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The Langlands Correspondence for Function Fields following Laurent Lafforgue

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In June 1999, Laurent Lafforgue proved the Langlands correspondence for GL_r over a function field. His proof follows the strategy introduced by V. Drinfeld, more than 25 years ago, in the rank 2 case. In this lecture, I explain Lafforgue's theorem. I also sketch some of his argmuments in the everywhere unramified case.

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1. The statement

Let X be a smooth, projective and geometrically connected curve over a finite field $\mathbb{F}_q$ with q elements and let F be its function field.

We denote by $\mathbb{A} = \prod'_x F_x$ the topological ring of adeles of F. Here x runs through the set of places of F, or equivalently the set $|X|$ of closed points of X, and F_x is the completion of F at x. For each x we denote by $\mathcal{O}_x = \{a_x \in F_x \mid x(a_x) \geq 0\}$ the ring of integers of the local field F_x and by $\deg(x)$ the degree of its residue field $\kappa(x)$ over $\mathbb{F}_q$. There is a degree map

$$\deg : \mathbb{A}^\times \to \mathbb{Z},\ a \mapsto \sum_{x\in|X|} \deg(x)x(a_x)$$

which vanishes on $F^\times$ and $\mathcal{O}^\times$. It is well known that, for each $a \in \mathbb{A}^\times$ whose degree is non zero, the quotient $F^\times\backslash\mathbb{A}^\times/\mathcal{O}^\times a^{\mathbb{Z}}$ is finite.

Let $r \geq 1$ be an integer.

We first consider the adelic group $\mathrm{GL}_r(\mathbb{A})$. As usual we identify its center (the subgroup of scalar matrices) with $\mathbb{A}^\times$. The space of *cuspidal automorphic forms*

$$L_{\mathrm{cusp}} = L_{\mathrm{cusp}}(\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}))$$

is by definition the space of complex functions φ on $\mathrm{GL}_r(\mathbb{A})$ which satisfy the following properties:

- - $\varphi(\gamma g) = \varphi(g)$, $\forall\gamma \in \mathrm{GL}_r(F)$, $\forall g \in \mathrm{GL}_r(\mathbb{A})$,
- - there exists a subgroup $K_\varphi \subset K := \mathrm{GL}_r(\mathcal{O}) = \prod_x \mathrm{GL}_r(\mathcal{O}_x)$ of finite index such that $\varphi(gk) = \varphi(g)$, $\forall g \in \mathrm{GL}_r(\mathbb{A})$, $\forall k \in K_\varphi$,
- - there exists $a \in \mathbb{A}^\times$ such that $\deg(a) \neq 0$ and $\varphi(ga) = \varphi(g)$, $\forall g \in \mathrm{GL}_r(\mathbb{A})$,

- - for every non trivial partition $r = r_1 + \cdots + r_s$ defining a standard parabolic subgroup $P = MU \subsetneq \mathrm{GL}_r$ with unipotent radical U and Levi component $M \cong \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_s}$, we have
$$\int_{U(F)\backslash U(\mathbb{A})} \varphi(ug)du = 0,\ \forall g \in \mathrm{GL}_r(\mathbb{A}),$$
where du is any Haar measure on $U(F)\backslash U(\mathbb{A})$.

The *Hecke algebra* $\mathcal{H} = C_c^\infty(\mathrm{GL}_r(\mathbb{A}))$ is the convolution algebra of locally constant functions with compact support on $\mathrm{GL}_r(\mathbb{A})$. It acts on L_{cusp} by right convolution.

The *cuspidal automorphic representations of* $\mathrm{GL}_r(\mathbb{A})$ are by definition the simple $\mathcal{H}$-modules which occur as subquotients of L_{cusp}. We will denote by $\mathcal{A}_r$ the set of (isomorphism classes of) these representations. Any $\pi \in \mathcal{A}_r$ admits a *central character* $\omega_\pi : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ which is of finite order as we have $\omega_\pi(a) = 1$ for some $a \in \mathbb{A}^\times$ of non zero degree.

The Hecke algebra is the restricted tensor product of local Hecke algebras $\mathcal{H}_x$. Accordingly any $\pi \in \mathcal{A}_r$ is a restricted tensor product of simple $\mathcal{H}_x$-modules π_x.

Let $e_{K_x} \in \mathcal{H}_x$ be the characteristic function of the standard maximal compact subgroup $K_x = \mathrm{GL}_r(\mathcal{O}_x) \subset \mathrm{GL}_r(F_x)$. For each $\pi \in \mathcal{A}_r$ the set N_π of *ramified places* of π is by definition the finite set of places x such that
$$\pi_x * e_{K_x} = (0).$$
For any $x \notin N_\pi$ the complex vector space $\pi_x * e_{K_x}$ is an irreducible module over the commutative algebra
$$e_{K_x} * \mathcal{H}_x * e_{K_x} \cong \mathbb{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}]^{\mathfrak{S}_r}.$$
Therefore, it is one dimensional and its isomorphism class is completely determined by an unordered r-tuple
$$(z_{x,1}(\pi), \ldots, z_{x,r}(\pi))$$
of complex numbers, the so-called *Hecke eigenvalues* of π at x, or equivalently by the sequence of power sums
$$S_x^{(n)}(\pi) = z_{x,1}(\pi)^n + \cdots + z_{x,r}(\pi)^n,\ n \geq 1,$$
or by the *local L factor*
$$L_x(\pi, s) = \frac{1}{\prod_{i=1}^r (1 - z_{x,i}(\pi)q^{-s\deg(x)})}.$$

We fix a separable closure $\overline{F}$ of F and we denote by Γ_F the Galois group of $\overline{F}$ over F. We fix some prime number ℓ distinct from the characteristic of $\mathbb{F}_q$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

A *ℓ-adic representation* of Γ_F of rank r is a group homomorphism $\sigma : \Gamma_F \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ which has the following properties:

- - there exists $g \in \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ and a finite extension E_λ of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ such that $g\sigma(\Gamma_F)g^{-1} \subset \mathrm{GL}_r(E_\lambda) \subset \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$,
- - $g\sigma g^{-1} : \Gamma_F \to \mathrm{GL}_r(E_\lambda)$ is continuous for the Krull topology on Γ_F and the ℓ-adic topology on $\mathrm{GL}_r(E_\lambda)$,

- \- for all but finitely many $x \in |X|$, σ is *unramified at* x, i.e. the restriction σ_x of σ to any decomposition subgroup $D_x \subset \Gamma_F$ at x is trivial on the inertia subgroup $I_x \subset D_x$, and thus factors through the quotient $D_x/I_x \cong \Gamma_{\kappa(x)} = \mathrm{Frob}_x^{\widehat{\mathbb{Z}}}$, where Frob_x is the geometric Frobenius element in the Galois group $\Gamma_{\kappa(x)}$ of $\kappa(x)$.

We consider the set $\mathcal{G}_r$ of (isomorphism classes of) irreducible ℓ-adic representations σ of Γ_F of rank r, the determinant of which is of finite order. For each $\sigma \in \mathcal{G}_r$ we denote by N_σ its finite set of ramified places. For any $x \notin N_\sigma$ we denote by

$$(z_{x,1}(\sigma), \dots , z_{x,r}(\sigma))$$

the unordered r-tuple of the eigenvalues of $\sigma_x(\mathrm{Frob}_x)$, by

$$S_x^{(n)}(\sigma) = z_{x,1}(\sigma)^n + \cdots + z_{x,r}(\sigma)^n,\ n \geq 1,$$

the corresponding sequence of power sums and by

$$L_x(\sigma, s) = \frac{1}{\prod_{i=1}^{r}(1 - z_{x,i}(\sigma)q^{-s\deg(x)})}$$

the corresponding *local L factor.*

We fix an isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. Each time the ℓ-adic topology of $\overline{\mathbb{Q}}_\ell$ plays no role we will freely use this isomorphism to identify $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$.

M AIN THEOREM (Langlands Correspondence). — *There exists a unique bijection*

$$\mathcal{A}_r \xrightarrow{\sim} \mathcal{G}_r,\ \pi \mapsto \sigma(\pi),$$

such that, for every $\pi \in \mathcal{A}_r$ *we have the equality of power sums*

$$S_x^{(n)}(\sigma(\pi)) = S_x^{(n)}(\pi),\ \forall n \geq 1,$$

or equivalently the equality of local L factors

$$L_x(\sigma(\pi), s) = L_x(\pi, s),$$

for all but finitely many places $x \notin N_{\sigma(\pi)} \cup N_\pi$.

