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A report on the proof of the Langlands conjectures for $GL(N)$ over p-adic fields

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In November 1999, I was invited to Cambridge, Mass. to give a talk at the conference on "Current developments in Mathematics". I lectured then on the work of Michael Harris and Richard Taylor [Ha T] on Shimura varieties, which provided in particular a proof of the Langlands conjectures for $\mathrm{GL}(N)$ over p-adic fields. I had written a preconference report, essentially a written account of the intended talk : you will find it below. Then it was my duty to write a more extensive report for the definitive proceedings. However there is already a very nice review by H. Carayol at the Bourbaki seminar [Ca]. It gives a rather extensive exposition of the geometric part of the work of Harris and Taylor, and also applications. Not being a specialist of algebraic geometry or Shimura varieties, I did not feel like duplicating, let alone emulating, Carayol. But it seemed to me that a report on the local-global principles behind the proof of the Langlands conjectures could be helpful, and I could talk there of things which are not very much explained elsewhere. Needless to say the reader should also consult [Ca] and [Ha T], especially the introduction, to get a more balanced overview. Laumon's talk at the same conference, on Lafforgue's work on the global Langlands conjectures for GL_n over function fields, is also relevant here.

I have kept the notes written for the talk as a first part A, thinking it provided a gentle general introduction to the topics to follow. On the other hand, I have taken the liberty of discussing in part B some more recent results like the proof of the Shimura-Taniyama-Weil conjecture, or the proof of the Artin conjecture for infinitely many degree 2 iscosahedral representations of the absolute Galois group of $\mathbb{Q}$.

Let me finally take the opportunity to thank the organizers of the "Current Developments in Mathematics" series for the invitation, M. Harris and R. Taylor for communications on the subject, and, above all, R.P. Langlands for inspiration.

Main references :

- [Ca] **H. CARAYOL.** — *Preuve de la conjecture de Langlands locale pour* GL_n *: travaux de Harris-Taylor et Henniart*, Sém. Bourbaki n^o 857, March 1999, Astérisque n^o 266, SMF, Paris (2000), 191-244.
- [Ha 1] **M. HARRIS.** — *Supercuspidal representations in the cohomology of Drinfeld's upper half-spaces ; elaboration of Carayol's program*, Invent. Math. 129 (1997), 75-120.
- [Ha 2] **M. HARRIS.** — *The local Langlands conjecture for* GL_n *over a p-adic field*, $n < p$, Invent. Math. 134 (1998), 177-210.
- [Ha T] **M. HARRIS, R.TAYLOR.** — *On the geometry and cohomology of some simple Shimura varieties*, Preprint n^o 227, Institut de Mathématiques de Jussieu, Paris, octobre 1999.
- [He] **G. HENNIART.** — *Une preuve simple des conjectures de Langlands pour* GL_n *sur un corps p-adique*, Invent. Math. 139 (1999), 439-455.

A) A general overview

As mentioned in the introduction, this is the account which I wrote for the conference, with only cosmetic changes. See also Carayol's report [Ca], my report at the Journées Arithmétiques [He JA] – and of course [Ha T].

A I) The main results, and the local-global principle

1. Introduction

The conjectures of Langlands that I will talk about relate two *a priori* rather different kinds of objects : Galois representations and automorphic representations. The conjectures come in two versions : global conjectures concern number fields, that is finite extensions of $\mathbb{Q}$, whereas local conjectures concern their completions, p-adic fields. In July 1998, M. Harris and R. Taylor announced a proof of the local conjectures, and I shall report on their proof. The principle is a local-global one : a weak result is obtained over some number fields, but this gives enough information for the full local result. The method is geometric : the two kinds of objects are found in the cohomology of some specially constructed algebraic varieties; as a byproduct, one gets a geometric model for the local correspondence.

There are also versions of the conjectures over fields of positive characteristics. There the local result has been known for some time, and very spectacular results on the global correspondence were recently obtained by L. Lafforgue. I refer you to Laumon's talk at this conference.

2. Galois representations

Let F be a number field, $\overline{F}$ an algebraic closure, G_F the Galois group of $\overline{F}$ over F. Galois representations are (continuous) homomorphisms $G_F \to \mathrm{Aut}(V)$ where V is some finite-dimensional vector space. There are two main sources of Galois representations.

The first comes upon taking a finite Galois extension E of F in $\overline{F}$ and representing the Galois group $\mathrm{Gal}(E/F)$ in some finite dimensional complex vector space. As $\mathrm{Gal}(E/F)$ is a quotient of G_F, one gets a **complex** representation of G_F. Since G_F is the inverse limit of such finite Galois groups $\mathrm{Gal}(E/F)$, studying those complex representations amounts to studying G_F, so to say, on the Fourier transform side. It is known often to be a good idea.

The second supply of Galois representation is geometry. Let X be a smooth projective variety over F. Then G_F acts on $X \otimes \mathrm{Spec}(\overline{F})$, and on the ℓ-adic etale cohomology groups $H^i(X \otimes \mathrm{Spec}(\overline{F}), \mathbb{Q}_\ell)$ for any prime number ℓ. Those are finite dimensional $\mathbb{Q}_\ell$-vector spaces. It is good to allow extension of coefficients to an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$, for example to decompose those cohomology spaces under the action of G_F, so one gets **ℓ-adic** representations $G_F \to \mathrm{Aut}(V)$, where V is a finite dimensional vector space over $\overline{\mathbb{Q}}_\ell$. Contrary to the complex representations above, their image is usually infinite. For example, one can take an elliptic curve over F and look at the representation of G_F on its Tate module at a prime number ℓ; this gives a 2-dimensional ℓ-adic representation of G_F.

Both types of representations – complex or ℓ-adic – share a crucial common property : they are **unramified almost everywhere**. Let me explain this; one looks at the different non-archimedean completions ("finite places") F_v of F. Choosing an extension to $\overline{F}$ of the completion yields an algebraic closure $\overline{F}_v$ of F_v and an embedding of the Galois group G_{F_v} of $\overline{F}_v$ over F_v into G_F; choosing a different extension gives a conjugate embedding. Each completion has a residue field and the residue field of F_v is a finite field, having the residue field of $\overline{F}_v$ as an algebraic closure. The group G_{F_v} has an inertia subgroup I_v, made out of the elements acting trivially on the residue field. The quotient G_{F_v}/I_v is a free profinite group with one prefered generator Frob_v, acting as $x \mapsto x^{1/q_v}$ on the residue field of $\overline{F}_v$, where q_v is the cardinality of the residue field of F_v.

Let $\Sigma : G_F \to \mathrm{Aut}(V)$ be a complex or ℓ-adic representation as above. Then for all but a finite number (we say "almost all") of finite places v of F, Σ is **unramified**, that is, trivial on I_v. The semisimplification of the restriction Σ_v of Σ to G_{F_v} is then completely characterized by the local L factor

$$L(\Sigma_v, s) = \det(1 - \Sigma(\mathrm{Frob}_v)q_v^{-s})^{-1}$$

seen, say, as a Dirichlet series.

Moreover, if Σ itself is semisimple, then it is determined by the data of $L(\Sigma_v, s)$ at almost all v's.

3. Class-field theory

Before stating the Langlands conjectures for GL_n, I have to recall the case $n = 1$, given by class field theory. One-dimensional complex representations of G_F, up to isomorphism, are simply characters (i.e. continuous homomorphisms) $G_F \to \mathbb{C}^\times$ (they have finite order). Class field theory gives a way to describe them all, purely in terms of the base field F. One needs the ring of **adeles** $\mathbb{A}_F$ of F, the subring of $\coprod F_v$ (the product is over all places v of F, including archimedean completions) made out of families (x_v) where x_v is in the ring of integers O_{F_v} for almost all v's. The group of units $\mathbb{A}_F^\times = GL_1(\mathbb{A}_F)$ carries a natural locally compact topology, and $F^\times$ sits diagonally in it as a discrete subgroup.

A character χ of $\mathbb{A}_F^\times$ is a continuous homomorphism into $\mathbb{C}^\times$. By continuity it is unramified (i.e. its restriction to $F_v^\times \subset \mathbb{A}_F^\times$ is trivial on the unit group of $F_v^\times$) for almost all finite v's; at such places its restriction to $F_v^\times$ is completely characterized by its value on a uniformizer ϖ_v of F_v.

One way to state class field theory is to say that it gives a one-to-one correspondence $\chi \mapsto \pi(\chi)$ between characters of G_F and finite order characters of $\mathbb{A}_F^\times$ **trivial on** $F^\times$. That correspondence is characterized by the fact that at almost all v's, where χ and $\pi(\chi)$ are unramified, one has

$$\pi(\chi)(\varpi_v) = \chi(\mathrm{Frob}_v).$$

An important feature is that the correspondence $\chi \leftrightarrow \pi(\chi)$ is determined by looking only at "good" unramified places. What happens at the other "bad" places is then forced, so one can hope to obtain something back at those bad places. This is indeed the case and gives local class field theory. If v is a finite place of F, there is a one-to-one correspondence $\chi \mapsto \pi(\chi)$ between characters of G_{F_v} and finite order characters of $F_v^\times$. That correspondence is the only one compatible with the global one in the sense that if χ is a character of G_F, then for any finite place v of F, $\pi(\chi_v)$ is the restriction of $\pi(\chi)$ to $F_v^\times$. But the local correspondence depends only upon F_v and not on the way we see it as a completion. This is what I call the **local-global principle of class-field theory**. We shall see it can be generalized to the context of the Langlands conjectures.

Note : If W_{F_v}, the Weil group of F_v, is the subgroup of G_{F_v} made out of elements acting on the residue field as an integral power of Frob_v, then there is a one-to-one correspondence between characters of W_{F_v} and (all) characters of $F_v^\times$. In the p-adic case it is often more convenient to work with the Weil group.

4. Automorphic representations

To generalize the above result from one-dimensional to n-dimensional Galois representations (where n is any integer ≥ 2), we need something on the other side. This is provided by **automorphic** forms and representations, which are an adelic generalization of classical modular forms and the action on them of Hecke operators.

If F is a number field as above, we can form the group $\mathrm{GL}_n(\mathbb{A}_F)$; as for $n = 1$ it carries a locally compact topology and $\mathrm{GL}_n(F)$ sits inside diagonally as a discrete subgroup. The group $\mathrm{GL}_n(\mathbb{A}_F)$ acts on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$ by right translations. Automorphic forms are complex functions on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$ with various smoothness and finiteness assumptions; the group $\mathrm{GL}_n(\mathbb{A}_F)$ acts on automorphic forms. To define the representations we are interested in, it is convenient to fix a character (unitary, say) ω of $F^\times\backslash\mathbb{A}_F^\times$ and look only at those functions where the centre of $\mathrm{GL}_n(\mathbb{A}_F^\times)$ (isomorphic to $\mathbb{A}_F^\times$) acts via multiplication by ω. The space $L^0(\omega)$ of cuspidal automorphic forms is then the space of functions on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$, transforming by ω under the centre, square integrable modulo the centre, and satisfying the cuspidality condition : for any proper parabolic subgroup P of GL_n over F with unipotent radical N, one has

$$\int f(ng)dn = 0 \text{ almost everywhere,}$$

where the integral is over $N(F)\backslash N(\mathbb{A}_F)$. (This is the generalization of the classical notion of parabolic modular form). The space $L^0(\omega)$ is the direct sum of irreducible inequivalent unitary representations of $\mathrm{GL}_n(\mathbb{A}_F)$. Such summands (for varying ω) are called **automorphic cuspidal representations** of $\mathrm{GL}_n(\mathbb{A}_F)$.

Let Π be such an automorphic cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then due to the topology of $\mathrm{GL}_n(\mathbb{A}_F)$, Π appears as a kind of completed tensor product $\otimes\Pi_v$, where for each finite place v, Π_v is a **smooth irreducible** representation of $\mathrm{GL}_n(F_v)$ i.e. a representation in some complex vector space, usually infinite dimensional, which is irreducible in the algebraic sense and verifies the strong continuity condition that each vector has an open stabilizer in $\mathrm{GL}_n(F_v)$. (At infinite places, the notion is slightly different but I shall speak abusively of smooth irreducible representations also at those places).

For almost all finite places v, Π_v is **unramified**, i.e. has a non-zero vector fixed under $\mathrm{GL}_n(\mathcal{O}_{F_v})$. Such an unramified Π_v is characterized by its L-factor $L(\Pi_v, s)$ which is of the form $P(q_v^{-s})^{-1}$, P a degree n complex polynomial such that $P(0) = 1$.

The automorphic cuspidal representation Π itself is determined by the data $L(\Pi_v, s)$ at almost all (unramified) places.

5. The global Langlands conjectures

Let $n \geq 2$ be an integer. We say that an n-dimensional **irreducible** complex Galois representation Σ of G_F and an automorphic cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ **correspond** if for almost all finite v's, we have $L(\Pi_v, s) = L(\Sigma_v, s)$. The hope expressed by the Langlands conjectures is that there are many corresponding pairs (Σ, Π).

If Σ is an irreducible **complex** representation of G_F, then certainly there should be a corresponding Π, but the Π's thus obtained are very special. An automorphic cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ will not generally correspond to a complex Galois representation. However, if the components of Π at infinity satisfy some algebraicity condition (smooth irreducible representations at the infinite places have been classified by Langlands), then Π should have, for any prime number ℓ, a corresponding ℓ-adic Galois representation of G_F. More precisely there should exist a half integer k (depending on n and the components of Π at infinite places), a number field E in $\mathbb{C}$ such that for almost all finite v, $L(\Pi_v, s+k)$ belongs to $E(q_v^{-s})$ and, for each finite place λ of E, a continuous homomorphism $\Sigma_\lambda : G_F \to \mathrm{GL}_n(E_\lambda)$ (where E_λ is the completion of E at λ), which, at almost all places v, is unramified and verifies

$$L(\Pi_v, s+k) = L(\Sigma_{\lambda,v}, s).$$

In the other direction, it is generally accepted that every ℓ-adic irreducible Galois representation coming from geometry as in section 2 should correspond to an automorphic cuspidal representation, in the sense just explained. An instance is provided by the recently proven theorem that every elliptic curve over $\mathbb{Q}$ is modular : indeed the statement is equivalent to saying that the ℓ-adic representation of $G_{\mathbb{Q}}$ on the Tate module of the elliptic curve corresponds to some automorphic cuspidal representation of GL_2 over $\mathbb{Q}$.

6. L-functions

Class field theory describes all finite abelian extensions of a number field F purely in terms of the base field F, using adeles, or, what amounts to the same, ideals and congruences. It is more difficult to say what is gained when a given Galois representation (of dimension $n \geq 2$) is shown to correspond to an automorphic cuspidal representation. One tangible gain is information on L-functions, to which we turn now.

