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Recent Developments in Algebraic K-theory

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The development of algebraic K-theory was initiated by J.H.C. Whitehead in his study of simple homotopy equivalence. He there defined, for any ring R, a group K_1R, whose values on the group rings of fundamental groups of spaces carries information about spaces homotopy equivalent to (but not simple homotopy equivalent to) a space with the given fundamental group (see [**8**] for an excellent introduction to this theory). C.T.C. Wall [**55**] used a group K_0R, also defined for any ring R, to measure an obstruction to finiteness for spaces which are *homologically finite*, i.e. for which the homology groups H_i vanish for i sufficiently large. It turns out that spaces X which are homologically finite need not have the homotopy type of a finite complex when X is not simply connected, and the determination of whether there exists such a complex is determined completely by the fundamental group of X. In the 1950's and 1960's, there was considerable work on these invariants, and there was a growing consensus that for any ring R there should be an infinite family of groups K_iR, and that these groups together should carry substantial information concerning the *homeomorphism* classification of manifolds homotopy equivalent to a given manifold. Such a family of groups would be called the ***higher algebraic** K-**theory*** of the ring. This direction of research was carried forward by [**32**], [**16**], and [**17**] with the introduction of a group K_2R and its application to the study of pseudoisotopies between homotopy equivalent manifolds.

By the early 1970's, there were a number of competing candidates for definitions of a higher algebraic K-theory (see, for example, [**52**], [**1**], and [**46**]). However, the model constucted by Quillen [**36**] in 1972 quickly became recognized as the "right model". This acceptance came about because of the many excellent formal properties enjoyed by Quillen's theory. These properties, which are in part analogous to properties of cohomology theories in topology, such as homotopy invariance and excision, allow algebraic K to be studied computationally and theoretically in a way which was not a priori possible with the other theories. Ultimately, it has been possible to show that in many cases, the earlier constructions agree with Quillen's construction.

Quillen's construction has permitted the further development of the geometric applications of algebraic K-theory in geometric topology. For example, Waldhausen [**53**] has developed his "algebraic K-theory of spaces", which generalizes greatly the possible areas of application of algebraic K-theoretic ideas, and which shows how

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all the higher algebraic K-groups are involved in the study of the homotopy type of automorphism groups of manifolds. An extremely fruitful new line of investigation has also opened up, beginning with the observation that the formal properties of Quillen's construction (specifically, the localization and homotopy properties) imply that algebraic K-theory can be thought of as a kind of *cohomology theory on schemes.* This observation has been used to obtain striking results about algebraic K-theory. Some of the notable examples include Bloch's work on the relationship between algebraic K-theory and the theory of cycles on algebraic varieties, Suslin's computation of the algebraic K-theory of algebraically closed fields, the work of Suslin-Merkurjev on the relationship between K_2 for fields with the Brauer group, and the recent striking work of Voevodsky on the so-called Milnor conjecture.

Our goal in this paper is to survey this recent algebraic geometric and arithmetic direction of research. We will begin with Quillen's construction, continue with discussions of some of the developments described above, as well as with a discussion of some of the important conjectures in the field, such as the conjectures of Quillen-Lichtenbaum, Bloch-Kato, and Beilinson-Lichtenbaum, and finish with some new developments relating the representation theory of the absolute Galois group of a field with its algebraic K-theory.

I. Quillen's Construction

We begin with the algebraic construction of K_0, K_1, and K_2. Recall that a **projective module** over a ring A is an A-module which is a direct summand of a free A-module. Note that direct sums of projective modules are again projective, and therefore that the collection of isomorphism classes of finitely generated projective A-modules forms a commutative monoid, which we'll denote by $\mathcal{P}(A)$.

DEFINITION I.1. *Let A be a ring. Then K_0A is defined to be the group completion of the monoid $\mathcal{P}(A)$.*

This definition is very closely related to the definition of topological K-theory of spaces, defined in terms of a group completion construction on a monoid of isomorphism classes of vector bundles on the space. The connection is made via the observation of Swan [**47**] that isomorphism classes of vector bundles over a base space X are identified with the isomorphism classes of finitely generated projective modules over the ring $\mathcal{C}(X)$ of continuous real valued functions on X. Here are some examples where K_0 is well understood, or is identified with familiar objects.

Example: If A is a local ring, or if A is a P.I.D. , then $K_0A \cong \mathbb{Z}$. A generator for this group is given by the isomorphism class of a free A-module of rank 1. This statement follows from the observation in these cases that any finitely generated projective module over A is in fact free. For fields, this result is immediate from the classification of vector spaces by their dimension. In the case of local rings, the result follows from the classification of modules over the residue class field by a standard lifting argument. The result for P.I.D.'s is immediate from the well-known classification of finitely generated modules over such rings.

Example: Suppose A is a *Dedekind domain.* Let $Cl(A)$ denote the *ideal class group* (see [**22**]) of A. This is a well known invariant of Dedekind rings, which carries important arithmetic information when A is the ring of integers in a number field, and which is the *Picard group* of a curve when we are dealing with modules over a curve. In this case, we find that

$$K_0A \cong \mathbb{Z} \oplus Cl(A)$$

Example: Let G be a finite group, and let $A = \mathbb{C}[G]$, the complex group ring of G. Then

$$K_0A \cong R[G]$$

where $R[G]$ denotes the *complex representation ring* or *complex character ring* of the group G. In this case, the computation comes from the theorem that any finite dimensional representation of G is isomorphic to a unique direct sum of irreducible representations.

In defining K_1, we first introduce the general linear group $GL_n(A)$ of invertible $n \times n$ matrices with entries in A. For each pair i, j, with $1 \leq i, j \leq n$, and $i \neq j$, and $a \in A$, we define the matrix $e_{ij}(a)$ to be the $M = \{m_{ij}\}$, with $m_{kk} = 1$ for all k, with $m_{ij} = a$, and with $m_{kl} = 0$ whenever $k \neq i$ or $l \neq j$. The matrices $e_{ij}(a)$ are referred to as *elementary matrices*, and the subgroup they generate is denoted by $E_n(A)$. When $n \geq 3$, $E_n(A)$ is equal to the commutator subgroup $[GL_n(A), GL_n(A)]$, and is *perfect* in the sense that $E_n(A) = [E_n(A), E_n(A)]$, where $[E_n(A), E_n(A)]$ denotes the commutator subgroup of $E_n(A)$. We have obvious inclusions $Gl_n(A) \to GL_{n+1}(A)$, which restrict to inclusions $E_n(A) \to E_{n+1}(A)$, and therefore induce homomorphisms

$$GL_n(A)/E_n(A) \to GL_{n+1}(A)/E_{n+1}(A)$$

DEFINITION I.2. *K_1A is defined to be the direct limit*

$$\underset{\rightarrow}{lim}\, GL_n(A)/E_n(A)$$

Example: For any commutative ring A, we have the determinant homomorphism $GL_n(A) \to A^*$, where A^* denotes the group of units of A. Since the determinant ble with the inclusions $GL_n(A) \hookrightarrow GL_{n+1}(A)$, we obtain a homomorphism $K_1A \to A^*$, also referred to as the determinant. This homomorphism is an isomorphism in a number of interesting cases, including fields, local rings, and rings of integers in numbers fields. The kernel of the determinant homomorphism is denoted by SK_1A, and is of importance in geometric topology. See [**33**] for information on SK_1A for group rings of finite groups.

We will also give Milnors definition of K_2A. In studying the group $E_n(A)$, we find that there are certain universal relations which hold among the generators $e_{ij}(a)$ for all rings. They are

1. $e_{ij}(a)e_{ij}(a') = e_{ij}(a + a')$
2. $[e_{ij}(a), e_{jl}(a')] = e_{il}(aa')$ for $i \neq l$
3. $[e_ij(a), e_{kl}(a')] = 1$ for $i \neq k$, $j \neq l$

We now let the *Steinberg group* $St_n(A)$ be the group generated by symbols $x_{ij}(a)$, where $1 \leq i, j \leq n$, and $a \in A$, subject to the relations

1. $x_{ij}(a)x_{ij}(a') = x_{ij}(a + a')$
2. $[x_{ij}(a), x_{jl}(a')] = x_{il}(aa')$ for $i \neq l$
3. $[x_ij(a), x_{kl}(a')] = 1$ for $i \neq k$, $j \neq l$

There is an obvious homomorphism $St_n(A) \to E_n(A)$, as well as homomorphisms $St_n(A) \to St_{n+1}(A)$, which are compatible with the evident inclusions $E_n(A) \hookrightarrow E_{n+1}(A)$.

DEFINITION I.3. *K_2A is the kernel of the homomorphism* $\varinjlim St_n(A) \to \varinjlim E_n(A)$

It turns out that K_2A is precisely the center of the infinite Steinberg group $St(A) = \varinjlim St_n(A)$.

Example: Milnor [**32**] proved that $K_2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$,

Example: Matsumoto [**26**] has given an explicit presentation for K_2F, where F is any field. We will talk about this in more detail later.

In his book [**32**], Milnor studies these three groups, performs some computations, and also obtains some interesting relations among the groups. In particular, he finds that in certain cases, there is a long exact sequence involving these K-groups which is analogous to the Mayer-Vietoris sequence for computing homology groups of topological spaces. Suppose we have a commutative diagram of ring homomorphisms

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

where both arrows are surjective ring maps. Let $\mathcal{D}$ denote the "pullback ring" given by

$$\mathcal{D} = \{(a,b) \in A \oplus B | f(a) = g(b)\}$$

Milnor shows that given this data, there is an exact sequence of K-groups

$$K_2\mathcal{D} \to K_2A \oplus K_2B \to K_2C \to K_1\mathcal{D} \to K_1A \oplus K_1B \to K_1C \to$$
$$\to K_0\mathcal{D} \to K_0A \oplus K_0B \to K_0C$$

This long exact sequence is of course very suggestive of the Mayer-Vietoris sequence in topology, and suggests the possibility that the algebraic K-groups should be defined as topological invariants of a topological space attached to the ring A. This is in fact Quillen's approach; it turns out that the right invariants to consider are homotopy groups. We will now outline his construction as well as its properties.

We will assume that the reader is familiar with the notions of simplicial sets, their morphisms, geometric realizations, etc. Good references are [**27**], [**9**], or [**13**]. There is an extremely useful construction, called the *nerve* construction, which constructs a space from a small category $\underline{C}$

DEFINITION I.4. *Let $\underline{C}$ be a small category. The* nerve *of $\underline{C}$, $N.\underline{C}$, is defined by letting $N_k\underline{C}$ consist of all k-tuples of composable morphisms*

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} x_{k-1} \xrightarrow{f_k} x_k$$

The 0-simplices are simply objects in $\underline{C}$. Face maps are obtained by composing various pairs of morphisms, and degeneracy maps are obtained be inserting identities.

The construction $N_{.}$ has many useful properties. For instance, it is functorial on the category of small categories (so functors induce morphisms of simplicial sets and consequently continuous maps of topological spaces on the nerves.) Also, natural transformations induce homotopies on the realizations of the nerves. From this result one can easily deduce that a category with an initial or terminal object has contractible nerve. Quillen's construction of higher algebraic K-theory now proceeds as follows. First, he constructs a category Q from the category $Proj(A)$ of finitely generated projective modules over a ring A. Next, he constructs the nerve of this category, and forms the geometric realization of the space. The higher K-groups of A are now given by the homotopy groups of this space, but with a dimension shift, in the sense that the i-th K-group of A is the $(i+1)$-st homotopy group of this space. In order to be able to work with and compare various spaces constructed as nerves of categories, Quillen proved two crucial technical theorems. The first gives a criterion which will guarantee that a functor between small categories induces a homotopy equivalence on the realizations of the nerves of the categories. Let $f : C \to D$ be a functor between small categories. For any object Y of D, let $Y\backslash f$ denote the category whose objects are pairs (X, v), where X is an object of C, and where $v : Y \to fX$ is a morphism in D, and where a morphism from (X, v) to (X', v') is a morphism $w : X \to X'$ so that the diagram

$$\begin{tikzcd}
Y \arrow[d, "v"'] \arrow[dr, "v'"] & \\
fX \arrow[r, "fw"] & fX'
\end{tikzcd}$$

commutes. We now have

Quillen's Theorem A *If the category $Y\backslash f$ has contractible nerve for every object Y of D, then the functor f induces a homotopy equivalence on nerves.*

Quillen proves a parallel result which gives a sufficient condition for the map induced by a functor to behave like a fibration. Note that for any morphism $\theta : Y \to X'$ in D, we obtain a functor $\theta^* : Y'\backslash f \to Y\backslash f$. In fact, $Y \to Y\backslash f$ gives a contravariant functor from the category D to the category of small categories. Let's also recall that the *homotopy fiber product* of a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "q"] \\
Z \arrow[r, "p"] & W
\end{tikzcd}$$

is defined to be the subspace of $Z \times W^I \times Y$ given by $\{(w, \phi, y) | \phi(0) = p(z)$ and $\phi(1) = q(y)\}$. There is a canonical map from X to the homotopy fiber product. We say the original diagram is *homotopy Cartesian* if this natural map is a weak equivalence. In the case when the space Z is contractible, this condition means that we obtain a long exact sequence of homotopy groups

$$\ldots \pi_{i+1}(W) \to \pi_i(X) \to \pi_i(Y) \to \pi_i(W) \to \pi_{i-1}(X) \to \ldots$$

Suppose $f : C \to D$ be a functor. For any object Y of D, we obtain a commutative square of categories

$$\begin{tikzcd} Y\backslash f \arrow[r] \arrow[d] & C \arrow[d, "f"] \\ Y\backslash id_D \arrow[r] & D \end{tikzcd}$$

Here, the horizontal arrows are given by $(X, v) \to X$ and $(Y', v') \to Y'$. We also observe that the nerve of the category $Y\backslash id_D$ is contractible, since it has the initial object (Y, id).

Then for any functor Quillen now proved the following.

Quillen's Theorem B Let $f\colon C \to D$ be a functor such that for every arrow $\theta\colon Y \to Y'$ in D, the corresponding functor θ^* induces a weak equivalence on nerves. Then applying the nerve construction to the above given square of categories gives a homotopy cartesian diagram of spaces. Consequently, there is a long exact sequence on homotopy groups

$$\ldots \pi_{i+1}(|N.D|) \to \pi_i(|N.Y\backslash f|) \to \pi_i(|N.C|) \to \pi_i(|N.D|) \to \pi_{i-1}(|N.Y\backslash f|) \to \ldots$$

Let's now proceed to the definition of the category whose nerve will be the K-theory space. It turns out that it is useful to construct the category Q from categories other than $Proj(A)$. In fact, a generalization of the construction is necessary in order to be able to reduce calculations to more tractable problems. Quillen's approach is to identify certain important properties of $Proj(A)$, and observe that he can construct a category Q from any category which has these properties. The relevant properties are as follows.

1. The category should be additive, i.e. the *Hom*-sets should all be abelian groups, and the composition operations should be bilinear in each of the arguments.
2. The category should admit direct sums. Clearly true for $Proj(A)$
3. The category should be equipped with a family of sequences of morphisms

$$M \xrightarrow{i} N \xrightarrow{j} P$$

referred to as the *exact* sequences. Any morphism which occurs as i in such a sequence is called an *admissible monomorphism*, and any morphism which occurs as j in such a sequence is called an *admissible epimorphism*. This is clearly the case for $Proj(A)$, since it has a well-defined notion of exact sequencs. The admissible monomorphisms correspond to the split monomorphisms, and the epimorphisms correspond to the usual epimorphisms.
4. Any sequence which is isomorphic to an exact sequence should itself be exact. This is clearly the case in $Proj(A)$.
5. Any sequence of the form

$$M \to M \oplus N \to N$$

is exact. Again, this is clearly true in $Proj(A)$
6. For any exact sequence as above, i should be a kernel for j and j should be a cokernel for i. Clearly true for $Proj(A)$.

