

## *K*-Theory in Quantum Field Theory

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ABSTRACT. We survey three different ways in which *K*-theory in all its forms enters quantum field theory. In Part 1 we give a general argument which relates topological field theory in codimension two with twisted *K*-theory, and we illustrate with some finite models. Part 2 is a review of pfaffians of Dirac operators, anomalies, and the relationship to differential *K*-theory. Part 3 is a geometric exposition of Dirac charge quantization, which in superstring theories also involves differential *K*-theory. Parts 2 and 3 are related by the Green-Schwarz anomaly cancellation mechanism. An appendix, joint with Jerry Jenquin, treats the partition function of Rarita-Schwinger fields.

Grothendieck invented *K*-Theory almost 50 years ago in the context of algebraic geometry, specifically in his generalization of the Hirzebruch Riemann-Roch theorem [BS]. Shortly thereafter, Atiyah and Hirzebruch brought Grothendieck's ideas into topology [AH], where they were applied to a variety of problems. Analysis entered after it was realized that the symbol of an elliptic operator determines an element of *K*-theory. Atiyah and Singer then proved a formula for the index of such an operator (on a compact manifold) in terms of the *K*-theory class of the symbol [AS1]. Subsequently, *K*-theoretic ideas permeated other areas of linear analysis, algebra, noncommutative geometry, etc. One of the pleasant surprises of the past few years has been the relevance of *K*-theory to superstring theory and related parts of theoretical physics. Furthermore, the story involves not only topological *K*-theory, but also the *K*-theory of  $C^*$ -algebras, the *K*-theory of sheaves, and other forms of *K*-theory.

Not surprisingly, this new arena for *K*-theory has inspired some developments in mathematics which are the subject of ongoing research. Our exposition here aims to explain three different ways in which topological *K*-theory appears in physics, and how this physics motivates the mathematical ideas we are investigating.

Part 1 concerns *topological quantum field theory*. Recall that an  $n$ -dimensional topological theory assigns a complex number to every closed oriented  $n$ -manifold and a complex vector space to every closed oriented  $(n - 1)$ -manifold. Continuing the superposition principle and ideas of locality to codimension two we are led to an extended notion which attaches to each closed oriented  $(n - 2)$ -manifold a special type of category, which we term a *K-module*. The '*K*' stands for *K*-theory, and so by this very general argument *K*-theory enters in codimension two, at least for

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topological quantum field theories. We illustrate these ideas with a finite topological quantum field theory, a “gauged  $\sigma$ -model” which generalizes the finite gauge theories studied in [F1]. The  $K$ -modules which enter here are a *twisted* form of ordinary  $K$ -theory. It should be noted that the appearance of twisted  $K$ -theory in the physics has spurred its development in mathematics.<sup>1</sup> The 2-dimensional version of the finite gauged  $\sigma$ -model was introduced in a different way in [T2]. Moore and Segal [M] make a general investigation of “boundary states” in 2-dimensional *topological* theories and are led to this same finite gauged  $\sigma$ -model. One observation here is that the category of boundary states, at least in 2-dimensional topological theories, is the  $K$ -module one encounters from general considerations in codimension two. From this point of view the appearance of  $K$ -theory is natural. Turning to the 3-dimensional theory, once the pure gauge theory for finite groups has been related to twisted  $K$ -theory—this connection was missed in [F1]—it is natural to conjecture a similar relationship for compact gauge groups of arbitrary dimension. In particular, we identify the “Verlinde algebra” in Chern-Simons theory as a particular twisted equivariant  $K$ -group. This set of ideas is being pursued jointly with M. Hopkins and C. Teleman. (See [F3] for a more detailed motivational account.)

Part 2 concerns *anomalies*; the main ideas go back to the mid 1980s. We set the framework with a geometric picture of anomalies, and then explain how the pfaffian line bundle of a family of Dirac operators encodes the anomaly associated to the functional integral over a fermionic field. The detailed construction depends on the dimension  $n$  of the theory modulo 8. One novelty here is the simultaneous presentation of all cases. Unfortunately, its salient feature is the lack of a unified approach which binds the different dimensions into a single picture—presumably held together with Bott periodicity. We have yet to find such a description. The topological anomaly is computed using the Atiyah-Singer index theorem, and it is through the Atiyah-Singer formula that topological  $K$ -theory enters.<sup>2</sup> In many cases, however, the anomaly is geometric—it is a smooth line bundle with hermitian metric and compatible connection. With this motivation we are led to believe that the geometric anomaly—the pfaffian line bundle with metric and connection—may be computed using *differential  $K$ -theory*, a version of  $K$ -theory which includes differential forms as “curvatures”. First notions of differential  $K$ -theory appear in [Lo], and a systematic development of differential cohomology theories in general begins in [HS]. Ongoing joint work with M. Hopkins and I. Singer continues these developments, in particular pursuing the connection between differential  $K$ -theory and geometric invariants of families of Dirac operators. The particular connection with the pfaffian is what is needed for anomalies.

Part 3 is an elementary exposition of *Dirac charge quantization*. In classical electromagnetism, and its generalizations in supergravity with forms of higher degree, charges take values in real cohomology. In quantum theories charges are constrained to lie in a full lattice inside real cohomology, and from a mathematical point of view it is natural that the appropriate lattice is determined by a generalized cohomology theory. Ordinary integral cohomology is the traditional choice, but in superstring theory it is  $K$ -theory—in some cases the real or quaternionic

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<sup>1</sup>Twisted  $K$ -theory was introduced in mathematics many years ago, both in topology [DK] and in  $C^*$ -algebras [R]. Recent references include [A2] and [BCMMS]. But a systematic development incorporating all of the properties needed here has yet to be written.

<sup>2</sup>There are notable exceptions in low dimensions, where the topological anomaly is computed by a cohomological formula.

version—which has proved relevant. The electric and magnetic currents in these theories must simultaneously encode local information—the positions and velocities of charges—as well as the global information of charge quantization. The geometric objects which accomplish this are cocycles for generalized differential cohomology theories, e.g., for differential  $K$ -theory. This provides further impetus for the development of differential cocycles. We emphasize the easy examples, where the geometry of charge quantization is more readily accessible. One important point is the anomaly in the electric coupling if there is simultaneous magnetic and electric current. We conclude Part 3 with some remarks explaining why  $K$ -theory quantizes Ramond-Ramond charge in superstring theory.

The occurrences of  $K$ -theory in Parts 2 and 3 are related. Namely, the anomaly in the electric coupling can cancel anomalies from fermions if the former may be computed in differential  $K$ -theory since, as explained above, the latter are conjecturally computed in differential  $K$ -theory using a refinement of the Atiyah-Singer index theorem. Therefore, the main ingredient in the *Green-Schwarz anomaly cancellation mechanism* [GS], extended to include global as well as local anomalies, is a geometric form of the Atiyah-Singer index theorem for families of Dirac operators. The superstring examples are explained from this point of view in [FH], [F5]. In ongoing work with J. Distler we consider generalizations in superstring theory, and with E. Diaconescu and G. Moore we are investigating similar questions in M-theory.

There is an appendix, joint with Jerry Jenquin, which is a pedagogical account of Rarita-Schwinger fields and their quantization to obtain spin 3/2 particles. The anomaly computation for these fields is a bit confusing, as it typically looks different in odd and even dimensions. Our treatment is uniform for all dimensions, chiralities, and multiplicities.