If Π is an automorphic cuspidal representation of GL_n over F, then its L-factors are in fact defined at every place, and give rise to a global L-function with nice analytic properties. More precisely, let F_0 be a p-adic field, and $\psi_0 : F_0 \to \mathbb{C}^\times$ a non-trivial additive character. Any smooth irreducible representation π of $\mathrm{GL}_n(F_0)$ has an L-factor $L(\pi, s)$ of the form $P(q_0^{-s})^{-1}$ where q_0 is the cardinality of the residue field of F_0 and P some complex polynomial of degree $\leq n$ with value 1 at 0; one also defines a factor $\varepsilon(\pi, s, \psi_0)$ which is a non-zero monomial in q_0^{-s}. There are also such definitions for archimedean fields such as $\mathbb{R}$ and $\mathbb{C}$. If Π is global as above, then the product $L(\Pi, s) = \coprod L(\Pi_v, s)$ (over all places v) converges for $\mathrm{Re}(s)$ big enough and extends to a meromorphic function of s (it is even an **entire** function because we have assumed $n \geq 2$) which satisfies a functional equation analogous to the functional equation for Riemann's zêta function

$$L(\Pi, s) = \varepsilon(\Pi, s) L(\Pi^\vee, 1-s).$$

(Here, and in the following, a superscript $^\vee$ indicates the dual or "contragredient" representation; indeed there is a natural notion of dual representation in those contexts). If we fix a non-trivial character Ψ of $\mathbb{A}_F$ trivial on F, the factor $\varepsilon(\Pi, s)$ factors as $\prod_v \varepsilon(\pi_v, s, \Psi_v)$, where almost all factors are identically 1 and Ψ_v is the component of Ψ at $v : \Psi_v : F_v \to \mathbb{A}_F \to \mathbb{C}^\times$.

So if an irreducible Galois representation Σ of G_F is shown to correspond to an automorphic cuspidal representation Π of GL_n over F, then one can complete the definition of L-factors of Σ from unramified places to all places, and get a global L-function for Σ with very nice analytic properties.

It is not straighforward to guess the bad factors $L(\Sigma_v, s)$ just by looking at Σ_v; indeed, if Σ is an ℓ-adic representation, the L-factors at infinite places presumably involve looking at the restriction of Σ to G_{F_λ}, where λ runs through **ℓ-adic** places of F. However, if Σ is complex and v is any place of F, or if Σ is ℓ-adic and v is a finite place of F prime to ℓ, then there is an intrinsic definition of $L(\Sigma_v, s)$ (determined by the local representation Σ_v) which goes back to E. Artin, and Langlands and Deligne have shown how to define local factors $\varepsilon(\Sigma_v, s, \psi_v)$ for any non-trivial additive character ψ_v of F_v.

It is expected, for example, that if Σ is an irreducible complex representation of G_F of degree ≥ 2 corresponding to an automorphic cuspidal representation Π, then the L and ε-factors of Σ and Π coincide at all places. In any case, the Artin L-function $L(\Sigma, s) = \prod_v L(\Sigma_v, s)$ is then entire, yielding **Artin's conjecture** for Σ.

Remark. There is even a notion of L-factors for pairs, which will be important in the next section. If F_0 is a p-adic field and $\psi_0 : F_0 \to \mathbb{C}^\times$ a non-trivial character, and if π, π' are smooth irreducible representations of $\mathrm{GL}_n(F_0)$ and $\mathrm{GL}_{n'}(F_0)$ respectively, then one defines factors $L(\pi \times \pi', s)$ and $\varepsilon(\pi \times \pi', s, \psi_0)$. If π, π' are unramified and $L(\pi, s) = \prod_i (1 - \alpha_i q_0^{-s})^{-1}$, $L(\pi', s) = \prod_j (1 - \beta_j q_0^{-s})^{-1}$ then $L(\pi \times \pi', s) = \prod_{i,j} (1 - \alpha_i \beta_j q_0^{-s})^{-1}$. Similar definitions exist for archimedean fields. If π' is the trivial character of $\mathrm{GL}_1(F_0)$ one recovers the previous definition of factors for π. If Π, Π' are automorphic cuspidal representations of GL_n and $\mathrm{GL}_{n'}$ respectively over a number field F, then the product over all places gives a meromorphic function $L(\Pi \times \Pi', s)$ (there are poles, but they are known) and a monomial $\varepsilon(\Pi \times \Pi', s) = \alpha N^s$ for some $\alpha \in \mathbb{C}^\times$ and $N \in \mathbb{N}$. There is a functional equation

$$L(\Pi \times \Pi', s) = \varepsilon(\Pi \times \Pi', s) L(\Pi^\vee \times \Pi'^\vee, 1-s)$$

If Π and Π' happen to correspond to complex Galois representations Σ and Σ' one expects of course the equalities

$$L(\Pi \times \Pi', s) = L(\Sigma \otimes \Sigma', s)$$
$$\varepsilon(\Pi \times \Pi', s) = \varepsilon(\Sigma \otimes \Sigma', s)$$

as it is the case for the components at unramified places : they should be the consequence of the analogous equalities for the components at the other places.

7. From global to local : the results

A local-global principle is emerging, generalizing the principle already present in class field theory. The basic idea is the following. The geometry of some special varieties introduced by Shimura yields correspondings pairs (Σ, Π) over some number fields F (in this geometric context, Σ is rather an ℓ-adic representation). If v is a finite place of F, prime to ℓ, one expects that the restriction Σ_v of Σ to G_{F_v} depends only on the component Π_v of Π at v, and conversely. Besides the local L and ε factors of Π_v and Σ_v should correspond. Moreover one hopes that all local representations occur that way. This is essentially what Harris and Taylor prove.

The first thing is to get corresponding pairs, matching at almost all places. The following is an improvement, due to Harris and Taylor, of a theorem of Clozel. To state it conveniently it is better to fix a prime number ℓ and an isomorphism of $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$.

THEOREM 1 (Global correspondence I). — *Let F be a* CM-*field with complex conjugation c. Let n be a positive integer and Π an automorphic cuspidal representation of* GL_n *over F. Assume*

(1) *The transform Π^c of Π by complex conjugation is isomorphic to $\Pi^\vee$*

(2) *Π is algebraic and regular enough at infinite places*

(3) *at some finite places, the matrix coefficients of Π_v are L^2-modulo the centre.*
Then there is a positive integer a and an ℓ-adic representation $\Sigma' : G_F \to \mathrm{GL}_{an}(\overline{\mathbb{Q}}_\ell)$ such that at almost all finite places v of F

$$L(\Sigma'_v, s) = L(\Pi_v, s + \frac{n-1}{2})^a .$$

Note. Condition (2) says more precisely (and technically) that Π_∞, the tensor product of the Π_v for v infinite, has the same infinitesimal character as some algebraic finite dimensional complex representation of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)$. Condition (3) is a technical condition, imposed by our present state of knowledge. A matrix coefficient for Π_v is a function on $\mathrm{GL}_n(F_v)$ of the form $g \mapsto \lambda(\Pi_v(g)x)$ where x is a vector in the space of Π_v and λ a linear functional on that space which is smooth with respect to the natural action of $\mathrm{GL}_n(F_v)$.

Of course, one expects that $\Sigma' = a\Sigma$ for some ℓ-adic representation $\Sigma : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$.

In fact Harris and Taylor first prove some special cases of the theorem above, which are enough for the expected local results; the theorem in its generality comes as an afterthought. Moreover, it can be **reinforced** by a statement at all finite places v of F prime to ℓ. Take such a place, possibly bad for Π i.e. with Π_v not unramified. Looking at the bad reduction of Shimura varieties at such a place (see Lecture II), Harris and Taylor show that if not Σ'_v then at least its semisimplification is of the form $a\Sigma_v$ where Σ_v is determined by Π_v and can be read from Π_v via a geometric local model. Seeing a given p-adic field F_0 as a completion of some suitable CM field, one gets a purely local result.

THEOREM 2 (Local correspondence). — *Let F_0 be a p-adic field and $\psi_0 : F_0 \to \mathbb{C}^\times$ a non-trivial additive character. Then for each positive integer n, there is a canonical map $\sigma \mapsto \pi(\sigma)$ between irreducible degree n complex representations* *of W_{F_0}, up to isomorphism, and smooth irreducible representations of $\mathrm{GL}_n(F_0)$, up to isomorphism. Those maps are injective and if σ, σ' are two irreducible complex representations of W_{F_0}, of degree n, n' respectively, then one has*

$$L(\pi(\sigma) \times \pi(\sigma'), s) = L(\sigma \otimes \sigma', s)$$
$$\varepsilon(\pi(\sigma) \times \pi(\sigma'), s, \psi_0) = \varepsilon(\sigma \otimes \sigma', s, \psi_0).$$

In fact the image is known : it consists of the supercuspidal representations (their matrix coefficients have compact support modulo the centre). Also the maps $\sigma \mapsto \pi(\sigma)$ can be characterized by the preservation of L and ε-factors above, and some other milder natural requirements, like $\pi(\sigma^\vee) = \pi(\sigma)^\vee$.

The inverse map of $\sigma \mapsto \pi(\sigma)$, written $\pi \mapsto \sigma(\pi)$, can be extended to a map $\pi \mapsto \sigma(\pi)$ which to each smooth irreducible representation of $\mathrm{GL}_n(F_0)$ associates a semisimple representation of W_{F_0} (this is no longer injective; to restore bijectivity one has to complicate things further by considering the Weil-Deligne group ...).

Thm. 2 proves the Langlands conjectures for GL_n over F_0, but Harris and Taylor do more : they provide a geometric local model for the correspondence. But I shall try and explain that Part A II. As mentioned above, Thm. 2, in turn, has global consequences.

THEOREM 3 (Global correspondence II). — *In the situation of theorem 1 above, let v be any finite place of F prime to ℓ. Then the semisimplification of the restriction of Σ_v to W_{F_v} is isomorphic to $\sigma(\Pi_v)$.*

There is some hope to determine Σ_v fully (up to isomorphism) from Π_v.

Remark. The methods used by Harris and Taylor are inspired by Deligne, who started the case $n = 2$. The theorems above when $n = 2$ were obtained by Carayol.

A II) The role of the geometry of Shimura varieties

8. Introduction

In this second part, I shall try to describe roughly how the geometry of Shimura varieties is used to get the main results. Of necessity, this part is more technical (even) that the previous one. I will not be able to give much detail, and one should not take the statements too literally. I simply hope to convey a roughly resembling picture of the methods and steps involved. Again, see Carayol's report [Ca] (52 pages) or the original paper [HaT].

In a nutshell, let me sketch the basic principle governing the use of geometry. Let X be a (smooth and proper, say) variety over a number field F, and assume X is in some (nice) way acted upon by $\mathrm{GL}_n(\mathbb{A}_F)$. Then on the etale ℓ-adic cohomology spaces of X over an algebraic closure $\overline{F}$ of F, we get two commuting actions of $\mathrm{GL}_n(\mathbb{A}_F)$ and $G_F = \mathrm{Gal}(\overline{F}/F)$. The best hope is that if Σ is an irreducible ℓ-adic representation of G_F and Π an automorphic cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\Sigma \otimes \Pi$ "occurs" in such an ℓ-adic cohomology space, then Σ and Π correspond : $L(\Pi_v, s+\frac{k}{2}) = L(\Sigma_v, s)$ for almost all places and some suitable integer k. Of course for that to be true we have to choose X carefully – and be lucky!

9. The model case : modular curves

The model for such a procedure is the case of modular curves $\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half-plane and Γ a congruence subgroup of $SL_2(\mathbb{Z})$. The complex curve $\Gamma\backslash\mathbb{H}$ is the set of complex points of an algebraic curve Y_Γ defined over some number field F; it is smooth if Γ is small enough and can be completed to a proper curve X_Γ/F. The cohomology spaces of X_Γ with constant or more general coefficients can be interpreted in terms of classical modular forms. We can also interpret this way the action of Hecke operators on modular forms. Indeed $\Gamma\backslash\mathbb{H}$ can be seen as $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/\mathbb{R}_+^\times K_\infty K_\Gamma$, where K_∞ is the maximal compact subgroup $0(2)$ of $\mathrm{GL}_2(\mathbb{R})$ and K_Γ is an open subgroup of the product over all prime numbers p of the groups $\mathrm{GL}_2(\mathbb{Z}_p)$. The group $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ does not act on such a quotient space, but there is an action of an algebra of Hecke operators : these are given, for each prime number p, by $\mathrm{GL}_2(\mathbb{Z}_p)$-double classes in $\mathrm{GL}_2(\mathbb{Q}_p)$ and such a class acts by a correspondence on X_Γ. Consequently it also acts on ℓ-adic etale cohomology spaces $H^i(X_\Gamma \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$. If we take the direct limit $H^i = \varinjlim H^i(X_\Gamma \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ over smaller and smaller Γ (we choose such $\Gamma's$ with X_Γ defined over $\mathbb{Q}$), we recover on H^i an action of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ (the restricted product of the $\mathrm{GL}_2(\mathbb{Q}_p)'$s for p prime); it is admissible in the sense that each vector has an open stabilizer (smoothness) and open subgroups have only finite dimensional spaces of fixed points. The action of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ on H^i commutes with that of $G_\mathbb{Q}$ and any natural decomposition of H^i with respect to $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ will preserve an action of $G_\mathbb{Q}$. In particular, we can isolate (on H^1) the part corresponding to a given weight 2 newform f; specifying the eigenvalues of Hecke operators on f amounts to fixing an irreducible admissible representation π_f^∞ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ (to get weights > 2 we have to allow non-constant coefficients). The part of H^1 where $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ acts via π_f appears as $\pi_f \otimes V$ where V is the space of a 2-dimensional ℓ-adic representation Σ of $G_\mathbb{Q}$ corresponding to

π_f, in the sense above that for almost all prime numbers p, $L(\pi_{f,p}, s-\frac{1}{2}) = L(V_p, s)$. (That last relation can be deduced from the Eichler-Shimura relation).