7. The class of admissible monomorphisms should be closed under composition, and so should the class of admissible epimorphisms. Clearly true for $Proj(A)$
8. The class of admissible monomorphisms should be closed under co-base change by any map in the category. This means that given a diagram of the form
$$\begin{tikzcd} M \arrow[r] \arrow[d, "i"'] & N \\ P & \end{tikzcd}$$
where i is an admissible monomorphism, there exists a universal pushout $\mathcal{D}$ fitting in the diagram
$$\begin{tikzcd} M \arrow[r] \arrow[d, "i"'] & N \arrow[d, "\hat{\imath}"] \\ P \arrow[r] & \mathcal{D} \end{tikzcd}$$
and $\hat{\imath}$ is itself an admissible monomorphism. This property holds for $Proj(A)$ since the pushout is given by the cokernel of the map $M \to N \oplus P$, which is a split monomorphism since $M \to P$ is. Similarly, the admissible epimorphisms should be closed under arbitrary base change, which is the evident dual condition.
9. Let $M \to M'$ be any morphism in our category which possesses a kernel. If there existsa map $N \to M$ such that the composite $N \to M \to M'$ is an admissible epimorphism, then $M \to M'$ is an admissible epimorphism. This condition is clear for $Proj(A)$. The dual condition should hold for admissible monomorphisms

DEFINITION I.5. *An exact category is a category, equipped with a family of exact sequences, satisfying all the above properties.*

This definition includes many categories which are not of the form $Proj(A)$ for any A, and which are not even module categories at all.

Example: The category of all finite abelian groups is an exact category, when equipped with its usual notion of exact sequence.

Example: Any abelian category is an exact category, where the exact sequences are exactly the exact sequences in the abelian category. All monomorphisms and epimorphisms are admissible.

Example: The category of coherent modules over a scheme is an exact category.

Example: The category of sheaves over any topological space is an exact category.

Quillen now attaches to any exact category E a new category $Q(E)$, whose nerve will be the space whose homotopy groups are the higher algebraic K-groups of the exact category. In the particular case where $E = Proj(A)$, it will give the higher algebraic K-groups of the ring. The objects of the category $Q(E)$ are the same as the objects of E. However, $Q(E)$ has a different morphism set. For objects X and Y in E, a morphism is an **isomorphism class** of diagrams of the form

$$
\begin{array}{ccc}
W & \xrightarrow{i} & Y \\
{\scriptstyle p}\downarrow & & \\
X & &
\end{array}
$$

where p is an admissible epimorphism and i is an admissible monomorphism. Two such diagrams are considered isomorphic if there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
{\scriptstyle p}\uparrow & & \uparrow{\scriptstyle p'} \\
W & \xrightarrow{\cong} & W' \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\
Y & \xrightarrow{id} & Y
\end{array}
$$

In the particular case $E = Proj(A)$, it is a useful exercise to check that a morphism from P to Q is exactly the same thing as an isomorphism from P to a *subquotient* of Q. In particular, the isomorphisms of E will be morphisms in $Q(E)$.

We now have

DEFINITION I.6. *The higher algebraic K-groups of a ring A are defined by*

$$K_i(A) \cong \pi_{i+1}(|N.Q(Proj(A))|)$$

For a general exact category E, we will write K_iE for $\pi_{i+1}(|N.Q(E)|)$. We will also write $\underline{K}(E)$ for the loop space $\Omega|N.Q(E)|$, so $K_i(E) \cong \pi_i(\underline{K}(E))$.

K-theory defined in this manner enjoys a number of useful and important properties.

Devissage

This is a very useful device for showing, that an inclusion of abelian categories $E \hookrightarrow E'$ induces a homotopy equivalence $\underline{K}(E) \to \underline{K}(E')$, and hence an isomorphism on homotopy groups. The statement is as follows.

THEOREM I.7. *Suppose that every object X of E' has a finite filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that X_j/X_{j-1} is in the category E for each j. Then the inclusion functor $Q(E) \to Q(E')$ induces a homotopy equivalence $\underline{K}(E) \to \underline{K}(E')$, and hence an isomorphism on K-groups.*

A typical application is given by considering the case where E' is the category of all finite abelian p-groups, and where E is the full subcategory of all finite abelian groups of exponent p. Since every finite abelian p-group admits a filtration whose subquotients have exponent p, the inclusion $E \hookrightarrow E'$ induces an equivalence $\underline{K}(E) \to \underline{K}(E')$. On the other hand, E is clearly equivalent to the category of modules over the finite field $\mathbb{F}_p$, so $K_i(E') \cong K_i(\mathbb{F}_p)$. This theorem is a consequence of Quillen's Theorem A, applied to the inclusion functor $Q(E) \hookrightarrow Q(E')$.

Reduction by resolution

This is another technique for comparing the K-theory spaces for two different exact categories. The devissage technique introduced above applies only to abelian categories. The localization sequence, which is the key technical tool we will introduce, also applies only to abelian categories. However, the categories $Proj(A)$, for A a ring, are not abelian, since kernels and cokernels of maps between projective modules are not necessarily abelian. The *reduction by resolution* technique, on the other hand, applies to categories which are not abelian. It will permit us to get information about K-groups of non-abelian categories using devissage and localization techniques by comparing the non-abelian category with an abelian one. Here is the statement of the theorem.

THEOREM I.8. *Let P be a full subcategory of an exact category E which is closed under extensions (i.e. if there is an exact sequence $X \to Y \to Z$ in E, and X and Z are both in P, then Y is also in P), and*

1. *For every exact sequence $X \to Y \to Z$ in E, if Y and Z are in P, then so is X.*
2. *Given an admissible epimorphism $j : X \to W$, with W in P, there exists an object W' in P and a commutative diagram*

$$\begin{tikzcd} W' \arrow[d] \arrow[dr, "p"] & \\ X \arrow[r, "j"] & W \end{tikzcd}$$

   where p is an admissible epimorphism.

Let E_∞ denote the full subcategory of all objects in E which admit finite resolutions by objects in P. (It is clear what is meant by a resolution in this case). Then the inclusion $P \hookrightarrow E_\infty$ induces an equivalence of K-theory spaces $\underline{K}(P) \to \underline{K}(E_\infty)$. In particular, if every object in E admits a finite resolution by objects of P, then the map $\underline{K}(P) \to \underline{K}(E)$ is a homotopy equivalence.

Example: Let E denote the exact category of finitely generated $\mathbb{Z}$-modules, and let P denote the full subcategory of finitely generated free $\mathbb{Z}$-modules. (P is not abelian.) Since every finitely generated modules admits a finite resolution (length one, in fact) by free modules, the above theorem implies that $\underline{K}(P) \to \underline{K}(E)$ is an equivalence. In other words, the K-theory of all finitely generated $\mathbb{Z}$-modules is isomorphic to the K-theory of the ring $\mathbb{Z}$. This example generalizes to show that for any Noetherian regular ring, the K-theory of the category of all finitely generated modules over the ring has the same K-theory as the category of all projective modules.

Localization

This is the key theorem which allows the decomposition of K-theoretic computations, in the way that the excision theorem in algebraic topology allows the computation of homology by successive reduction to the case of a point via Mayer-Vietoris sequences and the long exact sequences of pairs. Recall from **[45]** that if we have in inclusion $\underline{A} \hookrightarrow \underline{B}$ of abelian categories, with $\underline{A}$ a Serre subcategory, we may construct the quotient category $\underline{A}/\underline{B}$. $\underline{A}/\underline{B}$ is an abelian category in its own right.

(Recall that non-empty full subcategory$\underline{A}$ of and abelian category $\underline{B}$ is said to be a *Serre subcategory* if for any short exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\underline{B}$, any of the three objects is in $\underline{A}$ if and only if the other two are.) Moreover, it has the same objects as $\underline{B}$, but the morphisms are different. For instance, any morphism in $\underline{B}$ which factors through an objects in $\underline{A}$ is identified with the zero morphism in this construction. There is a canonical functor $\underline{B} \to \underline{B}/\underline{A}$.

Example: Let $\underline{B}$ be the category of all finitely generated abelian groups, and let $\underline{A}$ denote the full subcategory on the finite abelian groups. $\underline{A}$ is a Serre subcategory. Then $\underline{B}$ is the category of all $\mathbb{Z}$-modules, and the quotient category is equivalent to the category of finite dimensional $\mathbb{Q}$-vector spaces. This result generalizes to any localization of Noetherian rings, , i.e. if $R \hookrightarrow R_S$ is the inclusion of a Noetherian commutative ring R into its localization at a multiplicative subset S, and we let $\underline{B}$ denote the category of finitely generated R-modules, and $\underline{A}$ denote the full subcategory of $\underline{B}$ consisting of the modules M for which $M_S = 0$, the the quotient category $\underline{B}/\underline{A}$ is equivalent to the category of finitely generated R_S-modules.

The localization theorem describes the behavior of the K-theory functor on quotient categories.

THEOREM I.9. *Let $\underline{A}$ be a Serre subcategory of an abelian category $\underline{B}$. Then the sequence of maps*

$$\underline{K}(\underline{A}) \to \underline{K}(\underline{B}) \to \underline{K}(\underline{B}/\underline{A})$$

is a fibration up to homotopy in the sense that there is a canonical identification of $\underline{K}(\underline{A})$ with the homotopy fiber of the map $\underline{K}(\underline{B}) \to \underline{K}(\underline{B}/\underline{A})$ which commutes with the inclusions into $\underline{K}(\underline{B})$. (For any map of based topological spaces $f : X \to Y$, there exist functorial constructions $f \to F(f)$ and $f \to E(f)$ of new spaces, together with naturally defined maps $i : F(f) \to E(f)$, $p : E(f) \to Y$, and $\eta : X \to E(f)$, so that η is a weak homotopy equivalence,, so that the sequence

$$F(f) \to E(f) \to Y$$

is a fibration, and so that the diagram

$$\begin{tikzcd} X \arrow[d, "\eta"'] \arrow[dr, "f"] & \\ E(f) \arrow[r, "p"] & Y \end{tikzcd}$$

commutes. $F(f)$ is called the homotopy fibre of the original map f.)

It now follows that under the hypotheses of the theorem, we obtain a long exact sequence

$$\ldots \to K_{i+1}(\underline{B}/\underline{A}) \to K_i(\underline{A}) \to K_i(\underline{B}) \to K_i(\underline{B}/\underline{A}) \to K_{i-1}(\underline{A}) \to \ldots$$

Example: We consider the example from above, with $\underline{A}$ being the category of finite abelian groups, contained as a subcategory of $\underline{B}$, the category of all finitely generated abelian groups. The reduction by resolution theorem from above shows that $K_i(\underline{B}) \cong K_i(\mathbb{Z})$. On the other hand, the category of all finite abelian groups breaks up as a sum of the categories $\underline{A}_p$ consisting of all p-power order finite abelian groups, and it is easy to check from the definitions that $\underline{K}(\underline{A}) \cong \coprod_p \underline{K}(\underline{A}_p)$, where the coproduct sign indicates the subspace of the product where all but finitely many coordinates are at the base point. Devissage shows that $\underline{K}(\underline{A}_p) \cong \underline{K}(\mathbb{F}_p)$, so we have a long exact sequence

$$\ldots \to K_{i+1}(\mathbb{Q}) \to \bigoplus_p K_i(\mathbb{F}_p) \to K_i(\mathbb{Z}) \to K_i(\mathbb{Q}) \to \bigoplus_p K_{i-1}(\mathbb{F}_p) \to \ldots$$

Notice that we have reduced the calculation of $K_*(\mathbb{Z})$ to the calculation of the K-theory of the fields $\mathbb{F}_p$ and $\mathbb{Q}$, together with the analysis of a boundary map in an exact sequence and some potential extension problems. This is generally the case - devissage and localization in many cases permits the reduction of K-theory calculations to calculations for *fields*. This is analogous to the topological reduction of homology calculations to the homology of a point. In this case, though, the fields are not nearly as simple as one would like, so further ideas are required.

Gersten spectral sequence

There is a more explicit and systematic way to express the reduction of the calculation of K theory of rings to that of fields. Let's suppose we have a regular Noetherian ring A, of finite Krull dimension. Since the ring is Noetherian regular, we can apply the reduction by resolution ideas from above to conclude that in order to compute the K-theory of A, it will suffice to compute the K-theory of the exact category $Mod(A)$ of all finitely generated modules over A. It is a standard result from commutative algebra that every finitely generated A module admits a composition series in which the subquotients are all of the form $A/\wp$, where $\wp$ is a prime ideal of A. Let $Mod_p(A) \subseteq Mod(A)$ denote the full subcategory of all A modules which admit a composition series in which each of the subquotients is of the form $A/\wp$, where $\wp$ denotes a prime of codimension $\geq p$. (Recall that the codimension of a prime ideal $\wp$ is the Krull dimension of the local ring $A_\wp$) One can show that the subcategories $Mod_p(A)$ are all Serre subcategories of $Mod(A)$. Repeated use of the localization sequence now permits the construction of a spectral sequence whose E_1-term consists entirely of K-groups of the residue class fields $k(\wp)$ of the local rings of the various primes of A.

THEOREM I.10. *Let X_p denote the set of prime ideals of A of codimension $=p$. Then there is a spectral sequence with*

$$E_1^{p,q} = \coprod_{\wp \in X_p} K_{-p-q}k(\wp)$$

converging to $K_{-(p+q)}(A)$. The spectral sequence is concentrated in the region $p \geq 0$, $p+q \leq 0$. This spectral sequence was conjectured by Gersten to collapse at E_2 when A is a regular local ring. Quillen was able to show that it collapses at E_2 in the case of a semilocal algebra of finite type over a field, all of whose localizations at maximal ideals are regular.

Homotopy property

The preceding results together form the analogue of the excision property for algebraic K-theory. They show that the computation of algebraic K-theory can in many cases be reduced to case of "one point schemes", i.e. spectra of fields. One could ask the question, "What is the analogue of the homotopy property". There is actually more than one answer to that question, but one version proved by Quillen is particularly useful. The idea is to recognize that in topology, the real line is certainly contractible, i.e. that the inclusion of the origin in the line is a homotopy equivalence. The ring theoretic analogue of the affine line is the polynomial algebra on a single variable over a given ring A, $A[x]$.

THEOREM I.11. *Let A be a regular ring. Then the canonical inclusion $A \to A[x]$ induces an equivalence of K-theory spaces*

$$\underline{K}(A) \to \underline{K}(A[x])$$

and consequently $K_i(A) \cong K_i(A[x])$.

This result fails when the ring A is not regular. $K(A[x])$ is nevertheless understood, though. It contains the K-theory of the category of finitely generated modules of projective length one over A, equipped with a nilpotent endomorphism. See [**53**] for details.

II. Algebraic K- theory and stable homotopy theory

In the last section, we defined the higher algebraic K-groups of a ring (or more generally, of an exact category E), as the homotopy groups of a certain space $\underline{K}(E)$. It turns our that this space is equipped with additional structure, which is very useful in analyzing the stucture of the space, and in describing constructions on it. In order to describe this extra structure, we will need to make a brief digression into the area of stable homotopy theory.

A first motivational question to ask is, "What is the topological analogue of an abelian group?" One analogue is the notion of a topological abelian group. We can certainly study topological abelian groups, but it turns out that the possible homotopy types are very restricted. All the abelian groups are actually products of "Eilenberg-MacLane spaces", which are spaces with a single non-vanishing homotopy group. In fact, the homotopy category of topological abelian groups is equivalent to the homotopy category of chain complexes over $\mathbb{Z}$. Another, more interesting analogue, arises when we do not assume that the identites defining abelian groups hold "on the nose", but rather only up to homotopy. For instance, part of the structure of an abelian group A is a map $\mu\colon A \times A \to A$, so that $\mu T = \mu$, where $T\colon A \times A \to A \times A$ is the map which interchanges coordinates. So, a first step is to permit the identity $\mu T = \mu$ to be replaced by a homotopy $\mu T \simeq \mu$. However, the situation is not quite as simple as we might at first think. Once we have chosen a homotopy from μT to μ, we must also ask for a homotopy H from $\mu(\mu \times 1)\sigma$ to $\mu(\mu \times 1)$, where σ is an element of Σ_3, acting by permuting coordinates. Take, for example, the 3-cycle (123), which is written as composite in two different ways, as (13)(12) or as (12)(23). There are now two possible ways to generate such a homotopy from the H, namely (1) first using H to reverse the first two coordinates, next using it again to reverse the first and third, or (2) first using H to first reverse the last two coordinates, next using it againg to reverse the first and second coordinates. There is now reason to suppose that these two ways of generating a homotopy will agree, and it is reasonable to suppose that we must include *as part of the data* such a homotopy. In general, much more information is needed. Topologists and category theorists [**25**] have clarified what the required data is. They have shown that there is actually a hierarchy of different degrees of commutativity, one for each $n \geq 2$.