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## Part 1: Topological Quantum Field Theory in Codimension Two

**1.1. General Remarks.** Some essential features of a topological quantum field theory (TQFT) were abstracted in [A1]; refinements were discussed in [Q], [T1], [Wa] and other references as well. A TQFT gives algebro-topological invariants of manifolds, but unlike typical homotopical invariants is tied to a specific dimension  $n$  and may depend on the differentiable structure. The basic data is the *partition function*

$$(1.1) \quad X^n \longmapsto Z(X^n)$$

which assigns to every compact oriented  $n$ -manifold a complex number  $Z(X)$ . This is a topological invariant in the sense that the invariants of *diffeomorphic manifolds* agree. Also, if  $-X$  denotes the oppositely oriented manifold, then  $Z(-X) = \overline{Z(X)}$ . The quantum nature of  $Z$  manifests itself in its multiplicative properties. For example, the invariant of a disjoint union is the product of the invariants of the constituent manifolds:

$$(1.2) \quad Z(X_1 \sqcup X_2) = Z(X_1) Z(X_2).$$

This generalizes to a *gluing law* when a closed manifold  $X$  is cut along an oriented hypersurface  $Y$ . Assume for simplicity that  $Y$  separates  $X$  into two pieces  $X_1$  and  $X_2$ , as in Figure 1. Then the *local* nature of quantum field theory predicts that there should be an invariant  $Z(X_i)$  defined for the manifolds  $X_i$ , which now have a nonempty boundary, and that a multiplicative equation analogous to (1.2) should hold. In fact, field theory on a manifold with boundary requires boundary conditions for the fields, which leads not to a single complex-valued invariant, but rather a more complicated invariant which is in some sense a function of boundary conditions. The linearity of quantum mechanics leads us to suspect that the invariant is a vector

$$(1.3) \quad Z(X_i) = (Z(X_i)_1, \dots, Z(X_i)_N),$$

where  $Z(X_i)_j \in \mathbb{C}$  and the indices depend only on the boundary  $Y$ . In typical quantum field theories  $N = \infty$ , of course, but in special topological theories, such as the ones considered here, it is finite. We may go further and postulate an assignment

$$(1.4) \quad Y^{n-1} \longmapsto E(Y)$$

of a Hilbert space  $E(Y)$  to every closed oriented  $(n-1)$ -manifold—whether or not it is a boundary—and an assignment

$$(1.5) \quad X^n \longmapsto Z(X) \in E(\partial X)$$

to every compact oriented  $n$ -manifold of a vector in the Hilbert space of the boundary. Furthermore, (1.4) should be functorial in the sense that a diffeomorphism

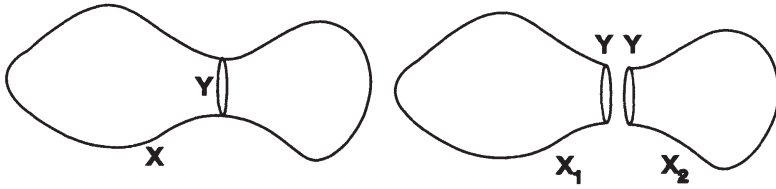


FIGURE 1. : Cutting a closed manifold in codimension one

$Y' \rightarrow Y$  induces an isomorphism  $E(Y') \rightarrow E(Y)$ . Finally, the gluing law which generalizes (1.2) in the situation of Figure 1 is

$$(1.6) \quad Z(X) = \langle Z(X_1), Z(X_2) \rangle_{E(Y)}.$$

These are the highlights of the standard axioms, but it is tempting to go further. Namely, of the two basic properties of the partition function (1.1)—functoriality and locality—we have only imposed functoriality on the invariant  $Y \mapsto E(Y)$  in (1.5); it is tempting to impose locality as well. Thus if  $Z^{n-2} \subset Y^{n-1}$  splits the closed oriented manifold  $Y$  into a union  $Y_1 \cup Y_2$ , we would like to factorize the vector space  $E(Y)$  as some sort of product of invariants associated to the  $Y_i$ , which are manifolds with boundary. Now in (1.3) we can consider  $Z(X_i)$  as a vector of complex numbers, so by analogy we expect that  $E(Y_i)$  should be a vector of Hilbert spaces. Further, we then expect an assignment

$$Z^{n-2} \mapsto \mathcal{E}(Z)$$

which attaches to a closed oriented  $(n-2)$ -manifold a “module” over the “ring” of Hilbert spaces. This module should have an “inner product” back to the ground “ring”, and then the gluing law analogous to (1.6) should state

$$Z(Y) = \langle Z(Y_1), Z(Y_2) \rangle_{\mathcal{E}(Z)}.$$

This extended notion of a TQFT can be made precise in various ways and occurs in different forms in the literature (e.g. [L]). We content ourselves here with a heuristic description as follows.

The collection of all finite dimensional Hilbert spaces forms a tensor category using the operations of direct sum and tensor product. The classifying space of this category is a product of classifying spaces for the general linear groups of all dimensions. Apparently what is needed here instead is a tensor category  $K$  whose classifying space is a classifying space for complex  $K$ -theory, i.e., is homotopic to  $\mathbb{Z} \times BU$ . So  $E(Y)$  is a category with an action of  $K$ , i.e., a  $K$ -module. We will use the category of finite dimensional Hilbert spaces as a heuristic for  $K$ , but a more careful account would substitute a correct model for  $K$  instead. Also, we will not always mention the inner product in  $K$ -modules, even though it exists in unitary theories and the finite theories we construct are unitary.

**Summary:** A TQFT assigns to a closed oriented manifold a complex number in the top dimension, a complex vector space in codimension one, and a  $K$ -module in codimension 2.

**1.2. The Groupoid of Fields in Finite TQFT.** Let  $G$  be a finite group and  $S$  a finite  $G$ -set, that is, a finite set with a  $G$ -action.<sup>3</sup> Let  $M$  be a topological space, which we will soon specialize to be a compact manifold. Let  $\mathcal{CM}$  be the category whose objects are pairs  $(P, \phi)$ , where  $P \rightarrow M$  is a principal  $G$ -bundle (Galois covering) and  $\phi : P \rightarrow S$  is a  $G$ -equivariant map. A morphism  $f : (P', \phi') \rightarrow (P, \phi)$  is a  $G$ -equivariant map  $f : P' \rightarrow P$  which commutes with the projections to  $S$  and such that  $\phi' = f \circ \phi$ . It is easy to see that every morphism is invertible:  $\mathcal{CM}$  is a groupoid. Let  $\overline{\mathcal{C}(M)}$  be the set of equivalence classes of objects in  $\mathcal{C}(M)$ . It is finite if  $M$  is compact.

The groupoid  $\mathcal{C}(M)$  is the collection of *fields* on  $M$ . A model with this set of fields is called a *gauged  $\sigma$ -model*. It is usually defined for  $G$  a Lie group and  $P, M, S$  smooth manifolds. If  $G$  has positive dimension, then an object in  $\mathcal{C}(M)$  also includes a connection on  $P \rightarrow M$  and morphisms are required to pullback connections. For  $G = \{1\}$  the collection of fields  $\mathcal{C}(M)$  is the set of maps  $\phi : M \rightarrow S$ , and the model is a  $\sigma$ -model with target  $S$ . If  $S$  is a point with trivial  $G$ -action, then we have a pure gauge theory with gauge group  $G$ . In the general case the  $G$  action on  $S$  determines a bundle  $S_P \rightarrow M$  with typical fiber  $S$  associated to a principal  $G$ -bundle  $P \rightarrow M$ , and the map  $\phi$  may be viewed as a section of this associated bundle.

It is convenient to replace  $\mathcal{C}(M)$  by an equivalent category. Fix a smooth universal  $G$ -bundle  $EG \rightarrow BG$ .<sup>4</sup> There is a category of triples  $(P, \gamma, \phi)$ , where  $P, \phi$  are as before and  $\gamma : P \rightarrow EG$  is a  $G$ -equivariant map; the quotient map  $\bar{\gamma} : M \rightarrow BG$  is a classifying map for  $P$ . A morphism  $f : (P', \gamma', \phi') \rightarrow (P, \gamma, \phi)$  is a map  $f : P' \rightarrow P$  where, as above,  $\phi' = f \circ \phi$ . Note that there is no condition on the classifying maps. The forgetful functor  $(P, \gamma, \phi) \rightsquigarrow (P, \phi)$  is an equivalence of categories.

