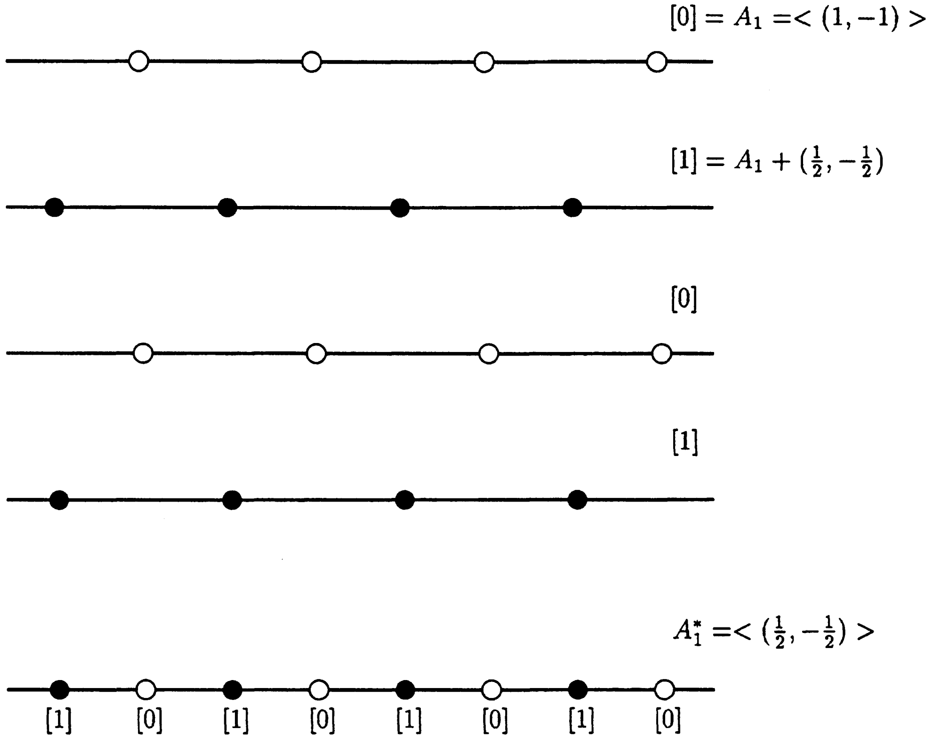


FIGURE 12. Applying Hypothesis 2 ...

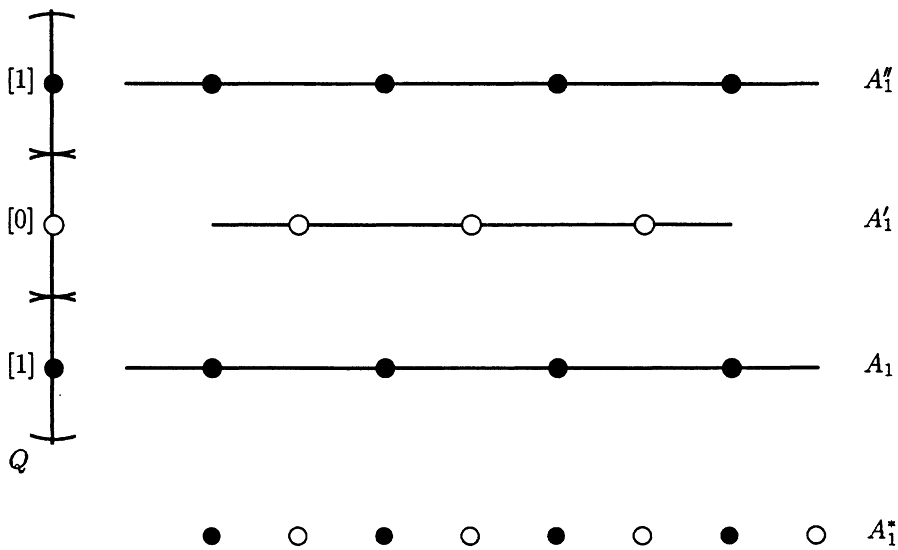


FIGURE 13. ... to prove Proposition 2

Proof. Hypothesis 2 tells us that any best packing in $\mathbf{R}^2$ fibers over A_1, the unique best packing in $\mathbf{R}^1$. The dual lattice A_1^* is the union of two cosets $[0] = A_1$ and $[1] = A_1 + (\frac{1}{2}, -\frac{1}{2})$. Each of the cosets consists of the deep holes of the other. The covering radius of A_1 is $1/\sqrt{2}$. The distance y between layers must be at least

$\sqrt{2-1/2} = \sqrt{3/2}$; this can be achieved only for the packing shown in Figure 13, in which alternate layers are the cosets [0] and [1] of A_1 in A_1^*. The packing Q is a rescaled A_1, in which the points are colored alternately [0] and [1]. Of course, this is a description of the root lattice A_2. □

As with the Barlow lattices, we can describe the *placements* of the layers of the A_2 lattice packing. The allowed sequences of possible placements corresponds to paths in the *placement graph*, in this case, merely the graph 2_1:

More generally, the layers in a "best" packing can only have a certain set of projections onto the starting layer; these are the placements. Two placements are joined by an edge in the placement graph just if they can correspond to layers at the minimal distance $\sqrt{2-R^2}$, that is, just if each point of either projects into a deep hole of the other.

Fibering over $\mathbf{A_2}$ Several times we will be fibering over A_2. The lattice A_2^* dual to A_2 consists of the three cosets

$$[0] = A_2, \quad [1] = A_2 + \frac{1}{3}(2,-1,-1), \quad [2] = A_2 + \frac{1}{3}(-2,1,1)$$

of A_2. The set of deep holes in any one of these cosets is the union of the other two cosets, and the covering radius is $\sqrt{\frac{2}{3}}$. Each of the cosets [0], [1], [2] and their union A_2^* is a version of the hexagonal lattice. This is, of course, precisely the illustration shown earlier in Figure 6.

Proposition 3 (using Hypotheses 2 and 3) *The "best" packings in* $\mathbf{R}^3$ *are the Barlow packings.*

We already have given a rather informal description of the Barlow packings as something like a fibering. Let's go through the process more carefully:

Proof. Let Π_3 be a best packing in R^3. Hypotheses 2 and 3 tell us that each layer in any Π_3 is a copy of A_2 and that the separation between any two layers is at least $\sqrt{2-2/3} = \sqrt{4/3}$. Now suppose adjacent layers are always at this distance, and choose any particular layer to be $[0] = A_2$. Then the next layer must be a copy of A_2 contained in $[1] \cup [2]$. However, since squared distances in A_2 are even integers, the next layer cannot contain points from both [1] and [2] (the squared distances between which have the form $(2n+2/3)$ and so must be all of [1] or all of [2], say [1]. The layer after this must be [0] or [2], etc. This is of course the way we described the Barlow lattices earlier. □

Now we use the hypotheses to make short work of the next several dimensions:

What are all the "best" packings in $\mathbf{R}^4$?