To give a reasonable idea about the significance of this hierarchy, we will need to talk about loop spaces. Recall from elementary topology that $\pi_2(X)$ is always abelian for any space X. The proof of this statement is actually such that it shows that the multiplication map on $\Omega^2 X$, the second loop space on X, is homotopy commutative in the sense that $\mu T \simeq \mu$. There are additional higher homotopies which hold on a second loop space. One is able to codify the necessary structures, so that there is an object called "the free second loop space on X", which we'll write $\mathcal{C}_2 X$. Just like the "free group functor" from sets to groups, this functor is a *monad* on the category of spaces, and a"second loop space structure" on a space X is just a map $\mathcal{C}_2 X \to X$, satisfying certain properties. There is a category of second loop spaces and second loop maps, and it is a reasonable candidate for the notion of a "homotopy abelian group". However, for every n, there is a notion of the "free n-fold loop space on a space X", written $\mathcal{C}_n X$, and a corresponding category of n-fold loop spaces and n-fold loop maps. Each of these categories is a reasonable candidate for a category of homotopy abelian groups, with one being"more commutative" than the one which precedes it. One can also pass to the limit with n, and obtain a category of"infinite loop spaces" and "infinite loop maps", and this is indeed the category we will be dealing with.

Example: For any based space X, we may construct the k-fold suspension of X, $\Sigma^k X$, and the k-fold loop space of $\Sigma^k X$, $\Omega^k \Sigma^k X$. Further, we have inclusions $\Omega^k \Sigma^k X \hookrightarrow \Omega^{k+1} \Sigma^{k+1} X$, obtained by suspending k-fold loops to obtain $k+1$-fold loops. The direct limit $\Omega^\infty \Sigma^\infty X$ is denoted by $Q(X)$. It is a very interesting space, whose homotopy groups are the *stable homotopy groups* of X. This space is a $\mathcal{C}_\infty$-space, but does *not* have the homotopy type of any topological abelian group.

Example: The infinite complex Grassmannian BU becomes a $\mathcal{C}_\infty$-space when we equip it with the multiplication operation we get by taking block sum of subspaces in $\mathbb{C}^\infty \cong \mathbb{C}^\infty \oplus \mathbb{C}^\infty$.

One can now for methods for constructing $\mathcal{C}_\infty$-spaces, analogous to the nerve construction which constructs spaces as the nerves of small categories. It turns out that when one is given a category with certain additional structure, one can construct a $\mathcal{C}_\infty$-space from it in a canonical way. The additional structure is that of a *symmetric monoidal category*, which means a category $\underline{C}$, together with a sum functor $\underline{C} \times \underline{C} \to \underline{C}$, together with certain natural isomorphisms which, among other things, means that the sum operation is commutative and associative up to a canonical isomorphism. See **[21]** for a discussion of these ideas.

Example: The category of finite sets, with sum functor given by disjoint union, is a symmetric monoidal category.

Example: For any ring A, the category of all finitely generated projective modules over A is a symmetric monoidal category when the sum functor is given by direct sums of modules.

It is not hard to see that the nerve of a symmetric monoidal category is equipped with a product operation induced by the sum operation, and that the product is homotopy commutative and asociative. It is not necessarily an infinite loop space, though, because the operation does not necessarily have inverses, even up to homotopy. However, it is possible to group complete the space in a canonical way to obtain an infinite loop space, and this operation is functorial from the category of symmetric monoidal categories to the category of infinite loop spaces. Such a functor is called an *infinite loop space machine*, and several exist [**40**], [**28**]. It turns out that K-theory can be constructed in this way.

THEOREM II.1. *For any ring A, the K-theory space $\underline{K}(A)$ is equivalent to the space constructed as above from the symmetric monoidal category of finitely generated projective A-modules. Moreover, $\underline{K}(E)$ is an infinite loop space for any exact category E.*

Remark: The reader may wonder what the value of Quillen's construction is when algebraic K-theory can be constructed so simply using the infinite loop space machinery (which did exist at the time of Quillen's work). It turns out that the important tools discussed in the previous section are not seen nearly as clearly in the infinite loop space machine context. Without the work of Quillen, it is doubtful that these theorems would have been discovered in this context. After Quillen's work, proofs have been found using infinite loop space theory, but they are still much more technically difficult.

Not only is it the case that infinite loop space machines produce infinite loop spaces, but the also produce *deloopings* of the space in question. That is, the infinite loop space machines start with a symmetric monoidal category and produce a whole family of spaces $\{X_i\}_{i\geq 0}$, together with identifications $X_i \to \Omega X_{i+1}$ for each i. Such a family of spaces is called a *spectrum*. Spectra are a very convenient tool for handling various questions in stable homotopy theory. Indeed, for any exact category E, the space $\underline{K}(E)$ is the zeroth space of a spectrum, which we will write $\{B^i\underline{K}(E)\}_i$. The K-theory spectrum (indeed any spectrum which is the output of an infinite loop space machine) is *connective* in the sense that $\pi_j(B^i\underline{K}(E)) = 0$ when $j < i$. It is, though, very useful to consider the possibility of *non-connective spectra*, and to construct a category (and a homotopy theory) of such objects. Inclusion of non-connective spectra means that the category is closed under various constructions, such as mapping spectra, homotopy pullbacks, etc.

Finally, recent work of [**11**], [**19**], and other has shown that it is possible to construct a full blown theory of ring spectra and module spectra over ring spectra, which permit Hom and $\otimes$ constructions. These constructions have much in common with the construction of algebras and modules in derived categories. Let us first consider the case of rings. A ring is a *monoid object* in the category $\underline{Ab}$ of abelian groups, in the sense that a ring is an abelian broup R together with a map $R \otimes R \to R$, satisfying certain properties. What is required to carry this definition through is that tensor product is a coherently associative and commutative operation on $\underline{Ab}$. What the above mentioned authors show is that the category of spectra also admits such an operation, called *smash product*, which is analogous to the tensor product on $\underline{Ab}$. Starting at this point, the authors are able to construct ring spectra and module spectra over ring spectra. For any ring spectrum R, and left R-module spectra M and N, there is a spectrum $Hom_R(M, N)$. Similarly, for any right R-module M and left R-module N, there is a spectrum $M \underset{R}{\wedge} N$, analogous to the tensor product construction. The key computational tool is a pair of spectral sequences for computing the homotopy groups of these constructions.

THEOREM II.2. *Let R be a ring spectrum, and let M and N be left R-modules. Then there is a spectral sequence with*

$$E_2^{p*} \cong Ext_{\pi_* R}^{-p}(\pi_* M, \pi_* N)$$

converging to $\pi_*(Hom_R(M,N))$. Similarly, if M is a right R-module, and N is a left R-module, there is a spectral sequence with

$$E_2^{**} \cong Tor_{**}^{\pi_* R}(\pi_* M, \pi_* N)$$

converging to $\pi_*(M \underset{R}{\wedge} N)$

These ideas will be very useful later when we formulate conjectures concerning the homotopy type of the K-theory spectrum of a field.

III. Theorems of Suslin and Gabber

We have earlier mentioned the homotopy property of algebraic K-theory. There is also a very deep theorem of A. Suslin [**43**] which describes the algebraic K-theory of an algebraically closed field. In order the describe the result, we will need to discuss "K-theory with coefficients", and the notion of completion in homotopy theory.

DEFINITION III.1. *Let E be an exact category, and let $\underline{K}(E)$ denote the K-theory space of E. Remember that $\underline{K}(E)$ is a topological monoid, and let $\times n \colon \underline{K}(E) \to \underline{K}(E)$ denote the n-th power map from $\underline{K}(E)$ to itself. It turns out that the group theoretic model for $\underline{K}(E)$ can be taken to be torsion free, so that $\times n$ is an inclusion. We define the K-theory space of E with* mod-n coefficients *to be the coset space attached to the image of the map $\times n$. We will denote it by $\underline{K}(E, \mathbb{Z}/n\mathbb{Z})$. Inclusion maps $\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Z}/mn\mathbb{Z}$ and projection maps $\mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ induce maps on K-theory spaces with coefficients and there are "Bockstein" fibration sequences.*

$$\underline{K}(E, \mathbb{Z}/m\mathbb{Z}) \to \underline{K}(E, \mathbb{Z}/mn\mathbb{Z}) \to \underline{K}(E, \mathbb{Z}/n\mathbb{Z})$$

and

$$\underline{K}(E) \xrightarrow{\times n} \underline{K}(E) \to \underline{K}(E, \mathbb{Z}/n\mathbb{Z})$$

which induce long exact sequences on homotopy groups. We write $K_i(E, \mathbb{Z}/n\mathbb{Z})$ for $\pi_i \underline{K}(E, \mathbb{Z}/n\mathbb{Z})$. Using the tower of coefficient homomorphisms

$$\cdots \to \mathbb{Z}/p^{i+1}\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^{i-1}\mathbb{Z} \to \cdots$$

and a homotopy theoretic version of the inverse limit construction, we can also constructed the p-completed K-theory space which we denote by $\underline{K}(E)\hat{}_p$ The homotopy groups of $\underline{K}(E)\hat{}_p$ are well related to the homotopy groups of $\underline{K}(E)$. There is a short exact sequence

$$0 \to \underset{\leftarrow}{lim}^1 \left(\pi_{i-1} K_{i-1}(E)/p^j K_{i-1}(E)\right) \to \pi_i(\underline{K}\hat{(E)}_p) \to K_i\hat{(E)}_p \to 0$$

where the left hand term is a derived functor of completion and the right hand term denotes the usual algebraic completion operation. The derived term vanishes when the groups in question are the sum of a finitely generated abelian group with a uniquely p-divisible group.

Consider the K-groups of an algebraically closed field k. In this case, $K_1 k$ is isomorphic to the group of units in k, which is *divisible* since every element has an n-th root. In fact, it is a direct sum of its torsion subgroup (isomorphic to $\mathbb{Q}/\mathbb{Z}$, the roots of unity), with a uniquely divisible group (i.e. a $\mathbb{Q}$-vector space) of infinite dimension. It appears difficult to obtain control of this large rational summand, but the roots of unity are present and isomorphic to $\mathbb{Q}/\mathbb{Z}$ for any field of characteristic zero. In the case of a field of characteristic p, the group of roots of unity is isomorphic to the quotient of $\mathbb{Q}/\mathbb{Z}$ by its p-torsion subgroup. Since the completion construction "does not see" the $\mathbb{Q}$-vector spaces, we migh conjecture that the K-spaces with finite coefficients prime to the characteristic might all be equivalent for algebraically closed fields. This is what Suslin was able to prove.

THEOREM III.2. *Let $k \subseteq K$ be an inclusion of algebraically closed fields. Then if $(n, char(k)) = 1$, the map $\underline{K}(k, \mathbb{Z}/n\mathbb{Z}) \to \underline{K}(K, \mathbb{Z}/n\mathbb{Z})$ is a weak equivalence. In particular, the maps $K_i(k, \mathbb{Z}/n\mathbb{Z}) \to K_i(K, \mathbb{Z}/n\mathbb{Z})$ are isomorphisms. By using standard completion techniques, it follows that if p is a prime, with $(p, char(k)) = 1$, then the map $\underline{K}(\hat{k_p} \to \underline{K}(K)\hat{}_p$ is an equivalence.*

In another paper [**44**], Suslin studies the algebraic K-theory of fields of characteristic zero. He considers $\underline{K}(\mathbb{C})$. He first observes that if the algebraic K-theory space construction is applied to a topological ring (such as $\mathbb{C}$), one can make a topologized K-theory space construction attached to that ring, written $\underline{K}^{top}$, and there is an evident map $\underline{K}(\mathbb{C}) \to \underline{K}^{top}(\mathbb{C})$. This construction can also be made with coefficients, or with a completion at a prime p. Moreover, it is direct from the constructions that $\underline{K}^{top}(\mathbb{C}) \cong BU \times \mathbb{Z}$, whose homotopy groups we understand completely due to the Bott periodicity theorem. Of course, the answer is that $\pi(BU \times \mathbb{Z}) \cong \mathbb{Z}$ when i is even, and $\cong 0$ when i is odd. Suslin now proves

THEOREM III.3. *The canonical maps*

$$\underline{K}(\mathbb{C}, \mathbb{Z}/n\mathbb{Z}) \to \underline{K}^{top}(\mathbb{C}, \mathbb{Z}/n\mathbb{Z})$$

and

$$\underline{K}(\mathbb{C})^{\wedge}_{p} \to \underline{K}^{top}(\mathbb{C})^{\wedge}_{p}$$

are equivalences.

As we have seen earlier, Quillen [**37**] has computed the algebraic K-groups of all finite fields. By passing to direct limits, he is able to compute the the algebraic K-theory of the algebraic closure of the prime field $\mathbb{F}_p$.

THEOREM III.4. *There is a canonical map $\underline{K}(\overline{\mathbb{F}}_p) \to BU \times \mathbb{Z}$, which induces equivalences $\underline{K}(\overline{\mathbb{F}}_p, \mathbb{Z}/n\mathbb{Z}) \to \underline{K}^{top}(\mathbb{C}, \mathbb{Z}/n\mathbb{Z})$ and $\underline{K}(\overline{\mathbb{F}}_p)^{\wedge}_{q} \to \underline{K}^{top}(\mathbb{C})^{\wedge}_{q}$ when $(n,p) = 1$ and p and q are distinct primes.*

We now have explicit computations of the K-theory spaces of some algebraically closed field for each possible characteristic, including 0. It now follows that we are able to compute the K-groups with finite coefficients relatively prime to the characteristic of any algebraically closed field, and also the homotopy groups of the completion at a prime relative to the characteristic of any algebraically closed field.

THEOREM III.5. *Let k be any algebraically closed field. Suppose $(n, char(k)) = 1$. Then $K_i(k, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ if i is even, and $\cong 0$ if i is odd. Similarly, let q be a prime different from the characteristic of k. Then $\pi_i(\underline{K}(k)^{\wedge}_p) \cong \mathbb{Z}_{(q)}$ when i is even, and $\cong 0$ when i is odd. Here $\mathbb{Z}_{(q)}$ denotes the q-adic integers.*

O. Gabber in **[12]** was able to "relativize" Suslin's theorem. We will first need to remind the reader about the notion of a Henselian local ring. Recall that for a complete discrete valuation ring A with maximal ideal m (such as the p-adic integers, or the power series ring in one variable over a field k), Hensel's lemma holds. This lemma asserts that if we have a monic polynomial f with coefficients in A, so that $\overline{f} = g_0 h_0$, where $\overline{f}$ denotes the reduction of f into a polynomial over A/m, and where g_0 and h_0 are coprime monic polynomials over A/m, then there is a factorization of $f = gh$, where g and h satisfy $\overline{g} = g_0$ and $\overline{h} = h_0$. We say a local ring A is *Henselian* if this result on polynomials holds for polynomials over A. There are a number of equivalent ways of defining this concept. Perhaps the most important one is that the category of Galois ring extensions of A are equivalent to the category of Galois ring extensions of A/m.

THEOREM III.6. *Let A be a Henselian local ring, with residue class field $k = A/m$. Let n be a number prime to the characteristic of k. Then the natural map*

$$\underline{K}(A, \mathbb{Z}/n\mathbb{Z}) \to \underline{K}(k, \mathbb{Z}/n\mathbb{Z})$$

is an equivalence. Also, if q is a prime distinct from $char(k)$, then the natural map

$$\underline{K}(A)^{\wedge}_q \to \underline{K}(k)^{\wedge}_q$$

is an equivalence.

This theorem permits, for instance, the computation of the K-theory (completed, or with finite coefficients) of power series rings of fields, analogous to the earlier described computations of the algebraic K-theory of polynomial rings. In this case, though, we need to work with finite coefficients prime to the characteristic or with completion at a prime distinct from the characteristic of the field.