10. Shimura varieties

If $G = \mathrm{GL}_n$ or more generally if G is any reductive algebraic group over a number field F, we easily get differentiable manifolds analogous to the above modular curves, by forming

$$X_K = G(F)\backslash G(\mathbb{A}_F)/ZK_\infty K^\infty,$$

where K_∞ is a maximal compact subgroup of the Lie group $G(F_\infty)$ (the product over infinite places v of $G(F_v)$), Z the centre of $G(F_\infty)$ and K^∞ is an open compact subgroup of $G(\mathbb{A}_F^\infty)$ (the restricted product over finite places v of the groups $G(F_v)$). It is actually a manifold only if K^∞ is small enough. To each algebraic finite dimensional representation ξ of G, we can attach a local system $\mathcal{L}_\xi$ on X_K and look at the (complex) cohomology spaces $H^i(X_K, \mathcal{L}_\xi \otimes \mathbb{C})$. By relating them to Lie algebra cohomology, the vector spaces $H^i(X_K \mathcal{L}_\xi \otimes \mathbb{C})$ can be interpreted in terms of automorphic forms on G. In particular, a cuspidal automorphic representation Π (the definition of such is similar to the one given above when $G = \mathrm{GL}_n$), with "algebraic" infinity components will usually appear, or rather a corresponding module for a Hecke algebra will appear, in a suitable space $H^i(X_K, \mathcal{L}_\xi)$.

The problem is that such manifolds are not generally algebraic varieties, so there is no way to get ℓ-adic representations. For example if $G = \mathrm{GL}_n$, $n \geq 3$, we do not get algebraic varieties this way. But long ago Shimura singled out a class of G's for which the manifolds X_K indeed have a natural algebraic structure and are defined over number fields. Moreover for most of them, the algebraic structure is obtained because X_K classifies over $\mathbb{C}$ abelian varieties endowed with a number of auxiliary data : this generalizes the case of modular curves, which classify elliptic curves with some extra structure. A similar moduli problem makes sense algebraically and this gives the structure over a number field E (that field is not necessarily equal to F, though). Moreover a variant of the moduli problem usually makes sense, at almost all places v of E, over the ring of integers $\mathcal{O}_{E_v}$ and the result is that X_K has good reduction there, its reduction to the residue field of E_v being again given by a moduli problem through which one can study it.

Again a cuspidal automorphic representation π of $G(\mathbb{A}_F)$, with suitable algebraicity conditions on its infinity part, will usually appear in some ℓ-adic etale cohomology space $H^i(X_K \otimes \overline{E}, \mathcal{L}_\xi \otimes \overline{\mathbb{Q}}_\ell)$, and the hope is to get a corresponding ℓ-adic representation of G_E by looking at the π^∞-isotypic component.

11. Some obstacles

There are many complications to that approach, though, besides the fact that it does not directly apply to GL_n. Firstly X_K as above is not usually proper so that to get a reasonable analysis of the cohomology, one would have to construct a smooth equivariant compactification, with possibly the boundary again given by moduli problems ... Secondly it is not so straightforward to analyze the cohomology spaces in terms of automorphic representations. Langlands has shown that in such an analysis, automorphic representations of groups other than G sneak in : this is the phenomenon of endoscopy. Thirdly it is not entirely clear where and how often a given cuspidal automorphic representation of G over F appears in cohomology and indeed such a multiplicity problem is responsible for the occurrence of the integer a in Theorem 1 of Lecture 1.

Moreover the technical complications are enormous. The standard approach, pioneered by Langlands, is to analyze the geometry of X_K via the Lefschetz-Grothendieck trace formula and the space of automorphic forms via the Arthur-Selberg trace formula. Two formidable tools, which have to be related to each other ...

12. Kottwitz's "simple" Shimura varieties

However R. Kottwitz singled out some "simple" Shimura varieties where most of the problems above can be solved or circumvented, and those varieties are used by Harris and Taylor. They are attached to unitary groups with respect to a quadratic extension F/F^+ of number fields, where F^+ is totally real and F totally imaginary. Let me describe the ones Harris and Taylor use which are "simple" in Kottwitz's sense. One chooses a division algebra B with center F and degree n^2 over F, with a positive involution of second kind $*$ (that is the restriction of $*$ to F is the non-trivial involution of F over F^+). There are other technical hypotheses, but the important requirement to get local results for our given p-adic field F_0, is the following : F_0 appears as a completion of F^+, and F and B are split at that place 0, so that F has two completions over F_0^+, one of which is F_0, and $B_0 = B \otimes_F F_0$ is isomorphic to $M_n(F_0)$, $B_0^\times$ to $\mathrm{GL}_n(F_0)$. At least for the given p-adic field F_0, we recover from $B^\times$ the group we want!

Conjugating the involution $*$ by a $*$-antisymmetric element β, we get another involution $\natural$ and a reductive group G over $\mathbb{Q}$ defined by

$$G(R) = \{g \in B^{op} \otimes R, bb^\natural \in R^\times\} \text{ for any } \mathbb{Q} - \text{algebra } R\ .$$

It is a form of the group of unitary similitudes of degree n over F. The element β is chosen so that group has signature $(n-1,1)$ at one infinite place of F, and $(n,0)$ at the other infinite places. If K^∞ is a compact open subgroup of $G(\mathbb{A}_\mathbb{Q}^\infty)$, we can form X_K as above, which is a compact manifold if K^∞ is small enough : it is in fact the union of a finite number of quotients $\Gamma\backslash\mathcal{X}$ where $\mathcal{X}$ is the unit ball in $\mathbb{C}^{n-1}$ and Γ a discrete cocompact subgroup of $G(\mathbb{R})$. In this case X_K is a proper algebraic variety over some number field, given by a moduli problem : it classifies abelian varieties up to isogeny, of dimension $[F^+ : \mathbb{Q}]n^2$, with an action of B (and, of course, other auxiliary data and compatibility requirements). Besides the group G presents no endoscopy problem. But, to expect Galois representations of G_F, it is better if X_K is defined over F! One way to obtain that is to assume $F = F^+E$, where E is a quadratic imaginary field (where p splits). We shall assume this from now on.

13. Results of Kottwitz

At almost all prime numbers p, X_K has good reduction, and a model for the reduction is given by a moduli problem classifying abelian varieties with an action of an order O_B in B, plus other data of course. Kottwitz analyzed enough of $X_K(k)$

for finite fields k to obtain, via the Grothendieck-Lefschetz and the Arthur-Selberg trace formulae, the following result.

Let Π be an automorphic cuspidal representation of $G(\mathbb{A}_Q)$ with algebraicity (and regularity) conditions on its infinity component Π_∞, and a mild requirement at some finite place. Then, for some algebraic finite dimensional representation ξ of G, the finite part Π^∞ of Π "occurs" in the etale ℓ-adic cohomology space $H^{n-1} = \varinjlim(X_K \times \overline{F}, \mathcal{L}_\xi \otimes \overline{\mathbb{Q}}_\ell)$. The space of $G(\mathbb{A}_\mathbb{Q}^\infty)$-embeddings of Π^∞ into H^{n-1} has finite dimension $a \cdot n$ for some positive integer a, and if Σ' is the action of G_F on that space, we have, at almost all places v of F

$$L(\Pi_v, s - \frac{n-1}{2}) = L(\Sigma'_v, s)^a .$$

(As for GL_n, Π is unramified at almost all finite places v of F; in fact for almost all prime numbers r, there is a non-trivial fixed vector under a compact subgroup of $G(\mathbb{Q}_r)$ of maximal volume; at places v of F above such r, Π_v has an L-factor of the form $P(q_v^{-s})^{-1}$, $P \in \mathbb{C}[X]$, $P(0) = 1$).

We are now close to Thm. 1, except that Π is an automorphic cuspidal representation of G, not of GL_n over F!

14. Results of Clozel

However, if we base change G from $\mathbb{Q}$ to the quadratic imaginary field E, we get a group close to the algebraic group defined by $B^{op\times}$, itself an interior form of GL_n/F. Such a close relationship between groups reflects on the side of automorphic representations : this is Langlands' theory of base change, or more generally Langlands' functoriality.

The situation at hand was explored by Clozel, and the result is roughly the following : the automorphic cuspidal representation Π of $G(\mathbb{A}_\mathbb{Q})$ above "base changes" to an automorphic cuspidal representation Π' of $\mathrm{GL}_n(\mathbb{A}_F)$ such that :
(1) $\Pi'^\vee = \Pi'^c$ where c is the complex conjugation of F over F^+;
(2) for almost all places v of F, $L(\Pi'_v, s) = L(\Pi_v, s)$. Conversely a cuspidal automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_F)$, satisfying (1) and the other conditions of Thm. 1, is such a base change. This gives a result slightly weaker than Thm. 1, at least in the case where $F = F^+E$, E quadratic imaginary.

15. Results of Harris and Taylor

After receiving the first version of [HT], I realized that using [Ha 2], the above case of Thm. 1 is enough to get the Langlands correspondence as stated in Thm. 2 [He]. But Harris and Taylor do much more : they actually get a geometric model for that local correspondence, and in turn this has global consequences as Thm. 3. That feat actually takes most of their manuscript, and I shall now give a (too) brief description of it. Let me mention though that, in doing so, they complete a program initiated by Carayol.

Most of the work consists in analyzing the Shimura variety X_K at finite places w of bad reduction (actually, Harris and Taylor also essentially reprove in their case Kottwitz's result for good reduction, by a different method).

At such a place w, X_K has an integral model over O_{F_w}, which is projective but not smooth, and is again given by a moduli problem classifying abelian varieties with an action of O_B. The special fiber has a filtration by closed subvarieties, indexed by a measure of "supersingularity" : this is analogous to the consideration of supersingular versus ordinary elliptic curves on the special fiber of modular curves. The most supersingular points form a finite subset of X_K. The etale ℓ-adic cohomology groups $H^i(X_K \otimes \overline{F}_w, \mathcal{L}_\xi \otimes \overline{\mathbb{Q}}_\ell)$ are studied via vanishing cycles on the special fiber : there is a spectral sequence of vanishing cycles

$$H^i(X_K \otimes \overline{k}_w, R\Phi^j \otimes \mathcal{L}_\xi \otimes \overline{\mathbb{Q}}_\ell) \Longrightarrow H^{i+j}(X_K \otimes \overline{F}_w, \mathcal{L}_\xi \otimes \overline{\mathbb{Q}}_\ell),$$

where the $R\Phi^j$'s are sheaves of vanishing cycles. They are analyzed stratum by stratum.

The main point (but for lack of time and expertise, I shall refrain from giving details here) is that the sheaves $R\Phi^j$ are related to purely local objects, which are spaces of ℓ-adic vanishing cycles attached to some specially constructed formal schemes. My last section will describe such formal schemes.

Remark. Berkovich's theory of ℓ-adic cohomology for formal schemes is absolutely crucial here. In fact Harris and Taylor needed a bit more than previously known, and that was provided by Berkovich : see the appendix of [Ha T].

16. Formal modules and Drinfeld's moduli varieties

Here the base field is our p-adic field F_0. Let $\mathcal{O}$ be its ring of integers, and ϖ a uniformizer of F_0. Fix the integer $n \geq 1$ and the prime number $\ell \neq p$.

Formal $\mathcal{O}$-modules, i.e. formal groups with an action of $\mathcal{O}$, have been used by Lubin and Tate to give a purely local proof of local class field theory. Drinfeld, using deformations of such objects, has constructed formal schemes with commuting actions of $\mathrm{GL}_n(F_0)$ and W_{F_0}. Berkovich's spaces of ℓ-adic vanishing cycles on such formal schemes then give commuting actions of $\mathrm{GL}_n(F_0)$ and W_{F_0} on ℓ-adic vector spaces, which implement the Langlands local correspondence. Let us be a little more precise.

We consider formal $\mathcal{O}$-modules of fixed height n (the height controls how ϖ acts in characteristic p). Over the algebraic closure $\overline{k}$ of the residue field of $\mathcal{O}$ there is a unique such formal module Σ up to isomorphism. Its endomorphism ring is the ring of integers $\mathcal{O}_D$ of the division algebra D with center F_0, degree n^2 over F_0 and invariant $1/n$ in the Brauer group of F_0. The functor of deformations of Σ is prorepresentable by a local complete algebra R, isomorphic in fact to $\widehat{\mathcal{O}}[[T_1, \dots T_{n-1}]]$, where $\widehat{\mathcal{O}}$ is the ring of integers of the completion of the maximal unramified extension of F_0 in $\overline{F}_0$. On the formal scheme $Spf(R)$, Σ has a universal deformation $\widetilde{\Sigma}$. As for elliptic curves, there is a notion of level m structure for such formal $\mathcal{O}$-modules. On the generic fiber it is simply an isomorphism of the ϖ^m-torsion points of $\widetilde{\Sigma}$ with the $\mathcal{O}$-module $(\varpi^{-m}\mathcal{O}/\mathcal{O})^h$. The general notion is due to Drinfeld. The functor classifying such level m-structures is prorepresentable by a local complete algebra R_m. On $Spf(R_m)$ the group $\mathrm{GL}_n(\mathcal{O}/\varpi^m\mathcal{O})$ acts.

17. Group actions on the vanishing cycles

Berkovich's theory of ℓ-adic cohomology for p-adic analytic spaces attaches vanishing cycle spaces Ψ^i_m to $Spf(R_m)$. Varying m, we get an inductive system of ℓ-adic vector spaces; write Ψ^i for the limit (in fact the union). On Ψ^i we have commuting actions of $\mathcal{O}_D^\times$ (by automorphisms on the base), of $\mathrm{GL}_n(\mathcal{O})$ (by automorphisms of the level structure), and of the inertia group I_{F_0}. From Ψ^i one constructs a slightly larger space $\widetilde{\Psi}^i$ with more group action : on $\widetilde{\Psi}^i$ we have an **admissible** action of $\mathcal{O}_D^\times \times \mathrm{GL}_n(F_0) \times W_{F_0}$. If τ is a given irreducible admissible representation of $\mathcal{O}_D^\times$ (such a τ is finite dimensional), we let $\widetilde{\Psi}^i(\tau)$ be the representation of $\mathrm{GL}_n(F_0) \times W_{F_0}$ on the τ-isotypic component of $\widetilde{\Psi}^i$. It is admissible of finite length and we put

$$\Psi^*(\tau) = (-1)^{n-1}\Sigma(-1)^i[\widetilde{\Psi}^i(\tau)]$$

where [ ] denotes a class in the Grothendieck group of (virtual) admissible finite length representations of $\mathrm{GL}_n(F_0) \times W_{F_0}$. The following theorem describes geometrically the local Langlands correspondence $\pi \mapsto \sigma(\pi)$ of Part A I), at least in the crucial case where π is supercuspidal.

THEOREM (Harris-Taylor). *Assume π is supercuspidal. Then there is a unique irreducible admissible representation τ of $\mathcal{O}_D^\times$ such that $\Psi^*(\tau) = [\pi \otimes (\sigma(\pi^\vee)| \, |^{\frac{n-1}{2}})]$.*

(Here $| \, |$ is the norm character of W_{F_0}).