Proposition 4 (using Hypotheses 2 and 4) *The unique best packing in* $\mathbf{R}^4$ *is the root lattice* D_4

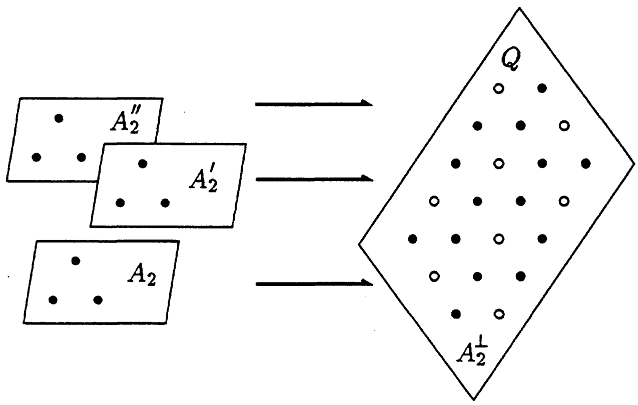


FIGURE 14. In the proof of Proposition 4, both fiber and quotient are two-dimensional

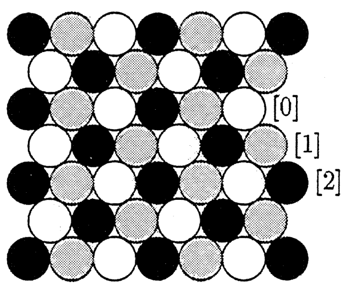


FIGURE 15. The unique three coloring of the hexagonal lattice packing

Recall that the points of D_n are the points in $\mathbf{Z}^n$ with the sum of the coordinates even.

Proof. Here both the layers and the quotient space are two-dimensional (Figure 14, and any best packing must come from a three-coloring of the two-dimensional packing Q, a rescaled version of A_2. However, this three-coloring is unique (Fig. 15) and is completely determined by the colors of any three mutually adjacent circles. The resulting lattice is D_4. □

The dual lattice D_4^* consists of four cosets

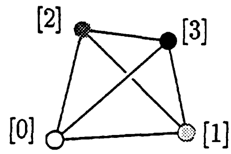
$$[0] = D_4, \quad [1] = D_4 + \frac{1}{2}(1,1,1,1), \quad [2] = D_4 + (0,0,0,1)$$
$$[3] = D_4 + \frac{1}{2}(1,1,1,-1)$$

and the set of deep holes in any one of these cosets is the union of the other three. The covering radius is 1.

What are all the "best" packings in $\mathbf{R}^5$?

Proposition 5 (using Hypotheses 2, 4 and 5) *The best packings in $\mathbf{R}^5$ are parametrized by paths along the edges of a tetrahedron. Just four of them are uniform packings, and just one of these is a lattice packing.*

Proof. The usual argument shows that any two layers must be separated by at least $\sqrt{2-1} = 1$, and that if two layers D_4 and D_4' are separated by just this distance, the points of D_4' must lie above the deep holes of D_4. In other words, D_4' projects into the union of the cosets [1], [2] and [3]. As before, this image cannot contain points from more than one of these cosets and so must be a single coset [1], [2] or [3]. So this time, we have four possible placements for each layer, and once again the only restriction is that we may not use the same placement twice in succession. The placement graph is thus a tetrahedron, 4_3:



Any "best" five-dimensional packing is given by a sequence of placements

$$\ldots abcde \ldots$$

which corresponds to a doubly infinite path along the placement graph.

For a uniform packing, there are several cases. First, if only two placements are used, say [0] and [1], we have the root lattice D_5 corresponding to the coloring

Λ_1^5 

The corresponding path, of course, winds back and forth across a single edge of the tetrahedron (shown at left in Figure 16). D_5 has quite a bit of symmetry and can be decomposed into layers in several such ways.

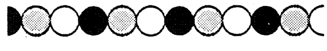
In any other uniform packing any three adjacent latters must be distinct. Moreover, the division into layers is characteristic: adjacent spheres A and B are in the same layer just if there is a third sphere C touching B antipodally to A.

We now consider the spheres within the second crystal ball of a given sphere, in layer c, say, among the five consecutive layers a, b, c, d, e, and we can ask the two questions

$$\text{Is } a = d? \qquad \text{Is } b = e?$$

The set of answers must be the same for every sphere and determines the packing.

Λ_2^5 NN 

Λ_3^5 NN 

Λ_4^5 NN 

In Figure 16 we see the paths in 4_3 that give rise to these four uniform best packings in $\mathbf{R}^5$.

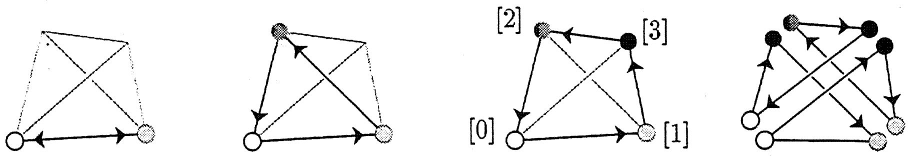


FIGURE 16. Paths in 4_3 giving rise to uniform best packings in $\mathbf{R}^5$

Thus there are four uniform packings, first found by Leech [**Le67**]. It turns out that Λ_5^n consists of n translates of a lattice packing $(n = 1, 2, 3, 4)$ □.

What are all the "best" packings in $\mathbf{R}^6$?

Proposition 6 (using Hypotheses 2, 4 and 6) *The best packings in* $\mathbf{R}^6$ *are parametrized by 4-colorings of the* A_2 *lattice. Just four of them are uniform.*

Proof. The main assertion goes as before.