For computations it is convenient to use a much smaller equivalent groupoid. For example, when  $M$  is a point we have

$$(1.7) \quad \mathcal{C}(\text{pt}) \approx S \text{ as a } G\text{-set.}$$

Recall that attached to any  $G$ -set  $S$  is a groupoid: the set of objects is  $S$  and each pair  $(g, s) \in G \times S$  corresponds to a morphism  $s \rightarrow g \cdot s$ . Consider next  $M = S^1$ . We rigidify  $\mathcal{C}S^1$  by fixing a basepoint  $* \in S^1$  and requiring that  $G$ -bundles  $P \rightarrow S^1$  have a basepoint in the fiber over  $*$ . Such a based bundle is determined up to unique isomorphism by its holonomy, which is an element of  $G$ . The map  $\phi : P \rightarrow S$  is determined by its value at the basepoint of  $P$ , which must be fixed by the action of the holonomy. So

$$(1.8) \quad \mathcal{C}(S^1) \approx G \text{ acting on } \{(s, g) : s \in S, g \in G, g \cdot s = s\}$$

The action of  $G$  translates the basepoint in the fiber over  $*$ . The morphism which corresponds to  $h \in G$  maps  $(s, g) \mapsto (h \cdot s, hgh^{-1})$ .

<sup>3</sup>It suffices to restrict to transitive  $G$ -sets, that is,  $S = G/H$  for a subgroup  $H \subset G$ , since any  $G$ -set decomposes into a disjoint union of such and the construction of TQFTs decomposes accordingly. But the analogy to continuous theories is clearer in the framework of arbitrary  $S$ .

<sup>4</sup>For example, embed  $G$  in a unitary group  $U(N)$  and take  $EG$  to be the Stiefel manifold of unitary maps of  $\mathbb{C}^N$  into a complex Hilbert space. In theories with  $\dim G > 0$  we also fix a connection on the universal bundle.

**1.3. Extended TQFTs from Functional Integrals.** This general discussion is taken from [F1], where it was applied to finite theories with  $S = \text{pt}$ .

In an  $n$ -dimensional quantum field theory the exponentiated classical action on a compact oriented  $n$ -manifold  $X$  is a function <sup>5</sup>

$$(1.9) \quad e^{iS_X} : \overline{\mathcal{C}(X)} \longrightarrow \mathbb{C}.$$

For finite theories, where  $\overline{\mathcal{C}(X)}$  is a finite set, a measure is simply a function  $\overline{\mathcal{C}(X)} \rightarrow \mathbb{R}^{>0}$ . Given a measure and classical action, the quantum partition function is the integral

$$(1.10) \quad Z(X) = \int_{\overline{\mathcal{C}(X)}} e^{iS_X(\varphi)} d\varphi.$$

For a finite theory (1.10) reduces to a finite sum.

The classical action (1.9) extends to compact oriented  $n$ -manifolds with boundary, but it is sometimes not valued in the complex numbers. Rather, in general its value on a field  $\varphi$  lies in a hermitian line which depends only on the restriction of  $\varphi$  to  $\partial X$ . It is natural, then to extend the notion of classical action to codimension one. That is, given a compact oriented  $(n-1)$ -manifold  $Y$  we assign to every field on  $Y$  a hermitian line. Recall that the fields  $\mathcal{C}(Y)$  form a groupoid, and so we require that morphisms act as isomorphisms on the hermitian lines. In other words, the action gives an equivariant hermitian line bundle over  $\mathcal{C}(Y)$ . The quantum Hilbert space  $E(Y)$  is usually described as the vector space of invariant sections of this line bundle, with an  $L^2$  metric relative to a measure on  $\overline{\mathcal{C}(Y)}$ . (In [F1] it is also described as the result of integrating the classical action over  $\overline{\mathcal{C}(Y)}$ .) There is then a natural extension of (1.10) to manifolds with boundary.

The story proceeds analogously in codimension two. In codimension one we can say that the classical action takes values in one-dimensional  $\mathbb{C}$ -modules. The extension to a compact oriented  $(n-2)$ -manifold  $Z$  assigns to every field a one-dimensional  $K$ -module, which we term a *K-line*.<sup>6</sup> Again morphisms in the groupoid  $\mathcal{C}(Z)$  act in a natural way, and  $\mathcal{E}(Z)$  is defined to be the  $K$ -module of “invariant” sections. As the quotes indicate, we must be careful to interpret the notion of invariance correctly. In codimension one, where the classical action on  $Y$  is a line bundle over  $\mathcal{C}(Y)$ , the set of invariants in the fiber at a field  $\varphi$  is either the entire fiber or zero, according as the action of the automorphism group at  $\varphi$  is trivial or not. But in codimension two the fiber is a category, not a set, so it is natural to interpret invariance under the automorphism group as being a representation of the automorphism group on the fiber. Thus if we trivialize the  $K$ -line at a particular field, the invariants under the automorphism group form the category of (finite dimensional) representations of the automorphism group.

This construction of a TQFT from an extended notion of a classical action allows us to deduce formally the axioms of the quantum theory from simple properties of the classical action and the measure.

<sup>5</sup>When we discuss anomalies in Part 2, we will see that in general quantum field theories the action is an element of a hermitian line, but in finite TQFT there is no anomaly.

<sup>6</sup>In a unitary theory the  $K$ -lines have a “hermitian structure” but we will not be careful about it. In [F2]  $K$ -modules (of arbitrary dimension) with hermitian structure are called “2-Hilbert spaces”.



**1.4. Finite TQFT.** Recall that the groupoids of fields in finite TQFT are determined by a finite group  $G$  and a finite  $G$ -set  $S$ . On any compact space  $M$  the groupoid of fields  $\mathcal{C}(M)$  is discrete with finite automorphism groups, and on such a groupoid there is a natural *counting measure*: To each object  $\varphi \in \mathcal{C}(M)$  we assign the positive number

$$(1.11) \quad \mu_M(\varphi) = \frac{1}{\#\text{Aut } \varphi}.$$

The counting measure clearly descends to  $\overline{\mathcal{C}(M)}$ .

The classical action in the  $n$ -dimensional finite TQFT is determined by a cocycle  $B$  for an element in the equivariant cohomology group  $H_G^n(S; \mathbb{R}/2\pi\mathbb{Z})$ . Concretely, we fix a singular cocycle of degree  $n$  with coefficients in  $\mathbb{R}/2\pi\mathbb{Z}$  on the total space of the bundle  $S_{EG} \rightarrow BG$ . The special case  $B = 0$  is already interesting and does not require the formalism explained in the next paragraph. Only the cohomology class of  $B$  matters in the sense that an  $(n-1)$ -cochain  $A$  determines an isomorphism between the theory defined by  $B$  and the theory defined by  $B + \delta A$ . Notice, since  $G$  and  $S$  are finite, that  $H_G^n(S; \mathbb{R}/2\pi\mathbb{Z}) \cong H_G^{n+1}(S; \mathbb{Z})$  for  $n \geq 1$ .