In fact τ is obtained from π by the Jacquet-Langlands correspondence, which is an instance of Langlands' functoriality between the local groups $\mathrm{GL}_n(F_0)$ and $D^\times$.

B) A local-global principle

In this part B, I want to illustrate several instances of the local-global principle already mentioned in part A, and in particular I want to show how it applies to give a proof of the Langlands conjectures for GL_n over p-adic fields, following [Ha2, HaT, He]. I shall assume some familiarity with number fields, their completions, and their adele rings and idele groups.

B.1 Class-field theory

B.1.1 Class-field theory describes all finite abelian extensions of numbers fields, i.e. finite extensions of $\mathbb{Q}$, and p-adic fields, i.e. finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers. This is an elaborate theory, its exposition requires books. I shall only give the salient facts refering to e.g., [Lg 1, Ne, We] for details. In the sequel, p is a prime number, F a number field, and K a p-adic field. We shall also refer here to number field as **global** fields and to p-adic fields as **local** fields.

All finite abelian extensions of $\mathbb{Q}$ or $\mathbb{Q}_p$ are cyclotomic, i.e. are subextensions of fields obtained by adjoining some roots of unity; this is the celebrated theorem of Kronecker-Weber. For general F and K the description is more complicated. For a number field F, it involves the adele ring $\mathbb{A}_F$ of F and its idele class group $F^\times\backslash\mathbb{A}_F^\times$ (it can also be expressed more classically in terms of ideals of the ring of integers $\mathcal{O}_F$ of F cf. [Lg 1, Jz].)

The adele ring $\mathbb{A}_F$ of F is defined in terms of the completions F_v of F at the different places v of F. There are the *infinite places* of F, the real ones which correspond to the different field embeddings of F into $\mathbb{R}$, and the complex ones which correspond to pairs of complex conjugate field embeddings of F into $\mathbb{C}$ as a dense subfield. On the other hand, there are the *finite* places of F, one for each maximal ideal v of the ring of integers $\mathcal{O}_F$ of F; if ℓ is the residue characteristic of v, i.e. the characteristic of the finite field $k_v = \mathcal{O}_E/v$, then the corresponding completion F_v is an ℓ-adic field; when $F = \mathbb{Q}$ and v is given by a prime number ℓ in $\mathbb{Z} = \mathcal{O}_{\mathbb{Q}}$, then the completion is none other than the field of ℓ-adic numbers $\mathbb{Q}_\ell$.

It is therefore natural to consider, alongside the number fields, also the p-adic fields for prime numbers p (and also the archimedean complete fields $\mathbb{R}$ and $\mathbb{C}$, but their finite abelian extensions are easy to describe). Such a p-adic field K has a prefered absolute value $|\ \ |_K$; its ring of integers $\mathcal{O}_K = \{x \in K \mid |x|_K \leq 1\}$ has a unique maximal ideal $P_K = \{x \in K \mid |x|_K < 1\}$, and the residue field $k_K = \mathcal{O}_K/P_K$ is a finite extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$; the ideal P_K is principal and its generators are called **uniformizers** of K. When $K = \mathbb{Q}_p$ we have $O_K = \mathbb{Z}_p$, $P_K = p\mathbb{Z}_p$. A given p-adic field K appears as a completion F_v of some number field F at a finite place v : that is to say, there is a choice of F and v such that F_v is topologically isomorphic to K. In fact there are very many different choices for F and v.

B.1.2 The adele ring $\mathbb{A}_F$ of F is defined as a union of subrings $\mathbb{A}_F^{(S)} = \Pi_{v\in S}F_v\Pi_{v\notin S}\mathcal{O}_{F_v}$, where S runs through finite sets of places of F containing the infinite places. It is a topological ring if we put the product topology on $\mathbb{A}_F^{(S)}$ and the inductive limit topology on $\mathbb{A}_F$. The field F then sits diagonally in $\mathbb{A}_F$ as a discrete subring, although of course, it is dense in each of its completions F_v. Similarly the *idele group* $\mathbb{A}_F^\times$ of units of $\mathbb{A}_F$ is a union of subgroups $\mathbb{A}_F^{\times(S)} = \Pi_{v\in S}F_v^\times\Pi_{v\notin S}\mathcal{O}_{F_v}^\times$; it is a topological group, again, if we put the product topology on $\mathbb{A}_F^{\times(S)}$ and the inductive limit topology on $\mathbb{A}_F^\times$, which turns $\mathbb{A}_F^\times$ into a locally compact topological group. This is not the topology on $\mathbb{A}_F^\times$ induced by the one on $\mathbb{A}_F$.

As before $F^\times$ sits diagonally in $\mathbb{A}_F^\times$ as a discrete subgroup. Also for each place v of F, $F_v^\times$ sits inside $\mathbb{A}_F^\times$, as the v-component. The quotient $C_F = \mathbb{A}_F^\times / F^\times$ is called the *idele class group* of F : When F is $\mathbb{Q}$, $\mathbb{A}_\mathbb{Q}^\times$ appears as the direct product of its diagonal subgroup $\mathbb{Q}^\times$ and the subgroup $\mathbb{R}_+^\times \Pi_{\ell\,\mathrm{prime}} \mathbb{Z}_\ell^\times$, where $\mathbb{R}_+^\times$ is the group of positive real numbers.

B.1.3 Let E/F be a finite extension of number fields. The norm homomorphism $N_{E/F}$ from $E^\times$ to $F^\times$ extends naturally to a norm homomorphism $N_{E/F}$ from $\mathbb{A}_E^\times$ to $\mathbb{A}_F^\times$. More precisely let v be a place of F; then there are finitely many places w of F *above* v (one writes $w|v$) : if v is infinite, this means the embeddings of E into $\mathbb{R}$ or $\mathbb{C}$ defining w extend those defining v, whereas if v is finite, this means the maximal ideal w of $\mathcal{O}_E$ contains v. The tensor product $F_v \otimes_F E$ splits naturally as a product of the finite extensions E_w/F_v where w runs through the places of E above v. The natural norm on the F_v-algebra $F_v \otimes_F E$ induces on each extension E_w the norm N_{E_w/F_v}, and the norm from $\mathbb{A}_E^\times$ to $\mathbb{A}_F^\times$ coincides with N_{E_w/F_v} on $E_w^\times$ and with $N_{E/F}$ on the diagonal $E^\times$.

We still write $N_{E/F}$ fo the induced norm map from C_E to C_F.

The main result of class-field theory for number fields is the following.

THEOREM 1. — *To each finite abelian extension E of the number field F associate its norm group $N_E = N_{E/F}(C_E)$. Then N_E is an open subgroup of finite index of C_F, and there is a canonical group isomorphism (reciprocity map) τ_E : $C_E/N_E \simeq \mathrm{Gal}(E/F)$. Moreover the map $E \mapsto N_E$ induces a bijection between finite abelian extensions E of F, up to isomorphism, and open subgroups of finite index of C_F.*

B.1.4 There is a similar result for p-adic fields, where the role of the idele class group is played by the multiplicative group.

THEOREM 2. — *To each finite abelian extension L of the p-adic field K associate its norm group $N_L = N_{L/K}(L^\times)$. Then N_L is an open subgroup fo finite index of $K^\times$, and there is a canonical group isomorphism (reciprocity map) τ_L : $K^\times/N_L \simeq \mathrm{Gal}(L/K)$. Moreover the map $L \mapsto N_L$ induces a bijection between finite abelian extensions L of K, up to isomorphism, and open subgroups of finite index of $K^\times$.*

As is to be expected the theorem for p-adic fields just stated and the theorem for number fields above are strongly related. Let me explain that now.

Let E be a finite *Galois* extension of the number field F, and let v be a place of F, w a place of E above v. Then E_w is Galois over F_v, and , as each F_v-automorphism of E_w induces an F-automorphism of E, we get an injective group homomorphism of $\mathrm{Gal}(E_w/F_v)$ into $\mathrm{Gal}(E/F)$. All the different places w of E above a given place v of F form only one orbit under the action of $\mathrm{Gal}(E/F)$, hence the subgroups $\mathrm{Gal}(E_w/F_v)$ are conjugate to each other; in particular, when E is abelian over F, they are all equal to the same group, called the decomposition subgroup at v.

The compatibility between the results above for number fields and for p-adic fields can be expressed in the following manner

THEOREM 3. — *Let v be a finite place of the number field F. Let E be a finite abelian extension of F and w a place of E above v. Then the following diagram is commutative*

$$\begin{array}{ccc} F_v^\times/N_{E_w/F_v}(E_w^\times) & \xrightarrow{\tau_{E_w}} & \mathrm{Gal}(E_w/F_v) \\ \downarrow & & \downarrow \\ C_F/N_{E/F}(C_E) & \xrightarrow{\tau_E} & \mathrm{Gal}(E/F), \end{array}$$

where the left-hand vertical map is induced by the embedding $F_v^\times \to \mathbb{A}_F^\times$, and the right-hand vertical map is the canonical inclusion of the decomposition group into $\mathrm{Gal}(E/F)$.

In fact, the exact same statement is true for an infinite place v. Also, the maps τ_{E_w} are the same for all places w of E above v.

B.1.5 Nowadays (see [We], [Ne]) class-field theory is first established for local fields, i.e. Theorem 2 is proved first. Then a homomorphism $\widetilde{\tau}_E : \mathbb{A}_F^\times \to \mathrm{Gal}(F/F)$ is constructed out the local maps τ_{E_w}, and Theorem 1 is proved, so that Theorem 3 holds essentially by definition. The main step there is to prove that the map $\widetilde{\tau}_E$ is trivial on the diagonal subgroup $F^\times$ of $\mathbb{A}_F^\times$.

However the classical approach (see [Lg 1]) is different. One proves first Theorem 2 in the easy case where the extension L/K is *unramified*, in which case $N_{L/K}(L^\times)$ contains the unit group $\mathcal{O}_K^\times$ (see below). If E is a fixed abelian extension of F, then for all finite places w of E except a finite number (it is customary to say "*for almost all*" places w of E), the corresponding extension E_w/F_v of local fields is unramified. In this way one can define a homomorphism $\widetilde{\tau}_E$ into $\mathrm{Gal}(E/F)$ but only on those elements of $\mathbb{A}_F^\times$ which have trivial component at the ramified (i.e. not unramified) places : indeed, for such an element $x = (x_v)$ of $\mathbb{A}_F^\times$, $\tau_{E_w}(x_v) = y_v$ is defined (because E_w/F_v is unramified) and is trivial for almost all v since x_v is a unit in $\mathcal{O}_{F_v}$ for almost all v; one then defines $\widetilde{\tau}_E(x)$ to be the product of the y_v's.

The most serious step is then to prove that this partially defined homomorphism $\widetilde{\tau}_E$ *can be extended* to a (continuous) homomorphism of $\mathbb{A}_F^\times$ into $\mathrm{Gal}(E/F)$ which is *trivial* on $F^\times$. This then defines a (continuous) homomorphism τ_E of C_F into $\mathrm{Gal}(E/F)$ and one shows Theorem 1.

Then it is possible to deduce Theorem 2 when the extension L/K is possibly ramified. Firstly, in the situation of Theorem 1, if v is a place of F and w a place of E above v (possibly ramified) one has to show that the image of $F_v^\times$ under τ_E is the subgroup $\mathrm{Gal}(E_w/F_v)$ of $\mathrm{Gal}(E/F)$, and τ_E does induce an isomorphism of $F_v^\times/N_{E_w/F_v}(E_w^\times)$ onto $\mathrm{Gal}(E_w/F_v)$. Secondly if K is a given p-adic field, and L a finite abelian extension of K, one can easily find a number field F, a finite abelian extension E of F, a place v of F and a place w of E above v, such that the extensions E_w/F_v and L/K are (topologically) isomorphic. By transport of structure, one then gets an isomorphism of $K^\times/N_{L/K}(L^\times)$ onto $\mathrm{Gal}(L/K)$. The final step is to show that this last isomorphism depends only the local situation L/K and not on the global situation E/F, $w|v$ we have embedded it in (and, of course, that we recover the right map when L/K is unramified).

So the classical approach would appear much more complicated than the modern one. However the principle behind it, which I like to call a local-global principle, can be applied in situations when there is no direct construction of the local theory, or no direct way to study the local situation. We shall see later instances of that principle.

B.1.6 But before we pursue, I want to explain what Theorem 2 means explicitely when L/K is unramified; and also what Theorem 1 says when F is $\mathbb{Q}$, where many abelian extensions (indeed all) are known, viz. the cyclotomic fields. In fact, it is important to point out that the knowledge of cyclotomic extensions is an essential ingredient in passing from Theorem 2 —at all places v of a number field or simply at the finite places which are unramified in a given abelian extension— to Theorem 1. However special the information appears to be, it is crucial in such local-global arguments to have, at least in *some* global situations, a precise control.

Let us first explain the unramified case of Theorem 2, then. A finite extension L of the p-adic field K is called *unramified* if the residue field degree $[k_L : k_K]$ is equal to the degree $[L : K]$; equivalently, if a uniformizer for K is also a uniformizer for L. Then L is obtained from K by adjoining the roots of $X^{|k_L|} - X$. The extension L/K is Galois : the Galois group $\mathrm{Gal}(L/K)$ acts naturally on the residue field k_L by k_K-automorphisms and, for L/K unramified, the induced morphism of $\mathrm{Gal}(L/K)$ into $\mathrm{Gal}(k_L/k_K)$ is an isomorphism; in particular L is cyclic over K and $\mathrm{Gal}(L/K)$ has a prefered generator, the arithmetic Frobenius element, which acts on the residue field as $x \mapsto x^{|k_K|}$.

On the other hand, every element from $\mathcal{O}_K^\times$ is a norm from $\mathcal{O}_L^\times$, and since uniformizers of K are also uniformizers of L, the group $K^\times/N_{L/K}(L^\times)$ is also cyclic of order $[L : K]$, which a prefered generator given by the class of uniformizers.

Class field theory for the unramified extension L/K is the simple statement that we have a canonical isomorphism

$$\tau_L : K^\times/N_{L/K}(L^\times) \simeq \mathrm{Gal}(L/K),$$

since both sides are cyclic of the same degree with a prefered generator! It is now customary to normalize the isomorphism so that the uniformizers of K, or rather their class in $K^\times/N_{L/K}(L^\times)$, correspond to the *geometric Frobenius elements* which are the inverses the arithmetic Frobenius elements.

Remark : In the case of (possibly ramified) extensions L/K of p-adic fields, the reciprocity maps τ of Theorem 2 can be characterized by what happens in the unramified case, plus some natural compatibilities when L varies. See [Ne] for such an approach.