Let us consider the the colors of six neighbors of a circle colored d in A_2. There are essentially just four possibilities for the colors:

abcabc
abacbc
ababab
ababac

For a uniform packing, every circle in the A_2 quotient must be surrounded in the same way as every other. It happens that each of the four types gives a unique uniform coloring, as shown in Figure 17, the first being the root lattice $\Lambda_6 = E_6$. Once again, the packing Λ_6^n consists of n translates of a lattice $(n = 1, 2, 3, 4)$. □

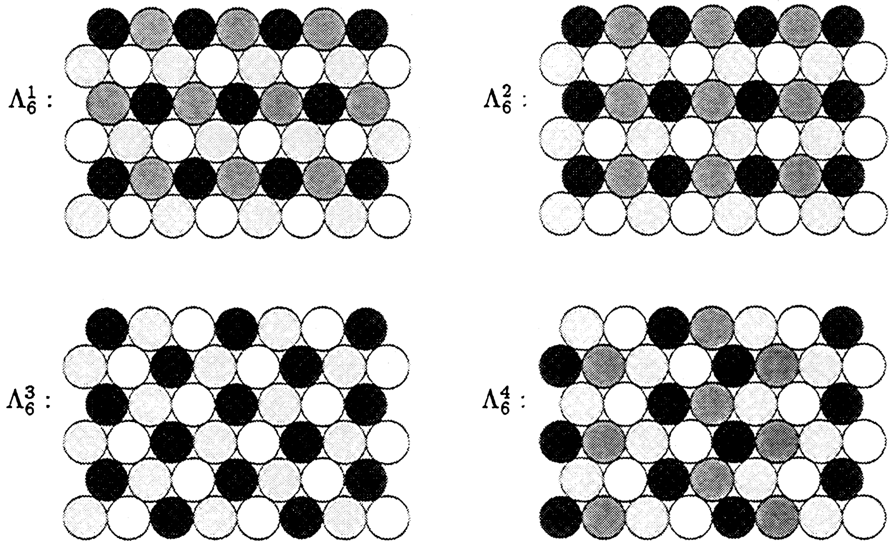


FIGURE 17. The four colorings that give the uniform best packings in $\mathbf{R}^6$

What are all the "best" packings in $\mathbf{R}^7$?

Proposition 7 (using Hypotheses 2, 3, 4 and 5) *The best packings in $\mathbf{R}^7$ fiber over D_5 and can be parametrized by choosing a Barlow packing and a "period 2 coloring" of one of its A_2 layers (as defined below). Alternatively, such a packing is specified by an ordered pair consisting of a doubly infinite path on a triangle 3_2 and one on a square 4_2. Just four of these packings are uniform.*

Proof. In the usual way, we find that any best packing Π_7 is determined by a best packing Π_3 in $\mathbf{R}^3$ and a 4-coloring of it. But here we have a new feature: there are infinitely many choices for Π_3.

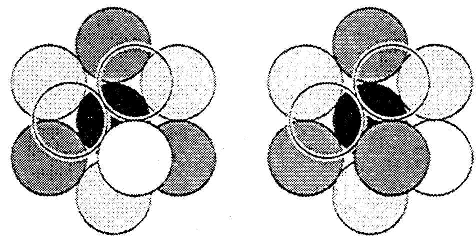


FIGURE 18. Two possible colorings of a neighborhood can be ruled out ...

However, in compensation, we find that any 4-coloring of any possible Π_3 is completely determined by its restriction to any layer (each sphere of the next layer touches three spheres in this one and so must be of the unique remaining color.)

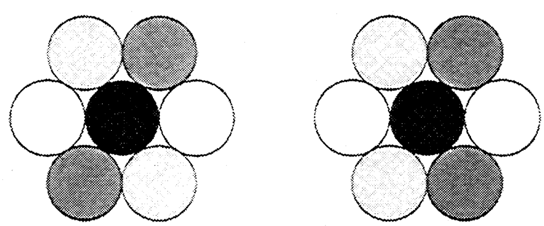


FIGURE 19. ... leaving two possibilities

Of the four ways to surround a sphere in a 4-coloring of the A_2 packing which form these layers, we find that we cannot use the cases *ababab* and *ababac*, since in each case, either choice for the next layer forces two contiguous spheres to have the same color (Figure 18). We have therefore shown that among the six neighbors of a sphere in any layer A_2 of Π_3, some pair of antipodal points has the same color, since the other two cases *abcabc* and *abacbc* have this property (Figure 19).

We next deduce that:

the A_2 can be decomposed into parallel A_1's, each of which uses just two colors.

For if every sphere in A_2 is surrounded as at left in Figure 19 then the coloring has period 2 in all directions. If any one sphere is surrounded as at right in Figure 19 then we ask: what is the pair of like-colored antipodal spheres around the white

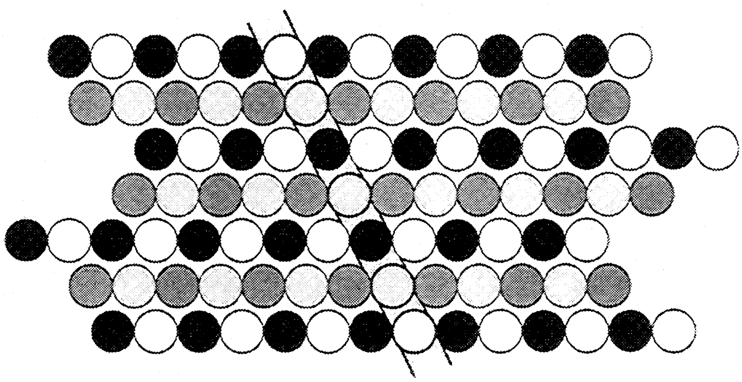
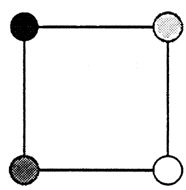


FIGURE 20. One may slide the rows; the coloring is parametrized by the colors along a strip

sphere at the right in Figure 19? The only possibility is that one of these spheres is the central black sphere. Repeating this argument, we obtain Figure 20, in which each row alternates between two of the colors.

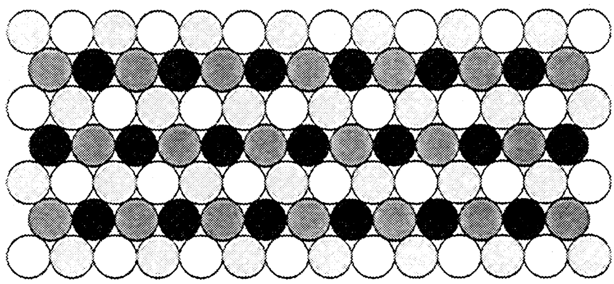
The most general Π_7 is therefore obtained as follows: We first choose any Barlow lattice Π_3. As described before this corresponds to a choice of doubly infinite path in a triangle 3_2. We next color one row of an A_2 layer in this Π_3 with two of the four colors, say 0 and 2. We then color the next row 1 and 3, the next row after that 0 and 2, and so forth. The coloring of this layer, and consequently that of our Π_3 itself, is completely specified by a doubly infinite path on the square graph 4_2



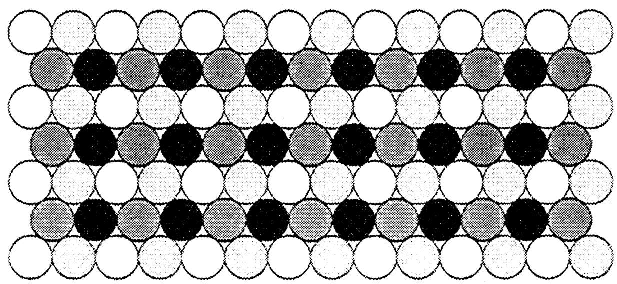
found from a sloping line such as that indicated in Figure 20. So altogether, as promised, our packing Π_7 is specified by a pair of doubly infinite paths, one in 3_2, the other in 4_2.