To write the classical action we integrate singular cocycles over compact oriented manifolds. What we need appears in [F1, Appendix], and as we explain later it is a special case of a more general integration theory for differential cocycles [HS].<sup>7</sup> We summarize briefly what we need. Let  $M$  be a compact oriented manifold and  $b$  a singular cocycle of degree  $n$  with coefficients in  $\mathbb{R}/2\pi\mathbb{Z}$ . In the simplest case, if  $M = X$  is closed (no boundary) of dimension  $n$ , then there is a cohomological pairing of the cohomology class of  $b$  with the fundamental class  $[X]$  of  $X$  which produces an element of  $\mathbb{R}/2\pi\mathbb{Z}$ . This is the integral of  $b$  over  $X$ . It is defined by evaluating  $b$  on any cycle representing  $[X]$ . Now suppose instead that  $M = Y$  is closed of dimension  $(n-1)$ . Then of course there is no cohomological meaning to the integral of  $b$  over  $Y$ . Instead, we claim that this integral may be interpreted as a hermitian line as follows. Let  $\mathcal{F}(Y)$  be the category whose objects are singular  $(n-1)$ -cycles which represent the fundamental class  $[Y]$ . Let a singular  $n$ -chain  $y$  determine a morphism  $x \rightarrow x + \delta y$  for all  $x$ . Hence  $\mathcal{F}(Y)$  is a groupoid which is connected—there is a morphism between any two objects. Over  $\mathcal{F}(Y)$  we consider a hermitian line bundle whose fiber at each object is  $\mathbb{C}$  with its canonical metric, and such that a morphism defined by  $y$  acts as multiplication by  $\exp(i\langle b, y \rangle)$ , where  $\langle \cdot, \cdot \rangle$  is the pairing of cochains and chains. The hermitian line  $\exp(i \int_Y b)$  is the hermitian line of invariant sections of this hermitian line bundle over  $\mathcal{F}(Y)$ . An analogous construction defines a  $K$ -line  $\exp(i \int_Z b)$  for  $M = Z$  a closed  $(n-2)$ -manifold. Namely, we attach the trivial  $K$ -line to each object of  $\mathcal{F}(Z)$  and to each morphism use a slight generalization of the previous construction to attach a hermitian line to each morphism in  $\mathcal{F}(Z)$ . These lines act on  $K$  by multiplication (tensor product). Then  $\exp(i \int_Z b)$  is the  $K$ -line of invariant sections. There are similar constructions for manifolds with boundary using cycles which represent the relative fundamental class.

Fix a cocycle  $B$  on  $S_{EG}$  as above. To construct the classical action of finite TQFT on a compact oriented manifold  $M$  of dimension  $\leq n$ , we use the category  $\mathcal{C}(M)$  of triples  $(P, \gamma, \phi)$ , where  $P \rightarrow M$  is a principal  $G$ -bundle,  $\gamma : P \rightarrow EG$

<sup>7</sup>The theory of differential cocycles is one context in which “ $K$ -line” and its generalizations acquire a precise mathematical meaning.

is a  $G$ -equivariant map, and  $\phi : M \rightarrow S_P$  is a section. Let  $\gamma_S : S_P \rightarrow S_{EG}$  be the classifying map associated to  $\gamma$ . Then set

$$e^{iS_M(\varphi)} = \exp\left(i \int_M \phi^* \gamma_S^* B\right).$$

As explained in the previous paragraph for  $M$  closed of dimension  $n$  this is a complex number, for  $M$  closed of dimension  $n - 1$  it is a hermitian line, etc.

Appropriate functoriality and gluing laws for the classical action follow from elementary facts about chains and cochains. It follows that (1.10) and the constructions which follow yield an extended TQFT. This follows the physicists' paradigm of the functional integral with two notable exceptions: (i) we have extended the construction down to codimension two, and (ii) the functional integrals here are over a finite measure space, so reduce to finite sums. We now analyze the theory for  $n = 1, 2, 3$ .

**1.5. The One-Dimensional Theory.** In general a one-dimensional TQFT attaches a Hilbert space  $E = E(\text{pt})$  to a point, the identity map to the closed interval, and the complex number  $\dim E$  to a circle. So we need only identify  $E$ . Recall that  $B$  is a cocycle for  $H_G^1(S; \mathbb{R}/2\pi\mathbb{Z}) \cong H_G^2(S; \mathbb{Z})$ . There is a natural equivalence of categories between such cocycles and the category of  $G$ -equivariant hermitian line bundles over  $S$ , so we fix such a hermitian line bundle  $L \rightarrow S$ . Recall from (1.7) that the groupoid of fields on  $\text{pt}$  is equivalent to  $S$  with its  $G$ -action, so it follows that  $E$  is the space of invariant sections of  $L \rightarrow S$ . The  $L^2$  metric is determined by the hermitian metric on  $L$  and the measure on  $S$  which assigns to  $s \in S$  the inverse of the cardinality of the stabilizer subgroup of  $s$ .

In general the finite TQFT splits as a sum of theories over the orbits of the  $G$ -action on  $S$ . For a transitive  $G$ -space  $S = G/H$  the cocycle which defines the theory is a character of  $H$ , and the Hilbert space is zero unless the character is trivial, in which case it is one-dimensional. It is instructive to compute the partition function of  $S^1$  in this case directly. One can identify  $\overline{C}(S^1) \approx H$ , and for the theory defined by the character  $\chi$  (1.10) reduces to

$$Z(S^1) = \sum_{h \in H} \chi(h) = \begin{cases} \#H, & \chi \text{ trivial;} \\ 0, & \text{otherwise.} \end{cases}$$

There is a variation of this theory where in addition to the cocycle  $B$  we also choose a cocycle for an element of  $H_G^0(S; \mathbb{Z}/2\mathbb{Z})$ . This amounts to replacing the line bundle over  $S$  with a  $\mathbb{Z}/2\mathbb{Z}$ -graded line bundle. The resulting quantum theory has a  $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert space. There are corresponding generalizations in  $n = 2, 3$  which we will not pursue here.

**1.6. Twisted  $K$ -Theory.** The theories in  $n = 2, 3$  dimensions involve  $K$ -theory in codimension two, as explained earlier, and the cocycle  $B$  leads to a twisted form of  $K$ -theory. We digress to introduce twisted  $K$ -theory.

Twistings are common in ordinary cohomology. For example, a flat real line bundle over a manifold  $T$  determines a twisting of its real cohomology. It can be easily described in de Rham theory. Similarly, a local system over a space twists its integral cohomology. In algebro-topological terms one can define twistings of any generalized cohomology theory. Briefly, such a theory is defined by a spectrum  $R$ ; the cohomology groups of a space  $T$  are the sets of homotopy classes of maps from  $T$  into loop spaces of  $R$ . A one-dimensional twisting over  $T$  is a cocycle

for  $H^1(T; GL_1(R))$ , and this leads to a fibration of spectra over  $T$  with typical fiber  $R$ . Now if  $R$  is a *ringed* spectrum, i.e., it leads to a *multiplicative* cohomology theory, then  $GL_1(R)$  is the group of units in  $R$ , which is interpreted as a space, more precisely an  $H$ -group. One can deduce the units  $GL_1(K)$  of  $K$ -theory from results in [DK], [S1], [ASe], [AP]. The relevant units for our purposes comprise the Eilenberg-MacLane space  $K(\mathbb{Z}, 2) \sim \mathbb{C}P^\infty$ . Intuitively, if we think of  $K$ -theory as the category of finite-dimensional complex vector spaces,<sup>8</sup> then the subcategory of complex lines—that is, one-dimensional complex vector spaces—is the collection of multiplicative units, where multiplication is by tensor product.<sup>9</sup> These twistings of  $K$ -theory over a space  $T$  are classified up to isomorphism by  $H^3(T; \mathbb{Z})$ .

The twisted  $K$ -theory depends on a choice of *cocycle*  $B$ , not just a cohomology class. We use the notation  $K^{\bullet+B}(T)$  to denote the  $K$ -theory groups of  $T$  twisted by  $B$ . Isomorphisms of cocycles lead to isomorphisms of the corresponding twisted  $K$ -theory groups. Thus,  $H^2(T; \mathbb{Z})$  acts as automorphisms on a twisted  $K$ -theory group  $K^{\bullet+B}(T)$ . More generally,  $K^{\bullet+B}(T)$  is a ( $\mathbb{Z}$ -graded) module over  $K^\bullet(T)$ ; the action of  $H^2(T; \mathbb{Z})$  is via its image in  $K^\bullet(T)$  realizing a degree two cohomology class as a complex line bundle.