For $r = 1$ this is a reformulation of the abelian class field theory in the function field case. For $r = 2$ the theorem has been proved by Drinfeld [3]. The general case $r \geq 3$ is due to Lafforgue [6].

For arbitrary r's but particular π's, some cases of the theorem had been proved earlier by Flicker and Kazhdan, and myself.

Remarks : (i) The uniqueness of the map $\mathcal{A}_r \xrightarrow{\sim} \mathcal{G}_r$ and its injectivity (assuming its existence) had been known for a long time. They respectively follow from the Čebotarev density theorem and from the strong multiplicity one theorem of Piatetski-Shapiro.

(ii) Let us fix an integer r. In order to prove the existence of the bijection $\mathcal{A}_{r'} \xrightarrow{\sim} \mathcal{G}_{r'}$ for $r' = 1, \dots , r$ it is sufficient to prove the following weaker statement for $r' = 1, \dots , r$:

(A)$_{r'}$ *For every* $\pi' \in \mathcal{A}_{r'}$ *there exists a Galois representation* $\sigma'(\pi') \in \mathcal{G}_{r'}$ *which satisfies the equality of local L factors*

$$L_x(\sigma'(\pi'), s) = L_x(\pi', s),$$

for all but finitely many places $x \notin N_{\pi'} \cup N_{\sigma'(\pi')}$.

Indeed, as was remarked by Deligne, if we have already proved the assertion $(\mathrm{A})_{r'}$ for $r' = 1, \dots, r-1$ the Grothendieck functional equation and the converse theorem of Hecke, Weil and Piatetski-Shapiro give for free the inverse maps

$$\mathcal{G}_{r'} \to \mathcal{A}_{r'},\ \sigma' \mapsto \pi'(\sigma'),$$

for $r' = 1, \dots, r$.

(iii) By standard techniques of L-functions one easily gets from the main theorem that $N_{\sigma(\pi)} = N_\pi$, that

$$L_x(\sigma(\pi), s) = L_x(\pi, s),$$

for all places $x \notin N_\pi$ and that, for each $x \in N_\pi$, the restriction of $\sigma(\pi)$ to any decomposition subgroup $D_x \subset \Gamma_F$ at x corresponds to π_x by the local Langlands correspondence. □

It is well known that the Jacquet-Shalika estimates of the Hecke eigenvalues of cuspidal automorphic representations and the main theorem imply the *Ramanujan-Petersson conjecture*:

T HEOREM (Drinfeld [2] for $r = 2$, Lafforgue [4], [6] for $r \geq 3$). — *For every* $\pi \in \mathcal{A}_r$ *and every place* $x \notin N_\pi$ *we have*

$$|z_{x,i}(\pi)| = 1,\ \forall i = 1, \dots, r.$$

□

There is a now standard strategy for constructing the map $\mathcal{A}_r \to \mathcal{G}_r$, $\pi \to \sigma(\pi)$.

The first step is to construct a "variety" V over F, equipped with an action of the Hecke algebra $\mathcal{H}$, so that its ℓ-adic cohomology

$$H^*_c(\overline{F} \otimes_F V, \overline{\mathbb{Q}}_\ell)$$

is a representation of the product of the Hecke algebra $\mathcal{H}$ and the Galois group Γ_F.

The second step is to compute the trace of this representation by the *Grothendieck-Lefschetz trace formula.*

The last step is to compare this geometric trace formula with the *Arthur-Selberg trace formula* in order to prove that the representation

$$\bigoplus_{\pi \in \mathcal{A}_r} \pi \otimes \sigma(\pi)$$

of $\mathcal{H} \times \Gamma_F$ that we are looking for occurs in $H^*_c(\overline{F} \otimes_F V, \overline{\mathbb{Q}}_\ell)$.

In the case we are considering there is an obstruction to the occurrence of the above direct sum representation into any ℓ-adic cohomology group. This strategy has thus to be slightly modified. Following Drinfeld it is the representation

$$\bigoplus_{\pi \in \mathcal{A}_r} \pi \otimes \sigma(\pi)^\vee \otimes \sigma(\pi)$$

of the product $\mathcal{H} \times \Gamma_F \times \Gamma_F$, where $\sigma^\vee$ is the contragredient representation of σ, which should occur in ℓ-adic cohomology.

Lafforgue proves the Langlands correspondence by induction on r. Assuming the Langlands correspondence $\mathcal{A}_{r'} \xrightarrow{\sim} \mathcal{G}_{r'}$ for all $1 \leq r' < r$ he constructs $\sigma(\pi)$ for each $\pi \in \mathcal{A}_r$.

Let $\mathcal{A}_r(K) \subset \mathcal{A}_r$ be the subset of everywhere unramified cuspidal automorphism representations π of $\mathrm{GL}_r(\mathbb{A})$ ($N_\pi = \emptyset$). *For simplicity we will only explain in this lecture Lafforgue's construction of $\sigma(\pi)$ for $\pi \in \mathcal{A}_r(K)$.*

2. Drinfeld shtukas

All the schemes (or stacks) that we will consider are over $\mathbb{F}_q$. We simply denote by $S \times T$ the product over $\mathbb{F}_q$ of two schemes (or stacks). If k is a field which contains $\mathbb{F}_q$ and S is a scheme (or a stack), we will also use the notation $k \otimes S = \mathrm{Spec}(k) \times S$. For each scheme (or stack) S we denote by Frob_S *its Frobenius endomorphism (relative to $\mathbb{F}_q$). For each scheme (or stack) S and each vector bundle $\mathcal{E}$ on $S \times X$ we define a new vector bundle ${}^\tau\mathcal{E}$ on $S \times X$ by*

$${}^\tau\mathcal{E} = (\mathrm{Frob}_S \times \mathrm{Id}_X)^*\mathcal{E}.$$

Let k be an algebraically closed field which contains $\mathbb{F}_q$.

A rank r vector bundle $\mathcal{E}$ on $k \otimes X$ equipped with an isomorphism ${}^\tau\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is nothing else than a rank r vector bundle on X. As it has been shown by Weil the set of isomorphism classes of rank r vector bundles on X is canonically isomorphic to the double coset space $\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A})/ \mathrm{GL}_r(\mathcal{O})$.

D EFINITION (Drinfeld [1]). — *A (right)* shtuka $\widetilde{\mathcal{E}}$ *of rank r over k is a diagram*

$$\mathcal{E} \xhookrightarrow{j_\infty} \mathcal{E}' \xhookleftarrow{j_o} {}^\tau\mathcal{E}$$

where:

- *- $\mathcal{E}$ and $\mathcal{E}'$ are two locally free $\mathcal{O}_{k\otimes X}$-Modules of rank r, or equivalently rank r vector bundles on $k \otimes X$,*
- *- j_∞ and j_o are two injective $\mathcal{O}_{k\otimes X}$-linear maps,*
- *- the torsion $\mathcal{O}_{k\otimes X}$-Modules* $\mathrm{Coker}(j_\infty)$ *and* $\mathrm{Coker}(j_o)$ *are of length* 1.

The supports $\infty, o \in X(k)$ of $\mathrm{Coker}(j_\infty)$ *and* $\mathrm{Coker}(j_o)$ *are called the* pole *and the* zero *of the shtuka.*

In other words a shtuka is a double modification of a rank r vector bundle $\mathcal{E}$,

$$\mathcal{E} \xhookrightarrow{j_\infty} \mathcal{E}' \xhookleftarrow{j'_o} \mathcal{E}''$$

(an elementary upper modification j_∞ at the point ∞ followed by an elementary lower modification j'_o at o), together with an isomorphism

$${}^\tau\mathcal{E} \xrightarrow{\sim} \mathcal{E}''.$$

The rank r shtukas are the points of a Deligne-Mumford algebraic stack Sht^r. The pole and the zero of the universal shtuka define a morphism

$$(\infty, o) : \mathrm{Sht}^r \to X \times X$$

which is smooth of pure relative dimension $2r - 2$.