Moreover, since each horizontal line in our diagrams represents a translate of D_5, all such packings fiber over D_5.

There are just four cases that give uniform packings. We can extend the coloring



to either the face centered cubic lattice packing (giving Λ_7^1) or the hexagonal close packing (Λ_7^2), or the coloring



to the face centered cubic lattice packing (Λ_7^3) or the hexagonal close packing (Λ_7^4).

The first of these is the root lattice $\Lambda_7 = E_7$ and for $n = 1, 2, 3, 4$ the packing Λ_7^n consists of 1,2,2,4 translates of a lattice. □

What are all the "best" packings in $\mathbf{R}^8$?

Proposition 8 (using Hypotheses 2, 4 and 8) *The best packing in* $\mathbf{R}^8$ *is unique, and is the root lattice* $\Lambda_8 = E_8$

Proof. A best packing Π_8 in $\mathbf{R}^8$ is determined by a 4-coloring of a (rescaled) Π_4, equivalently, of the lattice D_4^*. We show that this coloring is unique, and in fact assigns each vector of D_4^* to the coset of D_4 it determines.

Consider the spheres of D_4^* centered at the points shown in the table. We start by arbitrarily assigning colors $0, 1, 2$ to the three contiguous spheres A, B, C. Then D must be colored 3 since it touches each of these. In a similar way, we determine the colors of all the remaining spheres in the table. It follows that this (possibly partial) coloring is invariant under permutations of the coordinates. This is because any such permutation fixes A and B, and either fixes C or takes it to another sphere like E which has been assigned the same color.

The spheres mentioned in the table, and their permutations, show that the coloring is also invariant under changing the signs of any even number of coordinates. It is similarly invariant under the subtraction of $(1, 1, 0, 0)$. Since the images of

TABLE 1. The unique 4-coloring of D_4^*

Sphere	Coordinates	Color	Touches
A	$0\ 0\ 0\ 0$	0	
B	$\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$	1	A
C	$0\ 0\ 0\ 1$	2	A, B
D	$-\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$	3	A, B, C
E	$0\ 0\ 1\ 0$	2	A, B, D
F	$-\frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$	1	A, C, D
G	$-\frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}$	3	A, C, F
H	$-1\ 0\ 0\ 0$	2	A, D, F
I	$-\frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}$	1	A, G, H
J	$-1\ -1\ 0\ 0$	0	G, H, I

$(1, 1, 0, 0)$ under permutations and even sign changes generate D^4, all points of any coset of D_4 are assigned the same color, and hence the coloring and the packing are unique. □

What are all the "best" packings in $\mathbf{R}^9$?

It is astonishing that although Blichfeldt completed the solution of the lattice version of the sphere packing problem in dimensions up to eight before 1930 (cf. **[CS99]**), the intervening sixty years have seen essentially no progress on the nine-dimensional problem. So it is only to be expected that the non-lattice problem will have new features in nine dimensions. We now discuss more and more surprising putatively "best" packings in nine dimensions.

Needless to say, we are nearing the end of the usefulness of our Hypotheses. But for the moment, we suppose the truth of Hypothesis 9, and use also Proposition 8, which depended in turn on Hypotheses 2, 4 and 8.

Translation

It is known **[CS99]** that the deep holes in E_8 are members of the lattice $\frac{1}{2}E_8$. It follows that if all the layers of a best packing Π_9 differ by translations, then all the corresponding placements correspond to members of $\frac{1}{2}E_8$. However, since shifting by a member of E_8 has no effect, it is better to regard the placements as members of the quotient $\frac{1}{2}E_8/E_8$.

We recall from **[CS99]** the structure of this group. There are $2^8 = 256$ cosets of E_8 in $\frac{1}{2}E_8$, and the shortest vectors in a coset are:

(i) the zero vector (1 coset),
(ii) $\pm\frac{1}{2}u$ (120 cosets),
(iii) $\pm\frac{1}{2}v_1 \pm \ldots \pm \frac{1}{2}v_8$ (135 cosets),

where $u \in E_8$ is a norm 2 vector and $v_1, \ldots, v_8 \in E_8$ are mutually orthogonal norm 4 vectors that are congruent modulo $2E_8$. The deep holes correspond to the 135 type (iii) cosets. The placement graph for this problem, therefore, has as nodes these 256 cosets, and two nodes are joined by an edge whenever the difference of the corresponding cosets is of type (iii). This does indeed have valence 135. We have thus proved:

Proposition 9A. *The best packings in* $\mathbf{R}^9$ *in which the layers are all translates of one another are parametrized by doubly infinite paths on the above graph* 256_{135}.

Rotation

If we allow adjacent layers to be related by rotations as well as translations, there are many more possibilities. To find out how many, we first consider the relationship between two adjacent layers E and F. We suppose that E is the usual E_8 defined with respect to the standard basis $e_1, \ldots, e_8$.

Each sphere of F lies above a deep hole v of E and will touch 16 spheres of E. If we take $v = e_1 = 10000000$, these will be the E spheres centered at

$$v \pm e_1, \ldots, v \pm e_8,$$

and we remark that E contains all the vectors

$$z_1 e_1 + \ldots + z_8 e_8$$

for which the z_i are integers with even sum, and also all vectors

$$\pm\frac{1}{2}e_1 \pm \ldots \pm \frac{1}{2}e_8$$

for which the number of minus signs is even.

However, the relation between E and F is symmetrical! There will therefore be 16 F-spheres

$$\pm f_1 \pm \ldots \pm f_8$$

touching the E-sphere centered at the origin, and F will contain all the vectors

$$v + z_1 f_1 + \ldots z_8 f_8$$

for which the z_i are integers with even sum, and also either all vectors

$$\pm\frac{1}{2}f_1 \pm \ldots \pm \frac{1}{2}f_8$$

for which the number of minus signs is even, or all those for which this number is odd. The vectors $\pm f_1, \ldots, \pm f_8$ must be an orthonormal basis of deep holes in E, one of which is the particular vector 10000000. Up to symmetries of E_8 it turns out that there are just four possibilities for the doubled vectors $2f_1, .., 2f_8$, as shown in Table 2.