B.1.7 Let us now explain the content of Theorem 1 when F is $\mathbb{Q}$, and E is the cyclotomic field $\mathbb{Q}(\chi)$ generated by a primitive N-th root of unity, N being a fixed integer, $N \geq 2$. In that case, the Galois group $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ can be identified with $(\mathbb{Z}/N\mathbb{Z})^\times$ of an integer a prime to N acting as the automorphism of $\mathbb{Q}(\chi)$ sending χ to χ^a. The finite places of $\mathbb{Q}$ correspond to the prime numbers. All prime numbers p not dividing N are unramified in $\mathbb{Q}(\chi)$ (i.e. all places w of $\mathbb{Q}(\chi)$ above p have

$\mathbb{Q}(\chi)_w/\mathbb{Q}_p$ unramified), and the corresponding arithmetic Frobenius element acts as $\chi \mapsto \chi^p$ on χ, so is identified with the class of p in $(\mathbb{Z}/N\mathbb{Z})^\times$.
It is easy to verify that there is a unique homomorphism

$$\widetilde{\tau} : \mathbb{A}_{\mathbb{Q}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\wedge$$

which is

- trivial on the diagonal $\mathbb{Q}^\times$
- trivial on the factor $\mathbb{R}_+^\times$ at the infinite place,
- for a prime p dividing N, trivial on elements of $\mathbb{Q}_p^\times$ congruent to 1 mod $N\mathbb{Z}_p$
- for a prime p not dividing N, trivial on $\mathbb{Z}_p^\times$ and sending the uniformizer p to the class of p^{-1} in $(\mathbb{Z}/N\mathbb{Z})^\times$. Since $\widetilde{\tau}$ is trivial on $\mathbb{Q}^\times$ it induces a homomorphism $C_{\mathbb{Q}} \to (\mathbb{Z}/N\mathbb{Z})^\times$. One shows it is surjective (*Čebotarev's theorem*) and that its kernel is $N_{\mathbb{Q}(\chi)/\mathbb{Q}}(C_{\mathbb{Q}(\chi)})$, hence providing the reciprocity map

$$\tau_{\mathbb{Q}(\chi)} : C_{\mathbb{Q}}/N_{\mathbb{Q}(\chi)} \simeq \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times.$$

B.1.8 The reciprocity maps have some natural properties. An important one is when we consider a subextension of an abelian extension. In the global case, if E is a finite abelian extension of the number field F, and E' a subextension, the following diagram is commutative

$$\begin{array}{ccc} C_F/N_E & \xrightarrow[\sim]{\tau_E} & \mathrm{Gal}(E/F) \\ \downarrow & & \downarrow \\ C_F/N_{E'} & \xrightarrow[\sim]{\tau_{E'}} & \mathrm{Gal}(E'/F) \end{array}$$

where the left vertical map comes from the obvious inclusion $N_E \subset N_{E'}$ and the right vertical map is the canonical quotient obtained by restricting to E' the F-automorphisms of E. We have a similar commutativity statement in the local case.

This makes it possible to consider all finite abelian xtensions E of F at the same time. Let us fix an algebraic closure $\overline{F}$ of F. On the Galois group $\mathrm{Gal}(\overline{F}/F)$ of F-automorphisms of F put the topology of pointwise convergence on $\overline{F}$; then $\mathrm{Gal}(\overline{F}/F)$ is a compact topological group, and appears as a profinite group, more precisely, the projective limit of its finite (discrete) quotients $\mathrm{Gal}(E/F)$, where E runs through finite Galois extensions of F in $\overline{F}$.

One can restate the main theorem of global class field theory by saying that there is a canonical isomorphism of topological groups of the abelianized Galois group $\mathrm{Gal}^{\mathrm{ab}}(\overline{F}/F)$ (i.e. the quotient of $\mathrm{Gal}(\overline{F}/F)$ by the closure of its commutator group) and the quotient of C_F by some closed subgroup D_F, which actually turns out to be its connected component (see [AT, ch. 9]).

Similarly if $\overline{K}$ is an algebraic closure of the p-adic K, the abelianized Galois group $\mathrm{Gal}^{\mathrm{ab}}(\overline{K}/K)$ is identified with the completion of $K^\times$ with respect to the topology of finite index subgroups.

B.1.9 However, it is more fruitful to think in dual terms, going to the Fourier transform side. Global class field theory says that you have a canonical identification of quasicharacters [1] of $\mathrm{Gal}(\overline{F}/F)$ and finite order characters of C_F. Indeed, because $\mathrm{Gal}(\overline{F}/F)$ is profinite, a quasicharacter χ of $\mathrm{Gal}(\overline{F}/F)$ factorizes through a finite abelian quotient of $\mathrm{Gal}(E/F)$ hence corresponds via τ_E to a finite order character of $C_F/N_{E/F}(C_E)$. By (1.8.1) the induced character π_χ of C_F does not depend on the choice of E. Similarly local class field theory identifies quasicharacters of $\mathrm{Gal}(\overline{K}/K)$ (necessarily of finite order) and finite order characters of $K^\times$.

I shall now make those identifications more explicit since the Langlands conjectures suggest a generalization, involving not only quasicharacters of the Galois group i.e. representations of dimension 1, but representations of higher dimension.

If K is our p-adic field, the Galois group $\mathrm{Gal}(\overline{K}/K) = G_K$ has a canonical closed subgroup, the inertia group I_K , which is the subgroup fixing the union K^{nr} of the finite unramified extensions of K in $\overline{K}$. The quotient G_K/I_K is a topological group free with one topological generator Φ_K (the Geometric Frobenius element) acting as $x \mapsto x^{1/|k_K|}$ on the residue field of K^{nr}. If a quasicharacter χ of $\mathrm{Gal}(\overline{K}/K) = G_K$ is *unramified*, i.e. if χ is trivial on I_K, then the corresponding quasicharacter $\pi(\chi)$ of $K^\times$ is also *unramified* i.e. trivial on $U_K^\times$ and its value an a uniformizer $\widetilde{\omega}_K$ is $\pi(\chi)(\widetilde{\omega}_K) = \chi(\Phi_K)$.

If now F is our number field, and v a place of F, then any extension of v to the algebraic closure $\overline{F}$ of F yields an algebraic closure $\overline{F}_v$ of F_v and an embedding of $\mathrm{Gal}(\overline{F}_v/F_v)$ as a topological subgroup of $\mathrm{Gal}(\overline{F}/F)$. (The choice of another extension of v yields a conjugate embedding). If χ is a quasicharacter of $\mathrm{Gal}(\overline{F}/F)$ then it is *unramified almost everywhere*, meaning that its restriction χ_v to $\mathrm{Gal}(\overline{F}_v/F_v) = G_{F_v}$ is, for almost every finite v of F, trivial on the inertia group I_{F_v}, hence is characterized by its value on the Frobenius element $\chi_v(\Phi_{F_v})$.

Similarly, a quasicharacter λ of C_F (or, which amounts to the same but is more customary, a quasicharacter λ of $\mathbb{A}_F^\times$ trivial on $F^\times$) is *unramified almost everywhere*, meaning that for almost all finite places v of F, the restriction λ_v of λ to the subgroup $F_v^\times$ of $\mathbb{A}_F^\times$ is unramified.

If χ is a quasicharacter of $\mathrm{Gal}(\overline{F}/F)$ then the quasicharacter $\pi(\chi)$ attached to χ by class-field theory is the **unique one** such that, for every place v of F where χ_v is unramified, $\pi(\chi)_v$ is also unramified and verifies $\pi(\chi)_v(\varpi_{F_v}) = \chi_v(\Phi_{F_v})$ at uniformizers ϖ_{F_v} of F_v.

Of course then, at **any** place v of F, the quasicharacter $\pi(\chi)_v$ of $F_v^\times$ is associated by local class-field theory to the quasicharacter χ_v of $\mathrm{Gal}(\overline{F}_v/F_v)$.

B.1.10 R.P. Langlands advocated, to try and understand better the structure of $\mathrm{Gal}(\overline{F}/F)$, to look at higher dimensional representations, and associate them with automorphic representations of the groups $\mathrm{GL}_n(\mathbb{A}_F)$. It is our task in the following chapters to describe the conjectures (as before, there are variants over number fields and p-adic fields, strongly interrelated) and to describe also the local-global principle which has led Harris and Taylor ([Ha 2], [HT], see also [He]) to a complete proof of the conjectures for p-adic fields.

[1](For us a quasicharacter of a topological group is a continuous homomorphism into $\mathbb{C}^\times$; it is a character if its values have absolute value 1. A finite order quasicharacter is thus a character.)

B.2 Langlands' conjectures for GL(n) over number fields

B.2.1 So the idea is to try and understand better the Galois group $\mathrm{Gal}(\overline{F}/F)$ by investigating its representations on finite dimensional complex vector spaces. For a topological group like $\mathrm{Gal}(\overline{F}/F)$, the natural notion is that of continuous representations. Because $\mathrm{Gal}(\overline{F}/F)$ is profinite and $\mathrm{GL}_n(\mathbb{C})$ has a neighbourhood of 1 containing no non-trivial subgroup, a **continuous** representation $\sigma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{\mathbb{C}}(V)$ on a finite dimensional complex vector space V is the same as a representation which factorizes through a **finite** discrete quotient $\mathrm{Gal}(E/F)$, where E is a finite Galois extension of F in $\overline{F}$. So in effect we are looking at all finite Galois extensions E of F, and the (usual) complex representations of $\mathrm{Gal}(E/F)$.

Note that there is a natural obvious notion of isomorphism for such representations of $\mathrm{Gal}(\overline{F}/F)$, and the isomorphism classes of complex 1-dimensional representations of $\mathrm{Gal}(\overline{F}/F)$ correspond exactly to the quasicharacters $\mathrm{Gal}(\overline{F}/F) \to \mathbb{C}^\times$.

B.2.2 However there are other, arguably more important, sources of representations of $\mathrm{Gal}(\overline{F}/F)$. Indeed, consider an algebraic variety X over F, proper and smooth, say. If ℓ is a prime number, then the machinery of étale ℓ-adic cohomology gives finite-dimensional vector spaces over $\mathbb{Q}_\ell$, $H_\ell = H^i(X(\overline{F}), \mathbb{Q}_\ell)$, for integers i between 0 and $2\dim(X)$. On those vector spaces, the group $\mathrm{Gal}(\overline{F}/F)$ acts via its action on $X(\overline{F})$, and we get continuous homomorphisms ("ℓ-adic representations") $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(H^i_\ell)$.

Variants allow to put coefficient systems on X, to consider more general varieties or even motives [Mot], or to extend the coefficient-field $\mathbb{Q}_\ell$ to finite extensions or to an algebraic closure $\overline{\mathbb{Q}}_\ell$.

Such ℓ-adic representations usually have **infinite** image.

Example. Assume $F = \mathbb{Q}$ and take for X an elliptic curve E over $\mathbb{Q}$. In particular $E(\overline{\mathbb{Q}})$ is an abelian group. For each positive integer r we can consider the subgroup $E[\ell^r]$ of $E(\overline{\mathbb{Q}})$ made out of points killed by ℓ^r. As r varies, they form a projective limit $T_\ell(E)$, the ℓ-adic Tate module, which is a $\mathbb{Z}_\ell$-free module of rank 2. The $\mathbb{Q}_\ell$-vector space $V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ carries a (continuous) representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which is in fact a dual to $H^1(E(\overline{\mathbb{Q}}), \mathbb{Q}_\ell)$. That representation always has infinite image; indeed if you compose it with the determinant $\mathrm{Aut}_{\mathbb{Q}_\ell}.V_\ell(E)) \to \mathbb{Q}_\ell^\times$, you find the ℓ-adic cyclotomic character $\chi_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$, which is such that $\tau(\zeta) = \varepsilon^{\chi_\ell(\tau) \bmod \ell^n}$ for any root of unity ζ of order ℓ^n and any $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; the cyclotomic character has infinite image $\mathbb{Z}_\ell^\times$.

If moreover the elliptic curve E over $\mathbb{Q}$ has complex multiplication by an imaginary quadratic field I, then it is known that there exists a quasicharacter Θ of C_I, which is **not of finite order**, such that for any prime number ℓ the representation on $V_\ell(E)$ can be obtained from Θ (by an induction process, see § 3 on Weil groups) . (For elliptic curves the reader can consult [Lg 2, Hu, Kn, Si]).

All this shows not only that we should consider ℓ-adic representations besides complex representations, but also that some such representations could be connected with quasi-characters of idele class groups which are more general than finite order characters.

B.2.3 The continuous complex representations of $\mathrm{Gal}(\overline{F}/F)$, or its continuous ℓ-adic representations coming from geometry as above, share common properties.

In particular, if σ is such a representation on a space V, then for each place v of F, we can restrict σ to the subgroup $\mathrm{Gal}(\overline{F}_v/F_v)$ of $\mathrm{Gal}(\overline{F}/F)$, thus yielding a representation σ_v on the same space V, called the **component of** σ **ar** v. As the embedding of $\mathrm{Gal}(\overline{F}_v/F_v)$ into $\mathrm{Gal}(\overline{F}/F)$ is well-defined only up to inner automorphisms, only the **isomorphism class** of σ_v is well-defined – in fact we shall only really work with isomorphism classes of representations. Moreover, for almost all finite places v of F, σ_v is **unramified**, i.e. trivial on the inertia group I_{F_v}, in which case σ_v is determined by a unique automorphism $\sigma_v(\Phi_{F_v})$ of V – again only its conjugacy class in $\mathrm{Aut}(V)$ is well-defined. In fact it is known that if σ is semisimple then the conjugacy classes of $\sigma_v(\Phi_{F_v})$'s for almost all v (or even their traces) determine σ up to isomorphism [Se, I.2.3].

B.2.4 What Langlands proposed already some time ago [Ll 1], and which has been precised and refined over the years, is to relate Galois representations as above to automorphic forms; more precisely n-dimensional representations of $\mathrm{Gal}(\overline{F}/F)$ should be related to automorphic representations of the locally compact group $\mathrm{GL}_n(\mathbb{A}_F)$ (see [Corv], the standard reference on automorphic forms and representations).

The topology on $\mathrm{GL}_n(\mathbb{A}_F)$ is defined as in the case of $\mathbb{A}_F$ or $\mathbb{A}_F^\times$; note that $\mathbb{A}_F^\times$ is none other that $\mathrm{GL}_1(\mathbb{A}_F)$. For a finite set S of finite places of F, containing all infinite places, we put the product topology on the subgroups $\mathrm{GL}_n(\mathbb{A}_F^S) = \Pi_{v\in S}\,\mathrm{GL}_n(F_v)\Pi_{v\notin S}\,\mathrm{GL}_n(\mathcal{O}_{F_v})$, and $\mathrm{GL}_n(\mathbb{A}_F)$ is the union of such subgroups with the inductive limit topology. For each place v of F, $\mathrm{GL}_n(F_v)$ embeds continuous into $\mathrm{GL}_n(\mathbb{A}_F)$, and we also have the diagonal embedding of $\mathrm{GL}_n(F)$ into $\mathrm{GL}_n(\mathbb{A})$, which makes $\mathrm{GL}_n(F)$ into a discrete subgroup of $\mathrm{GL}_n(\mathbb{A})$.