The stack Sht^r has infinitely many components $(\mathrm{Sht}^{r,d})_{d\in\mathbb{Z}}$ which are indexed by the degree of the universal shtuka

$$\deg(\widetilde{\mathcal{E}}) = \deg(\mathcal{E}) = \deg(\mathcal{E}') - 1.$$

Example : For every integer d, the stack $\mathrm{Sht}^{1,d}$ is the fibered product

$$\begin{array}{ccc} \mathrm{Sht}^{1,d} & \longrightarrow & \mathrm{Bund}^{1,d} \\ \big\downarrow & \square & \big\downarrow L \\ X \times X & \xrightarrow[A]{} & \mathrm{Bund}^{1,0} \end{array}$$

where $\mathrm{Bund}^{1,d}$ is the Artin algebraic stack of line bundles of degree d on X, A is the Abel-Jacobi morphism which maps $(\infty, o) \in X(k) \times X(k)$ onto the line bundle $\mathcal{O}_{k\otimes X}(\infty - o)$, and L is the Lang "isogeny" which maps $\mathcal{L}$ onto $\mathcal{L}^{-1} \otimes_{\mathcal{O}_{k\otimes X}} {}^{\tau}\mathcal{L}$.

In particular, for every integer d the stack $\mathrm{Sht}^{1,d}$ is of finite type and admits a coarse moduli space which is a finite etale Galois covering of $X \times X$. □

But, except for $r = 1$ *none of the components* $\mathrm{Sht}^{r,d}$ *is of finite type.*

The Picard group $F^{\times}\backslash\mathbb{A}^{\times}/\mathcal{O}^{\times}$ of line bundles on X acts on the algebraic stack Sht^{r}: a line bundle $\mathcal{L}$ over X takes a rank r shtuka $\widetilde{\mathcal{E}}$ on $k \otimes X$ to

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \xhookrightarrow{\mathrm{Id}\otimes j} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E}' \xhookleftarrow{\mathrm{Id}\otimes t} \mathcal{L} \otimes_{\mathcal{O}_X} {}^{\tau}\mathcal{E} = {}^{\tau}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E}).$$

This action does not change the pole and the zero.

Any $g \in \mathrm{GL}_r(\mathbb{A})$ defines a *Hecke correspondence*

$$c = (c_1, c_2) : \mathrm{Sht}^r(g) \to \mathrm{Sht}^r \times_{X\times X} \mathrm{Sht}^r,$$

where $\mathrm{Sht}^r(g)$ is a Deligne-Mumford algebraic stack and c_1, c_2 are etale representable morphisms. This correspondence only depends on the double coset $KgK \subset \mathrm{GL}_r(\mathbb{A})$ and does not change the pole and the zero. If N_g is the finite set of places x such that $g_x \notin F_x^{\times}K_x \subset \mathrm{GL}_r(F_x)$, c_1 and c_2 are finite over $((X \setminus N_g) \times (X \setminus N_g)) \times_{X\times X} \mathrm{Sht}^r$.

If $a \in \mathbb{A}^{\times}$ is a central element in $\mathrm{GL}_r(\mathbb{A})$ the corresponding Hecke operator is nothing else that the action of the element $F^{\times}a\mathcal{O}^{\times}$ of the Picard group of X.

3. Truncations

From now on we fix $a \in \mathbb{A}^{\times}$ *such that* $\deg(a) \neq 0$ *and we assume that* $r \geq 2$.

The quotient stack

$$\mathrm{Sht}^r/a^{\mathbb{Z}} \cong \coprod_{d=1}^{r\deg(a)} \mathrm{Sht}^{r,d}$$

has finitely many components, but is not of finite type. To study its ℓ-adic cohomology we will need to *truncate* it.

As for vector bundles on Riemann surfaces it is not difficult to define the *Harder-Narasimhan polygon* of a rank r shtuka $\widetilde{\mathcal{E}}$ over an algebraically closed field $k \supset \mathbb{F}_q$.

A *subobject* $\widetilde{\mathcal{F}}$ of $\widetilde{\mathcal{E}}$ is a pair of $\mathcal{O}_{k\otimes X}$-submodules $(\mathcal{F} \subset \mathcal{E}, \mathcal{F}' \subset \mathcal{E}')$ such that

- - $j(\mathcal{F}) \subset \mathcal{F}'$ and $t({}^{\tau}\mathcal{F}) \subset \mathcal{F}'$,
- - $\mathcal{E}/\mathcal{F}$ and $\mathcal{E}'/\mathcal{F}'$ are locally free $\mathcal{O}_{k\otimes X}$-modules of the same rank.

A subobject has a *rank*

$$\mathrm{rk}(\widetilde{\mathcal{F}}) = \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F}')$$

and, for each $\alpha \in \mathbb{R}$ an *α-degree*

$$\deg_\alpha(\widetilde{\mathcal{F}}) = (1-\alpha)\deg(\mathcal{F}) + \alpha\deg(\mathcal{F}').$$

If $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are two subobjects of $\widetilde{\mathcal{E}}$ we say that $\widetilde{\mathcal{G}}$ is contained in $\widetilde{\mathcal{F}}$ and we write $\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{F}}$ if $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{G}' \subset \mathcal{F}'$.

Let $(0) = \widetilde{\mathcal{F}}_0 \subsetneq \widetilde{\mathcal{F}}_1 \subsetneq \cdots \subsetneq \widetilde{\mathcal{F}}_s = \widetilde{\mathcal{E}}$ be a filtration by subobjects of a shtuka. Its *α-polygon* is the continuous function

$$p : [0, r] \to \mathbb{R}$$

which vanishes at 0 and r, which is affine on the interval $[\mathrm{rk}(\widetilde{\mathcal{F}}_{j-1}), \mathrm{rk}(\widetilde{\mathcal{F}}_j)]$ for $j = 1, \ldots, s$, and which takes the values

$$p(\mathrm{rk}(\widetilde{\mathcal{F}}_j)) = \deg_\alpha(\widetilde{\mathcal{F}}_j) - \frac{\mathrm{rk}(\widetilde{\mathcal{F}}_j)}{r}\deg_\alpha(\widetilde{\mathcal{E}}),\ \forall j = 1, \ldots, s-1.$$

For a given $\alpha \in [0,1]$ and a given shtuka $\widetilde{\mathcal{E}}$ the set of the α-polygons of all the possible filtrations of $\widetilde{\mathcal{E}}$ admits a largest element $p_\alpha^{\mathrm{HN}}(\widetilde{\mathcal{E}})$, the so-called *Harder-Narasimhan polygon of index α* of the shtuka. It is a convex function.

The α's play a crucial role in Lafforgue's work. But in this lecture we will restrict ourself to $\alpha = 0$ and simply call *Harder-Narasimhan polygon* the polygon $p^{\mathrm{HN}} = p_0^{\mathrm{HN}}$.

We call *truncation parameter* any convex continuous function $p : [0, r] \to \mathbb{R}_{\geq 0}$ which vanishes at 0 and r and which is affine on each interval $[i-1, i]$ for $i = 1, \ldots, r$.

P ROPOSITION (Lafforgue [4]). — *For each truncation parameter p there exists a unique open substack*

$$\mathrm{Sht}^{r;\,\leq p} \subset \mathrm{Sht}^r$$

such that, for any algebraically closed field $k \supset \mathbb{F}_q$, we have

$$\mathrm{Sht}^{r;\,\leq p}(k) = \{\widetilde{\mathcal{E}} \in \mathrm{Sht}^r(k) \mid p^{\mathrm{HN}}(\widetilde{\mathcal{E}}) \leq p\}.$$

For every integer d and every truncation parameter p the open substack

$$\mathrm{Sht}^{r,d;\,\leq p} = \mathrm{Sht}^{r,d} \cap \mathrm{Sht}^{r;\,\leq p}$$

is of finite type. □

The open substacks $\mathrm{Sht}^{r;\,\leq p} \subset \mathrm{Sht}^r$ are obviously stable under the action of the Picard group of X. Therefore the algebraic stack $\mathrm{Sht}^r/a^{\mathbb{Z}}$ is an increasing union of open substacks of finite type

$$\mathrm{Sht}^{r;\,\leq p}/a^{\mathbb{Z}} \subset \mathrm{Sht}^r/a^{\mathbb{Z}}.$$

But none of the open substacks $\mathrm{Sht}^{r;\,\leq p}/a^{\mathbb{Z}}$ is stable under the action of the Hecke correspondences.