TABLE 2. All possibilities for orthonormal bases $\pm f_1, \ldots, \pm f_8$ of deep holes in E_8 containing e_1. + and − denote +1 and −1, and parentheses indicate that all cyclic permutations of the enclosed coordinates are to be applied. [The cases are named after the codes obtained by reducing their coordinates modulo 2 (see [**CP80**]).]

case	$2f_i$	group	index
d_0	(20000000)	$2^7 : S_7$	$2^0.1 = 1$
d_4	(2000) 0 0 0 0 0000(−+++)	$2^6 : S_3 \times S_4$	$2^1.35 = 70$
d_6	(20) 0 0 0 0 0 0 00(−+0++0)	$2^4 : 2^3 S_3$	$2^3.105 = 840$
e_7	(2) 0 0 0 0 0 0 0 0(−++0+00)	$2^7 : PSL_3(2)$	$2^6.30 = 1920$

Each of these cases leads to a unique choice for F, since F may not contain the vector $\frac{1}{2}f_1 + \ldots + \frac{1}{2}f_8 = \frac{1}{2}e_1 + \ldots + \frac{1}{2}e_8$, which is already in E. We omit the arguments proving that these cases survive, and that the list is complete. We next compute the number of possibilities for F. For each case the table shows the structure of the group that fixes or negates the leading vector 20000000, and its index in the subgroup $2^7 : S_7$ of all automorphisms of E that do this. We conclude that the total number of such frames that contain the vector 20000000 is

$$1 + 70 + 840 + 1920 = 2831$$

and so the total number of choices for F is

$$2831 \cdot \frac{2160}{16} = 382185$$

since there are 2160 norm 4 vectors we could use in place of 20000000 and each of the frames contains 16 such vectors. From this we obtain:

Lemma 9B. *The best packings in $\mathbf{R}^9$ whose layers differ by arbitrary rotations and translalions are parametrized by doubly infinite paths on a placement graph of the form* ∞_{382185}.

The reason there are infinitely many possible placements is really that the automorphism group of E_8 is a maximal finite subgroup of the eight-dimensional orthogonal group, and two placements may differ by a rotation not in this subgroup. We can for instance find a best packing of $\mathbf{R}^9$ in which every pair of alternate layers is related by the matrix

$$\frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & - & + & + & 0 & + & 0 & 0 \\ 0 & 0 & - & + & + & 0 & + & 0 \\ 0 & 0 & 0 & - & + & + & 0 & + \\ 0 & + & 0 & 0 & - & + & + & 0 \\ 0 & 0 & + & 0 & 0 & - & + & + \\ 0 & + & 0 & + & 0 & 0 & - & + \\ 0 & + & + & 0 & + & 0 & 0 & - \end{pmatrix}$$

(corresponding to case e_7), whose characteristic polynomial

$$(X-1)^2(X^2+\frac{3}{2}X+1)^3$$

shows it to be of infinite order. Since there are infinitely many placements but just 382,185 possibilities for a neighbor of a given one, the placement graph does indeed have type ∞_{382185}.

Flotation

It seems that of all these packings, only the lattice packing $\Lambda_9^1 = \Lambda_9$, was previously known. This was first described by Korkine and Zolotareff **[KZ73]** in 1873 and is their packing T_9 **[CS99]**. We describe it in some detail, because it can be modified in interesting ways.

Λ_9 consists of the vectors

$$(x_1, \dots, x_8, 2n) \quad \text{and} \quad (x_1, \dots, x_8 + 1, 2n+1),$$

where $(x_1, \dots, x_8)$ is a typical vector of E_8 in the standard coordinate system, and n is any integer.

However, we can look at these vectors in another way! The ones with integral coordinates constitute the lattice D_9, and so

$$\Lambda_9 = D_9 \cup (D_9 + v), \quad \text{where } v = \left((\frac{1}{2})^8, 0\right)$$

This leads us to ask:
Are there other vectors w for which the points of the "fluid diamond packing"

$$D_9(w) = D_9 \cup (D_9 + w)$$

have minimal distance $\sqrt{2}$?

Lemma 9C. *The answer to the above question is "Yes"! In fact there are uncountably many choices for w!*

The condition is just that the squared distance of w from the nearest point of D_9 should be at least 2, and this is achieved for instance if any eight of the coordinates of w are halves of odd integers, the ninth then being completely free. Let us write

$$D_9^{\theta+} = D_9 + \left((\frac{1}{2})^8, \frac{1}{2}\theta\right).$$

Then $D_9^{0+} = \Lambda_9$ and $D_9^{1+} = D_9^+$ is the nine dimensional diamond packing **[CS99]**.

Gold and Silver Among the Diamonds. We say that a sphere in a fluid diamond packing $D_9(w)$ is "golden" if its center is in D_9, and "silver" if its center is at a point of $D_9 + w$. Then if the squared distance of w from D_9 is strictly greater than 2, no silver sphere touches any golden one. So the packing Λ_9 has the remarkable flotation property described in the following theorem.

Flotation Theorem. *Let X and Y be two silver spheres in Λ_9. Keeping the golden spheres fixed, the silver spheres can be collectively moved so that the final position of X is the original position of Y. Usually the silver spheres do not touch the golden ones, and at no time do any two spheres ouerlap. At every stage in the motion the packing is of type $D_9^{\theta+}$ for $0 \leq \theta \leq 1$.*

Proof. Let $(x_1, \ldots, x_9)$ and $(y_1, \ldots, y_9)$ be the centers of X and Y.Then x_9 and y_9 are integers, but the other coordinates x_i and y_i are halves of odd integers. The desired motion is performed in 10 stages. In stage 0 we fix $x_1, \ldots, x_8$ and increase x_9 smoothly by $\frac{1}{2}$. Then at stage n, $1 \leq n \leq 9$, we move the nth coordinate to y_n keeping the others fixed. At all times in this motion, eight of the coordinates are halves of odd integers, and so the spheres at no time overlap. Only at the instants when some coordinate of w is an integer does any silver sphere touch a golden one. □

Unfortunately the existence of these "floating packings" violates our Hypothesis 9. We therefore simply weaken that Hypothesis to:

Hypothesis 9*. *Every best nine-dimensional packing* either *fibers over* $\mathrm{II}_8 = E_8$, or *is a fluid diamond packing.*

This is not quite so despicable as it seems, since at least Hypothesis 9* entails that every tight nine-dimensional packing is *isotopic* to one that fibers over II_8. (Two best packings of spheres of a given radius are said to be *isotopic* if one can be continuously deformed into the other through best packings of spheres of the same radius.)

We summarize our beliefs in:

Proposition 9 (depends on Hypotheses 2, 4, 8, and 9*) *The best nine-dimensional packings are of two kinds:*

(i) Packings containing E_8, which are parametrized by random walks on the graph ∞_{382185}.

(ii) FIuid diamond packings $D_9(w)$, parametrized by the vector w.

There is an uncountably infinite collection of uniform packings.