Automorphic forms for GL_n over F are functions on the homogeneous space $\mathrm{GL}_n(F)\backslash\,\mathrm{GL}_n(\mathbb{A}_F)$ which have some degree of smoothness and satisfy some growth conditions : see [BJ] for the exact requirements. The group $\mathrm{GL}_n(\mathbb{A}_F)$ acts naturally on fonctions on $\mathrm{GL}_n(F)\backslash\,\mathrm{GL}_n(\mathbb{A}_F)$, by right translations; strictly speaking however, it does not act on automorphic forms : the subgroup $\mathrm{GL}_n(\mathbb{A}_F)_f$ of elements with trivial component at infinite places does act, but at infinite places only the Lie algebra $\mathfrak{G}_\infty$ of the Lie group $\Pi_{v\text{ infinite}}\,\mathrm{GL}_n(F_v)$, and a fixed maximal compact subgroup K_∞, act.

B.2.5 Automorphic representations are (nearly) the irreducible representations of $\mathrm{GL}_n(\mathbb{A})$ occurring as subquotients of the representation on the space of automorphic forms; more precisely, since the group $\mathrm{GL}_n(\mathbb{A})$ does not act on automorphic forms, we have to consider only the joint action of $\mathrm{GL}_n(\mathbb{A}_F)_f$ and $\mathfrak{G}_\infty$, K_∞, and take an irreducible subquotient for that joint action.

Because $\mathrm{GL}_n(\mathbb{A}_F)$ is rather close to the product of the groups $\mathrm{GL}_n(F_v)$, an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ can be decomposed [Fl] as a kind of generalized tensor product $\Pi = \bigotimes\limits_v \prod\limits_v$, where Π_∞ the tensor product of the Π_v's for infinite v is a $(\mathfrak{G}_\infty, K_\infty)$-module, whereas, for a finite place v, Π_v is a **smooth irreducible** representation of $\mathrm{GL}_n(F_v)$.

Then Π_v is called the component of Π at v; it is well-defined only up to isomorphism.

Here a **smooth** representation of $\mathrm{GL}_n(F_v)$ on a complex vector space W is a homomorphism of $\mathrm{GL}_n(F_v)$ into $\mathrm{Aut}_{\mathbb{C}}(W)$ such that every vector w in W has open stabilizer in $\mathrm{GL}_n(F_v)$. (Of course, the same definition applies to representations of $\mathrm{GL}_n(K)$ for our p-adic field K). This smoothness condition is very strong and makes possible a nearly completely algebraic investigation of smooth representations see, e.g [BZ, Ct, Z]. The smooth representation is irreducible if W is non zero and has no subspace, other than W and $\{0\}$, invariant under $\mathrm{GL}_n(F_v)$.

If $\Pi = \bigotimes_v \prod_v$ is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ as above, then for almost all finite places v of F, Π_v is **unramified**, meaning that there is a non-zero vector in the space of Π_v which is fixed under the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_{F_v})$ of $\mathrm{GL}_n(F_v)$. Such smooth irreducible unramified representations of $\mathrm{GL}_n(F_v)$ have been classified by Satake (see [Ct]) : their isomorphism classes are parametrized by n-tuples $(\alpha_1, \ldots, \alpha_n)$ of complex numbers, up to ordering.

Note : most automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, and most smooth irreducible representations of $\mathrm{GL}_n(F_v)$, are infinite dimensional when $n > 1$. When $n = 1$ however they all have dimension 1, and, for example, the isomorphism classes of automorphic representations of $\mathrm{GL}_1(\mathbb{A}_F)$ can be identified with the quasicharacters of $\mathbb{A}_F^\times$ trivial on $F^\times$; the component at the place v is simply the restriction to $F_v^\times$.

B.2.6 Semisimple representations of $\mathrm{Gal}(\overline{F}/F)$ are direct sums of irreducibles. In an analogous, but more complicated manner, all automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ can be obtained from building blocks, the **cuspidal** automorphic representations of $\mathrm{GL}_r(\mathbb{A}_F)$ for $r \leq n$: see [Ll 3]. A cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ is characterized, up to isomorphic, by giving Π_v at almost all finite place v of F [JS 1].

Example : Let N be a positive integer. A classical modular form f of some weight $k \geq 1$ and level N [Lg 3, Kn] can be translated into an automorphic form on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ [Ge]; if f is an eigenform for the Hecke operators T_p, $p \nmid N$, then the corresponding automorphic form gives rise to an automorphic representation $\Pi = \Pi_\infty \otimes \bigotimes_p \prod_p$ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$; the component Π_∞ is determined by the weight k, whereas Π_p is unramified at primes $p \nmid N$, the parameters for Π_p being determined by the eigenvalue of T_p on f. The automorphic representation Π is cuspidal if and only if the modular form f is parabolic.

B.2.7 We can now express the conjectures of Langlands for GL_n over F, at least for complex Galois representations.

Conjecture 1. Let Σ be an irreducible, n-dimensional, complex representation of $\mathrm{Gal}(\overline{F}/F)$. Then there should exist an automorphic cuspidal $\Pi = \Pi(\Sigma)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ **associated** to Σ in the sense that, for almost all finite places v of F, where both Σ_v and Π_v are unramified, the n-tuple of complex numbers $(\alpha_1(v), \ldots, \alpha_n(v))$ parametrizing Π_v is the n-tuple of eigenvalues of $\Sigma_v(\Phi_{F_v})$.

Note that Π, if it exists at all, is determined up to isomorphism, and that two irreducible complex representations Σ and Σ' yielding the same automorphic cuspidal representation $\Pi(\Sigma) = \Pi(\Sigma')$ are isomorphic.

B.2.8 Of course, when $n = 1$, the conjecture is true because of class field theory. There are relatively few cases, when $n > 1$, where Conjecture 1 is known.

Most of those cases are for $n = 2$. When n is 2, the image of Σ in the projective linear group can be dihedral, or isomorphic to one of $\mathfrak{a}_4$, $\mathfrak{S}_4$, $\mathfrak{a}_5$. When that image is dihedral, the conjecture was proved long ago [JL]; when it is $\mathfrak{a}_4$, the conjecture was proved by Langlands [Ll 4] and for $\mathfrak{S}_4$ by Langlands and Tunnell [Tu]. Only recently has there been progress in the case where the projective image is $\mathfrak{a}_5$. Indeed recent work of Buzzard, Dickinson, Shepherd-Barron and Taylor gives the following result, when $F = \mathbb{Q}$.

THEOREM [BDST]. — *Let* $\Sigma : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ *be a continuous representation with projective image* $\mathfrak{a}_5$. *Assume that* Σ *is odd, i.e. that* $\det \Sigma$ *takes the value* -1 *on complex conjugations in* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Assume further either of* a) *or* b)

a) Σ_3 *and* Σ_5 *are unramified and* $\Sigma_5(\Phi_{\mathbb{Q}_5})$ *has order* 2 *in* $\mathrm{PGL}_2(\mathbb{C})$.

b) Σ_2 *and* Σ_3 *are unramified and* $\Sigma_2(\Phi_{\mathbb{Q}_2})$ *has order* 3 *in* $\mathrm{PGL}_2(\mathbb{C})$.

Then there is an automorphic Π *of* $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ *associated to* Σ, *which is in fact attached to a parabolic modular form of weight* 1 *as above.*

The only general case where $\Pi(\Sigma)$ is known to exist, for $n > 2$, is when the image of Σ is nilpotent [AC, III].

B.2.9 It is clear from class field theory that one should expect a local counterpart to Conjecture 1. Let us first consider finite places; we then get the following conjecture for the p-adic field K.

Conjecture 2 . There exists a **natural** way to attach, to each isomorphism class σ of n-dimensional complex representations of $\mathrm{Gal}(\overline{K}/K)$, an isomorphism class $\pi = \pi(\sigma)$ of smooth irreducible representations of $\mathrm{GL}_n(K)$.

Note that we have not imposed any irreducibility condition on σ : indeed if Σ is an irreducible complex representation of $\mathrm{Gal}(\overline{F}/F)$, of dimension > 1, then Σ_v will in general be reducible, for example when it is unramified. In fact σ should be irreducible if and only if $\pi = \pi(\sigma)$ is supercuspidal i.e. has coefficients compactly supported mod. centre. As for number fields the supercuspidal representations are the building blocks which the construction of all smooth irreducible ones [BZ, Ct, Z].

That conjecture 2 is rather vague : we have to give a meaning to "natural". There are several ways to do that. Firstly, if σ is unramified, then certainly $\pi(\sigma)$ should be the class of smooth irreducible unramified representations of $\mathrm{GL}_n(K)$ with parameters given by the eigenvalues of $\sigma(\Phi_K)$. Seccondly, we want an even greater compatibility with conjecture 1 : if K is isomorphic to the completion F_v of F at some finite place v and Σ, $\Pi = \Pi(\Sigma)$ are as in conjecture 1, then the class of representations of $\mathrm{GL}_n(K)$ obtained from $\Pi(\Sigma)_v$ by transport of struccture should be $\pi(\sigma)$ where σ is the class of representations of $\mathrm{Gal}(\overline{K}/K)$ obtained from Σ_v.

B.2.10 In fact this suggests a way to prove conjecture 2, by a local-global principle : first prove conjecture 1 for as many Σ's as we can; then try and prove that the assignment of $\Pi(\Sigma)_v$ to Σ_v at finite places is purely local i.e. depends only on F_v and Σ_v and not on F and Σ. Then show that this allows to produce the natural map expected in conjecture 2.

This is indeed what works in [Ha 2, Ha T, He], but using ℓ-adic representations Σ instead of complex ones; indeed as explained above the supply of complex representations Σ for which $\Pi(\Sigma)$ is known to exist is too small; even, the existence of $\Pi(\Sigma)$, as in the theorem B.2.8, is often proved by a detour via ℓ-adic representations!

We shall come to ℓ-adic representations shortly, but we have to remark that it would be better, in conjecture 2, to express the naturality condition in purely local terms, rather than by the implied compatibility with any potential case of conjecture 1 (of course it would be even nicer to have a purely local proof of conjecture 2, but that is still a long way, I guess). Harris and Taylor [Ha T] in fact show that the cohomology of varieties constructed by Drinfeld provide a geometric model where one can see the correspondence $\sigma \mapsto \pi(\sigma)$ of conjecture 2, at least when σ is irreducible (see Part A, II)). They also show following [Ha 2], that the correspondence can be characterized in non-geometrical terms, by the preservation of some local invariants, the L and ε factors. Our paper [He] offers a simpler approach to that last characterization.

B.2.11 Before we turn to ℓ-adic representations, we should say what happens at the infinite places of F. At such an infinite place v, the local component of an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is not exactly a representation of $\mathrm{GL}_n(F_v)$; it is rather what is called an admissible irreducible $(\mathfrak{g}, K)$-module (see [Wa]). Langlands long ago classified [Ll 2] the admissible irreducible $(\mathfrak{g}, K)$-modules : for GL_n over the archimedean field F_v, their isomorphism classes are parametrized exactly by the isomorphism classes of n-dimensional representations of the **Weil group** W_{F_v}. The Weil group is bigger than the Galois group $\mathrm{Gal}(\overline{F}_v/F_v)$ (as we shall see later, there are versions over number fields and p-adic fields). When F_v is isomorphic to $\mathbb{C}$, the Weil group is simply $F_v^\times$; when F_v is isomorphic to $\mathbb{R}$, it is a non-trivial extension

$$1 \to \overline{F}_v^\times \to W_{F_v} \to \mathrm{Gal}(\overline{F}_v/F_v) \to 1$$

where $\mathrm{Gal}(\overline{F}_v/F_v)$ acts naturally on $\overline{F}_v^\times$.

Many irreducible admissible $(\mathfrak{g}, K)$-modules can occur as components at infinite places of automorphic cuspidal representations Π of $\mathrm{GL}_n(\mathbb{A}_F)$. However, if Π is associated to an irreducible complex representation Σ of $\mathrm{Gal}(\overline{F}/F)$ as in conjecture 1, then for each infinite place v of F, Π_v should correspond to Σ_v seen as a representation of W_{F_v} via the quotient map $W_{F_v} \to \mathrm{Gal}(\overline{F}_v/F_v)$, and indeed this is true whenever conjecture 1 has been established.

In any case, complex representations of $\mathrm{Gal}(\overline{F}/F)$ only account for those automorphic cuspidal representations with very special components af infinity.

B.2.12 Let us now turn to ℓ-adic representations of $\mathrm{Gal}(\overline{F}/F)$. For us here, an ℓ-adic representation of $\mathrm{Gal}(\overline{F}/F)$ will be a continuous homomorphism $\Sigma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$, where V is a finite dimensional $\overline{\mathbb{Q}}_\ell$ vector space, satisfying the following properties

(i) V is defined over a finite extension of $\mathbb{Q}_\ell$;

(ii) V is unramified at almost all finite places of F (see [Ra] for the necessity of that last hypothesis).

Certainly the representations coming from ℓ-adic étale cohomology of smooth and proper varieties over F satisfy those properties.

Isomorphism, irreducibility, semisimplicity have the usual meaning. A semisimple ℓ-adic representation Σ of $\mathrm{Gal}(\overline{F}/F)$ is determined by the data of the traces $\mathrm{tr}\,\Sigma_v(\Phi_{F_v})$, for almost all (unramified) places v of F.

One would like to relate such an ℓ-adic representation Σ, at least when irreducible, to an automorphic cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ by relating, at unramified places, the eigenvalues of $\Sigma_v(\Phi_{F_v})$ to the parameters for Π_v. However, several difficulties arise :

(i) The eigenvalues of $\Sigma_v(\Phi_{F_v})$ are in $\overline{\mathbb{Q}}_\ell$ whereas the parameters for Π_v are complex numbers;

(ii) The irreducible ℓ-adic representations Σ with an associated Π should satsify some condition bearing on the restriction Σ_v at the places v of F **above** ℓ.

(iii) Accordingly the automorphic cuspidal representations Π associated to a Σ should satisfy some condition bearing on the components **at infinite places** of F.

B.2.13 Here I can give only brief indications. For details and precisions on (i) and (ii) see the work of Clozel [Cl 1] and for (iii) see, e.g., the report of Fontaine and Mazur [FM].