4. Lefschetz numbers

If ∞ and o are two closed points in X the finite subscheme $\infty \times o \subset X \times X$ has exactly $\delta(\infty, o)$ closed points, where $\delta(\infty, o)$ is the greatest common divisor of $\deg(\infty)$ and $\deg(o)$. For each $\xi \in \infty \times o$ the residue field $\kappa(\xi)$ is a composed extension of $\kappa(\infty)$ and $\kappa(o)$, and its degree $\deg(\xi)$ over $\mathbb{F}_q$ is thus the least common multiple

$$\mu(\infty,o)=\frac{\deg(\infty)\deg(o)}{\delta(\infty,o)}$$

of $\deg(\infty)$ and $\deg(o)$.

If ξ is a closed point in $X\times X$ we denote by $\mathrm{Sht}^r_\xi/a^{\mathbb{Z}}$ the fiber at ξ of the canonical projection $\mathrm{Sht}^r/a^{\mathbb{Z}}\to X\times X$. It is a smooth algebraic stack of pure dimension $2r-2$ over the finite field $\kappa(\xi)$. We denote by

$$\mathrm{Frob}_\xi:\mathrm{Sht}^r_\xi/a^{\mathbb{Z}}\to\mathrm{Sht}^r_\xi/a^{\mathbb{Z}}$$

its geometric Frobenius endomorphism relative to $\kappa(\xi)$.

Let us fix $g\in\mathrm{GL}_r(\mathbb{A})$. Let ∞ and o be two closed points in $X\setminus N_g$, let ξ be a closed point in $\infty\times o$, let n be a positive integer and let $p:[0,r]\to\mathbb{R}$ be a truncation parameter. We denote by

$$c_\xi=(c_{1,\xi},c_{2,\xi}):\mathrm{Sht}^r_\xi(g)/a^{\mathbb{Z}}\to\mathrm{Sht}^r_\xi/a^{\mathbb{Z}}\times_{\kappa(\xi)}\mathrm{Sht}^r_\xi/a^{\mathbb{Z}}$$

the fiber at ξ of the Hecke correspondence which is defined by g.

D EFINITION The Lefschetz number

$$\mathrm{Lef}(g\times\mathrm{Frob}^n_\xi,\mathrm{Sht}^{r;\leq p}_\xi/a^{\mathbb{Z}})$$

is the sum

$$\sum_y\frac{1}{|\mathrm{Aut}(y)|}$$

where y runs through the set of (isomorphism classes of) points in $\mathrm{Sht}^r_\xi(g)/a^{\mathbb{Z}}$ such that

$$c_{1,\xi}(y)=\mathrm{Frob}^n_\xi(c_{2,\xi}(y))\in\mathrm{Sht}^{r;\leq p}_\xi/a^{\mathbb{Z}}\subset\mathrm{Sht}^r_\xi/a^{\mathbb{Z}},$$

and where $\mathrm{Aut}(y)$ is the finite automorphism group of the fixed point y.

We say that a truncation parameter is *convex enough* if, for every $i=1,\dots,r-1$ the slope of p on the interval $[i-1,i]$ is much bigger than the slope of p on the interval $[i,i+1]$.

Using Drinfeld's adelic description of shtukas, the particular case of the *fundamental lemma* proved by Drinfeld and the *Arthur-Selberg trace formula*, Lafforgue has shown:

. — *If* $\deg(\infty)$ *and* $\deg(o)$ *are large enough with respect to* g *and if* p *is convex enough with respect to* g*, the average Lefschetz number*

$$\frac{1}{\delta(\infty,o)}\sum_{\xi\in\infty\times o}\mathrm{Lef}(g\times\mathrm{Frob}^n_\xi,\mathrm{Sht}^{r;\leq p}_\xi/a^{\mathbb{Z}}).$$

is equal to the spectral expression

$$\sum_{\substack{\pi\in\mathcal{A}_r(K)\\ \omega_\pi(a)=1}}\mathrm{Tr}_\pi(f_g)q^{(r-1)n\mu(\infty,o)}S_\infty^{\left(-\frac{n\mu(\infty,o)}{\deg(\infty)}\right)}(\pi)S_o^{\left(\frac{n\mu(\infty,o)}{\deg(o)}\right)}(\pi)$$
$$+\sum_{\substack{1\leq r'<r\\ 1\leq r''<r}}\sum_{\substack{\pi'\in\mathcal{A}_{r'}(K)\\ \pi''\in\mathcal{A}_{r''}(K)}}\mathrm{Tr}^{\leq p}_{\pi',\pi''}(f_g,n\mu(\infty,o))S_\infty^{\left(-\frac{n\mu(\infty,o)}{\deg(\infty)}\right)}(\pi')S_o^{\left(\frac{n\mu(\infty,o)}{\deg(o)}\right)}(\pi'')$$

where

- - f_g the characteristic function of $KgK \subset \mathrm{GL}_r(\mathbb{A})$ and $\mathrm{Tr}_\pi(f_g)$ is the trace of the operator $\pi(f_g)$,
- - $m \mapsto \mathrm{Tr}^{\leq p}_{\pi',\pi''}(f_g, m)$ is a complex function of the integer m, which does not depend on the places $o, \infty \in X \setminus N_g$ and on the integer n, and which is of the form
$$\sum_{\lambda\in\Lambda} P_\lambda(m)\lambda^m$$
for some finite subset $\Lambda \subset \mathbb{C}^\times$ and some family $(P_\lambda(T))_{\lambda\in\Lambda}$ of polynomials in $\mathbb{C}[T]$,
- - the function $m \mapsto \mathrm{Tr}^{\leq p}_{\pi',\pi''}(f_g, m)$ is identically zero for all but finitely many pairs (π', π'').

(Recall that
$$S_x^{(m)}(\pi) = z_{x,1}(\pi)^m + \cdots + z_{x,r}(\pi)^m$$
is the m-th power sum of the Hecke eigenvalues of π at x.) □

If the open substacks $\mathrm{Sht}_\xi^{r;\leq p}/a^{\mathbb{Z}}$ *of* $\mathrm{Sht}_\xi^r/a^{\mathbb{Z}}$ *were stable under the action of the Hecke operators, the main theorem would easily follows from the above result and would have been proved many years ago.*

5. Compactifications

Let the torus $\mathbb{G}_{\mathrm{m}}^{r-1} = \mathrm{Spec}(\mathbb{F}_q[t_1, t_1^{-1} \ldots, t_{r-1}, t_{r-1}^{-1}])$ act on the standard affine space $\mathbb{A}^{r-1} = \mathrm{Spec}(\mathbb{F}_q[u_1, \ldots, u_{r-1}])$ by
$$(t_1, \ldots, t_{r-1}) \cdot (u_1, \ldots, u_{r-1}) = (t_1u_1, \ldots, t_{r-1}u_{r-1}).$$
The quotient stack $[\mathbb{A}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}]$ is an Artin algebraic stack which is smooth of dimension 0. Its closed substack
$$\{u_1 \cdots u_{r-1} = 0\} = \{u_1 = 0\} \cup \cdots \cup \{u_{r-1} = 0\} \subset [\mathbb{A}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}]$$
is the union of $r-1$ smooth divisors with normal crossings. The complementary open substack
$$\{u_1 \cdots u_{r-1} \neq 0\} = [\mathbb{G}_{\mathrm{m}}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}] \subset [\mathbb{A}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}]$$
is reduced to one point. For each partition $\mathbf{r} = (r_1 + \cdots + r_s = r)$ of r (into a sum of positive integers) the intersection
$$[\mathbb{A}_{\mathbf{r}}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}] = \bigcap_{i\in I}\{u_i = 0\} \cap \bigcap_{i\notin I}\{u_i \neq 0\}.$$
where we have set $I = \{r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_{s-1}\}$, is smooth of dimension $1 - s$. When $\mathbf{r}$ runs through the set of partitions of r the locally closed substacks $[\mathbb{A}_{\mathbf{r}}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}]$ form a stratification of $[\mathbb{A}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}]$.