IV. Descent

In the preceding section we saw that a good deal of information is available about the K-theory spaces of algebraically closed fields. In particular, we can describe completely the p-adically completed homotopy type of $\underline{K}(\overline{k})$ whenever p is relatively prime to the characteristic of k. It remains to analyze the homotopy type of the K-theory spaces attached to non-algebraically closed fields. Let k be a field, and let $G = Gal(\overline{k}/k)$ denote the absolute Galois group of k. Of course, $k = (\overline{k})^G$, where $(-)^G$ denotes "fixed point set". Because of functoriality of the K-theory construction, we obtain an evident action of G on $\underline{K}(\overline{k})$. Further, it is not difficult to see that the fixed point set of this action is precisely $\underline{K}(k)$. Perhaps the easiest way to see this is to observe that we have an obvious action of G on the category

$Q(Vect(\bar{k}))$, whose fixed point category is $Q(Vect(k))$, where $Vect(F)$ denotes the category of finite dimensional vector spaces over F, for any field F. We now ask whether there is a way to make use of the fact that $\underline{K}(k)$ is actually the fixed point space of group action to obtain concrete computational information about it. More specifically, we ask if there is a way of computing the homotopy groups of $\underline{K}(F)^\wedge_p$ using only the G-action on the homotopy groups of $\underline{K}(\overline{F})^\wedge_p$

Homotopy fixed point sets

Suppose that we have a topological space X with group action by a group G. We have the fixed point subspace X^G, but we can also consider a "homotopy theoretic version". In the same way as we have earlier considered topological groups which are only homotopy theoretically commutative, so we can consider a space where points are not required to be fixed on the nose, but only up to homotopy. This vague notion is made precise in the following definition.

DEFINITION IV.1. *Let EG denote a contractible G-space on which G acts* freely, *i.e. for which $gx = x$ occurs only if $g = 1$. Such G-spaces exist and are unique up to equivariant homotopy equivalence. Then the* homotopy fixed point set *of X is the space$F^G(EG, X)$ of equivariant maps from EG to X, and it will be denoted by X^{hG}.*

To understand this definition, note that EG is in a sense a homotopy theoretic version of a point, since it has (non-equivariantly) the underlying homotopy type of a point. In fact, we have the G-map $EG \to pt$, which is an equivariant map which is non-equivariantly a homotopy equivalence. The G-space X can be regarded as the function space $F(pt, X)$, with the group G acting on the space by conjugating maps. We can also construct the function space $F(EG, X)$, and permit G to act by conjugation. The fixed point set of this action is the space of equivariant maps, i.e. X^{hG}, and so this space can be regarded as a homotopy theoretic version of the fixed point set. We also obtain the pullback map $\eta\colon X^G = F(pt, X)^G \to F(EG, X)^G = X^{hG}$.

Example: Consider any space with an action by $G = \mathbb{Z}$, i.e. a space equipped with an automorphism. In this case, the space EG can be taken to be $\mathbb{R}$, with G action given by translation $x \to x + 1$. An equivariant map ϕ from $\mathbb{R}$ to X is determined by its restriction to $[0, 1]$, and further $t\phi(0) = \phi(1)$, where t is a generator for G. So, X^{hG} can be identified with the space of all maps from $\phi\colon [0, 1] \to X$, for which $t\phi(0) = \phi(1)$. Note that for any $x \in X^G$, the constant map ϕ_x with value x lies in X^{hG}, and that the map $x \to \phi_x$ is exactly the map η described above.

Remark: We do not generally expect the map η to be an equivalence. For instance, consider any space X with trivial $\mathbb{Z}$-action. Then the fixed point set is X, but X^{hG} is the space of all maps of the circle into X, i.e. the *free loop space* of X. These two are rarely, if ever, homotopy equivalent. However, there are interesting cases when the map η becomes an equivalence after p-adic completion, notably the spaces BU (see **[2]**), $Q(S^0)$ (see **[6]**), with certain finite group actions, and any finite G-complex (see **[30]**, **[7]**). Results of this type are typically deep and difficult, though.

Remark: Homotopy fixed sets can also be defined for spectra, as well as for spaces, and we will refer to them as *homotopy fixed point spectra*. They are defined *levelwise*, i.e. by applying the homotopy fixed set construction to every space making up the spectrum.

The value of the homotopy fixed set is that its homotopy groups can often be computed by spectral sequence techniques. Specifically, we have the following theorem.

THEOREM IV.2. *Let X be a G-space. Then there is a spectral sequence with $E_2^{pq} \cong H^{-p}(G, \pi_q(X))$ converging to $\pi_{p+q}(X^{hG})$. The same result holds for spectra.*

The significance of this result is that it is possible (modulo computing differentials and extension problems in a spectral sequence) to compute the homotopy groups of the homotopy fixed set from knowing the homotopy groups of the space X, and the action of G on them. This is exactly what we would like to be able to carry out for the space $\underline{K}(\overline{k})$. So, one possible strategy for computing the homotopy groups of $\underline{K}(k)$ is to compare it via the map η with the space $\underline{K}(\overline{k})^{hG}$. If, for instance, we knew that η is an equivalence after p-adic completion, we would have a spectral sequence converging to the K-groups of k.

Remark: There is a technical issue which comes up in this program, which we haven't mentioned yet. Absolute Galois groups are profinite groups, and we haven't made clear what role the topology on the profinite group G plays in the construction of the homotopy fixed set. There are actually two distinct constructions of homotopy fixed sets for profinite groups. Only one of them produces a reasonable spectral sequence, with $E_2^{p,q} \cong H^{-p}_{cont}(G, \pi_q(X))$, where H^p_{cont} denotes the *continuous cohomology* of the profinite group in question.

Unfortunately, η is not generally an equivalence for K-theory spaces, even after p-adic completion.

The Quillen-Lichtenbaum conjecture

We now examine an example to see if there is a possibility that the fixed point set and the homotopy fixed sets are equivalent for completed K-theory.

Example: For any algebraically closed field k of characteristic 0 , let $k((x))$ denote the power series field in a single variable over k. It follows from the localization theorem and the result of Gabber from the previous section that $\pi_i\underline{K}(k((x)))^\wedge_p \cong \pi_i\underline{K}(k)^\wedge_p \oplus \pi_{i-1}\underline{K}(k)^\wedge_p$. We can iterate this calculation to obtain

$$\pi_i\underline{K}(k((x))((y)))^\wedge_p \cong \pi_i\underline{K}(k)^\wedge_p \oplus \pi_{i-1}\underline{K}(k)^\wedge_p \oplus \pi_{i-1}\underline{K}(k)^\wedge_p \oplus \pi_{i-2}\underline{K}(k)^\wedge_p$$

(Since k is algebraically closed of characteristic 0, we have that $\pi_i\underline{K}(k)^\wedge_p \cong \mathbb{Z}_p$ if i is even, and $\cong 0$ if i is odd). This means that we have here a complete calculation of $\pi_i\underline{K}(k((x))((y)))^\wedge_p)$ .On the other hand, the absolute Galois group of $k((x))$ is a free profinite group on a single generator, which we denote by $\hat{\mathbb{Z}}$ (this depends on the fact that the field of *Puiseux series* is the algbraic closure of the power series field. See [**51**]). The absolute Galois group of $k((x))((y))$ can easily be seen to be $\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$. In computing the spectral sequence, we find that in the portion of the E_2-term which will eventually compute $\pi_0\underline{K}(k((x))((y)))^\wedge_p$, there are two groups which contribute, namely $H^0_{cont}(G, \pi_0\underline{K}(\overline{k((x))((y))})^\wedge_p)$ and $H^2_{cont}(G, \pi_2\underline{K}(\overline{k((x))((y))})^\wedge_p)$. We have $H^2_{cont}(\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}, \mathbb{Z}_p) \cong \mathbb{Z}_p$, which follows from the Kunneth formula. It is an easy computation with the spectral sequence that both terms survive to E_∞, which shows that $\pi_0((\underline{K}(\overline{k((x))((y))})^{\wedge}_p)^{hG}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. The formula above shows, though, that $\pi_0\underline{K}(k((x))((y)))^{\wedge}_p \cong \mathbb{Z}_p$, so the fixed point set is not equivalent to the homotopy fixed set. Counterexamples of this type were first pointed out by S. Bloch.

Although the example above shows that one cannot hope that fixed set and homotopy fixed set are equivalent in in this case, it also suggests a weaker conjecture which appears plausible. Note that the formula describing $\pi_i\underline{K}(k((x))((y)))^{\wedge}_p$ shows that $\pi_i\underline{K}(k((x))((y)))^{\wedge}_p \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ for $i \geq 1$. It is also easy to check that the descent spectral sequence collapses, and that therefore

$$\pi_i\underline{K}(\overline{k((x))((y))})^{\wedge}_p)^{hG} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$$

for all $i \geq 0$. Consequently, the groups coincide for $i \geq 1$, and differ for $i = 0$. This says that for *sufficiently large* i, the homotopy groups of the fixed point space and the homotopy fixed space agree, or that the descent spectral sequence converges in *sufficiently high degrees.* That a similar result should hold for all fields was now suggested by Quillen [**38**] and Lichtenbaum [**24**].

Conjecture *For any field F, the map*

$$\underline{K}(F)^{\wedge}_p \to (\underline{K}(\overline{F})^{\wedge}_p)^{hG}$$

induces an isomorphism on homotopy groups for $i > d$, where d is the cohomological dimension of the absolute Galois group of F. In particular, the descent spectral sequence converges to $K_*(F)$ in this same range of dimensions.

This conjecture has been studied intensively over the last 30 years or so. R.W. Thomason [**48**] worked with a periodic version of algebraic K-theory, obtained by inverting a suitable Bott element, actually has a convergent descent spectral sequence. The relationship between the periodic theory and actual K-theory was unclear, though. Thomason also defined *etale K-theory*, and proved that it had descent. Again, the relationship between the etale theory and actual K-theory remained unclear. Dwyer, Friedlander, Snaith, and Thomason [**10**] were able to show that the map from K-theory to etale K-theory is surjective in sufficiently high degrees. Hesselholt and Madsen [**18**] proved the conjecture for finite extensions of the p-adic numbers. Rognes and Weibel [**39**], using the powerful motivic techniques developed by Voevodsky and Suslin-Voevodsky, are also able to prove this conjecture at $p = 2$ for rings of integers of number fields.

Finally, Beilinson and Lichtenbaum have proposed a modified form of the Quillen-Lichtenbaum conjecture, which is expected to hold in arbitrary dimensions.

Conjecture: *There is a spectral sequence with $E_2^{pq} \cong H^{-p}_{cont}(G_F, \pi_q\underline{K}(\overline{F})^{\wedge}_p)$ for $q \geq 2p$, and $\cong 0$ otherwise, converging to $\pi_{p+q}\underline{K}(F)^{\wedge}_p$.*

The E_2 term of this spectral sequence is identical to the E_2-term for the original spectral sequence, except that is truncated in a particular way in certain low degree. The conjecture does not propose any particular geometric way to construct the spectral sequence. The so-called *motivic spectral sequence*, which we will discuss below, is conjectured to be this spectral sequence.

V. Milnor K-theory and the Bloch-Kato conjecture

In this section we are going to look at a part of algebraic K-theory of a field, the so-called *Milnor K-theory*, which has a purely algebraic definition, with no topology involved. Let's remember from above that when a ring A is commutative the K-theory space is actually a commutative homotopy ring space. It follows that the groups $K_*(A) = \pi_* \underline{K}(A)$ form a graded commutative ring. Let F be a field. In this case, the determinant map $K_1(F) \to F^*$ is an isomorphism. This follows because of Gaussian elimination, which shows that after suitable multiplication by upper and lower triangular matrices, any matrix can be reduced to a diagonal one. We wish to determine the kernel of the product map $F^* \otimes F^* \cong K_1(F) \otimes K_1(F) \to K_2(F)$. We first have two obvious families of relations. The first is that for any $u, v \in F^*$, $u \otimes v + v \otimes u$ is in the kernel, because the K-theory product is "graded commutative". The second family of relations are given by $u_1u_2 \otimes v - u_1 \otimes v - u_2 \otimes v$ and $u \otimes v_1v_2 - u \otimes v_1 - u \otimes v_2$. These relations simply state the distributivity property for the multiplication in the K-groups. There is a final relation, which is less formal, which we need to discuss.

In order to describe this final family of relations, we must first have a better model of the product map $K_1(F) \otimes K_1(F) \to K_2(F)$. Recall that we have explicit algebraic descriptions for both K_1 and K_2. For any elements $u, v \in F^*$, we construct the product $u \cup v \in K_2(F)$ as follows. First, construct the matrices $U = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}$ in $GL_3(A)$. They commute, and are both in the subgroup $E_3(F)$ of elementary matrices over F. Therefore, we can lift both matrices to elements $\tilde{U}$ and $\tilde{V}$ of $St_3(F)$. The commutator $[\tilde{U}, \tilde{V}]$ is now an element of the kernel of the natural homomorphism $St_3(F) \to E_3(F)$, i.e. an element of $K_2(F)$. This process describes the cup product in terms of the algebraic description of these K-groups. (The fact that this construction agrees with the space level construction is non-trivial, and we will not go into it here). What this construction now permits is the calculation of the cup product $u \cup (1-u)$ for any $u \in F^*$. This cup product turns out to be trivial in $K_2(F)$; this is demonstrated by complicated calculations in the Steinberg group. See [**32**] for details.

We now have three families of relations among the cup products of elements of $K_1(F)$. Matsumoto has proved that this is a complete set of relations for K_2 of any field F. See [**32**] for a proof.

THEOREM V.1. *The abelian group $K_2(F)$ has a presentation, in terms of generators and relations, as follows. The generators are given by symbols $\{u, v\}$, with $u, v \in F^*$. The relations are*

1. $\{u, 1-u\} = 1$ *for* $u \neq 0, 1$
2. $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$
3. $\{u, v_1v_2\} = \{u, v_1\}\{u, v_2\}$

The skew symmetry can be shown to follow from these relations.

From the above disussion, it may not be so clear where the relation (1) comes from. We will show that it occurs quite naturally in some familiar arithmetic situations. This discussion is taken from [**32**].

DEFINITION V.2. *Let F be a field. By a* Steinberg symbol *on F with values in an abelian group A, we mean a bilinear map $\beta\colon F^* \times F^* \to A$ so that $\beta(u, 1-u) = 0$ for any $u \in F$ with $u \neq 0, 1$. It is now clear from Matsumoto's theorem that a Steinberg symbol on F is the same thing as a homomorphism from $K_2(F)$ to A.*

Steinberg symbols can often be constructed arithmetically. We first recall the definition of the *Brauer group* of a field. Recall from Wedderburn's theorem that any finite dimensional central simple algebra over F is isomorphic to a matrix ring over a division algebra with center F. We say two central simple algebras Λ_1 and Λ_2 over F are *similar* if there exist m and n so that $M_m(\Lambda_1) \cong M_n(\Lambda_2)$. Similarity is an equivalence relation , and the similarity classes of central simple algebras form an abelian group under tensor product of central simple algebras. This group is known as the Brauer group of F, written $Br(F)$, and it is an extremely interesting invariant of F. See [**56**] for a thorough discussion. We will now construct a Steinberg symbols on F with values in $Br(F)$. For any two elements $a, b \in F$,, $a, b \neq 0, 1$, we may construct the *generalized quaternion algebra* over F attached to a and b, $\mathcal{Q}(a, b)$, as follows. A basis for $\mathcal{Q}(a, b)$ is given by $\{1, x, y, xy\}$, and defining relations are given by $x^2 = a$, $y^2 = b$, and $xy = -yx$. It is not hard to check that it is a central simple algebra over F. We now claim that the map $q\colon F^* \times F^* \to Br(F)$ given by $q(a, b) = [\mathcal{Q}(a, b)]$, where $[-]$ denotes similarity class, is a Steinberg symbol. The bilinearity is a straightforward check. The Steinberg relation $q(a, 1-a) = 0$ is verified by showing that the quaternion algebra $\mathcal{Q}(a, 1-a)$ is actually a matrix algebra. The isomorphism is given by $x \to \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ and $y \to \begin{pmatrix} 1 & a \\ -1 & -1 \end{pmatrix}$. We can make two further observations. The first is that the generalized quaternion algebras always represent elements in $Br(F)$ whose square is zero, i.e. $[\mathcal{Q}(a, b)] \in Br(F)_2$, the two torsion part of the Brauer group. The second is that since the Steinberg symbol induces a homomorphism $K_2(F) \to Br(F)_2$, we obtain a homomorphism

$$\theta_2\colon K_2(F)/2K_2(F) \to Br(F)_2$$

Given a field which contains the n-th roots of unity, it is possible to construct Steinberg symbols q_n on F which take their values in the n-torsion subgroup of the Brauer group. Here, instead of constructing quaternion algebras, one constructs central simple algebras of dimension n^2 over F. This gives homomorphisms

$$\theta_n\colon K_2(F)/nK_2(F) \to Br(F)_n$$

These homomorphisms were studied intensively by Merkurjev and Suslin, who succeeded in proving the following striking result.