The space of all sections of the bundle of spectra over  $T$  determined by  $B$  is a  $K$ -module whose Grothendieck group is twisted  $K$ -theory. For the application to finite TQFT the space  $T$  is a finite set and we will describe such  $K$ -modules in different terms. The  $K$ -theory spectrum has concrete realizations, for example as spaces of Fredholm operators, and this leads to more concrete pictures of twisted  $K$ -theory [A2], [BCMMS].

We encounter twistings of *equivariant*  $K$ -theory as well. Recall that for a compact Lie group  $G$  the equivariant group  $K_G^0(\text{pt})$  is isomorphic to the representation ring of  $G$ ; the odd equivariant  $K$ -theory of a point vanishes. The category of cocycles for the equivariant cohomology group  $H_G^3(\text{pt})$  is equivalent to the category of central extensions  $\tilde{G}$  of  $G$  by the circle group  $\mathbb{T}$ :

$$1 \longrightarrow \mathbb{T} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

The twisted  $K$ -group  $K_G^{0+\tilde{G}}(\text{pt})$  is then the set of equivalence classes of finite-dimensional  $\mathbb{Z}/2\mathbb{Z}$ -graded complex representations of  $\tilde{G}$  on which the center  $\mathbb{T}$  acts by the standard representation. It is a module over  $R(G)$ . A heuristic model for  $K_G$  is the category of all finite-dimensional  $\mathbb{Z}/2\mathbb{Z}$ -graded representations of  $G$ . Then the category of finite-dimensional  $\mathbb{Z}/2\mathbb{Z}$ -graded representations of  $\tilde{G}$  on which the center is standard is a  $K_G$ -module whose Grothendieck group is the twisted equivariant  $K$ -theory.

**1.7. The Two-Dimensional Theory.** The Hilbert space  $E = E(S^1)$  attached to the circle in a two-dimensional TQFT is a commutative associative algebra with unit and inner product such that the trilinear form  $x, y, z \mapsto \langle xy, z \rangle$  is totally symmetric. This structure, called a *Frobenius algebra*, is derived from

<sup>8</sup>As we pointed out earlier, the classifying space of this category is *not* that of  $K$ -theory. Nonetheless, it is useful as a heuristic approximation.

<sup>9</sup>If we pursue the variation mentioned at the end of the previous section, then we encounter another type of twisting classified on a space  $T$  by  $H^1(T; \mathbb{Z}/2\mathbb{Z})$ . The corresponding group of all units—including those which lead to  $K(\mathbb{Z}, 2)$  above—is heuristically the subcategory of  $\mathbb{Z}/2\mathbb{Z}$ -graded lines in the category of  $\mathbb{Z}/2\mathbb{Z}$ -graded complex vector spaces, where we use the latter as a heuristic model for  $K$ . There are other units in  $K$ -theory which have not (yet?) appeared in quantum field theory.

the functional integral over simple surfaces—spheres with disks removed. There are many accounts in the literature; [D] is one of the earliest. For mathematical accounts, see [Sa], [Ab]. If this algebra is semisimple, as it is in the examples we encounter, it can be diagonalized and then there is an explicit formula for the partition function of an oriented surface. It is also true that the field theory is determined by the Frobenius algebra. An extended two-dimensional TQFT, of the type we are considering, has a more refined invariant: the  $K$ -module  $\mathcal{E} = \mathcal{E}(\text{pt})$  attached to a point. Then  $E$  may be computed in terms of  $\mathcal{E}$  as

$$(1.12) \quad E \cong \bar{\mathcal{E}} \otimes \mathbb{C},$$

where  $\bar{\mathcal{E}}$  is the Grothendieck group of  $\mathcal{E}$ . Although (1.12) is a  $\mathbb{C}$ -algebra, there is no natural  $K$ -algebra structure on  $\mathcal{E}$ . In particular, there is not necessarily a ring structure on  $\bar{\mathcal{E}}$  which induces the  $\mathbb{C}$ -algebra structure on  $E$ .

The starting data for finite TQFT in two dimensions is a cocycle  $B$  for  $H_G^2(S; \mathbb{R}/2\pi\mathbb{Z}) \cong H_G^3(S; \mathbb{Z})$ . As explained above this gives a twisting of the  $G$ -equivariant  $K$ -theory of  $S$ . We can interpret  $B$  as providing a  $K$ -line attached to each point of  $S$  together with a lifting of the  $G$ -action, i.e., an  $G$ -equivariant  $K$ -line bundle over  $S$ . Indeed, this must be so to be consistent with (1.7). Note that for  $S = \text{pt}$  the cocycle  $B$  assigns a hermitian line to each  $g \in G$ , since lines are the units in  $K$ , and the action determines a group law on the union of the lines, which is a central extension  $\tilde{G}$  of  $G$  by  $\mathbb{T}$ . The relationship between integral  $H^3$  and  $\mathbb{T}$ -gerbes is explained in many recent works, e.g. [B] and [H]. On the other hand  $\mathbb{T}$ -gerbes are equivalent to  $K$ -lines, just as  $\mathbb{T}$ -torsors are equivalent to (hermitian)  $\mathbb{C}$ -lines.

The  $K$ -module  $\mathcal{E}$  is the space of invariant sections of this  $K$ -line bundle. Recall the notion of “invariant section” explained earlier. At a point of  $S$  the action of the stabilizer group  $H \subset G$  determines a central extension  $\tilde{H}$  of  $H$  by  $\mathbb{T}$ , and the  $K$ -module of invariant sections at that point is the category of representations of  $\tilde{H}$  on which the center acts by scalar multiplication. We can describe an object in  $\mathcal{E}$  more explicitly if, following [F1, §8], we identify the  $K$ -line attached to each point of  $S$  with the trivial  $K$ -line. Then the action of  $G$  on the trivialized  $K$ -line bundle over  $S$  attaches a  $\mathbb{C}$ -line  $L_{(s,g)}$  to every  $s \in S$ ,  $g \in G$ . It acts by tensor product as a morphism from the trivial  $K$ -line at  $s$  to the trivial  $K$ -line at  $g \cdot s$ , and there is a composition law for the action. An object of  $\mathcal{E}$  is then a  $\mathbb{Z}/2\mathbb{Z}$ -graded complex vector space  $W_s$  attached to each  $s \in S$  together with an isomorphism  $L_{(s,g)} \otimes W_s \rightarrow W_{g \cdot s}$  for every  $s \in S$ ,  $g \in G$ . For the theory with  $B = 0$  this specializes to the category of  $G$ -equivariant  $\mathbb{Z}/2\mathbb{Z}$ -graded complex vector bundles over  $S$ .

The Grothendieck group  $\bar{\mathcal{E}}$  is the twisted equivariant  $K$ -theory group  $K_G^{0+B}(S)$ .

The Hilbert space  $E$  attached to  $S^1$  may be computed directly from the description (1.8) of the groupoid of fields on  $S^1$ . Note that the action attaches a hermitian line  $L_{(s,g)}$  to each pair  $(s, g)$  with  $g \cdot s = s$ ; this line is a special case of the hermitian lines considered above, and since  $s$  is fixed by  $g$  it does not depend on a trivialization of the  $K$ -line at  $s$ . The group  $G$  acts on the set  $\mathcal{P}$  of pairs  $(s, g)$  with  $g \cdot s = s$ , and  $E$  is the space of invariant sections of the line bundle  $L \rightarrow \mathcal{P}$ . Note that  $\mathcal{P}$  is a groupoid, and therefore there is a natural product on

$$(1.13) \quad E \cong \left[ \bigoplus_{(s,g): g \cdot s = s} L_{(s,g)} \right]^G.$$

It maps  $L_{(s,g)} \otimes L_{(s',g')}$  to zero unless  $s' = s$ , in which case it lands in  $L_{(s,gg')}$ . The product is not imposed by hand, but rather is computed from the functional integral over the sphere with three holes. (See [F1, §6], for the case  $S = \text{pt}$  and  $B = 0$ . Beware that the counting measure (1.11) figures in the product, which therefore involves rational numbers.)