Let us fix a truncation parameter p which is convex enough with respect to X. Let us also fix an integer d.

T HEOREM (Drinfeld [3] for $r = 2$, Lafforgue [5] for $r \geq 3$). — *There exists an Artin algebraic stack* $\overline{\mathrm{Sht}}^{r,d;\leq p}$ *and a stack morphism*
$$\overline{(\infty, o, \varepsilon)} : \overline{\mathrm{Sht}}^{r,d;\leq p} \to X \times X \times [\mathbb{A}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}]$$
with the following properties:

- - *all the automorphisms groups of* $\overline{\mathrm{Sht}}^{r,d;\leq p}$ *are* finite *(but not necessarily unramified),*
- - $\overline{(\infty,o,\varepsilon)}$ *is* smooth *of pure relative dimension* $2r-2$ *and*
$$\overline{(\infty,o)}=\mathrm{pr}_{X\times X}\circ\overline{(\infty,o,\varepsilon)}:\overline{\mathrm{Sht}}^{r,d;\leq p}\to X\times X$$
(which is also smooth of pure relative dimension $2r-2$*) is* proper,
- - *the restriction of* $\overline{(\infty,o,\varepsilon)}$ *over the open substack*
$$X\times X=X\times X\times[\mathbb{G}_{\mathrm{m}}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}]\subset X\times X\times[\mathbb{A}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}]$$
is nothing else than the stack morphism $(\infty,o):\mathrm{Sht}^{r,d;\leq p}\to X\times X$. Λ

It follows from the theorem that the Artin algebraic stack $\overline{\mathrm{Sht}}^{r,d;\leq p}$ is proper and smooth of pure dimension $2r$, and that it contains the Deligne-Mumford algebraic stack $\mathrm{Sht}^{r,d;\leq p}$ as a dense open substack. The closed complementary substack is the union of $r-1$ divisors
$$\overline{\mathrm{Sht}}^{r,d;\leq p}\setminus\mathrm{Sht}^{r,d;\leq p}=\bigcup_{i=1}^{r-1}\overline{(\infty,o,\varepsilon)}^{-1}(\{u_i=0\}),$$
which are smooth with relative normal crossings over $X\times X$.

When $\mathbf{r}$ runs through the set of partitions of r the locally closed substacks
$$\overline{\mathrm{Sht}}_{\mathbf{r}}^{r,d;\leq p}=\overline{(\infty,o,\varepsilon)}^{-1}([\mathbb{A}_{\mathbf{r}}^{r-1}/\mathbb{G}_{\mathrm{m}}^{r-1}])$$
which are smooth of pure relative dimension $2r-1-s$ over $X\times X$, form a stratification of $\overline{\mathrm{Sht}}^{r,d;\leq p}$.

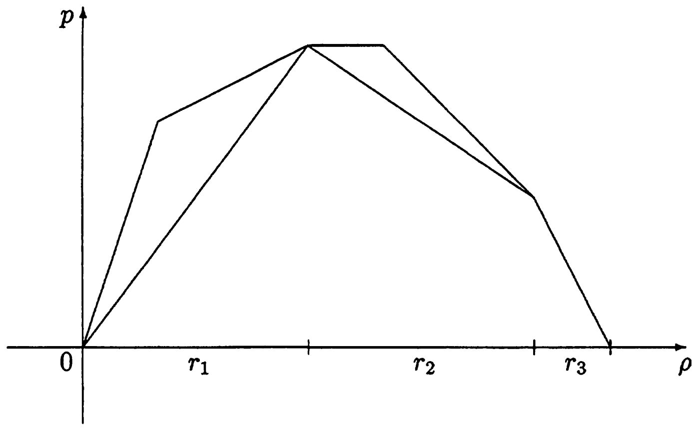
For each partition $\mathbf{r}=(r_1+\cdots+r_s=r)$ of r we also consider the Deligne-Mumford algebraic stack
$$\mathrm{Sht}^{\mathbf{r}}=\mathrm{Sht}^{r_1}\times_X\mathrm{Sht}^{r_2}\times_{X,\mathrm{Frob}_X}\mathrm{Sht}^{r_3}\times_{X,\mathrm{Frob}_X}\cdots\times_{X,\mathrm{Frob}_X}\mathrm{Sht}^{r_s}$$
which classifies the families $(\widetilde{\mathcal{E}}_1,\widetilde{\mathcal{E}}_2,\dots,\widetilde{\mathcal{E}}_s)$ of shtukas of ranks $r_1,\dots,r_s$ such that the zero o_1 of $\widetilde{\mathcal{E}}_1$ is equal to the pole ∞_2 of $\widetilde{\mathcal{E}}_2$ and that, for $j=2,\dots,s-1$, the zero o_j of $\widetilde{\mathcal{E}}_j$ is equal to the image by the Frobenius endomorphism Frob_X of the pole ∞_{j+1} of $\widetilde{\mathcal{E}}_{j+1}$. By construction we have a smooth morphism of pure relative dimension $2r-2s$
$$(\infty_1,o_1=\infty_2,o_2=\mathrm{Frob}_X(\infty_3),\dots,o_{s-1}=\mathrm{Frob}_X(\infty_s),o_s):\mathrm{Sht}^{\mathbf{r}}\to X\times X^{s-1}\times X.$$
Therefore $\mathrm{Sht}^{\mathbf{r}}$ is smooth of pure relative dimension $2r-s-1$ over $X\times X$.

For each $i=0,1,\dots,r$ let $\widetilde{p}(i)$ be the unique integer in the length 1 interval
$$]p(i)+\frac{i}{r}d-1,p(i)+\frac{i}{r}d].$$
We set $d_1=\widetilde{p}(r_1)$ and we denote by $p_1:[0,r_1]\to\mathbb{R}$ the truncation parameter which takes the values
$$p_1(i_1)=\widetilde{p}(i_1)-\frac{i_1d_1}{r_1},\ \forall i_1=1,\dots,r_1-1.$$
For each $j=2,\dots,s$ we set
$$d_j=\widetilde{p}(r_1+\cdots+r_{j-1}+r_j)-\widetilde{p}(r_1+\cdots+r_{j-1})-1$$

and we denote by $p_j : [0, r_j] \to \mathbb{R}$ the truncation parameter which takes the values

$$p_j(i_j) = \widetilde{p}(r_1 + \cdots + r_{j-1} + i_j) - \widetilde{p}(r_1 + \cdots + r_{j-1}) - 1 - \frac{i_j d_j}{r_j}, \ \forall i_j = 1, \dots, r_j - 1.$$

(The p_j's are essentially the normalized restrictions of p to the intervals $[r_1 + \cdots + r_{j-1}, r_1 + \cdots + r_{j-1} + r_j]$'s as in the figure below.



In particular all the p_j's are automatically convex enough with respect to X as soon as this is the case for p.)

We define an open substack

$$\mathrm{Sht}^{\mathbf{r},d;\leq p} \subset \mathrm{Sht}^{\mathbf{r}}$$

by requiring that, for $j = 1, \dots, s$, the degree of the shtuka $\widetilde{\mathcal{E}}_j$ is equal to d_j and its Harder-Narasimhan polygon $p^{\mathrm{HN}}(\widetilde{\mathcal{E}}_j)$ is bounded above by p_j.

P ROPOSITION (Lafforgue [5]). — *For each non trivial partition* $\mathbf{r}$ *of* r *there exists a canonical morphism of stacks*

$$\overline{\mathrm{Sht}}_{\mathbf{r}}^{r,d;\leq p} \to \mathrm{Sht}^{\mathbf{r},d;\leq p}$$

which is the composition of a gerb whose structural group is finite, flat and radicial, and of a radicial representable morphism. Λ

Lafforgue calls *iterated shtukas* the points in $\overline{\mathrm{Sht}}^{r,d;\leq p}$. To each iterated shtuka is associated a partition $\mathbf{r} = (r_1, \dots, r_s)$ of r and a family of "small" shtukas of ranks $r_1, \dots, r_s$. Their zeros and poles $o_1 = \infty_2, o_2 = \mathrm{Frob}_X(\infty_3), \dots, o_{s-1} = \mathrm{Frob}_X(\infty_s)$ are the *degenerators* of the iterated shtuka. The pole ∞_s and the zero o_1 are the *pole* and the *zero* of the iterated shtuka.