THEOREM V.3. *Let F be any field containing a primitive n-th root of unity. Then the homomorphism $\theta_n\colon K_2(F)/nK_2(F) \to Br(F)_n$ is an isomorphism.*

Another way to look at this theorem is via Galois cohomology. Recall (see [**42**]) that the Brauer group can be interpreted cohomologically. There is a canonical isomorphism $Br(F) \cong H^2_{cont}(G_F, \overline{F}^*)$. If we now consider the short exact coefficient sequence $0 \to \mu_p \to \overline{F}^* \overset{\times p}{\to} \overline{F}^* \to 0$, where μ_p denotes the p-th roots of unity, we obtain the long exact sequence on cohomology

$$0 \to H^0(G_F, \mu_p) \to H^0(G_F, \overline{F}^*) \overset{\times p}{\to} H^0(G_F, \overline{F}^*) \to$$
$$\cdots \to H^1(G_F, \mu_p) \to H^1(G_F, \overline{F}^*) \overset{\times p}{\to} H^1(G_F, \overline{F}^*) \to$$
$$\cdots \to H^2(G_F, \mu_p) \to H^2(G_F, \overline{F}^*) \overset{\times p}{\to} H^2(G_F, \overline{F}^*) \to \cdots$$

Using the identfications $H^0(G_F, \overline{F}^*) \cong F^*$ and $H^2(G_F, \overline{F}^*) \cong Br(F)$, as well as the standard fact that $H^1(G_F, \overline{F}^*) = 0$ (Hilbert's theorem 90), we find that we have short exact sequences

$$F^* \overset{\times p}{\to} F^* \to H^1(G_F, \mu_p) \to 0$$

and

$$0 \to H^2(G_F, \mu_p) \to Br(F) \overset{\times p}{\to} Br(F)$$

so that $H^1(G_F, \mu_p) \cong F^*/pF^*$ and $H^2(G_F, \mu_p) \cong Br(F)_p$, where for any abelian group A, A_p denotes the subgroup $\{a \in A | pa = 0\}$. The Merkurjev-Suslin statement that $K_2(F)/pK_2(F) \cong Br(F)_p$ can be rewritten as

$$K_2(F)/pK_2(F) \cong H^2(G_F, \mu_p)$$

In the earlier discussion, we assumed that the field contained a primitive p-the root of unity. Choosing such a root is equivalent to choosing a generator for $H^0(G_F, \mu_p)$. By using the cup product

$$H^0(G_F, \mu_p) \otimes H^2(G_F, \mu_p) \to H^2(G_F, \mu_p \otimes \mu_p) \cong H^2(G_F, \mu_p^{\otimes 2})$$

so

$$K_2(F)/pK_2(F) \cong H^2(G_F, \mu_p^{\otimes 2})$$

Similarly, the statement that $K_1(F) \cong F^*$ now permits us to write

$$K_1(F)/pK_1(F) \cong H^1(G_F, \mu_p) \cong H^1(G_F, \mu_p^{\otimes 1})$$

On may now be tempted to conjecture that for general n, we have

$$K_n(F)/pK_n(F) \cong H^n(G_F, \mu_p^{\otimes n})$$

but this is false. However, a related conjecture is expected to be true, and is already known for many cases. We first define Milnor K-theory.

DEFINITION V.4. *We define the* Milnor K-theory *of F, $K_*^M(F)$, to be the graded ring generated by F^*, and subject to Matsumoto's relations given above. There is of course a natural map $K_*^M(F) \to K_*(F)$, but the map fails to be surjective.*

As we have seen, there are evident ismomorphisms $\theta_1 \colon K_1^M(F)/pK_1^M(F) \to H^1(G_F, \mu_p^{\otimes 1})$ and $\theta_2 \colon K_2^M(F)/pK_2^M(F) \to H^2(G_F, \mu_p^{\otimes 2})$. It is not hard to check that these isomorphisms respect products, i.e. that $\theta_2(u \cup v) = \theta_1(u) \cup \theta_1(v)$. Since $K_*^M(F)$ is defined by relations in $K_1(F) \otimes K_1(F)$, it now follows easily that there is a map of graded rings

$$K_*^M(F)/pK_*^M(F) \longrightarrow \bigoplus_{n=0}^{\infty} H^n(G_F, \mu_p^{\otimes n})$$

where the product on the right is given by the external pairings

$$H^n(G_F, \mu_p^{\otimes n}) \otimes H^m(G_F, \mu_p^{\otimes m}) \to H^{m+n}(G_F, \mu_p^{\otimes (m+n)})$$

Milnor (for $p = 2$) and Bloch and Kato (all primes), on the basis of extensive calculations, made the following conjecture.

Milnor-Bloch-Kato Conjecture There is an isomorphism of graded rings

$$K_*^M(F)/pK_*^M(F) \cong \bigoplus_{n=0}^{\infty} H^n(G_F, \mu_p^{\otimes n})$$

This conjecture has been proved in a numer of cases. The case $n = 2$ is due to Merkurjev and Suslin [**29**]) and for all n with $p = 2$ by Voevodsky [**50**].

There is another conjecture concerning $K_*^M(F)/pK_*^M(F)$ proposed by Positelskii and Vishik ([**35**]). A graded algebra A_* over a field k is called *quadratic* if it is a quotient of the graded tensor algebra $T(A_1)$ on A_1 by an ideal generated by a subspace $R \subseteq A_1 \otimes A_1 = T(A_1)_2$. The k-vector spaces $Ext^n_{A_*}(k,k)$ inherit a grading from that of A_*, Moreover, the sum $H^{**}(A) = \bigoplus_{n=0}^{\infty} Ext^n_{A_*}(k,k)$ becomes a bigraded algebra, taking into account the homological degree. The quadratic algebra A_* is now called a *Koszul duality algebra* if $H^{**}(A)$ is generated by elements in bidegree $(1,1)$. Positselskii and Vishik now make the following conjecture.

Conjecture *The graded algebra $K_*^M(F)/pK_*^M(F)$ is a Koszul duality algebra.*

They go on to show that this conjecture actually implies the Bloch-Kato conjecture, and hence that the Koszul duality property will also hold for $\bigoplus_{n=0}^{\infty} H^n(G_F, \mu_p^{\otimes n})$

VI. The Bloch-Lichtenbaum spectral sequence

Recall that earlier, we discussed the Gersten spectral sequence, which computes the algebraic K-theory of a ring A from the algebraic K-theory of the fraction fields of the residue class rings of the various prime ideals of A. In this section, we will discuss a spectral sequence constructed by Bloch and Lichtenbaum [**4**], which in a sense works in the reverse direction, and computes the K-theory of a field k from the algebraic K-theory of varieties over k. We begin with a construction originally due to C. Weibel, the so-called homotopy K-theory. It relies on the homotopy property (see Section **I**) for algebraic K-theory of regular rings.

For any ring A, we define an algebra $\Delta_n(A)$ over A by

$$\Delta_n(A) = A[X_0, \dots X_n]/(\Sigma X_i - 1)$$

We first observe that $\Delta_n(A) \cong A[X_1, \dots X_n]$, since the only relation can be written as $X_0 = 1 - \Sigma_{i=1}^n X_i$. This implies, by the use of the homotopy property, that $\underline{K}(\Delta_n(A)) \cong \underline{K}(A)$. A second observation is that if we let $\underline{G}(R)$ denote the K-theory space of the exact category of all finitely generated R-modules, for a Noetherian ring R, then the natural map $\underline{K}(R) \to \underline{G}(R)$ is an equivalence. This follows by the reduction by resolution equivalence (again, see Section **I**). We now define a simplicial spectrum $CW_.$ by $CW_n = \underline{G}(\Delta_n(A))$. The face map d_i is induced by the ring map $X_j \to X_j$ for $j \leq i$, and by $X_j \to X_{j-1}$ for $j > i$. The degeneracy s_i is induced by the ring homomorphism given by $X_j \to X_j$ for $j \leq i$, and $X_j \to X_{j+1}$ for $j > i$. There is a canonical map from the constant simplicial spectrum with value $\underline{G}(A)$ in to $CW.$, which induces an equivalence on total spectra, since the homotopy property tells us that it is an equivalence in each level. So, $|CW.|$ is a model for $\underline{G}(A)$. The idea of Bloch and Lichtenbaum is now to observe that there is a natural filtration of $CW.$ by simplicial subspectra, which cannot be seen in any obvious way from the standard model for $\underline{G}(A)$.

We will now recall that for any noetherian commutative ring A, any finitely generated module A-module M admits a composition series $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$, so that each subquotient M_i/M_{i-1} is isomorphic to a module of the form $A/\wp$, for a prime ideal $\wp$ of A. Recall also that the *codimension* of a prime ideal $\wp$ of A is the Krull dimension of the local ring $A_\wp$. More generally, the codimension of any ideal I is the minimum of the codimensions of the prime ideals containing I. At this point we assume that A is a field k, and we consider certain subcategories in the categories of finitely generated modules over the rings $\Delta_n(k)$. Each of the face maps is a projection $\Delta_n(k) \to \Delta_{n-1}(k)$, which corresponds to a variety which is defined by one of the coordinate ideals (X_i) for some i. More generally, each iterated face map corresponds to an ideal of the form $I_S = (x_{i_1}, x_{i_2}, \dots, x_{i_{s-1}}, x_{i_s})$ for some subset $S = \{i_i, i_2, \dots, i_{s-1}, i_s\} \subset \{0, 1, \dots, n-1, n\}$. Call any ideal of the form I_S a *face ideal.* We will say that a prime $\wp$ of $\Delta_n(k)$ is in ***good position*** if the ideal $(\wp + I_S)/\wp$ has codimension s for any face ideal I_S, where $\#S = s$. Let the set of primes of $\Delta_n(k)$ in good position, and of codimension $\leq l$, be denoted by $\mathcal{V}^l(\Delta_n)$. The subcategory of modules whose support lies in $\mathcal{V}^l(\Delta_n)$ will be denoted by $\mathcal{M}^l(\Delta_n)$. It is an exact subcategory of the category of all finitely generated modules, and we may compute its K-theory. Further, $d_i(\mathcal{M}^l(\Delta_n)) \subseteq \mathcal{M}^l(\Delta_{n-1})$ and $s_i(\mathcal{M}^l(\Delta_n)) \subseteq \mathcal{M}^l(\Delta_{n+1})$, so we obtain a simplicial exact subcategory and consequently a simplicial subspectrum $CW.^l$ of $CW.$. These spectra form a filtration of the spectrum $CW.$, and hence give rise to a spectral sequence converging to $\pi_*(|CW.|) \cong K_*(k)$.

Bloch and Lichtenbaum analyze this spectral sequence. They find that the subquotient spectra $CW.^l/CW.^{l+1}$ are generalized Eilenberg-MacLane spectra, associated to a certain simplicial abelian group $\mathcal{Z}.^l(k)$ defined as follows. $\mathcal{Z}^l_n(k)$ is equal to the free abelian group on the prime ideals of $\Delta_n(k)$ which are in good position, and which have codimension exactly equal to l. One can construct face maps on these abelian groups by forming intersections of te associated varieties with the associated faces. Bloch **[3]** has defined these simplicial abelian groups, and has called their homotopy groups ***higher Chow groups.*** He denotes by $CH^r(k,n)$ the homotopy group $\pi_n(\mathcal{Z}^r(k))$. The reason for the term "higher Chow group" should be clear. In their paper, Bloch and Lichtenbaum *define* motivic cohomology in terms of these higher Chow groups.

What is not clear from the Bloch-Lichtenbaum construction, though, is what the relationship is between this spectral sequence and the descent spectral sequence which involves Galois cohomology. It is not clear from the construction that the E_2-term is related to anything which could be called a cohomology theory on schemes.

VII. Motivic cohomology

We saw in the last section that there is a spectral sequence whose E_2-term consists of Bloch's "higher Chow groups", converging to the algebraic K-theory of a field k. From the Bloch-Lichtenbaum description of these groups, though, it is also very difficult to compute these groups. In particular, there appears to be no straightforward way to compare these groups with the Galois cohomology groups which appear in the E_2-term of the descent spectral sequence. Galois cohomology groups (and more generally, the etale cohomology groups which are attached to rings and schemes, not just to fields) are regarded as relatively computable. The reason for this computability is that these groups have good excision and Mayer-Vietoris properties, and these properties come about because etale cohomology groups are defined using derived functors on certain categories of sheaves. Here, sheaves are not necessarily just defined on the category of open subsets of a topological space, but are rather generalized sheaves on a *site*. See [**31**] for a thorough discussion of etale cohomology. So, one approach to making the higher Chow groups more computable is to identify them with sheaf cohomology groups in an appropriate setting. This approach has been carried out by Voevodsky [**50**], building on earlier ideas of Suslin. We will outline this approach in this section.

Correspondences

Let k be a field. We will first need to discuss *correspondences.* The idea of a correspondence is to generalize the notion of a map of schemes. Let X be any connected scheme over k, and let Y be a separated scheme over k. Then by an *elementary correspondence* from X to Y, we mean an irreducible closed subset $W \subseteq X \times Y$ whose associated integral subscheme is finite and surjective over X. If X is not connected, and elementary. If we are thinking only of varieties over k, an elementary correspondence is simply a subvariety of $X \times Y$ whose projection to X is surjective and finite to one. For any map $f\colon X \to Y$ of varieties, the graph Γ_f of the map is therefore an elementary correspondence, because its projection onto X is one-to-one and onto. We define an abelian group $Cor(X,Y)$ to be the free abelian group generated by the elementary correspondences from X to Y. Elements of $Cor(X,Y)$ will be callen *finite correspondences.*

The first idea will now be to define an additive category Cor_k whose objects are the smooth separated schemes over k, and so that $Hom_{Cor_k}(X,Y) = Cor(X,Y)$. In order to do this, we must define composition of correspondences. This is done by a scheme theoretic version of a pullback operation. Its topological version is the following. Suppose we have spaces X, Y, and Z, together with subspaces $W \subseteq X \times Y$ and $U \subseteq Y \times Z$. Then we write $W \underset{Y}{\times} U$ for the pullback $\{(w,u)|\pi_Y(w) = \pi_Y(u)\}$. If both $\pi_X|W$ and $\pi_Y|U$ are surjective and finite to one, it is elementary to check that $W \underset{Y}{\times} U$ is too. This gives a way to compose correspondences. We can now extend by lineaity to obtain a bilinear composition pairing. The resulting category will be called Cor_k. If Sm/k denotes the category of smooth schemes over k, we have an embedding $Sm/k \hookrightarrow Cov_k$. Cor_k also possesses a tensor product, which on objects amounts to product of schemes over k. Cor_k is in fact a symmetric monoidal category under tensor product.

Presheaves with transfers

Motivic cohomology will be constructed as derived functors of certain objects called *presheaves with transfers*. These are not sheaves in the usual sense. Sheaves on topological spaces are defined as functors from the category of open sets into abelian groups, satisfying certain properties. Sheaves in the etale topology are defined ale site attached to the scheme in question. The etale site has as objects schemes which are etale over open subsets of the schemes in question. The sheave which occur in computing motivic cohomology are obtained as restrictions of functors on Cor_k with values in abelian groups to the Zariski site, i.e. the category of open sets on the scheme in question. Cor_k has a quite different character from categories of open sets and from the etale site. One important difference is that Cor_k is an additive category. For this reason, the category of all contravariant abelian group valued functors on Cor_k is called the category of *presheaves with transfers*. We will denote it by **PST** or $\mathbf{PST}(k)$. The reason for this terminology is that a transfer in the usual topological setting can be viewed as a correspondence. Suppose that we have a space X, and a covering space $\tilde{X} \to X$. Then $\tilde{X} \overset{(p,id)}{\hookrightarrow} X \times \tilde{X}$ itself can be viewed as a correspondence from X to $\tilde{X}$. In ordinary topological theory, one way to construct transfers is to add correspodences to the usual continuous maps as morphisms in ones category. In fact, the notion of transfer embodied in this definition is much less restrictive than that required for the transfer as usually studied in algebraic topology. There, transfers are typically attached only to covering maps. The kind of transfer envisioned in the motivic theory requires only a finite surjective map, like for instance a branched covering. It is more like the so-called *Oliver transfer* [**34**], which applies to homology of orbit projections. Here are some examples of presheaves with transfers.