In case  $S = G/H$  cocycles  $B$  for  $H_G^3(G/H; \mathbb{Z}) \cong H_H^3(\text{pt}; \mathbb{Z})$  are equivalent to central extensions  $\tilde{H}$  of  $H$  by  $\mathbb{T}$ . Then  $\mathcal{E}$  is the category of  $\mathbb{Z}/2\mathbb{Z}$ -graded representations of  $\tilde{H}$  on which the center acts by scalar multiplication, and by (1.12)  $E$  is the complexified Grothendieck group of equivalence classes. The description (1.13) is via characters of representations, which are sections of the complex line bundle over  $H$  associated to  $\tilde{H} \rightarrow H$ . The algebra structure is convolution. It is a semi-simple algebra which, by classical results of Frobenius, is diagonalized by the equivalence classes of irreducible representations. Notice that before complexification there is no product on the Grothendieck group  $\bar{\mathcal{E}}$  of  $\mathcal{E}$  (due to the rational numbers in the counting measure).

The theory with  $S = G$ , where  $G$  acts by conjugation, occurs as the dimensional reduction of the  $n = 3$  finite TQFT considered in the next section. In that case the cocycle for a class in  $H_G^3(G; \mathbb{Z})$  is transgressed from a cocycle for  $H_G^4(\text{pt}; \mathbb{Z})$ , and there is an integral ring which refines the complex Frobenius algebra of the two-dimensional theory.

The theory with  $G = \{1\}$  is a finite version of the perturbative string model. The space  $S$  plays the role of spacetime, and the cocycle  $B$  is the usual  $B$ -field of the theory. The category  $\mathcal{E}$  plays the role of the category of *boundary states* or *D-branes* in two-dimensional conformal field theories.

**1.8. The Three-Dimensional Theory.** The  $K$ -module  $\mathcal{E} = \mathcal{E}(S^1)$  attached to the circle in an extended three-dimensional TQFT has additional structure due to the functional integral over spheres with disks removed. We summarize all of this structure in the assertion that  $\mathcal{E}$  is a *Frobenius algebra over  $K$* .<sup>10</sup> A Frobenius algebra over  $K$  determines a three-dimensional TQFT [T1], much as a Frobenius algebra over  $\mathbb{C}$  determines a two-dimensional TQFT. The Grothendieck group  $\bar{\mathcal{E}}$  of  $\mathcal{E}$  is a ring and is known as the *Verlinde ring* or *Verlinde algebra* of the three-dimensional TQFT. The Frobenius algebra associated to the dimensional reduction of this theory to two dimensions is the complexification of the Verlinde algebra.

The most well-known example of a three-dimensional TQFT is the quantum Chern-Simons theory associated to a compact group  $G$  and a cocycle for a class in  $H_G^4(\text{pt}; \mathbb{Z})$ . It was introduced by Witten [W1] (for connected groups) and subsequently studied by many mathematicians. The theory for finite groups  $G$  was studied in [FQ], [F1], and other places as well. It is exactly the three-dimensional case of the finite TQFT described here for  $S = \text{pt}$ . In fact, the generalization to arbitrary  $S$  does not produce anything new: if  $S = G/H$  we recover Chern-Simons theory on  $H$ . Consider, then, the theory with  $S = \text{pt}$  and  $G$  an arbitrary finite group. An element of  $H_G^4(\text{pt}; \mathbb{Z})$  determines a Chern-Simons functional on principal  $G$ -bundles over closed oriented 3-manifolds, and the choice of a representing cocycle  $B$  leads to the extended classical action described earlier. In particular, there is an equivariant  $K$ -line bundle over the groupoid  $\mathcal{C}(S^1)$  of fields on  $S^1$ . Recall

<sup>10</sup>It usually goes by other names, such as *modular tensor category*. Of course, in a two-dimensional TQFT functional integrals over the same spaces lead to the Frobenius algebra structure on the complex vector space  $E(S^1)$ .

from (1.8) that this is equivalent to a  $G$ -equivariant  $K$ -line bundle on  $G$ , where  $G$  acts on itself by conjugation. It may be considered as a cocycle  $\hat{B}$  for an element of  $H_G^3(G; \mathbb{Z})$ , obtained from  $B$  by a cochain version of the transgression

$$H_G^4(\text{pt}; \mathbb{Z}) \longrightarrow H_G^3(G; \mathbb{Z}).$$

Therefore  $\mathcal{E}$  is the  $K$ -module of invariant sections, and its Grothendieck group, the Verlinde ring, is a twisted equivariant  $K$ -theory group:

$$(1.14) \quad \text{Verlinde}(G, B) \cong K_G^{0+\hat{B}}(G).$$

It was computed explicitly in [F1], but no connection was made there with twisted  $K$ -theory. The main point of that paper was to use the extended notions of classical action and quantum invariants to directly relate Chern-Simons TQFT with quantum groups, at least for finite gauge groups.

The connection (1.14) with twisted  $K$ -theory was realized recently, and it was natural to conjecture that (1.14) holds for all compact Lie groups  $G$  and cocycles  $B$ . There are different precise mathematical definitions of the left hand side of (1.14). One possibility is the free  $\mathbb{Z}$ -module generated by positive energy representations of a central extension (determined by  $B$ ) of the loop group of  $G$  endowed with the fusion product. The proof of (1.14) in this form is ongoing joint work with Michael Hopkins and Constantin Teleman; see [F3] for further motivating remarks and sample computations. Details will appear elsewhere.

## Part 2: Anomalies and Pfaffians of Dirac Operators

The geometric interpretation of the anomaly in the partition function of a chiral spinor field, or chiral Rarita-Schwinger field,<sup>11</sup> was developed in the mid 1980s. First, a link with the Atiyah-Singer index theorem was discovered; see [AS4] for the first mathematical account. Witten's global anomaly formula [W3], together with Quillen's work [Qn] on the determinant of the  $\bar{\partial}$ -operator on Riemann surfaces, inspired the development [BF], [F4] of a geometric structure on the determinant line bundle of a family of Dirac operators. See [C], [Si2] for other interpretations and proofs of Witten's formula. The scope of mathematical and physical investigations into anomalies during that period may be gleaned from the conference proceedings [BW].

In the past 15 years there have been many refinements, extensions, and variations of these ideas. We begin here with a general picture of anomalies, explain why the partition function of a fermionic field is anomalous, and give a formula for the anomaly in terms of  $K$ -theory. The expression of the full anomaly—including its geometric structure—in *differential K-theory* is ongoing joint work with M. Hopkins and I. Singer.

**2.1. Actions in Euclidean QFT.** Consider an  $n$ -dimensional Euclidean quantum field theory formulated on a Riemannian  $n$ -manifold  $X$ . For simplicity we take  $X$  to be compact; if it is not compact we need to impose conditions on the fields at the ends of  $X$ . The space of fields  $\mathcal{C}(X)$  is in general a groupoid, due to possible gauge symmetries. As before, we denote the set of equivalence classes of fields by  $\overline{\mathcal{C}(X)}$ . We treat  $\overline{\mathcal{C}(X)}$  as if it is a smooth manifold. (In general there are singularities due to nontrivial automorphisms in the groupoid, and in a more careful treatment they would be made explicit.)

The main character in a lagrangian field theory is the action  $S_X$ . In some theories only the *exponentiated action*<sup>12</sup>

$$(2.1) \quad e^{-S_X} : \overline{\mathcal{C}(X)} \longrightarrow \mathbb{C}$$

is defined;  $S_X$  is then not a complex-valued function, but rather a function with values in the quotient  $\mathbb{C}/2\pi i\mathbb{Z}$ . This suffices as the *quantum partition function* is defined by the formal expression

$$(2.2) \quad Z(X) = \int_{\overline{\mathcal{C}(X)}} e^{-S_X}(\varphi) \text{“}d\varphi\text{”}.$$

<sup>11</sup>The appendix, joint with Jerry Jenquin, explains some basics about the partition function of the Rarita-Schwinger field. These are necessary for the proper computation of its anomaly. We remark that *nonchiral* spinor and Rarita-Schwinger fields also have (global) anomalies.