For each partition $\mathbf{r}$ of r we set

$$\overline{\mathrm{Sht}}_{\mathbf{r}}^{r;\leq p} = \coprod_{d\in\mathbb{Z}} \overline{\mathrm{Sht}}_{\mathbf{r}}^{r,d;\leq p} \subset \overline{\mathrm{Sht}}^{r;\leq p} = \coprod_{d\in\mathbb{Z}} \overline{\mathrm{Sht}}^{r,d;\leq p}$$

and

$$\mathrm{Sht}^{r;\leq p} = \coprod_{d\in\mathbb{Z}} \mathrm{Sht}^{r,d;\leq p}.$$

These algebraic stacks are naturally equipped with an action of the Picard group $F^\times\backslash\mathbb{A}^\times/\mathcal{O}^\times$ of X. In particular we may form the algebraic stack

$$\overline{\mathrm{Sht}}^{r;\leq p}/a^{\mathbb{Z}} = \coprod_{d=1}^{r\deg(a)} \overline{\mathrm{Sht}}^{r,d;\leq p}$$

which is a smooth compactification of $\mathrm{Sht}^{r;\leq p}/a^{\mathbb{Z}}$ over $X\times X$. It is stratified by the locally closed substacks $\overline{\mathrm{Sht}}_{\mathbf{r}}^{r;\leq p}/a^{\mathbb{Z}}$ which are "homeomorphic" to the Deligne-Mumford algebraic stacks $\mathrm{Sht}^{\mathbf{r};\leq p}/a^{\mathbb{Z}}$.

6. r-negligible Galois representations

In this section and the next one we will forget the action of the Hecke operators and concentrate on the Galois action on the ℓ-adic cohomology of the shtuka moduli varieties.

We denote by E the fraction field of $F\otimes F$ (the field of rational functions on the surface $X\times X$). We fix an algebraic closure $\overline{E}$ of E and an embedding $\overline{F}\otimes_{\overline{\mathbb{F}}_q}\overline{F}\hookrightarrow\overline{E}$ over the embedding $F\otimes F\hookrightarrow E$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$ in $\overline{F}$.

We have thus defined a geometric point $\overline{\delta}:\mathrm{Spec}(\overline{E})\to X\times X$ over the generic point $\delta=\mathrm{Spec}(E)$ of $X\times X$. The images of $\overline{\delta}$ by the two canonical projections of $X\times X$ booth factors through the geometric point $\overline{\eta}:\mathrm{Spec}(\overline{F})\to X$ over the generic point η of X.

The Grothendieck fundamental group $\pi_1(X,\overline{\eta})$ is the quotient of $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$ which classifies the finite extensions of F in $\overline{F}$ which are unramified everywhere. It admits as quotient the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Similarly the Grothendieck fundamental group $\pi_1(X\times X,\overline{\delta})$ is a quotient of the Galois group $\mathrm{Gal}(\overline{E}/E)$ and admits $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ as quotient.

L EMMA The homomorphism

$$\pi_1(X\times X,\overline{\delta})\to\pi_1(X,\overline{\eta})\times\pi_1(X,\overline{\eta}),$$

which is induced by the two canonical projections of $X\times X$, is injective. Its image is the group of pairs of elements $\gamma',\gamma''\in\pi_1(X,\overline{\eta})$ which have the same images in the quotient $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Λ

In particular, any irreducible ℓ-adic representation of $\pi_1(X\times X,\overline{\delta})$ is a direct factor of a semi-simple ℓ-adic representation of the form

$$\lambda\otimes(\sigma'\boxtimes\sigma'')$$

where λ is an ℓ-adic character of $\pi_1(X\times X,\overline{\delta})$ which factors through the quotient $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)\cong\widehat{\mathbb{Z}}$, and where σ' and σ'' are two irreducible ℓ-adic representations of $\pi_1(X,\overline{\eta})$ with determinants of finite order. We may identify λ to an ℓ-adic unit (its value at the geometric Frobenius element) and we may view the tensor product by λ as a generalized Tate twist.

We will simply call *virtual* $\pi_1(X \times X, \overline{\delta})$*-module* a formal linear combination $\sum_\rho m_\rho[\rho]$ where ρ runs through the set of (isomorphism classes of) irreducible ℓ-adic representations of $\pi_1(X \times X, \overline{\delta})$ and where the m_ρ's are rational numbers which are equal to 0 for all but finitely many ρ's. We say that ρ *occurs in* $\sum_\rho m_\rho[\rho]$ if its *multiplicity* m_ρ is non zero. The *trace* of $\sum_\rho m_\rho[\rho]$ is the function

$$\mathrm{Tr}_{\sum_\rho m_\rho[\rho]} : \pi_1(X \times X, \overline{\delta}) \to \overline{\mathbb{Q}}_\ell,\ \gamma \mapsto \sum_\rho m_\rho \,\mathrm{Tr}(\rho(\gamma)).$$

Any graded ℓ-adic representation H^* of $\pi_1(X \times X, \overline{\delta})$ defines a virtual $\pi_1(X \times X, \overline{\delta})$-module

$$[H^*] = \sum_\rho \sum_\nu (-1)^\nu m_\rho^\nu [\rho]$$

where m_ρ^ν is the number of times that ρ occurs in any Jordan-Hölder filtration of H^ν.

. — *A ℓ-adic representation of $\pi_1(X \times X, \overline{\delta})$ is said* r-negligible *if all its irreducible subquotients are direct factors of ℓ-adic representations of the form $\lambda \otimes (\sigma' \boxminus \sigma'')$ as above, where σ' and σ'' are both of dimension $\leq r-1$.*

A virtual $\pi_1(X \times X, \overline{\delta})$-module is said r-negligible *if any ρ which occurs in it is r-negligible.*

We now fix a truncation parameter which is convex enough with respect to X. We denote by $\mathrm{Sht}_{\overline{\delta}}^{r;\leq p}$, $\overline{\mathrm{Sht}}_{\overline{\delta}}^{r;\leq p}$ and $\mathrm{Sht}_{\overline{\delta}}^{\mathbf{r};\leq p}$ the fibers at the geometric point $\overline{\delta}$ of the morphisms (∞, o), $\overline{(\infty, o)}$ and (∞_1, o_s), and we consider their ℓ-adic cohomologies

$$H_c^*(r; \leq p) = H_c^*(\mathrm{Sht}_{\overline{\delta}}^{r;\leq p}/a^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell),$$

$$\overline{H}^*(r; \leq p) = H^*(\overline{\mathrm{Sht}}_{\overline{\delta}}^{r;\leq p}/a^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell),$$

$$H_\partial^*(r; \leq p) = H^*((\overline{\mathrm{Sht}}_{\overline{\delta}}^{r;\leq p} \setminus \mathrm{Sht}_{\overline{\delta}}^{r;\leq p})/a^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell)$$

and

$$H_c^*(\mathbf{r}; \leq p) = H_c^*(\mathrm{Sht}_{\overline{\delta}}^{\mathbf{r};\leq p}/a^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell).$$

It follows from the results of the previous section that:

- - the natural continuous actions of the Galois group $\mathrm{Gal}(\overline{E}/E)$ on these ℓ-adic cohomologies factor through the fundamental group $\pi_1(X \times X, \overline{\delta})$,
- - there is a long exact sequence

  $$\cdots \to H_c^\nu(r; \leq p) \to \overline{H}^\nu(r; \leq p) \to H_\partial^\nu(r; \leq p) \to H_c^{\nu+1}(r; \leq p) \to \cdots$$

  and a spectral sequence

  $$E_1^{s,\nu} = \bigoplus_{\mathbf{r}=(r_1,\dots,r_{s+2})} H_c^\nu(\mathbf{r}; \leq p) \Rightarrow H_\partial^{\nu+s}(r; \leq p).$$

The virtual $\pi_1(X \times X, \overline{\delta})$-module

$$[H_c^*(r; \leq p)]$$

is thus equal to the virtual $\pi_1(X \times X, \overline{\delta})$-module

$$[\overline{H}^*(r; \leq p)] - \sum_{\mathbf{r}} [H_c^*(\mathbf{r}; \leq p)]$$

where $\mathbf{r}$ runs through the set of non trivial partitions of r.