Example: The sheaf of global units $\mathcal{O}^*$ is a presheaf with transfers. To see this, we must first observe that for any finite morphism $f\colon X \to Y$ of schemes, then there is a norm map $N\colon \mathcal{O}^*(X) \to \mathcal{O}^*(Y)$. On affine schemes, this map is simply restriction of scalars from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules. Given a correspondence $W \hookrightarrow X \times Y$, the homomorphism $\mathcal{O}^*(Y) \to \mathcal{O}^*(X)$ is the composite

$$\mathcal{O}^*(Y) \to \mathcal{O}^*(W) \xrightarrow{N} \mathcal{O}^*(X)$$

Example: For any object $X \in Cor_k$, we can define the functor represented by X, $\mathbb{Z}_{tr}(X)$, by $\mathbb{Z}_{tr}(X)(U) = Hom_{Cor_k}(U, X)$. It follows by the Yoneda lemma that $Hom_{PST}(\mathbb{Z}_{tr}(X), F) \cong F(X)$, and it follows that $\mathbb{Z}_{tr}(X)$ is a projective object in $\mathbf{PST}(k)$.

In addition to the representable presheaves with transfers,we will want to study a reduced counterpart. Note first that the category $\mathbf{PST}(k)$ is in fact an abelian category, and so admits kernels and cokernels. Suppose (X, x) is a pointed scheme over k. Then we define the reduced presheaf with transfers $\mathbb{Z}_{tr}(X, x)$ to be the cokernel of the natural transformation $\mathbb{Z} = \mathbb{Z}_{tr}(point) \to \mathbb{Z}_{tr}(X)$ induced by the inclusion $Spec(k) \to X$. More generally, we can speak of a *smash product* construction. Given pointed schemes (X_i, x_i), define $\mathbb{Z}_{tr}((X_1, x_1) \wedge (X_2, x_2) \wedge \ldots \wedge (X_n, x_n))$ to be the cokernel

$$coker(\bigoplus_i \mathbb{Z}_{tr}(X_1 \times X_2 \times \cdots \times X_{i-1} \times x_i \times X_{i+1} \times \cdots \times X_n) \to \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n))$$

It is not hard to check that $\mathbb{Z}_{tr}((X_1, x_1) \wedge (X_2, x_2) \wedge \ldots \wedge (X_n, x_n))$ is a direct summand in $\mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)$. We will also permit the notation $\mathbb{Z}_{tr}((X,x)^n)$ when we perform this smash product construction with n copies of the same pointed scheme. This construction will be particularly useful for us in the case when $X = \mathbb{A}^1(k) - \{0\}$. We will write $\mathbb{G}_m$ for this scheme; it is canonically pointed with the point 1.

Complexes and homotopy invariance in PST

We will also need to construct complexes of objects in **PST**. We first construct a cosimplicial scheme $\Delta^\bullet$ over k. We set

$$\Delta^n = \text{Spec } k[x_0, \ldots, x_n]/(\Sigma_{i=0}^n x_i = 1)$$

Coface maps are given by inclusion on the subspaces $x_j = 0$, and codegeneracy maps are given by summing coordinates. For any contravariant functor F from Sm/k to abelian groups, $F(\Delta^\bullet)$ becomes a simplicial abelian group, which we write $C_.(F)$. The corresponding chain complex, obtained by taking alternating sums of face maps, is denoted $C_*(F)$. We will say that a contravariant functor F from Sm/k to abelian groups is *homotopy invariant* if the canonical map $F(X) \to F(X \times \mathbb{A}^1)$ induced by the projection $p\colon X \times \mathbb{A}^1 \to X$ is an isomorphism. It is not difficult to prove

THEOREM VII.1. *The abelian group valued functors $H_n C_*(F))$ are homotopy invariant.*

It follows easily from this fact that the projection induces an a chain homotopy equivalence $C_*(\mathbb{Z}_{tr}(X \times \mathbb{A}^1)) \to C_*(\mathbb{Z}_{tr}(X))$.

Motivic complexes

Motivic cohomology will be defined as the *hypercohomology groups* of certain complexes of sheaves on the scheme in question. Recall that if we are given a chain complex of sheaves C_* on a topological space, the hypercohomology of the chain complex can is defined as the cohomology of an injective resolution of the complex, i.e. a complex of injective sheaves $\mathcal{I}_*$ for which there is a map $C_* \to \mathcal{I}_*$ inducing an isomorphism of homology sheaves. Hypercohomology can typically be computed via spectral sequences starting with the sheaf cohomology of the homology sheaves.

We now define complexes of presheaves with transfers $\mathbb{Z}(q)$ by the equation

$$\mathbb{Z}(q) = C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))[-q]$$

where $\mathbb{G}_m^{\wedge q}$ denotes the m-fold smash product of copies of $\mathbb{G}_m$ based at 1, and where $[-q]$ denotes a downward dimension shift of q. It can be shown that the restriction of these presheaves with transfers to the Zariski site are actually sheaves, so we obtain a chain complex of sheaves in the Zariski topology. It now turns out that the E_2-term of the Bloch-Lichtenbaum spectral sequence can be identified in terms of the hypercohomology of these complexes of sheaves.

THEOREM VII.2. *There is a spectral sequence with E_2-term given by $E_2^{p,q} \cong \mathbf{H}^p_{Zar}(Spec(k), \mathbb{Z}(q))$ converging to $K_{2q-p}(k)$. It coincides with the Bloch-Lichtenbaum spectral sequence.*

Voevodsky has analyzed this spectral sequence carefully. In particular, he has proved

THEOREM VII.3. *For any field F, we have*

$$H^n_{Zar}(Spec(F), \mathbb{Z}(n)) \cong K^M_n(F)$$

This result is proved using a careful analysis of the complexes $C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))$. Voevodsky [**50**] then went on to make a comparison of motivic and etale cohomology in this dimension, to arrive at a proof of the Milnor conjecture (Milnor-Bloch-Kato conjecture at the prime 2). Stronger comparison and vanishing theorems for motivic cohomology should also imply the Quillen-Lichtenbaum and Beilinson-Lichtenbaum conjectures. This has been carried out for number fields at the prime 2 in the work of Rognes-Weibel [**39**].

VIII. A representation theoretic model

In the preceding sections, we have described methods for constructing spectral sequences converging to algebraic K-theory, and some conjectures (notably the Beilinson-Lichtenbaum conjecture) concerning these spectral sequences. The conjecture identifies the E_2-term for the spectral sequence in terms of etale cohomology, and would imply the original Quillen-Lichtenbaum conjecture. Both the Beilinson-Lichtenbaum and Quillen-Lichtenbaum conjectures are aimed at the problem of computing the algebraic K-theory of a field in terms of two ingredients:

- The algebraic K-theory space of the algebraic closure $\overline{F}$ of F
- The action of the absolute Galois group G_F on $\underline{K}(\overline{F})$

Instead of asking directly for the spectral sequence, though, one might ask instead for a model for the homotopy type of $\underline{K}(F)$, which is built out these two ingredients. Of course, the homotopy fixed point set construction $\underline{K}(\overline{F})^{hG_F}$ is built out of the correct ingredients, but although its homotopy groups are conjectured to agree with $K_*(F)$ in high degrees, it is known not to be homotopy equivalent to $\underline{K}(F)$. In order to describe such a model, we will need to introduce some more material on spectra and stable homotopy theory.

Equivariant spectra

Recall from our earlier discussion in Section **III** that the algebraic K-theory spaces are actually a part of a spectrum, which is a family of based spaces $\{X_i\}_{i\geq 1}$ together with identifications $X_i \to \Omega X_{i+1}$. The theory of these objects is known to be extremely powerful in the topological setting. For instance, spectra give rise to *generalized cohomology theories* (such as K-theory or bordism theories) on spaces, and every generalized cohomology theory satisfying mild hypotheses is obtained from a spectrum. It is a reasonable question to ask how the theory extends to the theory of spaces with G-action, where G is a finite group. This question has been answered by G.B. Segal [**41**]. It turns out that when one is dealing with equivariant topology, one should no longer consider **the** n-sphere, but rather one should study spheres with different linear actions on them. Given a representation space V of the group G, we form its one-point compactification S^V, which is a sphere with G-action. An action which arises this way is called a *linear action*. The content of the statement that a space is X_0 for some spectrum $\{X_i\}_{i\geq 1}$ is that it admits n-fold *deloopings* X_n, i.e. spaces X_n so that $\Omega^n X_n \cong X$. Note that $\Omega^n Z \cong Z^{S^{V_n}}$, where V_n denotes an n-dimensional real vector space. We therefore make the definition, for any finite dimensional representation space V of G, of the G-space $\Omega^V Z \cong Z^{S^V}$, where the group action is given by conjugation of maps. The requirement for a G-space to be the zeroth space of a G-spectrum is that it admit V-deloopings for every representation V. In summary, a G-spectrum is a family of G-spaces, parametrized by the representations of the groups in question. They admit various useful constructions, most notably the existence of a an H-fixed point spectrum for every subgroup $H \subseteq G$. The H-fixed point spectrum becomes a $W_G(H)$-spectrum, where $W_G(H)$ enotes the *Weyl group* of H, $N_G(H)/H$. When E/F is a finite Galois extension of fields, the space $\underline{K}(E)$ is the zeroth space of a G-spectrum. The study of these spectra has been useful in many contexts in homotopy theory, particularly in the analysis of homotopy fixed point problems.

The analogue of homotopy groups in the setting of G-equivariant spectra takes values inside the abelian category of *Mackey functors.* Mackey functors for a group G are defined as contravariant abelian group valued additive functors from a certain addtiive category $\mathcal{B}_G$ associated to G. The objects of $\mathcal{B}_G$ are the finite G-sets, and functors are said to be additive if they carry disjoint unions to direct sums. When we are given a G-spectrum X, then $\pi_*(X)$ will denote the Mackey functor which takes the value $\pi_*((Z_+ \wedge X)^G)$. Here, $(-)_+$ denotes adding a disjoint base point. Let $\mathcal{M}_G$ denote the category of Mackey functors over the group G. Then there is an analogue to the tensor product pairing on the category of abelian groups, called the *box product*, written $\square$. Given such a product, it is now possible to define the analogues of rings, which are called *Green functors.* G. Lewis originally developed this theory; see [**5**] for a clear development.

Ring and module spectra

The theory of ring and module spectra described in Section **III** extends naturally to a theory of ring and module spectra with G-action. The spectral sequences converging to the homotopy groups of smash products and Hom-spectra now involve derived functors of the box product and of Mackey functor Hom.

Completions

There is also an analogue of the commutative algebraic notion of completion. We will first describe this construction in the non-equivariant situation, and then indicate the changes necessary in the equivariant context. Recall that in the case of rings, we define the completion of a ring A at an ideal I as the inverse limit $\varprojlim A/I^n$. In the context of ring spectra, we will need to use a homotopy theoretic version of the inverse limit construction, based on cosimplicial spaces. A *cosimplicial space* is a *covariant* functor from the simplicial category Δ of finite ordered sets to spaces. Just as a simplicial set (or a simplicial space) is given a an increasing filtration by skeleta, so there is a construction of a *total space* $Tot(X^\cdot)$ which is the inverse limit of a tower of fibrations. Cosimplicial spaces have many properties dual to those of simplicial spaces. The Tot construction has a spectrum version as well. We will define the *homotopy theoretic completion* or *derived completion* of any map of commutative ring spectra $f\colon R \to S$ as the total spectrum of a cosimplicial spectrum, whose spectrum in codimension k is the iterated smash product $S \underset{R}{\wedge} S \underset{R}{\wedge} \ldots \underset{R}{\wedge} S \underset{R}{\wedge} S$, where there are $k+1$ factors. The coface maps are given by smashing the inclusion

$$S \cong S \underset{R}{\wedge} R \xrightarrow{id \wedge f} S \underset{R}{\wedge} S$$

with the identity maps on the remaining smash factors. The codegeneracies are constructed using the multiplication map $S \underset{R}{\wedge} S \to S$. We will write $R_f^\wedge$ for this competion. It is easy to check that there is an evident natural map $\eta\colon R \to R_f^\wedge$. Here are some of the properties enjoyed by this construction.

- Suppose that $f\colon R \to S$ is a homomoprhism on commutative ring spectra, and that $\pi_0(f)$ is an isomorphism and $\pi_1(f)$ is surjective. Then $\eta\colon R \to R_f^\wedge$ is an equivalence of spectra.
- Any ring (in the usual algebraic sense) can be regarded as a ring spectrum via constructing the corresponding Eilenberg-MacLane spectrum. The derived completion of the ring will now be a ring spectrum, and it will in general have higher homotopy, and hence will not be the Eilenberg-MacLane spectrum attached to a ring. However, if A is a commutative Noetherian ring, $I \subseteq A$ is an ideal, and $\pi\colon A \to A/I$ is the evident projection, then $\pi_i(A_\pi^\wedge) = 0$ for $i \neq 0$, and $\pi_0(A_\pi^\wedge)$ is isomorphic to the algebraic completion of A at I. So, $A_\pi^\wedge$ is in this case the Eilenberg-MacLane spectrum attached to a ring. In general,we obtain a notion of derived completion on the category of rings, which can be extended to graded rings.
- For a homomorphism $f\colon R \to S$ of commutative ring spectra, there is a spectral sequence whose E_2-term is given by $\pi_*(R)_{\pi_*(f)}^\wedge$, and which converges to $\pi_*(R_f^\wedge)$. In other words, the spectral sequence starts from the derived completion of the homotopy groups and converges to the derived completion of the original spectrum.
- If we are given a commutative diagram of maps of ring spectra

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ R' \arrow[r, "f'"] & S' \end{tikzcd}$$

  there is an induced map $R_f^\wedge \to R'^\wedge_{f'}$, which makes derived completion a functor on the category of ring homomorphisms $R \to S$.

Example: Let R be the complex representation ring of the group of p-adic integers (defined as the direct limit of the representations of the subquotients $\mathbb{Z}/p^n\mathbb{Z}$). Since for abelian groups, the complex representation ring is simply the group ring of the character group, we find that $R \cong \mathbb{Z}[\mathbb{Z}/p^\infty\mathbb{Z}]$, where $\mathbb{Z}/p^\infty\mathbb{Z}$ denotes the p-torsion part of $\mathbb{Q}/\mathbb{Z}$. Let $f\colon R \to \mathbb{F}_p$ denote the augmentation $R \to \mathbb{Z}$ followed by mod p reduction. Then the derived completion $R_f^\wedge$ has as its zeroth space the p-adic group ring of the simplicial group attached to the circle group. In particular, $\pi_i(R_f^\wedge) = \mathbb{Z}_p$ when $i = 0, 1$, and $= 0$ otherwise.

Remark: The above example suggests a relationship between the derived representation ring of certain groups and a more geometrically defined representation theory. Specifically, let Γ denote a discrete group. For each $k \geq 0$, let $\mathcal{R}_k(\Gamma)$ denote the set of isomorphism classes of continuous actions of Γ on $\Delta^k \times V$, where V is a complex vector spaces, which respect the projection $\Delta^k \times V \to \Delta^k$, and which are linear on each fiber. These sets form a simpicial set, and the groups $K_0(\mathcal{R}_k(\Gamma))$ form a simplicial abelian group, in fact a simplicial ring. We refer to this simplicial ring as the *deformation representation theory* of Γ, and write it as $\mathcal{R}^{def}_{\cdot}(\Gamma)$. If we let $\Gamma^{\wedge}_p$ denote the p-profinite completion of Γ, then we obtain a map from the derived completion of the representaion ring of $\mathcal{R}_k(\Gamma)$ at p to the derived completion of $\mathcal{R}^{def}_{\cdot}(\Gamma)$ at p. This map becomes an equivalence for the group $\Gamma = \mathbb{Z}$, as the above example allows one to prove. It is interesting to make conjectures about the class of groups for which this is true. It is true for abelian groups. It tends to fail as stated for many non-abelian groups, but an improved notion of completion, involving Green functors and modules over them seems to enlarge the class of groups for which the relationship holds.