The theory of ℓ-adic representations of $\mathrm{Gal}(\overline{K}/K)$, when K is a p-adic field with $p \neq \ell$, is very close to the theory of complex representations of $\mathrm{Gal}(\overline{K}/K)$ [De 2, Ta 2]. The situation is entirely different when $p = \ell$, essentially because $\mathrm{Gal}(\overline{K}/K)$ has then a very large pro-ℓ-normal subgroup, the wild ramification group. There is a whole hierarchy of ℓ-adic representations for ℓ-adic fields; we refer to [Per-p] and [FM]. In any case, the condition in (ii) above is that at places v of F above ℓ, Σ_v should be a **potentially semistable** representation.

On the other hand, the condition in (iii) above is that, for any infinite place v of F, Π_v should be **algebraic** i.e. parametrized by an algebraic complex representation $\sigma(\Pi_v)$ of W_{F_v} ; here $\sigma(\Pi_v)$ is **algebraic** if its restriction to the subgroup $\overline{F}_v^\times \simeq \mathbb{C}^\times$ is a sum of algebraic characters of the form $z \mapsto z^a \overline{z}^b$ for $z \in \mathbb{C}^\times$ with integers a, b. In fact one expects a very close relationship between the components of Π at infinite places (e.g. the exponents a, b in the algebraic characters occurring in $\sigma(\Pi_v)$) and the components of Σ at places above ℓ (e.g. the Hodge-Tate numbers). I refer to [Cl 1] for such considerations. Actually this is not quite $\sigma(\Pi_v)$ which should be algebraic but a twist of it by some positive real valued character. I shall neglect that complication here, although it can be annoying sometimes cf. [Ha T § 11], [He SA § 5].

B.2.14 Suppose now that Π is an automorphic cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is algebraic at infinite places in the above sense. Then it is expected that Π is defined over a number field; more precisely there should be a number field E in $\mathbb{C}$ such that the tensor product Π_f of the Π_v's, over finite places v, has a model over E. This is in fact proved by Clozel [Cl 1] when Π_v verifies some further regularity condition at infinity. Moreover, there should be, for any finite place λ of E of residue characteristic ℓ, say, a continuous representation $\Sigma_\lambda = \Sigma(\Pi)_\lambda : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E_\lambda)$ which is associated to Π is the sense that at every finite place v of F, not above ℓ, where Π_v is unramified, $\Sigma_{\lambda,v}$ is also unramified, the characteristic polynomial of $\Sigma_{\lambda,v}(\Phi_{F_v})$ being in $E[T]$ and equal (in $\mathbb{C}[T]$) to $\Pi_{i=1}^n(1-\alpha_i(v)T)$ where $(\alpha_1(v), \ldots, \alpha_n(v))$ in the n-tuple parametrizing Π_v. (Again, to be entirely correct, there is a slight twist which I shall neglect here). Of course one also expects that $\Sigma_{\lambda,v}$ is potentially semistable at finite places v above the residue characteristic ℓ of E_λ, and that there is some precise relationship between $\Sigma_{\lambda,v}$ and Π_v, Π_∞, at such a place v.

Conversely any ℓ-adic representation Σ as in B.2.12, which is moreover potentially semistable at places above ℓ, should be obtained from some $\Sigma(\Pi)_\lambda$ as above, by extending the scalars from E_λ to $\overline{\mathbb{Q}}_\ell$.

B.2.15 In fact it is presumed that the converse statement is much more difficult than the direct one. For example, when $n = 2$, it has been known for some time how to attach ℓ-adic representations to classical modular forms of weight $k \geq 2$ [De 1] or even $k = 1$ [DS]. But it is only recently, through the works of Wiles [Wi], Taylor-Wiles [TW] and Breuil-Conrad-Diamond-Taylor [BCDT], that the Shimura-Tamiyama-Weil conjecture was proved, saying that each elliptic curve over $\mathbb{Q}$ is **modular**. In our terms :

THEOREM. — *Let E be an elliptic curve over $\mathbb{Q}$. Then there exists a modular parabolic newform f of weight 2, with rational, even integral, coefficients, such that the cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to f has for each prime number ℓ an associated ℓ-adic representation $\Sigma(\Pi)_\ell$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which is simply given by the action on $V_\ell(E) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$.*

In more concrete terms, if $f(x) = \Sigma_{n \geq 1} a_n q^n$ is the development of f in $q = \exp(2\pi i z)$, then for each prime number p not dividing the level of f, E has good reduction at p and has exactly $p + 1 - a_p$ points over $\mathbb{Z}/p\mathbb{Z}$.

Remark. When $n = 1$, the conjectures of B.2.14 are true and can be deduced, with some work, from class field theory. See [Se Chap. II and He 1] for those and more generally for abelian ℓ-adic representations.

B.2.16 The result of Harris and Taylor quoted in part A (I.7, Global correspondence), which generalizes some earlier resultz of Kottwitz and Clozel [Ko, Cl 2, 3, CL], is of the sort looked for in B.2.15. Indeed, if F is a CM-field, and Π is an automorphic cuspidal representation of $GL_n(\mathbb{A}_F)$ satisfying some conditions – admittedly rather stringent but not precluding many examples – then there are associated Galois representations $\Sigma(\Pi)_\lambda$. The only catch is that there is an integer $a \geq 1$ such that dim $\Sigma(\Pi)_\lambda$ is *an* instead of the expected dimension n, and accordingly the characteristic polynomial of $\Sigma(\Pi)_\lambda(\Phi_{F_v})$ for unramified $\Sigma(\Pi)_{\lambda,v}$, is the a-th power of what it should be. There is some hope of getting Σ_λ's of the right dimension, by an argument of Taylor cf. [Ha 1 § 4]. In any case, in the application to the local correspondence, one can get Σ_λ's with the right dimension n (see, e;g. [He JA § 5]). We shall turn to that application in B.4, but only after discussing Weil groups for number fields and p-adic fields, and also L and ε-factors, which are all essential ingredients to the proof of the Langlands conjectures for GL_n over p-adic fields.

B.3. L and ε factors

B.3.1 It is now appropriate to say more about L and ε factors. They are essential invariants for Galois or automorphic representations, and they also provide crucial ingredients in the proofs. Emil Artin, already in the twenties, defined L-functions for continuous complex representations of $\mathrm{Gal}(\overline{F}/F)$. Nowadays, that can be presented as follows (see [Ta 2], for example).

If σ is a continuous representation of $\mathrm{Gal}(\overline{K}/K)$ (where K is our p-adic field), on a finite dimensional complex vector space V, then on the subspace V^{I_K} of vectors fixed by I_K, we get an unramified representation σ_I. The L factor of σ is defined, as a function of a complex parameter s, by $L(\sigma, s) = \det(\mathrm{Id} - \sigma_I(\Phi_K)|k_K|^{-s})^{-1}$ (this is the determinant of an endomorphism of V^{I_K}). This L factor is a rational function in $|k_K|^{-s}$, with complex coefficients.

Similarly it is possible to define L factors for continuous complex representations of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ or $\mathrm{Gal}(\mathbb{C}/\mathbb{C})$; such factors involve the gamma function : for example the factor for the trivial character of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is $(2\pi)^{-s/2}\Gamma(s/2)$.

If F is our number field, and Σ a continuous complex representation of $\mathrm{Gal}(\overline{F}/F)$, we can form the **Artin L function**, given by the Euler product

$$L(\Sigma, s) = \Pi_v L(\Sigma_v, s), \quad \text{product over all places } v\ .$$

As a function of a complex variable, it converges absolutely for $\mathrm{Re}(s) > 1$. When F is $\mathbb{Q}$ and Σ is the trivial character, the product over prime numbers is none other than Riemann's zeta function. More generally if Σ is a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, it factorizes through the Galois group of a cyclotomic extension, hence corresponds (cf. B.1.7) to a character χ of $(\mathbb{Z}/N\mathbb{Z})^\times$ for some positive integer N, and the product over prime numbers is simply Dirichlet's L function $\Pi_{p\nmid N}(1 - \chi(p)p^{-s})^{-1}$.

B.3.2 Now it is well known that Riemann's zeta function, as well as Dirichlet's L functions, extend to meromorphic functions of s, with a functional equation relating s to $1-s$. The same is true for Artin L functions : for a general Σ as above, $L(\Sigma, s)$ extends to a meromorphic function on the whole complex plane, and if $\Sigma^\vee$ denotes the contragredient representation to Σ (the natural representation of $\mathrm{Gal}(\overline{F}/F)$ on the space dual to that of Σ), there is a functional equation

$$L(\Sigma; s) = \varepsilon(\Sigma, s)L(\Sigma^v, 1 - s),$$

where $\varepsilon(\Sigma, s)$ is a monomial αN^{-s} for some complex number α and some positive integer N. The way such things are proved is the following :

1) If Σ is one-dimensional, it corresponds by class field theory to a quasicharacter $\Pi(\Sigma)$ of $F^\times\backslash\mathbb{A}_F^\times$. For such a quasicharacter, an L-function can be defined which is known to have those properties (see below B.3.3); the equality $L(\Sigma, s) = L(\Pi(\Sigma), s)$, actually the equality of the corresponding factors at all places, implies the above properties for Σ.

2) If E is a finite extension of F in $\overline{F}$, and Σ' is a continuous complex finite dimensional representation of $\mathrm{Gal}(\overline{F}/E)$, then we can induce Σ' to $\mathrm{Gal}(\overline{F}/F)$ to get a representation $\Sigma = \mathrm{Ind}\,\Sigma'$ of $\mathrm{Gal}(\overline{F}/F)$. By construction, $L(\mathrm{Ind}\,\Sigma', s) = L(\Sigma, s)$;

there is even an equality for the local factors, corresponding to the extensions E_w/F_v of local fields.

3) By a theorem of Brauer, each continuous complex representation Σ of $\mathrm{Gal}(\overline{F}/F)$ is a linear combination $\Sigma = \Sigma_i n_i \,\mathrm{Ind}(\Sigma_i)$, where n_i is an integer, F_i a finite extension of F in $\overline{F}$, Σ_i a finite order character of $\mathrm{Gal}(\overline{F}/E_i)$. Accordingly we have

$$L(\Sigma, s) = \Pi_i L(\Sigma_i, s)^{n_i} \ ,$$

and meromorphicity and the functional equation follow.

Artin **conjectured** that $L(\Sigma, s)$ is even entire when Σ is **irreducible** of dimension > 1. As we shall see it is essentially a consequence of conjecture 1 of B.2.7.

B.3.3 I have just mentioned that a quasicharacter χ of $F^\times \backslash A_F^\times$ has an L-function with the right properties. In his thesis [CF, Lg 1], Tate gave a proof of that, where the global functional equation appears as a consequence of local functional equations for the component χ_v at all places v. This was progressively extended to L-function for automorphic representations of GL_n over F [GJ, JPSS 1], and even for a pair π, π' of automorphic representations of GL_n, $\mathrm{GL}_{n'}$ respectively [JPSS 2, JS 1, JS 2] see also [Sh 1, 2].

The pattern is the same in all situations. The first step is the functional equation for a p-adic field K (there is also a similar theory for infinite places, a bit more complicated). Let us describe it briefly for a pair π, π' of smooth irreducible representations of $\mathrm{GL}_n(K)$, $\mathrm{GL}_{n'}(K)$ respectively (to get the case of a single π, omit π'; to get the case of a quasicharacter of $K^\times$, take $n = 1$).

B.3.4 A space of integrals is defined, called zeta integrals, which depends on π, π' and auxiliary data. They are rational functions in q^{-s}, where $q = |k_K|$, and they generate a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ containing the constants. Such a fractional ideal has a unique generator of the form $P(q^{-s})^{-1}$ where $P \in \mathbb{C}[T]$, $P(0) = 1$. One puts $L(\pi \times \pi', s) = P(q^{-s})^{-1}$.

For each zeta integral Z as above, there is a "dual" zeta integral $\widehat{Z}$, and there is a non-zero monomial in q^{-s}, $\varepsilon(\pi \times \pi', s, \psi)$, such that for all Z we have

$$\frac{\widehat{Z}}{L(\pi^\vee \times \pi'^\vee, 1 - s)} = \varepsilon(\pi \times \pi', s, \psi) \frac{Z}{L(\pi \times \pi', s)} \ .$$

Here ψ is a non-trivial character of the additive group K, and the process $Z \mapsto \widehat{Z}$ is some kind of Fourier transform involving ψ. Also $\pi^\vee$ denotes the representation contragredient to π i.e. the smooth irreducible representation of $\mathrm{GL}_n(K)$ on the space of linear functionals on the space of π which are **smooth**, i.e. fixed under an open subgroup of $\mathrm{GL}_n(K)$ (this is the natural duality for smooth representations).

B.3.5 The global functional equation reads, for Π, Π' automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, $\mathrm{GL}_{n'}(\mathbb{A}_F)$ respectively,

$$L(\Pi \times \Pi', s) = \varepsilon(\Pi \times \Pi', s) L(\Pi^\vee \times \Pi'^\vee, 1 - s).$$

It is obtained as a consequence of the local functional equation and some kind of Poisson summation formula. The monomial $\varepsilon(\Pi \times \Pi', s)$, as the L factor, factorises as a product over all places

$$\varepsilon(\Pi \times \Pi', s) = \Pi_v \varepsilon(\Pi_v \times \Pi'_v, \Psi_v),$$

for any choice of non-trivial additive character Ψ of $\mathbb{A}_F$ trivial on F, Ψ_v being its restriction to F_v. A very important fact is that such L functions are very often **entire**; for example $L(\Pi, s)$ is entire if Π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, $n \geq 2$.

B.3.6 If χ is a quasicharacter of $\mathrm{Gal}(\overline{K}/K)$ corresponding to the quasicharacter $\pi(\chi)$ of $K^\times$ by class field theory, then we have $L(\chi, s) = L(\pi(\chi), s)$. Similarly over the number field F if χ is a quasicharacter of $\mathrm{Gal}(\overline{F}/F)$ corresponding to the quasicharacter $\pi(\chi)$ of $F^\times \backslash \mathbb{A}_F^\times$, then $L(\chi, s) = L(\pi(\chi), s)$ because of the equality of the factors at each place v. This gives step 1 in B.3.2.