Proof. Only the last assertion remains to be proved. The fluid diamond packing $D_9(w)$ is seen to be uniform because it has the symmetries

$$x \mapsto v + x$$

$$x \mapsto w + v - x$$

for every $v \in D_9$, and these act transitively on the spheres. □

Our real reason for believing this proposition is not that we find Hypothesis 9* inherently plausible (indeed, the flotation property initially made it seem extremely implausible), but rather that we have tried and failed many times to produce any other packings which are at least as good as those described there. The reader who finds our arguments unconvincing is invited to produce a putatively best packing not covered by the proposition!

What are the "best" packings in $\mathbf{R}^{10}$

Now we have come to the end of the useful life of the Hypotheses. It is not hard to find the best packings in dimensions $8+n$, $0 \leq n \leq 8$ that *do* fiber over E_8. They are parametrized by what we might call∞_{382185}-colorings of the best packings Π_n. In other words, the "color" assigned to each sphere of Π_n, is a node of the placement graph ∞_{382185} of Lemma 9B, and adjacent spheres must be colored by nodes that are adjacent in this graph. However, one such packing is the laminated lattice Λ_{10} whose density is strictly exceeded by the 10-dimensional packing found by Malcolm R. Best and briefly described below.

So we have:

Proposition 10 (depends on Hypotheses 2, 4, and 8) *Hypothesis 10 is false, even upon being weakened up to isotopy. The best known packing in* $\mathbf{R}^{10}$ *is due to Best.*

Best's packing is a uniform packing, for which Conway and Sloane have recently given a very simple construction [CS94].

The Pentacode consists of all cyclic shifts of the four vectors

$$01112, 03110, 21310, 21132,$$

and their negatives, where the digits are integers modulo 4. (These eight words are all the words of the form $c-d, b, c, d, b+c$, where b, c, d are odd.) We obtain the centers of Best's 10-dimensional packing by replacing each digit of a word of the pentacode by two integers according to the following scheme:

0 → even, even
1 → even, odd
2 → odd, odd
3 → odd, even

Our assertions about this packing follow from [CS94].

Proposition 10 suggests that an appropriate modification of Hypothesis 10 might be:

Hypothesis 10* ("Best is best"). *Best's packing is the unique best packing in 10 dimensions.*

The authors are of $2 + \epsilon$ minds about the possible truth of this hypothesis.

Hypotheses 11, 12 and 13 fail as dramatically as Hypothesis 10 does. However, there may be some truth to Hypotheses 14, 15 and 16, and so some value in understanding the packings Π_{8+n} that do fiber over E_8.

Unfortunately the description in terms of ∞_{382185}-colorings of Π_n, does not make it clear *a priori* that there is more than one such Π_{8+n} However, we remark that the graph ∞_{382185}, and indeed its subgraph 256_{135} contains a copy of the complete graph 16_{15} on 16 points. This is because E_8 can be embedded in a scaled copy E_8^+ of itself having *half* the minimal norm [**CS99**]. Then E_8^+ consists of 16 cosets of E_8,and we can take those to be the desired 16 placements, since any two of them differ by a deep hole vector. This shows us that there is a particular packing Π_{8+n} corresponding to any 16-coloring of any tight packing Π_n.

We estimate that this method gives us more than 10^7 distinct, putatively best, 16-dimensional packings having the same density as the Barnes-Wall lattice Λ_{16}. This lattice Λ_{16} itself is one of these packings. It corresponds to a 16-coloring of E_8 in which the colors correspond to the cosets in E_8 of a sublattice E_8^- that is a scaled copy of E_8 at *twice* the minimal norm. The automorphisms of E_8 permute these 16 colors in just $|16.GL_4(2)| = 8.8!$ ways. So by applying all 16! color permutations we may expect to obtain at least $16!/8.8! > 10^7$ best packings Π_{16}.

Best Known Packings in Dimensions Greater Than 10

In higher dimensions, there is little reason to suspect that the best known packings are the best possible packings. We offer only one more hypothesis, Hypothesis 12* below.

For the rest of this paper we will take a quick look at some of the current best known packings in higher dimensions.

The Laminated Lattice Packings

Most of the best known packings fall into this sequence, which is defined inductively. The n-dimensional laminated lattices having a given minimum being supposed to be known, one obtains those of dimension $n + 1$ by selecting the densest lattices of this dimension that contain an n-dimensional laminated lattice and have the same minimum. So for example, by laminating $\Lambda_1 = A_1$, we have the hexagonal lattice $\Lambda_2 = A_2$; laminating this we have the face-centered cubic lattice $\Lambda_3 = A_3$, and so on.

The particular cases are easiest discussed in terms of the Leech Lattice, whose discovery revolutionized this subject in 1969, so we'll define that first.

The Leech Lattice consists of the vectors in $\mathbf{R}^{24}$ whose coordinates are all congruent modulo 2, say to m, and have sum congruent to modulo 8 to $4m$, while the set of coordinates taking a given value modulo 4 is the support of a word in the binary Golay (24, 12, 8) code, for which see page 84 of [**CS99**].

We shall describe the laminated lattices of dimension up to 24 as sections of the Leech Lattice, supposing that the coordinates of the latter are grouped in a "sextet" of six tetrads the union of any two of which is a special octad—the support of a Golay codeword of weight 8. Then according to the dimension we take the vectors of the Leech Lattice that are supported on suitable subsets of the coordinates and satisfy the following conditions:

$\Lambda_1 \cong \mathbf{Z} \cong A_1$: two coordinates of the first tetrad whose sum is zero.
$\Lambda_2 \cong A_2$: three coordinates of the first tetrad whose sum is zero.
$\Lambda_3 \cong A_3 \cong D_3$: three coordinates of the first tetrad.
$\Lambda_4 \cong D_4$: the first tetrad.
$\Lambda_5 \cong D_5$: first and second tetrads, the 4 coordinates of the latter being equal.
$\Lambda_6 \cong E_6$: first and second tetrads, the last 3 coordinates of the latter being equal.
$\Lambda_7 \cong E_7$: first and second tetrads, the last 2 coordinates of the latter being equal.
$\Lambda_8 \cong E_8$: the first and second tetrads.
Λ_9: first three tetrads, the coordinates of the third being equal.
Λ_{10}: first three tetrads, the last 3 coordinates of the third being equal.
Λ_{11}: first three tetrads, the last 2 coordinates of the third being equal.
Λ_{12}: the first three tetrads.
Λ_{13}: first three tetrads, and two coordinates of the fourth having zero sum.
Λ_{14}: first three tetrads, and three coordinates of the fourth having zero sum.
Λ_{15}: first three tetrads, and three coordinates of the fourth.
Λ_{16}: the first four tetrads.
Λ_{17}: first four tetrads, and two coordinates of the fourth having zero sum.
Λ_{18}: first four tetrads, and three coordinates of the fourth having zero sum.
Λ_{19}: first four tetrads, and three coordinates of the fourth.
Λ_{20}: the first five tetrads.
Λ_{21}: all six tetrads, the 4 coordinates of the sixth being equal.
Λ_{22}: all six tetrads, the last 3 coordinates of the sixth being equal.
Λ_{23}: all six tetrads, the last 2 coordinates of the sixth being equal.
Λ_{24}: all six tetrads.