The constructions above generalize easily to the equivariant situation. The notion of derived completion of rings is now replaced by the derived completion of Green functors at homomorphisms of Green functors.

Completion in the K-theory of fields

Consider a field F. Its K-theory spectrum (which we'll denote by $\underset{\sim}{K}(F)$) becomes a ring spectrum because the category of finite dimensional F-vector spaces admits not only a direct sum operation (which gives the spectrum structure) but also a tensor product, which gives rise to a multiplication map of spectra $\underset{\sim}{K}(F) \wedge \underset{\sim}{K}(F) \to \underset{\sim}{K}(F)$, which turns out to be the multiplication map in a ring structure on the spectrum $\underset{\sim}{K}(F)$. Let $E = \overline{F}$ be the algebraic closure of F, and denote the absolute Galois group of F. We will need some definitions.

DEFINITION VIII.1. *By a G_F-semilinear vector space, we will mean a finite dimensional E-vector space V equipped with a G_F-action which is semilinear in the sense that for $\gamma \in G_F$, $v \in V$, and $e \in E$, we have*

$$\gamma(ev) = e^{\gamma}\gamma$$

where e^{γ} denotes the result of the the Galois action on e. Let $\mathcal{C}_{G_F}(E)$ denote the category of G_F-semilinear vector spaces, and equivariant E-linear maps. We have the forgetful functor $\mathcal{C}_{G_F}(E) \to Vect(E)$, which forgets the group action, as well as the fixed point functor $V \to V^{G_F}$ from $\mathcal{C}_{G_F}(E)$ to $Vect(F)$. It is a standard result from descent theory **[14]** *that the fixed point functor is an equivalence of categories, so that the semilinear E-vector spaces are equivalent to F-vector spaces. $\mathcal{C}_{G_F}(E)$ also admits an operation $\underset{E}{\otimes}$, which turns its K-theory spectrum into a ring spectrum, which is equivalent to the ring spectrum $\underset{\sim}{K}(F)$.*

DEFINITION VIII.2. *For any field L and profinite group G, we will define the representation category of G over L, $Rep_L[G]$, as the symmetric monoidal category of* **continuous** *L-linear representations of G. (In this setting,continuity amounts to the fact that the representation should factor through one of the finite quotients of G). $Rep_L[G]$ admits a tensor product operation $\underset{L}{\otimes}$, which means that the K-theory spectrum of $Rep_L[G]$ becomes a ring spectrum, which we will denote by $\underset{\sim}{K}^G(L)$. Let E and F be as above, then there is a functor $\alpha\colon Rep_F[G_F] \to \mathcal{C}_{G_F}(E)$, given* *by $W \to E \underset{F}{\otimes} W$, where the group action is given on the tensor product by the Galois action on the first factor and by the given representation on the second. This functor respects the tensor product operations on both these categories, and therefore induces a map of ring spectra after taking K-theory spectra. In other words, $\underset{\sim}{K}\ (\mathcal{C}_{G_F}(E)) \cong \underset{\sim}{K}\ (F)$ is a $\underset{\sim}{K}^G(F)$-algebra spectrum.*

Remark: The homotopy groups of $\underset{\sim}{K}^G(L)$ are easy to understand when $K_*(L)$ is understood, and when the roots of unity are included in the field L. In this case, we have

$$\pi_* \underset{\sim}{K}^G(L) \cong R[G] \otimes \pi_* \underset{\sim}{K}(L)$$

where $R[G]$ denotes the complex representation ring of G, computed as the direct limit of the representation rings of the finite quotients of G.

We are now going to complete these ring spectra. Let $H(\mathbb{F}_p)$ denote the mod p Eilenberg-MacLane spectrum. We have maps p: $\underset{\sim}{K}^G(F) \to H(\mathbb{F}_p)$ and q: $\underset{\sim}{K}\ (\mathcal{C}_{G_F}(E)) \to H(\mathbb{F}_p)$ so that the diagram

$$\begin{tikzcd}
\underset{\sim}{K}^G(F) \arrow[r, "\alpha"] \arrow[dr, "p"'] & \underset{\sim}{K}\ (\mathcal{C}_{G_F}(E)) \arrow[d, "q"] \\
& H(\mathbb{F}_p)
\end{tikzcd}$$

commutes. From the functoriality statement in the discussion of the derived completion, we now obtain a map $\underset{\sim}{K}^G(F)^\wedge_p \to \underset{\sim}{K}\ (\mathcal{C}_{G_F}(E))^\wedge_q$. It follows from the properties of the derived completion described above, as well as from the equivalence $\underset{\sim}{K}\ (\mathcal{C}_{G_F}(E)) \cong \underset{\sim}{K}\ (F)$, that $\underset{\sim}{K}\ (\mathcal{C}_{G_F}(E))^\wedge_q \cong \underset{\sim}{K}(F)^\wedge_p$, the p-adic completion of $\underset{\sim}{K}\ (F)$. For simplicity, let us now assume that F contains an algebraically closed subfield k. Then we have a composition

$$\beta \colon \underset{\sim}{K}^G(k)^\wedge_{pi} \to \underset{\sim}{K}^G(F)^\wedge_p \to \underset{\sim}{K}\ (\mathcal{C}_{G_F}(E))^\wedge_q \cong \underset{\sim}{K}\ (F)$$

where i: $\underset{\sim}{K}^G(k) \to \underset{\sim}{K}^G(F)$ is the evident inclusion. Let's consider the map β, for the special case when $F \cong k((x))$, the power series field in a single variable over the field k. In this case, we know that the K-theory spectrum of $k((x))$ is equivalent to $\underline{K}(k) \vee \Sigma \underline{K}(k)$, so that the homotopy groups of $\underline{K}(k((x))$ are given by $\pi_i(\underline{K}((x))) \cong \pi_i \underline{K}(k) \oplus \pi_{i-1}\underline{K}(k)$ for $i \geq 1$, and $\pi_0(\underline{K}((x))) \cong \mathbb{Z}$. This means that the p-adic completion of $\underline{K}(k((x))$ has homotopy groups given by $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for $i \geq 1$, and by $\mathbb{Z}_p$ for $i = 0$. On the other hand, the spectral sequence for the homotopy groups of the completion together with the calculation of the derived completion of the representation ring $R[\mathbb{Z}_p]$ shows that the homotopy groups of $\underset{\sim}{K}^G(k)^\wedge_{pi}$ are given by $\pi_i(\underset{\sim}{K}^G(k)^\wedge_{pi}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ for $i \geq 1$, and $\cong \mathbb{Z}_p$ for $i = 0$. Thus the calculation are consistent, and it is not hard to show that β is an equivalence for this example.

Remark: In fact, the spectral sequence which computes the homotopy of the derived completion and the motivic spectral sequence appear identical. Under this identification, the mod-p homotopy groups of the derived completion of the representation ring appear to correspond to the Milnor K-groups of the field, mod p. We expect that this will be general phenomenon.

Calculations suggest that for abelian absolute Galois groups, the proposition that β is an equivalence is consistent with known calculations. For non-abelian groups, though we must make the equivariant generalization. To describe this, we must first talk about equivariant versions of Eilenberg-MacLane spectra. Recall that for any abelian group, one may construct the corresponding Eilenberg-MacLane spectrum. In the equivariant world, it was shown in [**23**] that one can construct an Eilenberg-MacLane spectrum corresponding to any Mackey functor, as discussed above. We define the Mackey functor $\mathcal{H}_G(\mathbb{F}_p)$ to have value equal to the number of orbits on any G-set X, and so that any projection $G/H \to G/K$ induces the identity map on $\mathbb{F}_p$. This turns out to determine the Mackey functor. Both $\underset{\sim}{K}^G(k)$ and $\underset{\sim}{K}\ (\mathcal{C}_{G_F}(E))$ are in fact fixed point spectra of equivariant spectra, which we denote by $\mathcal{A}$ and $\mathcal{B}$, respectively. There is now a corresponding diagram of equivariant spectra

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\alpha"] \arrow[dr, "p"'] & \mathcal{B} \arrow[d, "q"] \\ & \mathcal{H}_G(\mathbb{F}_p) \end{tikzcd}$$

and the conjecture that the corresponding map $\mathcal{A}_p \to \mathcal{B}_q$ is an equivalence for fields F containing an algebraically closed subfield is consistent with known calculations.

Remark: There is a less direct but more general construction which suggests that there should be an equivalence

$$\mathcal{A}_p^\wedge \to \mathcal{B}_q^\wedge$$

whenever the the field F contains the p-power roots of unity. This depends on a more indirect way of proceeding, since one cannot in this case produce a map β. It depends instead on an *Ascent conjecture*, which describes how the K-theory of the algebraic closure of a field is built out of the algebraic K-theory of the ground field.

Remark: There is also a *twisted* version of the representation theoretic construction, which allows one to formulate a version of the conjecture for any field with torsion free absolute Galois group. Fields with torsion (necessarily 2-torsion) such as $\mathbb{R}$ are more problematic, although particular calculations there suggest that there should be a version which works.

IX. Rings of operations

As usual, let F be a field, $E = \overline{F}$ the algebraic closure of F, and $G_F = Gal(E/F)$ the absolute Galois group of F. Consider the K-theory spectrum of E. The group G_F acts on $\underline{K}(E)$, with $\underline{K}(F)$ as fixed point set. This means that the twisted group ring $\underline{K}(E)\langle G_F\rangle$ acts on $\underline{K}(E)$, i.e. $\underline{K}(E)$ is a $\underline{K}(E)\langle G_F\rangle$-module. Since $\underline{K}(F)$ is fixed by the action of G_F, it is clear that we have an inclusion

$$i\colon \underline{K}(F) \hookrightarrow Hom_{\underline{K}(E)\langle G_F\rangle}(\underline{K}(E), \underline{K}(E)) = End_{\underline{K}(E)\langle G_F\rangle}(\underline{K}(E))$$

There is the dual statement that we have a homomorphism

$$j\colon \underline{K}(E)\langle G_F\rangle \hookrightarrow End_{\underline{K}(F)}(\underline{K}(E))$$

Recall that there is a spectral sequence for the homotopy groups of Hom-spectra, whose E_2-term is a suitable Ext-group. Suppose that we wish to compute the homotopy groups of the spectrum $End_{\underline{K}(E)\langle G_F\rangle}(\underline{K}(E))$. The spectral sequence in question would have E_2-term

$$Ext_{\pi_*\underline{K}(E)\langle G_F\rangle}(\pi_*\underline{K}(E), \pi_*\underline{K}(E))$$

Now, $\pi_*(\underline{K}(E)\langle G_F\rangle)$ is isomorphic to the *algebraic* twisted tensor product $\pi_*(\underline{K}(E))\langle G_F\rangle$. This means that the above mentioned E_2-term is isomorphic to

$$\begin{aligned} Ext_{\pi_*(\underline{K}(E))\langle G_F\rangle}(\pi_*\underline{K}(E), \pi_*\underline{K}(E)) &\cong Ext_{\mathbb{Z}[G_F]}(\mathbb{Z}, \pi_*\underline{K}(E)) \\ &\cong H^*(G_F, \pi_*\underline{K}(E)) \end{aligned}$$

This last group is the E_2-term of the spectral sequence for the homotopy fixed spectrum of the G_F-action on $\underline{K}(E)$. Since we know that this homotopy fixed spectrum is not equivalent to $\underline{K}(F)$, we see that the inclusion i is proper, i.e. $\underline{K}(F)$ does not account for all the endomorphisms of $\underline{K}(E)$ over $\underline{K}(E)\langle G_F\rangle$. Dually, this suggests that j is also a proper inclusion, i.e. that $\underline{K}(E)\langle G_F\rangle$ does not account for all the $\underline{K}(F)$-linear endomorphisms of $\underline{K}(E)$. We are therefore interested in $\pi_*End_{\underline{K}(F)}(\underline{K}(E))$.

In order to give the reader an idea of the kind of new operations which are present, beyond the $K_*(E)$-linear combinations of group elements, we will do an easy calculation. Suppose we have an element $\gamma \in G_F$. It acts on the spectrum $\underline{K}(E)^\wedge_p$, and in the context of spectra, I am allowed to add and subtract maps. So, we may consider the map $id - \gamma\colon \underline{K}(E)^\wedge_p \to \underline{K}(E)^\wedge_p$. We also have a map of spectra $\pi\colon \underline{K}(E)^\wedge_p \to H_{\mathbb{Z}_p}$, where $H_{\mathbb{Z}_p}$ denotes the Eilenberg-MacLane spectrum for the p-adic integers. This map is simply the first stage in the Postnikov system for $\underline{K}(E)^\wedge_p$. Because of the identification of $\underline{K}(E)^\wedge_p$ with the p-adic completion of the space BU via Suslin's theorem, and the Bott periodicity theorem (which asserts that $BU \times \mathbb{Z} \cong \Omega^2 BU$), we find that the homotopy fibre of the map π (which is usually denoted by $\underline{K}(E)^\wedge_p[2,3,4,+\infty)$) is in fact equivalent to $\Sigma^2\underline{K}(E)^\wedge_p$. We now have the following picture.

$$\begin{tikzcd} & \Sigma^2\underline{K}(E)^\wedge_p \arrow[d, "q"] & \\ \underline{K}(E)^\wedge_p \arrow[r, "id-\gamma"] & \underline{K}(E)^\wedge_p \arrow[r, "\pi"] & H_{\mathbb{Z}_p} \end{tikzcd}$$

Note that the vertical map can be regarded as multiplication by the Bott generator in dimension 2,which be write as B. Since the composite $\pi(id-\gamma)$ is nullhomotopic, $id - \gamma$ will lift to a map $\theta\colon \underline{K}(E)^\wedge_p \to \Sigma^2 K(E)^\wedge_p$.

$$
\begin{tikzcd}
 & \Sigma^2\underline{K}(E)^\wedge_p \arrow[d, "\cup B"] & \\
\underline{K}(E)^\wedge_p \arrow[ur, "\theta"] \arrow[r, "id-\gamma"] & \underline{K}(E)^\wedge_p \arrow[r, "\pi"] & H_{\mathbb{Z}_p}
\end{tikzcd}
$$

Consequently, we see that the $1-\gamma$ is divisible by the element B in the ring of operations $\pi_*End_{\underline{K}(F)^\wedge_p}(\underline{K}(E)^\wedge_p)$. The element θ is not in $\pi_{-2}\underline{K}(E)^\wedge_p\langle G_F\rangle$. We can then ask for a conjecture about the structure of $\pi_*End_{\underline{K}(F)^\wedge_p}(\underline{K}(E)^\wedge_p)$.

Conjecture IX.1. *Let F be a field containing the p-power roots of unity and having a torsion free, p-profinite absolute Galois group G_F. Form the group ring $\varprojlim \mathbb{Z}_p[x][G/K]$, where the inverse limit is over the finite index subgroups defining G_F as a profinite group, and where x has degree 2. (x represents the Bott element in $\pi_2(BU)$). Call the resulting algebra Λ_F. There is a natural homomorphism $\Lambda_F \to \mathbb{Z}_p[x]$, the augmentation in the group ring. Let $I_F \subseteq \Lambda_F$ denote the kernel of this ideal, referred to as the augmentation ideal. Let $M_F \subseteq \Lambda_F$ denote the subring $\Sigma_n \frac{1}{x^n} I_F^n \subseteq \Lambda_F$, where I_F^0 is conventionally given by Λ_F. Let $\mathcal{M}_F$ denote the completion of M_F in the augmentation ideal topology. Then the conjecture is that*

$$\pi_*End_{\underline{K}(F)^\wedge_p}(\underline{K}(E)^\wedge_p) \cong \mathcal{M}_F$$

There is a twisted form of the conjecture, when the p-power roots of unity are not in the ground field F.

Remark: Note that the inclusion $\pi_*\underline{K}(E)^\wedge_p\langle G_F\rangle \to \pi_*End_{\underline{K}(F)^\wedge_p}(\underline{K}(E)^\wedge_p)$ would be identified with the inclusion $\Lambda_F \hookrightarrow \mathcal{M}_F$.