In the general setting, if π is a smooth irreducible **unramified** representation of $\mathrm{GL}_n(K)$ with parameters $(\alpha_1, \dots, \alpha_n)$ then $L(\pi, s) = \Pi_{i=1}^n (1 - \alpha_i |k_K|^{-s})^{-1}$. If π' is another smooth irreducible **unramified** representation of $\mathrm{GL}_{n'}(K)$ with parameters $(\alpha'_1, \dots, \alpha'_n)$ then $L(\pi \times \pi', s) = \Pi_{i,1}^n (1 - \alpha_i \alpha'_j |k_K|^{-s})^{-1}$. Consequently, if σ (resp. σ') is an unramified complex representation of $\mathrm{Gal}(\overline{K}/K)$ such that the eigenvalues of Φ_K are $(\alpha_1, \dots, \alpha_n)$ (resp. $(\alpha'_1, \dots, \alpha'_n)$) then we have

$$L(\sigma, s) = L(\pi, s) \ , \ L(\sigma \otimes \sigma', s) = L(\pi \times \pi', s).$$

Therefore conjecture 1 in B.2.7 can be restated by saying that if Σ is an irreducible continuous complex representation of $\mathrm{Gal}(\overline{F}/F)$ of dimension n, there should exist a cuspidal automorphic representation $\Pi = \Pi(\Sigma)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ with

$$L(\Sigma_v, s) = L(\Pi_v, s)$$

at almost all finite places v.

Of course one should expect that this holds at all places. In particular $L(\Sigma_v, s) = L(\Pi_v, s)$ should then be **entire** when $n \geq 2$, thus yielding **Artin's conjecture**, when one can associate $\Pi(\Sigma)$ to Σ. This is indeed the case for the 2-dimensional representations mentioned in B.2.8, for which Artin's conjecture is true.

B.3.7 In the context of conjecture 2 of B.2.9, it is to be expected that the correspondence $\sigma \mapsto \pi(\sigma)$ should satisfy $L(\pi(\sigma), s) = L(\sigma, s)$ and even

$$L(\pi(\sigma) \times \pi(\sigma'), s) = L(\sigma \otimes \sigma', s)$$

whenever σ, σ' are complex representations of $\mathrm{Gal}(\overline{K}/K)$.

Over the p-adic field K, the L factor are very often identically 1, hence do not give very informative invariants. However, from Tate's thesis, Langlands guessed that there are subtler invariants $\varepsilon(\sigma, s, \psi)$, depending on a non-trivial character ψ of K, and conjectured that they should be preserved under correspondence of conjecture 2

$$e(\pi(\sigma) \times \pi(\sigma'), s, \psi) = \varepsilon(\sigma \otimes \sigma', s, \psi).$$

Langlands' definition is very subtle.

Certainly, when σ is one dimensional, we can define $\varepsilon(\sigma, s, \psi)$ to be $\varepsilon(\pi(\sigma), s, \psi)$ where $\pi(\sigma)$ is the quasicharacter of $K^\times$ corresponding to σ. Also we ask

$$\varepsilon(\sigma \oplus \sigma', s, \psi) = \varepsilon(\sigma, s, \psi)\varepsilon(\sigma', s, \psi),$$

so that $\sigma \mapsto \varepsilon(\sigma, s, \psi)$ extends to a homomorphism from the Grothendieck group $R\mathcal{G}(K)$ of finite dimensional continuous complex representations of $\mathrm{Gal}(\overline{K}/K)$ to the multiplicative group of monomials in $|k_K|^{-s}$. Langlands then imposed the following further condition, inspired by step 2) in B.3.2, and called "inductivity in degree 0" : If L is a finite extension of K and $\rho \in \mathrm{R}\mathcal{G}(L)$ has degree 0, and if $\sigma = \mathrm{Ind}\ \rho$ is the element of $R\mathcal{G}(K)$ obtained by inducing ρ, then we should have

$$\varepsilon(\mathrm{Ind}\ \rho, s, \psi) = \varepsilon(\rho, s, \psi \circ \mathrm{tr}_{K/F}).$$

From Brauer's induction theorem already mentioned in B.3.2, it is clear that there is at most one way of defining such factors with the required properties. Langlands' existence proof was purely local, but was never published. Deligne found a much shorter proof, by a local-global principle [De, Ta 1]. This principle is based on the fact that if Σ is a complex representation of $\mathrm{Gal}(\overline{F}/F)$ then $\varepsilon(\Sigma, s)$ is the product of the local factors $\varepsilon(\Sigma_v, s, \Psi_v)$ where Ψ is as in B.3.5. A similar local-global principle lies behind the recent proofs of the local correspondence for GL_n i.e. of a precise version of conjecture 2 of B.2.9.

B.4. Weil groups, and the local correspondence

B.4.1 Before we give a precise – and proved – version of conjecture 2 of B.2.9, it is convenient to introduce the Weil group of the p-adic field K – Weil groups for number fields are used in the proofs, see below. For Weil groups and their representations, see [AT, § 15, De, Ta 2].

The Weil group W_K is a subgroup of $\mathrm{Gal}(\overline{K}/K)$ containing the inertia group I_K ; it is made out of the elements with image modulo I_K an integral power of the Frobenius Φ_K ; we give W_K the group topology for which I_K (with its topology as a Galois group) is open. The Weil group W_K is very close to $\mathrm{Gal}(\overline{K}/K)$ – it is dense in $\mathrm{Gal}(\overline{K}/K)$ – but it is slightly more flexible. Any representation of $\mathrm{Gal}(\overline{K}/K)$ gives a representation of W_K by restriction, but W_K has slightly more representations : in particular its quasicharacters are not necessarily of finite order, and by class field theory they are in bijection with **all** quasicharacters of $K^\times$.

Introducing the Weil group has the advantage of allowing us to state the Langlands conjectures for GL_n over K in terms of bijective maps, as we shall do now. To any continuous complex representation σ of W_K we attach an L factor $L(\sigma, s)$ and an ε factor $\varepsilon(\sigma, s, \psi)$, extending in the same manner the definition for representations of $\mathrm{Gal}(\overline{K}/K)$.

B.4.2 For each positive n, consider the set $\mathcal{G}_K^0(n)$ of isomorphism classes of irreducible continuous complex representations of W_K, of dimension n, and the set $\mathfrak{a}_K^0(n)$ of isomorphism classes of smooth irreducible supercuspidal representations of $\mathrm{GL}_n(K)$. (We have already mentioned in B.2.9 that, in a way similar to the global case, the supercuspidal representations are the building blocks for all smooth irreducible representations [BZ, Z]).

THEOREM [Ha T, He]. — *There is a unique family of bijections $\mathcal{G}_K^0(n) \to \mathfrak{a}_K^0(n)$, $\sigma \mapsto \pi(\sigma)$, satisfying the following properties :*

(i) *for $n = 1$, $\sigma \mapsto \pi(\sigma)$ is given by class field theory;*

(ii) $\pi(\sigma^\vee) = \pi(\sigma)^\vee$ *for $\sigma \in \mathcal{G}_K^0(n)$;*

(iii) *for* $\chi \in \mathcal{G}_K^0(1)$ *and* $\sigma \in \mathcal{G}_K^0(n)$, $\pi(\chi \otimes \sigma) = (\pi(\chi) \circ \det) \otimes \pi(\sigma)$ *;*

(iv) *for* $\sigma \in \mathcal{G}_K^0(n)$ *and* $\sigma' \in \mathcal{G}_K^0(n')$, *then*

$$L(\pi(\sigma) \times \pi(\sigma'), s) = \lambda(\sigma \otimes \sigma', s)$$
$$\varepsilon(\pi(\sigma) \times \pi(\sigma'), s, \psi) = \varepsilon(\sigma \otimes \sigma', s, \psi)$$

for any non-trivial additive character ψ *of* F.

B.4.3 That theorem is proved by the local-global principle sketched in B.2.10, using the cases of the global correspondence – between ℓ-adic representations of $\mathrm{Gal}(\overline{F}/F)$ and automorphic cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_F)$ – stated in A.1.7 and B.2.16. In the last part of this paper, I shall briefly sketch the argument, refering to [Ca] and [He JA] for more detail, and of course to the original papers [Ha 2, Ha T, He].

We start from the particular case of Theorem A.1.7 obtained from the analysis of the good reduction of Shimura varieties [Ko, Cl 2, Cl 3, CL] : the full theorem is a consequence of that special case and the local correspondence, see [Ha T § 11].

So we choose a totally real number field F_+ with some completion isomorphic to our p-adic field K; we also choose a quadratic imaginary field I split at p, and we consider the CM field $F = F_+ I$, which has a completion F_v isomorphic to K. By theorem A.1.7 we have a supply of cuspidal automorphic representations Π of $\mathrm{GL}_n(\mathbb{A}_F)$ which have **corresponding** ℓ-adic Galois representations $\Sigma_\ell : Gal(\overline{F}/F) \to \mathrm{GL}_{an}(\overline{\mathbb{Q}}_\ell)$, where a is a positive integer, hopefully equal to one. For each such cuspidal automorphic Π we would like to associate the class of the component Π_v to the class of the component $\Sigma_{\ell,v}$. The questions are as follows :

1) Is the assignment $\Pi_v \mapsto \Sigma_{\ell,v}$ purely local?

2) Does it preserve L and ε factors as in the theorem above?

3) Have we got enough global Π to get information on all of $\mathfrak{a}_K^0(n)$ and $\mathcal{G}_K^0(n)$?

Moreover there is the problem in questions 2) & 3) that Σ_ℓ is an ℓ-adic representation, and not a complex one!

B.4.4 Question 1 already is very serious. In [Ha T] it is proved, using geometry, that the assignment $\Pi_v \mapsto [\Sigma_{\lambda,v}]$, where $[\Sigma_{\lambda,v}]$ is the class of $\Sigma_{\lambda,v}$ in the Groghendieck group of finite dimensional ℓ-adic representations of W_{F_v}, is local. For a supercuspidal Π_v this is true because the correspondence $\Pi_v \mapsto [\Sigma_{\lambda,v}]$ can be seen in Drinfeld's geometric local model, which is local, and for general Π_v it comes out of the description of the bad reduction of Shimura varieties, which description form the essential part of [Ha T].

Another way of dealing with question 1 [He] is to show that if L and ε factors are preserved (question 2) and we have enough of the proposed local correspondance (question 3) then question 1 is also solved.

So we concentrate on questions 2 and 3, where Weil groups for number fields are used.

B.4.5 The only known way of dealing with question 2 is via the global functional equation for L functions. However, if L functions of automorphic representations – or pairs of such – are known to satisfy a functional equation (B.3.5), it is not known how to define L functions for ℓ-adic representations (especially the factors at places above ℓ and infinite places) so that they satisfy a functional equation. For ℓ-adic representations with finite image, one can use the complex L-functions, because a finite group has the "same" representation theory over all algebraically closed fields of characteristic zero, but ℓ-adic representations with finite image never appear in theorem A.I.7! So we need a larger supply of L functions with functional equations. This is provided by the (complex) representations of the Weil groups, to which we turn now.

B.4.6 A **Weil group** W_F for F is obtained as follows [AT § 15, Ta 2]. For any finite Galois extension E of F in $\overline{F}$ we have a relative Weil group $W(E/F)$, given by an extension of topological groups

$$1 \to C_E \to W(E/F) \to \mathrm{Gal}(E/F) \to 1 ,$$

with the natural action of $\mathrm{Gal}(E/F)$ on the idele class group C_E, and W_F appears as a projective limit of the $W(E/F)$'s as E varies. An essential point here is that W_F mixes the Galois groups and the ideal class groups. In particular the quasicharacters of W_F are, as in the local case, in one-to-one correspondence with the quasicharacters of C_F. Also $\mathrm{Gal}(\overline{F}/F)$ is a quotient of W_F so that representations of $\mathrm{Gal}(\overline{F}/F)$ give rise to representations of W_F.

For each place v of F, finite or infinite, there is an embedding of the local Weil group W_{F_v} into W_F, well-defined up to – essentially – inner automorphism. If Σ is a continuous complex (finite dimensional) representation of W_F, we can define its components Σ_v by restriction to W_{F_v}, and Σ_v is unramified for almost all finite places v. We also define the global L function $L(\Sigma, s)$ as the product of $L(\Sigma_v, s)$ over all v's and its converges for $Re(s) >> 0$.

Of course if Σ comes from a representation of $\mathrm{Gal}(\overline{F}/F)$ we recover the previous definition. Now it is easy to see that Σ factorizes through some relative Weil group $W(E/F)$, and, using the fact that C_E is an abelian subgroup of finite index, that $L(\Sigma, s)$ is a quotient of products of factors $L(\chi, s)$ where χ is a quasicharacter of $C_{F'}$, F' an extension of F in E. It follows that $L(\Sigma, s)$ has a global functional equation $L(\Sigma, s) = \varepsilon(\Sigma, s)L(\Sigma^\vee, 1-s)$, where $\varepsilon(\Sigma, s)$ is the product of local ε factors $\varepsilon(\Sigma_v, s, \Psi_v)$ as in B.3.5.

B.4.7 It is only natural to say that a continuous complex representation Σ of W_F, of dimension n, and a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ **are associated** (or **correspond**) if for almost all places v of F, where Σ_v and Π_v are unramified, we have $L(\Sigma_v, s) = L(\Pi_v, s)$.

A crucial fact here is that if (Σ, Π), (Σ', Π') are two such corresponding pairs then [He 2 § 4, He]

$$L(\Sigma_v \otimes \Sigma'_v, s) = L(\Pi_v \times \Pi'_v, s)$$

$$\varepsilon(\Sigma_v \otimes \Sigma'_v, s, \Psi_v) = \varepsilon(\Pi_v \times \Pi'_v, s, \Psi_v)$$

at **all** finite places v.

Another important fact is that if Σ is **algebraic** (if Σ factorizes through $W(E/F)$ as above this means that its restriction to C_E is a sum of quasicharacters with algebraic component at infinity cf. B.2.13) then there is a number field E and for each finite place λ of E, a continuous representation $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(E_\lambda)$

which has the same characteristic polynomial of Frobenius at almost all places [Se, Ch. 3].

B.4.8 M. Harris was the one to remark that Theorem A.I.7 provides many instances where a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ has associated ℓ-adic representations which in fact come from a complex continuous representation Σ of W_F, of dimension n. It follows that Σ and Π are associated.

Moreover Harris showed [Ha 2 § 4] that we can choose such pairs (Σ, Π) so that the classes $[\Sigma_v]$ we obtain generate, in the Grothendieck group of complex representations of W_{F_v}, a subgroup containing all representations factoring through a fixed finite Galois extension E_w/F_v. It follows that the assignment $\Pi_v \to [\Sigma_v]$ preserves L factors and ε factors as in Theorem B.4.2. With the geometric results of [Ha T] this proves the correspondence (theorem B.5.2) see [Ha T § 11]. My papers [He] and [He JA] offer a variant which uses only Theorem A.I.7 and not the hard geometry of [Ha T].

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