7. The induction

Lafforgue proves the main theorem and the following proposition by a simultaneous induction on r.

P ROPOSITION (Lafforgue [6]). — *For any truncation parameter p which is convex enough with respect to X, the ℓ-adic representations $H_c^\nu(r; \leq p)$ of $\pi_1(X \times X, \overline{\delta})$ is $(r+1)$-negligible for every integer ν.*

From now on we thus assume that the main theorem in rank $< r$ is already proved and that, for every truncation parameter p which is convex enough with respect to X, every positive integer $r' < r$ and every integer ν the $\pi_1(X \times X, \overline{\delta})$-module $H_c^\nu(r'; \leq p)$ is r-negligible.

Let p be a truncation parameter which is convex enough with respect to X and let $\mathbf{r} = (r_1, \ldots, r_s)$ be a partition of r. Then it follows from the structure of the stratum $\mathrm{Sht}^{\mathbf{r}; \leq p}$ and the Künneth formula that, for each integer ν the ℓ-adic representation $H_c^\nu(\mathbf{r}; \leq p)$ of $\pi_1(X \times X, \overline{\delta})$ is r-negligible. In fact the virtual $\pi_1(X \times X, \overline{\delta})$-module $[H_c^*(\mathbf{r}; \leq p)]$ is a linear combination of virtual modules of the form

$$[\mu \otimes H_c^* \left(\overline{\mathbb{F}}_q \otimes_{\mathbb{F}_q} X^{s-1}, \Theta_{j=1}^{s-1}(\sigma_j'' \otimes \sigma_{j+1}')\right) \otimes (\sigma_1' \boxtimes \sigma_s'')] = \sum_\lambda m_\lambda \left[\lambda \otimes (\sigma_1' \boxtimes \sigma_s'')\right]$$

where μ and λ are ℓ-adic characters of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, where σ_j' and σ_j'' are irreducible ℓ-adic representations of dimension $\leq r_j$ of $\pi_1(X, \overline{\eta})$ with determinants of finite order, and where the m_λ's are integers which are all zero except for finitely many λ's.

Therefore the induction hypothesis and the spectral expression for the average Lefschetz number

$$\frac{1}{\delta(\infty, o)} \sum_{\xi \in \infty \times o} \mathrm{Lef}(\mathrm{Frob}_\xi^n, \mathrm{Sht}_\xi^{r; \leq p} / a^{\mathbf{Z}}).$$

$(g = 1)$ given in Section 4 imply that:

P ROPOSITION (i) For each integer ν the kernel and the cokernel of the canonical homomorphism

$$H_c^\nu(r; \leq p) \to \overline{H}^\nu(r; \leq p)$$

and the virtual $\pi_1(X \times X, \overline{\delta})$-module

$$[\overline{H}^*(r; \leq p)] - [H_c^*(r; \leq p)]$$

are r-negligible.

(ii) There exists a virtual $\pi_1(X \times X, \overline{\delta})$-module $[H_{\mathrm{cusp}}^*(r; \leq p)]$ such that the difference

$$[H_{\mathrm{cusp}}^*(r; \leq p)] - \frac{1}{r!} \sum_{n=1}^{r!} [(\mathrm{Frob}_X^n \times \mathrm{Id}_X)^* H_c^*(r; \leq p)]$$

is r-negligible, and such that, for each pair (∞, o) of distinct closed points in X, each closed point $\xi \in \infty \times o$ and each positive integer n we have

$$\mathrm{Tr}_{[H^*_{\mathrm{cusp}}(r;\leq p)]}(\mathrm{Frob}^n_\xi) = \sum_{\substack{\pi \in \mathcal{A}_r(K) \\ \omega_\pi(a)=1}} q^{(r-1)n\deg(\xi)} S_\infty^{\left(-n\frac{\deg(\xi)}{\deg(\infty)}\right)}(\pi) S_o^{\left(n\frac{\deg(\xi)}{\deg(o)}\right)}(\pi).$$

Λ

Taking into account the purity of the cohomology groups $\overline{H}^*(r; \leq p)$, which follows from the Weil conjecture proved by Deligne, Lafforgue deduces by L-function arguments:

COROLLARY

. — (i) *All the irreducible ℓ-adic representations of $\pi_1(X \times X, \overline{\delta})$ which occur in $[H^*_{\mathrm{cusp}}(r; \leq p)]$ occur with a positive multiplicity and are pure of weight $2r-2$. Moreover none of them is r-negligible.*

(ii) *The ℓ-adic representations $H^\nu_{\mathrm{c}}(r; \leq p)$, $\nu \neq 2r-2$, of $\pi_1(X \times X, \overline{\delta})$ and the virtual $\pi_1(X \times X, \overline{\delta})$-module*

$$[H^*_{\mathrm{cusp}}(r; \leq p)] - \frac{1}{r!}\sum_{n=1}^{r!}[(\mathrm{Frob}^n_X \times \mathrm{Id}_X)^* H^{2r-2}_{\mathrm{c}}(r; \leq p)]$$

are all r-negligible. Λ

Now it is also easy to deduce from Drinfeld's study of the horocycles on Sht^r and the induction hypothesis that:

L EMMA For every truncation parameters $p \leq q$ which are convex enough with respect to X the kernel and the cokernel of the canonical homomorphism

$$H^{2r-2}_{\mathrm{c}}(r; \leq p) \to H^{2r-2}_{\mathrm{c}}(r; \leq q)$$

are r-negligible. Λ

Therefore the direct limit

$$H^{2r-2}_{\mathrm{c}}(r) = \varinjlim_p H^{2r-2}_{\mathrm{c}}(r; \leq p) = H^{2r-2}_{\mathrm{c}}(\mathrm{Sht}^r_{\overline{\delta}}/a^{\mathbb{Z}}),$$

which is an infinite dimensional representation of $\pi_1(X \times X, \overline{\delta})$, has the following property:

. — *There exists a unique finite filtration*

$$F^\bullet = ((0) = F^0 \subset F^1 \subsetneq F^2 \subsetneq \cdots \subsetneq F^{2u+1} \subsetneq F^{2u} \subsetneq \cdots \subsetneq F^T = H^{2r-2}_{\mathrm{c}}(r))$$

such that:

- *- for any non negative integer u such that $2u+1 \leq T$, F^{2u+1}/F^{2u} is the sum of all the finite dimensional ℓ-adic subrepresentations of $H^{2r-2}_{\mathrm{c}}(r)/F^{2u}$ which are r-negligible,*
- *- for any non negative integer u such that $2u+2 \leq T$, F^{2u+2}/F^{2u+1} is the sum of all the finite dimensional ℓ-adic subrepresentations of $H^{2r-2}_{\mathrm{c}}(r)/F^{2u+1}$ which do not admit any r-negligible subquotient,*
- *- if p a truncation parameter which is convex enough with respect to X and if we denote by $F^\bullet(\leq p)$ the filtration on $H^{2r-2}_{\mathrm{c}}(r; \leq p)$ which is induced by $F^\bullet$ then, for any non negative integer u the embedding*

$$F^{2u+2}(\leq p)/F^{2u+1}(\leq p) \hookrightarrow F^{2u+2}/F^{2u+1}$$

is an isomorphism. Λ

We have thus proved:

P ROPOSITION The direct sum

$$H^{2r-2}_{\text{cusp}} = \bigoplus_{u\geq 0} F^{2u+2}/F^{2u+1}$$

is a finite dimensional ℓ-adic representation of $\pi_1(X\times X,\overline{\delta})$ and, for any truncation parameter which is convex enough with respect to X we have the equality of virtual $\pi_1(X\times X,\overline{\delta})$-modules

$$[H^{*}_{\text{cusp}}(r;\leq p)] = [H^{2r-2}_{\text{cusp}}].$$

Λ

8. A fixed point formula

Let us now consider again the action of the Hecke operators. They act on $H^{2r-2}_{c}(r) = H^{2r-2}_{c}(\mathrm{Sht}^{r}_{\overline{\delta}}/a^{\mathbb{Z}})$ and they necessarily stabilize the above canonical filtration $F^{\bullet}$. Therefore they also act on the finite dimensional ℓ-adic representation H^{2r-2}_{cusp} of $\pi_1(X\times X,\overline{\delta})$.