Remark: It appears likely that the validity of this conjecture would imply the Quillen-Lichtenbaum and Beilinson-Lichtenbaum conjectures. It also appears likely that the validity of the Positselskii-Vishik strong form of the Bloch-Kato conjecture would imply this result. This will be the subject of the final section.

X. Groups and Koszul Duality

In this section, we will consider a relationship between the cohomological Koszul duality property conjectured by Positselskii and Vishik and the conjecture in the previous section. The discussion will be speculative. We first remind the reader about the Eilenberg-Moore spectural sequence.

Theorem X.1. *Let*

$$X \xrightarrow{f} B \xleftarrow{p} E$$

be a diagram of spaces, where the arrow p is a fibration. Consider the pullback $X \underset{B}{\times} E = \{(x,e) \in X \times E | f(x) = p(e)\}$. If the space B is simply connected, then there is a spectral sequence with

$$E_2^{p,q} = Tor^{-p,-q}_{H^*(B)}(H^*(X), H^*(E))$$

converging to $H^{-p-q}(X \underset{B}{\times} E)$. *There is a grading on these Tor groups coming from the graded algebra and module structures on the cohomology groups. Similarly, there is a spectral sequence with*

$$E_2^{p,q} = Ext_{H^*(B)}^{-p,q}(H^*(X), H^*(E))$$

converging to $H_{p+q}(X \underset{B}{\times} E)$. *In the special case when* X *is a single point, and* E *is contractible, the space* $X \underset{B}{\times} E$ *is the loop space* ΩB, *which admits the loop sum (Pontrjagin) multiplication. This means that* $H_*(\Omega B)$ *is a graded algebra.* $E_2^{*,*}$ *is a bigraded algebra, since Ext is equipped with Yoneda mulitplication. The Eilenberg Moore spectral sequence is a spectral sequence of algebras, and the multiplication on* E_∞ *is the associated graded version of the Pontrjagin multiplication on* $H_*(\Omega B)$.

Let G be a group. We will be thinking of the case of G_F for a field F, but keep in mind the case of a discrete group. We would like to consider the special case of the diagram

$$pt \longrightarrow BG \longleftarrow EG$$

where EG is a contractible free G-space, and where $BG = EG/G$ is the classifying space for G. Of course, $pt \underset{BG}{\times} EG \cong G$. If an Eilenberg-Moore spectral sequence converged in this setting, we would obtain a spectral sequence converging to the homology of the group G, in other words the *group ring* of the group G. Note that in this case, If we consider coefficients in a commutative ring k, the E_2-term has the form

$$Ext_{H^*(BG,k)}^{**}(k, k)$$

For G a discrete group, the spectral sequence does exist, but it does not converge to the group ring $k[G]$, but rather to the *derived completion* $k[G]_\epsilon^\wedge$ of the group ring attached to the augmentation $\varepsilon : k[G] \to k$, as defined in Section **VIII**.

Now let us suppose that the group $H^*(BG, k)$ is a Koszul duality algebra, in the sense of section **V**. This means that the bigraded algebra $Ext_{H^*(BG,k)}^{p,q}(k, k)$ vanishes for $p \neq q$. So $E_\infty^{p,q}$ of the Eilenberg-Moore spectral sequence vanishes for $p + q \neq 0$, and consequently $\pi_i(k[G]_\epsilon^\wedge) = 0$ for $i \neq 0$. It is not hard to check that $\pi_0(k[G]_\epsilon^\wedge)$ is isomorphic to the actual algebraic completion of $k[G]$, and that the filtration induced by the spectral sequence is just the augmentation ideal filtration on $k[G]_\epsilon^\wedge$. Moreover, the spectral sequence converges, so we obtain that

$$(1) \qquad Ext_{H^*(BG,k)}^{p,p}(k, k) \cong E_2^{p,-p} \cong E_\infty^{p,-p} \cong I^p(G)/I^{p+1}(G)$$

where $I^k(G)$ denotes the k-the power of the augmentation ideal $I(G) = Ker(\varepsilon)$.

Next let us consider the endomorphism ring spectrum

$$\mathcal{E}_F \cong Hom_{KF}(K\overline{F}, K\overline{F})$$

for a field F which contains an algebraically closed subfield. (There is a twisted version of the discussion which follows). It is standard that there is a spectral sequence converging to $\pi_*(\mathcal{E}_F)$ with

$$(2) \qquad E_2^{p,q} = Ext_{\pi_*KF}^{-p,q}(\pi_*K\overline{F}, \pi_*K\overline{F})$$

converging to $\pi_{p+q}(\mathcal{E}_F)$. For fields as above, the motivic spectral sequence predicts that the p-completed algebraic K-theory of the field F will be given by

$$K_*(F) \cong K_*^M(F)_p^\wedge \underset{\mathbb{Z}_p}{\otimes} \mathbb{Z}_p[\beta]$$

where β denotes the "Bott element". Of course, Suslin's theorem tells us that the homotopy groups of the p-adic completion of $K_*\overline{F}$ are equal to $\mathbb{Z}_p[\beta]$. Now 2 above tells us that the E_2-term for the spectral sequence converging to $\pi_*(\mathcal{E}_F)$ is given by

$$Ext_{K_*^M(F)_p^\wedge \underset{\mathbb{Z}_p}{\otimes} \mathbb{Z}_p[\beta]}(\mathbb{Z}_p[\beta], \mathbb{Z}_p[\beta]) \tag{3}$$

A standard "change of rings" argument shows that 3 is actually isomorphic to

$$Ext_{K_*^M(F)_p^\wedge}(\mathbb{Z}_p, \mathbb{Z}_p[\beta]) \tag{4}$$

The Bloch-Kato conjecture asserts that $K_i^M(F)/pK_i^M(F) \cong H^i(G_F, \mathbb{F}_p)$. In the situation where the p-th roots of unity are in the ground field, it is expected that it can be strengthened to the completed integral statement

$$K_*^M(F)_p^\wedge \cong H^*(G_F, \mathbb{Z}_p) \tag{5}$$

Combining 4 and 5, we obtain that the E_2-term 3 is actually isomorphic to

$$Ext_{H^*(G_F,\mathbb{Z}_p)}(\mathbb{Z}_p, \mathbb{Z}_p[\beta]) \cong Ext_{H^*(G_F,\mathbb{Z}_p)}(\mathbb{Z}_p, \mathbb{Z}_p) \underset{\mathbb{Z}_p}{\otimes} \mathbb{Z}_p[\beta] \tag{6}$$

If we now assume the Positselskii-Vishik form of the Bloch-Kato conjecture, and use 1 , we will find that 6 can be rewritten as

$$\hat{gr}_*(\mathbb{Z}_p[G_F]) \underset{\mathbb{Z}_p}{\otimes} \mathbb{Z}_p[\beta] \tag{7}$$

where $\hat{gr}_*(\mathbb{Z}_p[G_F])$ denotes the graded vector space defined by

$$\hat{gr}_{-2k}(\mathbb{Z}_p[G_F]) \cong I^k(G_F)/I^{k+1}(G_F)$$

for $k \geq 0$, and $= 0$ otherwise.

Remark: The grading on $\hat{gr}$ comes out the way it does because of the nature of the grading on the Milnor K-theory. In the Eilenberg-Moore spectral sequence, the ring $H^*(G_F)$ is given a *negative* grading, whereas the Milnor K-theory is *positively* graded. This difference in grading (which amounts to a difference of $2k$ in grading k), accounts for the result.

Remark: We observe that if we assume the collapse of this spectral sequence, it is completely consistent with the conjecture IX.1 from the previous section. This suggests that the Positselskii-Vishik conjecture will completely determine the structure of the graded ring $\pi_*(\mathcal{E}_F)$. Complete knowledge of this ring should give convergent descent spectral sequences for K-theory, which should hold "on the nose" rather than only in sufficiently high degrees.

References

[1] D. Anderson, M. Karoubi, J. Wagoner *Higher algebraic K-theories* Trans. Amer. Math. Soc. **226** (1977), 209–225

[2] M.F. Atiyah, *Characters and cohomology of finite groups*, Inst. Hautes Etudes Scient. Publ. **9** (1961), 23-64.

[3] S. Bloch, *Algebraic K-theory, motives, and algebraic cycles* Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 43- 54, Math. Soc. Japan, Tokyo, 1991.

[4] S. Bloch and S. Lichtenbaum, *A spectral sequence for motivic cohomology*, preprint, available on the K-theory server at U. of Illinois, Urbana Champaigne.

[5] S. Bouc, *Green functors and G-sets* Lecture Notes in Mathematics, **1671** Springer-Verlag, Berlin, 1997.

[6] G. Carlsson, *Equivariant stable homotopy theory and Segal's Burnside ring conjecture*, Annals of Mathematics **120** (1984) 189-224.

[7] G. Carlsson, *Equivariant stable homotopy and Sullivan's conjecture*, Inventiones Mathematicae **103** (1991) 497-525.

[8] M.M. Cohen **A course in Simple Homotopy Theory**, Springer Verlag 1973.

[9] Edward B. Curtis, *Simplicial homotopy theory* Advances in Math. **6** (1971) 107–209

[10] W. Dwyer, E. Friedlander, V.P. Snaith, and R.W. Thomason, *Algebraic K-theory eventually surjects onto topological K-theory*, Invent. Math. 66 (1982), no. 3, 481–491.

[11] A. D. Elmendorf, I. Kriz, M.A. Mandell, and J.P. May, **Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole.** Mathematical Surveys and Monographs, 47. American Mathematical Society, Providence, RI, 1997.

[12] O. Gabber, *K-theory of Henselian local rings and Henselian pairs* Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), 59–70, Contemp. Math., 126, Amer. Math. Soc., Providence, RI, 1992.

[13] P. Goerss and J.F. Jardine, **Simplicial homotopy theory**, Progress in Mathematics, 174. Birkhuser Verlag, Basel, 1999.

[14] A. Grothendieck, et al, with M. Artin and J.-L. Verdier, *Theorie des topos et cohomologie etale des schemas*, Lecture Notes in Mathematics **269**, **270**, **305**, Springer Verlag, 1972-3.

[15] A. Grothendieck, *Le groupe de Brauer. I. Algebres d'Azumaya et interpretations diverses, II. Theorie cohomologique, III. Exemples et complements*, in **Dix Exposes sur la Cohomologie des schemas**, NOrth-Holland, Amsterdam, 1968, 46-188.

[16] A. Hatcher, *Pseudo-isotopy and K_2* Lecture Notes in Mathematics **342**, Springer Verlag, 1973.

[17] A.Hatcher and J. Wagoner *Pseduo-isotopies of compact manifolds*, Asterisque **6** (1973)

[18] L. Hesselholt and I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), no. 1, 29–101.

[19] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra* J. Amer. Math. Soc. 13 (2000), no. 1, 149–208.

[20] K. Kato, *Generalized class field theory* Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 419–428, Math. Soc. Japan, Tokyo, 1991.

[21] G. M. Kelly and S. Mac Lane, *Coherence in closed categories* J. Pure Appl. Algebra **1** (1971) no. 1, 97–140.

[22] S.Lang, **Algebraic Number Theory**, Addison Wesley, 1970.

[23] G. Lewis, J.P. May, and J. McClure, *Ordinary $RO(G)$-graded cohomology* Bull. Amer. Math. Soc. (N.S.) **4** (1981), no. 2, 208–212.

[24] S. Lichtenbaum,*On the values of zeta and L-functions I* Ann. of Math. (2) **96** (1972), 338–360.

[25] S. Mac Lane, *Natural associativity and commutativity*, Rice Univ. Studies **49** 1963 no. 4, 28–46.

[26] H. Matsumoto, *Sur les sous-groupes arithmetiques des groups semi- simples Deployes*, Ann. Sci. Ex. Norm. Sup. 4th series, 2 (1969), 1-62.

[27] J.P. May, **Simplicial Objects in Algebraic Topology** Van Nostrand 1967.

[28] J.P. May, *Geometry of iterated loop spaces*, Lecture Notes in Mathematics **271**, Springer Verlag, 1972

[29] A.S. Merkurjev, *Algebraic K-theory and Galois cohomology* First European Congress of Mathematics, Vol. II (Paris, 1992), 243–255, Progr. Math., 120, Birkhuser, Basel, 1994.

[30] H.R. Miller, *The Sullivan conjecture on maps from classifying spaces*, Annals of Mathematics **120** (1984), 39-87.

[31] J.S. Milne, **Etale cohomology** Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980.

[32] J. Milnor, **An Introduction to Algebraic K-Theory** Annals of Mathematics Studies, **1972**, Princeton University Press, 1971.

[33] R. Oliver, SK_1 **for finite group rings. I** Invent. Math. **57** (1980), no. 2, 183–204

[34] R. Oliver, *A proof of the Conner conjecture.*Ann. of Math. (2) **103** (1976), no. 3, 637–644.

[35] L. Positselski and A. Vishik, *Koszul duality and Galois cohomology* Math. Res. Lett. 2 (1995), no. 6, 771–781.

[36] D.G. Quillen *Higher algebraic K-theory I* Lecture Notes in Mathematics **341**, Springer Verlag 1973, 85-147

[37] D.G. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Annals of Mathematics **96** (1972), 552-586.

[38] D.G. Quillen, *Higher algebraic K-theory*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, pp. 171–176. Canad. Math. Congress, Montreal, Que., 1975.

[39] J. Rognes, and C. Weibel, *Two-primary algebraic K-theory of rings of integers in number fields* Appendix A by Manfred Kolster. J. Amer. Math. Soc. **13** (2000), no. 1, 1–54.

[40] G.B. Segal, *Configuration spaces and iterated loop spaces*, Inventiones Mathematicae **21** (1973) 213-221.

[41] G.B. Segal, *Equivariant stable homotopy theory* Actes du Congrs International des Mathmaticiens (Nice, 1970), Tome 2, pp. 59–63. Gauthier-Villars, Paris, 1971.

[42] J.P. Serre **Groupes algbriques et corps de classes** (French) Deuxieme edition. Publication de l'Institut de Mathematique de l'Universite de Nancago, No. VII. Actualites Scientifiques et Industrielles, No. 1264. Hermann, Paris, 1975.

[43] A.A. Suslin, *On the K-theory of algebraically closed fields*, Inventiones Mathematicae **73**, 1983, 241-245.

[44] A.A. Suslin, *On the K-theory of local fields* Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983). J. Pure Appl. Algebra 34 (1984), no. 2-3, 301–318.

[45] R.G. Swan, *Algebraic K-theory*, Lecture Notes in Mathematics **76**, Springer Verlag, 1968.

[46] R. G. Swan, *Algebraic K-theory.* Actes du Congrs International es Mathmaticiens (Nice, 1970), Tome 1, pp. 191–199. Gauthier-Villars, Paris, 1971.

[47] R . G. Swan, *Vector bundles and projective modules* Trans. Amer. Math. Soc. **105** (1962) 264–277.

[48] R. W. Thomason, *Algebraic K-theory and tale cohomology* Ann. Sci. cole Norm. Sup. (4) **18** (1985), no. 3, 437–552.

[49] V. Voevodsky, A. Suslin, and E.M. Friedlander, **Cycles, transfers, and motivic homology theories** Annals of Mathematics Studies, **143** Princeton University Press, Princeton, NJ, 2000.

[50] V. Voevodsky, *The Milnor conjecture*, preprint, available at the K-theory server, Univ. of Illinois, Urbana-Champaigne.

[51] H. Volklein, **Groups as Galois groups. An introduction.** Cambridge Studies in Advanced Mathematics, 53. Cambridge University Press, Cambridge, 1996

[52] I.A. Volodin *Algebraic K-theory* Uspehi Mat. Nauk. no. 4 and 5, 1972.

[53] F. Waldhausen, *Algebraic K-theory of topological spaces I*, in **Algebraic and Geometric Topology** Proc. Symp. Pure Math. XXIII, part 1, American Mathematical Society 1978.

[54] F. Waldhausen, *Algebraic K-theory of generalizes free products*, Annals of Mathematics **108** (1978) 135-256.

[55] C.T.C. Wall, *Finiteness obstructions for CW-complexes I* Annals of Mathematics **81** (1965) 56-59.

[56] Andre Weil, **Basic Number Theory**, Grund. math. Wiss., 3rd edition, Springer Verlag, 1974

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