This is all summarized neatly in the Figure 21, reproduced from [**CS99**].

We should mention that in dimension 11 there are two distinct laminated lattices, and in dimension 12 and 13, three: the ones described above are those with maximal kissing number. However, the laminated lattices in all other dimensions below 25 are unique. The sequence has been continued to dimension 48, there being known to be just 23 distinct laminated lattices of dimension 25, and conjecturally many millions in dimension 26 and higher.

The alternative "Kappa" sequence is summarized in the Figure 22 reproduced from [**CS99**]. This sequence diverges from the above "Lambda" sequence in dimensions 6 through 18.

The best lattice packings known in dimensions 1 through 29 are all among those in these two sequences, as is summarized in the following chart reproduced from [**CS99**]. Note that the Kappa sequence provides denser packings only in dimensions 11, 12, and 13.

Though we cannot know the final status of the question of which packings are best in general, we feel relatively at ease proposing the following hypothesis:

Hypothesis 12* *The laminated lattice* K_{12} *is the best packing in* $\mathbf{R}^{12}$.

The History of the Laminated Lattices.

Of course the sphere packings determined by the laminated lattices in dimensions up to 3 have long been familiar, even to greengrocers. Several proofs of the optimality of the hexagonal lattice in the 2-dimensional case were offered in the nineteenth century. They are now superceded by that of Fejes Toth, which also shows that this lattice is optimal in many other senses.

The 3-dimensional laminated lattice in the face-centered cubic lattice. The question of proving its optimality is attributed to Kepler since he mentioned it in his little book "The Six-Fingered Snowflake" of 1613. In 1831 Gauss established (in a book review!) the optimality among 3-dimensional lattices, and the work of Hales on which we have already commented seems at last to have established this optimality among all 3-dimensional packings.

The St. Petersburg school of mathematicians founded by Chebyshev has had a great influence on this subject. For example, the laminated lattices in dimensions up to 9 were discovered and conjectured to be optimal by Khorkine and Zolotarev before 1873, and they and Voronoi established their optimality among lattices in dimensions up to 5. The analogous proof for dimensions 6,7 and 8 was given by Blichfeldt in the 20s and later simplified by Mordell and Vetchinkin. This is still the boundary of our positive knowledge in these matters, although in 1946 Chaundy gave a purported proof that two particular lattices were the optimal lattices in dimensions 9 and 10.

What Chaundy effectively did was to establish that the two lattices he discussed were in fact the 9- and 10-dimensional laminated lattices. In 1982, Conway and Sloane gave the formal definition of this notion of laminated lattice, and worked out all such lattices in dimensions up to 25, and the densities of all laminated lattices in dimensions up to 48. It would be incredibly difficult to continue the sequence beyond this dimension, and in any case there is little point in doing so, because denser lattices than the laminated ones are now known in the dimensions from 30 upwards.

Denser Packings than the Laminated ones.

The conjecturally "best" packings described in the first portion of this paper have the same densities as the those derived from the laminated lattices (of which they are generalizations) in all dimensions up to 9. The smallest dimension in which the laminated lattices can be beaten is 10, where they are beaten by Best's packing. This is also the smallest dimension in which a non-lattice packing appears to be optimal.

In dimensions 11,12,13 the Kappa sequence gives the best known lattices, but there are better non-lattice packings in dimensions 11 and 13, already known to the first edition of *Sphere Packings, Lattices and Groups* [**CS99**]. In all dimensions from 14 to 25 the laminated lattices (also in that book) still hold the record among lattice packings.

However, there are new, non-lattice, packings in dimensions 18 (Bierbrauer and Edel, 1998), 20 (Vardy, 1995), and 22 (Conway and Sloane, 1996). The nicest of these is the 20-dimensional packing found by Alexander Vardy in 1995. This came as a great surprise since we had considered improvements near 24 dimensions very unlikely because the Leech lattice is so spectacularly efficient there. It uses what Conway and Sloane called "the Antipode Construction", analogous to the better known "Anticode Construction" of coding theory.

We obtain Vardy's 20-dimensional packing as follows. In the Leech lattice, take all the vectors that, projected onto the 4-space spanned by the first tetrad of coordinates, fall onto one of the vertices of a regular tetrahedron of the shortest possible edge-length. Then project these vectors instead onto the orthogonally complementary 20-space. The result is not a lattice, because the sum of two vectors of the tetrahedron is not another, but it is in fact the union of four sublattices that project onto the different vertices of the tetrahedron. Conway and Sloane find its group, which is indeed obtained by projecting all the symmetries of the Leech lattice that fix the tetrahedron (as a whole) onto its orthogonal space.

They also discussed the analogous packing in 22 dimensions, where the tetrahedron in 4-space is replaced by an equilateral triangle in 2-space. It has similar properties; in particular its group is obtained from that of the Leech lattice by a rule like the one above. More recently, by a different method, Bierbrauer and Edel have found a non-lattice packing that beats the laminated lattice of dimension 18—this being the only other improvement on the first edition of *Sphere Packings, Lattices and Groups* in 25 dimensions or below. There have also been particular improvements on the upper bounds in many dimensions beyond 25, for which we refer the reader to the third edition of that book.

"Stop Press" news about lower bounds

The only new "final" result is the value of the optimal density in the 3-dimensional case. However, Henry Cohn and Noam Elkies have made some equally exciting new progress, which might soon yield the answers in 8 and 24 dimensions [**Co00**].

Some time ago, Odlyzko, Bannai and Sloane [**OdSl79**],[**BaSl81**] solved the kissing number problem in those dimensions, using an analytic method developed by Levenshtein [**Le79**]. (The only other dimensions for which that problem was solved previously were 0,1,2,3.) Cohn and Elkies' method gives a lower bound that depends on a certain function whose best value is to be determined. This function will be a linear combination of certain "radial Fourier eigenfunctions" $g_1, g_3, \dots$, and Cohn and Elkies' new bounds are achieved by finding the best combinations of the first $2n$ of these. It appears that they improve on the Rogers bound in all dimensions above 2 (where the Rogers bound is the exact answer). It also seems that in the dimensions 8 and 24 the appropriate linear combination of all the g_n will establish the optimality of the E_8 lattice and the Leech lattice, since the bounds obtained by taking larger and larger finite numbers of the g_n appear to be tending to the correct limits.