<sup>12</sup>The Euclidean action (2.1) differs from (1.9), which models the action in Minkowski spacetime, by a factor of  $i$ . See [F5], Appendix A for a discussion of the Wick rotation which relates them. We have set  $\hbar = 1$ , where  $\hbar$  is Planck's constant.

More generally, Euclidean quantum field theory consists of integrals of the exponentiated action times local functionals on the space of fields, called *correlation functions* of the local functionals. For ordinary quantum field theories  $\overline{\mathcal{C}(X)}$  is infinite-dimensional and much of the mystery of quantum field theory is hidden in the symbol “ $d\varphi$ ”. For our purposes we treat “ $d\varphi$ ” as a measure on  $\overline{\mathcal{C}(X)}$ , so do not discuss the analytic issues hidden in (2.2). Instead, we focus on a variation of (2.1) which leads to a geometric issue.

One cannot always define the exponentiated action as a complex-valued function. Rather, in general it is a product

$$e^{-Sx} = \prod_i e^{-S_X^{(i)}}$$

where

$$e^{-S_X^{(i)}} \text{ is a section of a geometric line bundle } L_X^{(i)} \longrightarrow \overline{\mathcal{C}(X)}.$$

By “geometric line bundle” we understand a complex line bundle endowed with a metric and compatible unitary connection. The geometric line bundles  $L_X^{(i)}$  are part of the data of the classical action. Thus

$$e^{-Sx} \text{ is a section of a geometric line bundle } L_X = \bigotimes_i L_X^{(i)}.$$

The integral in (2.2) no longer makes sense, as we are attempting to sum a function which does not take values in a fixed vector space. To complete the definition of the quantum theory we need to specify

$$1_X = \text{a trivialization of } L_X.$$

The partition function is then

$$Z(X) = \int_{\overline{\mathcal{C}(X)}} \frac{e^{-Sx}}{1_X}(\varphi) \text{ “}d\varphi\text{”},$$

which makes sense since the ratio  $e^{-Sx}/1_X$  is a complex-valued function on  $\overline{\mathcal{C}(X)}$ . We require that  $1_X$  be a geometric trivialization, i.e., it must trivialize both the metric and connection:

$$\begin{aligned} |1_X| &= 1 \\ \nabla 1_X &= 0. \end{aligned}$$

This raises existence and uniqueness questions. The obstruction to finding a trivialization of  $L_X$  is called the *anomaly* of the exponentiated action  $e^{-Sx}$ . Notice that the collection of geometric line bundles over  $X$  forms a groupoid; we denote the set of equivalence classes by  $\check{H}^2(X)$ . (The reason for this notation will emerge at the end of §3.5.) The set of equivalence classes of *topological* line bundles is the integer cohomology group  $H^2(X; \mathbb{Z})$ , and there is a natural forgetful map

$$(2.3) \quad \check{H}^2(X) \longrightarrow H^2(X; \mathbb{Z}).$$

By definition the anomaly is the equivalence class of  $L_X$  in  $\check{H}^2(X)$ . More specifically, there is an anomaly associated to each factor in the exponentiated action:

$$(2.4) \quad \text{Anomaly}(e^{-S_X^{(i)}}) = [L_X^{(i)}] \in \check{H}^2(X).$$



We say the anomaly cancels if the total anomaly—the sum of  $[L_X^{(i)}]$ —is zero. Sometimes there is a *canonical* trivialization of the tensor product of some  $L_X^{(i)}$ .<sup>13</sup>

The equivalence class of a geometric line bundle  $L$  is determined by the holonomy around every smooth loop in  $X$ . The holonomy around a contractible loop is determined from the curvature of  $L$ , and the holonomy around a loop of finite order in  $H_1(X)$  from the curvature and topological class of  $L$ . Therefore, the topological class and/or curvature are often used as approximations to the anomaly (2.4). In the physics literature the curvature is called the *local anomaly* and the holonomy the *global anomaly*.

If the anomaly vanishes, then the set of possible trivializations  $1_X$  is a torsor for  $H^0(\mathcal{C}(X); \mathbb{T})$ , the group of locally constant  $\mathbb{T}$ -valued functions on  $\mathcal{C}(X)$ . (We may divide by the group of constant functions, since an overall phase does not affect the quantum theory.) Just as the exponentiated action  $e^{-Sx}$  is constrained by functoriality and locality, so too is the trivialization  $1_X$ . In particular, there are gluing laws which relate trivializations on different manifolds. (See [W2] for an example in which these constraints play a role.)

**2.2. The Partition Function of a Spinor Field.** In Minkowski spacetime  $\mathbb{M}^n$  a spinor field is a function  $\psi : \mathbb{M}^n \rightarrow S$  with values in a *real*<sup>14</sup> spinor representation of the Lorentz group  $\text{Spin}(1, n-1)$ . The complexification  $S_{\mathbb{C}}$  carries a representation of the complex spin group  $\text{Spin}(n; \mathbb{C})$  whose restriction to the Euclidean spin group  $\text{Spin}(n)$  is in general not real. So in Euclidean field theory it makes no sense to impose a reality condition on the spinor fields. Instead, the reality condition on spinor fields in Minkowski spacetime ensures that the Euclidean partition function—the pfaffian of a Dirac operator—makes sense.<sup>15</sup> The details depend on  $n \pmod{8}$ , so cry out for a unified treatment using the Bott periodicity of the orthogonal group (or real Clifford algebras). We have yet to find a coherent picture along these lines.

Consider a Euclidean field theory on a compact Riemannian  $n$ -manifold  $X$ . We assume that  $X$  is endowed with an orientation and spin structure. Then the choice of a real spin representation  $S$  in Minkowski spacetime corresponds to fixing the rank of a vector bundle  $E$  in the Euclidean theory. We consider spinor fields with coefficients in  $E$ . Either  $E$  is fixed to be the trivial bundle of the specified rank, or  $E$  is variable in which case a connection on  $E$  is a field in the theory. The vector bundle may carry a real ( $\mathbb{R}$ ) or quaternionic ( $\mathbb{H}$ ) structure or it may not be self-conjugate ( $\mathbb{C}$ ). This depends on  $n \pmod{8}$ , as indicated in Table 1. In dimensions  $n = 2$  and  $n = 6$  there are two vector bundles, corresponding to the fact that there are two inequivalent irreducible real spinor representations of the Lorentz groups in those dimensions. Notice that the entries in Table 1 also correspond to the reality conditions of the irreducible complex spinor representation in Lorentz signature [De].

For example, the Lorentz group in dimension  $n = 3$  is isomorphic to  $SL(2; \mathbb{R})$  and up to isomorphism there is one irreducible real spin representation  $S$ , which

<sup>13</sup>An example occurs at the end of the appendix. A less trivial example is the Green-Schwarz anomaly cancellation mechanism, in which case the canonical trivialization follows from the conjectural index theorem in differential  $K$ -theory.

<sup>14</sup>The reality of  $S$  is essentially the requirement of CPT invariance.

<sup>15</sup>The fact that the pfaffian of the complexification of a real matrix equals the pfaffian of the matrix implies that nothing is lost by complexification.