To finish the induction on r, at least for the everywhere unramified representations, Lafforgue proves:

. — *For each $g\in \mathrm{GL}_r(\mathbb{A})$, each pair (∞,o) of distinct closed points in $X\setminus N_g$, each closed point $\xi\in\infty\times o$ and each positive integer n we have the equality of traces*

$$\mathrm{Tr}_{H^{2r-2}_{\text{cusp}}}(g\times\mathrm{Frob}^{n}_{\xi}) = \sum_{\substack{\pi\in\mathcal{A}_r(K)\\ \omega_\pi(a)=1}} \mathrm{Tr}_{\pi}(f_g)q^{(r-1)n\deg(\xi)}S_{\infty}^{\left(-n\frac{\deg(\xi)}{\deg(\infty)}\right)}(\pi)S_{o}^{\left(n\frac{\deg(\xi)}{\deg(o)}\right)}(\pi).$$

C OROLLARY For each $\pi\in\mathcal{A}_r(K)$ such that $\omega_\pi(a)=1$ there exists an everywhere unramified Galois representation $\sigma(\pi)\in\mathcal{G}_r$ such that $L_x(\pi,s) = L_x(\sigma(\pi),s)$ for all but finitely many places x. Λ

Remark : In fact, we have $L_x(\pi,s)=L_x(\sigma(\pi),s)$ for all places x, and we have the equality of virtual modules over $\mathcal{H}_K\times\pi_1(X\times X,\overline{\delta})$

$$[H^{2r-2}_{\text{cusp}}] = \sum_{\substack{\pi\in\mathcal{A}_r(K)\\ \omega_\pi(a)=1}} [\pi^K\,\Theta\,(\sigma(\pi)^\vee\,\Theta\,\sigma(\pi))(1-r)].$$

where $\mathcal{H}_K = e_K*\mathcal{H}*e_K$ is the commutative algebra of K-biïnvariant functions with compact support on $\mathrm{GL}_r(\mathbb{A})$ and $\pi^K=\pi*e_K$ is the one dimensional $\mathcal{H}_K$-module associated with the everywhere unramified $\mathcal{H}$-module π. Λ

The proof of the proposition is based on the following variant of a conjecture of Deligne on the Grothendieck-Lefschetz trace formula, in the way it has been formulated and proved by Pink ([8]).

Let $\kappa\supset\mathbb{F}_q$ be a finite field and k be an algebraic closure of κ. We simply denote

$$H^*(S) = H^*(k\otimes_\kappa S,\overline{\mathbb{Q}}_\ell)$$

the ℓ-adic cohomology of a separated scheme of finite type S over κ. The Galois group $\mathrm{Gal}(k/\kappa)$ acts on $H^*(S)$ and we denote by Frob_κ the endomorphism of $H^*(S)$ which is induced by the geometric Frobenius element in $\mathrm{Gal}(k/\kappa)$.

Let S be a proper and smooth scheme of pure dimension d over κ and $U \subset S$ be a dense open subset. We denote by $\mathrm{Frob}_S : S \to S$ the Frobenius endomorphism with respect to κ, and by $\mathrm{Frob}_U : U \to U$ its restriction to U

Let $c_U = (c_{U,1}, c_{U,2}) : V \to U \times_\kappa U$ be a finite morphism. We assume that $c_{U,1} : V \to U$ is etale, so that V is also smooth of pure relative dimension d over κ. For any positive integer n we consider the Lefschetz number

$$\mathrm{Lef}(c_U \times \mathrm{Frob}_U^n) = |\{t \in V \mid c_1(t) = \mathrm{Frob}_U^n(c_2(t))\}|.$$

If $U = S$ this Lefschetz number admits a well known cohomological interpretation. The generalized codimension d cycle $c_U : V \to U \times_\kappa U$ has a cohomology class $[c_U] \in H^{2d}(U \times_\kappa U)(d)$. Moreover, by Poincaré duality any class z in

$$H^{2d}(U \times_\kappa U)(d) = \bigoplus_{i=0}^{2d} H^i(U) \otimes_{\overline{\mathbb{Q}}_\ell} H^{2d-i}(U)(d) = \bigoplus_{i=0}^{2d} H^i(U) \otimes_{\overline{\mathbb{Q}}_\ell} (H^i(U))^\vee$$

may be viewed as an endomorphism of $H^*(U)$.

. — *If $U = S$ then, for any positive integer n we have the Lefschetz trace formula*

$$\mathrm{Lef}(c_U \times \mathrm{Frob}_U^n) = \mathrm{Tr}_{H^*(S)}([c_U] \times \mathrm{Frob}_\kappa^n).$$

Λ

If $U \subsetneq S$ we may extend c_U to a finite morphism $c = (c_1, c_2) : T \to S \times_\kappa S$ by normalizing $S \times_\kappa S$ in V. The κ-scheme T is normal, proper and of pure dimension d. The morphism $c_1 : T \to S$ is generically finite and proper, but it is not necessarily finite.

We view c as a geometric correspondence on S. Its fixed point set is the closed subset

$$\mathrm{Fix}(c) = \{t \in T \mid c_1(t) = c_2(t)\}$$

of T. More generally, for each non negative integer n we set

$$\mathrm{Fix}(c \times \mathrm{Frob}_S^n) = \{t \in T \mid c_1(t) = \mathrm{Frob}_S^n(c_2(t))\} \subset T.$$

D EFINITION We say that the correspondence c stabilizes $U \subset S$ in a neighborhood of its fixed points if there exists an open subset $W \subset T$ containing $\bigcup_{n \geq 0} \mathrm{Fix}(c \times \mathrm{Frob}_S^n)$ such that

$$c_2(c_1^{-1}(U) \cap W) \subset U.$$

. — *Let us assume that U is the complementary open subset in S of a divisor with normal crossings which is the union of a finite family $(S_i)_{i \in \Delta}$ of smooth divisors. For each $I \subset \Delta$ let us set*

$$S_I = \bigcap_{i \in I} S_i.$$

(By hypothesis S_I is proper and smooth of dimension $d - |I|$ over κ.)

Then, if c stabilizes $U \subset S$ in a neighborhood of its fixed points there exist a positive integer n_0 and cohomology classes

$$z_I \in H^{2(d-|I|)}(S_I \times_\kappa S_I)(d - |I|),\ \forall \emptyset \neq I \subset \Delta,$$

such that, for any integer $n \geq n_0$ *we have the Lefschetz trace formula*

$$\mathrm{Lef}(c_U \times \mathrm{Frob}_U^n) = \mathrm{Tr}_{H^*(S)}([c] \times \mathrm{Frob}_\kappa^n) + \sum_{\substack{I \subset \Delta \\ I \neq \emptyset}} (-1)^{|I|} \mathrm{Tr}_{H^*(S_I)}(z_I \times \mathrm{Frob}_\kappa^n).$$

Moreover, if (S, U, c) *varies in an algebraic family which satisfies the obvious relative variant of the above hypotheses, the integer* n_0 *and the cohomology classes* z_I *can be chosen in a uniform way.* Λ

The hypotheses of the theorem are satisfied by the Hecke correspondences. More precisely let us fix $g \in \mathrm{GL}_r(\mathbb{A})$ and let ξ be a "general enough" closed point in $X \times X$. Let us take

$$S = \overline{\mathrm{Sht}}_\xi^{r;\leq p} / a^{\mathbb{Z}} \supset U = \mathrm{Sht}_\xi^{r;\leq p} / a^{\mathbb{Z}}$$

and

$$V = c_\xi^{-1}(\mathrm{Sht}_\xi^{r;\leq p} / a^{\mathbb{Z}} \times_{\kappa(\xi)} \mathrm{Sht}_\xi^{r;\leq p} / a^{\mathbb{Z}}) \subset \mathrm{Sht}_\xi^r(g)/a^{\mathbb{Z}}.$$

T HEOREM (Drinfeld [3] for $r = 2$; Lafforgue [6] for $r \geq 3$). — *The correspondence* $T \to S \times_\kappa S$ *which is obtained by normalizing* $S \times_\kappa S$ *in* V *stabilizes* $U \subset S$ *in a neighborhood of its fixed points.* Λ

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