If so, all that will be required to establish these two answers will be the "divination" of the function corresponding to the correct infinite linear combination of the g_n, and the verification that it satisfies certain inequalities. So we might soon have a solution in 8 and 24 dimensions that is very much shorter than the one in 3 dimensions! But since this depends on somebody's having a bright idea it is hard to know just how long it will take.

All our other so-called "progress" is at the level of conjecture. But we can now say that we *think* we know all the "best" packings in dimensions 1-10, and although this is merely a conjectural answer to an undefined question, this is a kind of "knowledge" that we didn't previously have!

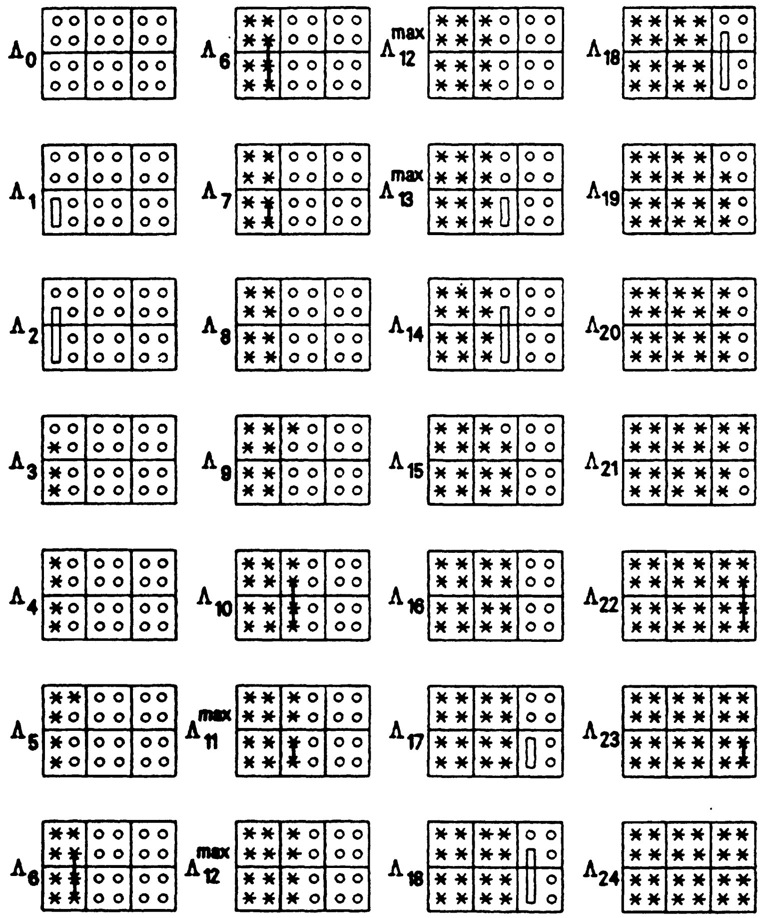


FIGURE 21. Coordinates for $\Lambda_0, \ldots, \Lambda_n$ as sections of the Leech Lattice. A small circle represents a zero coordinate, a hollow loop is a set of coordinates adding to zero, and asterisk is a free coordinate, and a line of asterisks is a set of equal coordinates.

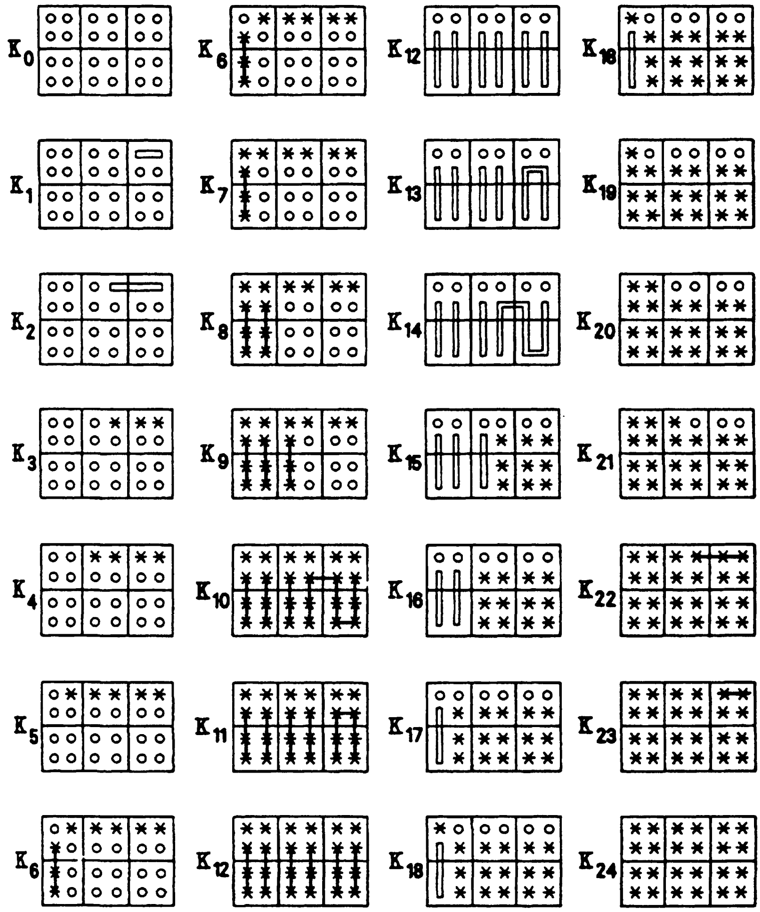


FIGURE 22. Coordinates of the lattices $K_0, \ldots, K_n$ as sections of the Leech Lattice.

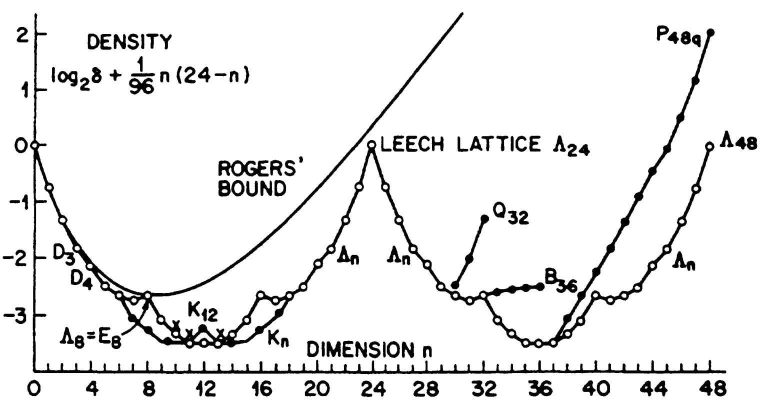


FIGURE 23. The densest sphere packings known in dimensions $n \leq 48$. See [CS99].

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