$n \pmod{8}$	vector bundle $E$
0	$\mathbb{C}$
1	$\mathbb{R}$
2	$\mathbb{R} \oplus \mathbb{R}$
3	$\mathbb{R}$
4	$\mathbb{C}$
5	$\mathbb{H}$
6	$\mathbb{H} \oplus \mathbb{H}$
7	$\mathbb{H}$

Table 1: Reality conditions on  $E$ 

has dimension 2. Thus any real spin representation of the Lorentz group has the form  $S \otimes_{\mathbb{R}} E$  for a real vector space  $E$ . Its complexification  $S_{\mathbb{C}} \otimes_{\mathbb{C}} E_{\mathbb{C}}$  is a representation of the Euclidean spin group, which is isomorphic to  $SU(2)$ . As another example, in dimension  $n = 6$  the Lorentz spin group is isomorphic to  $SL(2; \mathbb{H})$ , and there are two inequivalent irreducible 2-dimensional quaternionic spinor representations. The underlying 8-dimensional real representations  $S_1, S_2$  are irreducible, and any irreducible real spinor representation has the form

$$(2.5) \quad S_1 \otimes_{\mathbb{R}} F_1 \oplus S_2 \otimes_{\mathbb{R}} F_2$$

for some real vector spaces  $F_i$ . Now the complexification of  $S_i$  can be written as  $T_i \otimes W$ , where  $T_i$  is a 4-dimensional complex vector space and  $W$  is the quaternions, thought of as a 2-dimensional complex vector space. Note that  $T_i$  carries a representation of the Euclidean spin group, which is isomorphic to  $SU(4)$ . Thus as a representation of the Euclidean spin group we identify the complexification of (2.5) as

$$T_1 \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} (F_1)_{\mathbb{C}} \oplus T_2 \otimes W \otimes_{\mathbb{C}} (F_2)_{\mathbb{C}} \cong T_1 \otimes_{\mathbb{C}} E_1 \oplus T_2 \otimes E_2$$

for quaternionic vector spaces  $E_i$  whose complexification is  $W \otimes_{\mathbb{C}} (F_i)_{\mathbb{C}}$ .

A theory with a spinor field  $\psi$  often involves many other fields  $f$ . We assume that the spinor field enters the action only<sup>16</sup> through a term of the form

$$(2.6) \quad \int_X \frac{1}{2} \psi D_f \psi.$$

The Dirac operator  $D$  depends on the metric on  $X$  and perhaps also a connection on  $E$ . The metric and connection may be included among the fields  $f$ , or more generally depend on  $f$ . A mathematical interpretation of (2.6) involves supermanifolds, since  $\psi$  is a fermionic field, but here we only need know that the formal integral of the exponential of (2.6) over  $\psi$  is

$$(2.7) \quad e^{-S_{\text{fermi}}(f)} = \text{pfaff } D_f.$$

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<sup>16</sup>Higher degree polynomials in  $\psi$  are easily handled as perturbations and do not affect anomalies.

This is a factor in the exponentiated effective action<sup>17</sup> for the fields  $f$ . For fixed  $f$  (2.7) makes sense as an element of a hermitian line

$$(2.8) \quad L_{\text{fermi}}(f) = \text{Pfaff } D_f,$$

and as  $f$  varies these lines fit together into a smooth geometric line bundle  $L_{\text{fermi}}$ , i.e., a hermitian line bundle with unitary connection. The exponentiated effective action (2.7) is a section of  $L_{\text{fermi}}$ .

We recall the formal motivation for (2.7) as the integral of (2.6) over  $\psi$ . Replace  $\psi$  by a one-dimensional real variable  $x$  and the quadratic form (2.6) by the quadratic form  $\frac{1}{2}\lambda x^2$ ; then the integral of the exponentiated action over  $x$  is

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}\lambda x^2} \sim \frac{1}{\sqrt{\lambda}}.$$

In arbitrary finite dimensions  $\lambda$  is replaced by a quadratic form  $Q$  and the right hand side by a multiple of  $1/\sqrt{\det Q}$ . (The measure is used to define the determinant of a quadratic form.) For a fermionic variable the answer is the reciprocal of the ordinary integral—it is the square root of the determinant of  $Q$ , i.e., the pfaffian of  $Q$ .

In *odd* dimensions the pfaffian line bundle  $L_{\text{fermi}} = \text{Pfaff } D$  carries a real structure, and the metric and connection are compatible with it. Hence over a space of fields  $f$  parametrized by a manifold  $T$  the line bundle  $\text{Pfaff } D$  is classified up to isomorphism by an element in  $H^1(T; \mathbb{Z}/2\mathbb{Z})$ . In other words, in odd dimensions the curvature vanishes and the anomaly is topological. We caution that this topological interpretation includes the real structure. The forgetful map which omits the real structure is the Bockstein homomorphism

$$H^1(T; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(Z; \mathbb{Z}),$$

and the Bockstein of  $L_{\text{fermi}}$  equals the image of  $L_{\text{fermi}}$  under (2.3).

In even dimensions  $L_{\text{fermi}}$  is usually geometrical, so its equivalence class in  $\check{H}^2(T)$  is not determined by passing to a topological cohomology group.

**2.3. Construction of Pfaff  $D$ .** As mentioned earlier there are different constructions of the pfaffian line bundle depending on the dimension  $n$  modulo 8. We sketch the cases  $n = 1$  and  $n = 2$  as illustrative examples. A more detailed discussion of the odd dimensional cases appears in [Si1] and [MW, §4.1]. In dimensions  $n \equiv 0, 4 \pmod{8}$  the pfaffian reduces to an ordinary determinant. An extra square root is required for  $n \equiv 2, 6 \pmod{8}$ . These even dimensional cases are described in [F4] and the references therein.

We begin with a finite dimensional analogy. Let  $V$  be a finite dimensional vector space, real or complex,<sup>18</sup> and

$$(2.9) \quad D : /V \longrightarrow V^*$$

a skew-adjoint operator:  $D^* = -D$ . This means that under the isomorphism  $\text{Hom}(V, V^*) \cong V^* \otimes V^*$  the operator  $D$  corresponds to  $\omega_D \in \bigwedge^2 V^*$ . Now the

<sup>17</sup>The field on  $X$  may be written as an infinite dimensional (odd) vector bundle  $\overline{\mathcal{C}(X)} \rightarrow \overline{\mathcal{C}_{\text{bos}}(X)}$ ; the fibers are the fermionic fields, the base the bosonic fields (denoted  $f$  in the text). The effective action is obtained by integration (2.2) over the fibers of this map.

<sup>18</sup>The real case is a model for the Dirac operator in dimensions  $1, 5 \pmod{8}$ ; the complex case for the Dirac operator in dimensions  $2, 6 \pmod{8}$ . Other finite dimensional models are relevant for the remaining dimensions.

pfaffian of  $D$  vanishes if  $D$  is not invertible, and the invertibility of  $D$  requires that  $V$  be even dimensional. Assuming  $\dim V = 2r$ , we define

$$(2.10) \quad \text{pfaff } D = \frac{\omega_D^r}{r!} \in \text{Det } V^*,$$

where the determinant line of  $V^*$  is  $\text{Det } V^* = \bigwedge^{2r} V^*$ . In this finite dimensional case the pfaffian line  $\text{Pfaff } D$  is  $\text{Det } V^*$ , which is independent of  $D$ . The determinant  $\det D : \text{Det } V \rightarrow \text{Det } V^*$  is the map on highest exterior powers induced from (2.9). It is the square of the pfaffian:

$$(\text{pfaff } D)^{\otimes 2} = \det D \in (\text{Det } V^*)^{\otimes 2}.$$

If  $V$  is real, and  $\text{Det } V^*$  is endowed with a metric (i.e.  $V$  has a volume form), then  $(\text{Det } V^*)^{\otimes 2}$  is trivial. In this case the determinant may be identified with a number, but not the pfaffian. Without the metric there is a preferred contractible space of trivializations of  $(\text{Det } V^*)^{\otimes 2}$ , so in a family of operators parametrized by a manifold  $T$  the line bundle  $(\text{Det } V^*)^{\otimes 2} \rightarrow T$  is topologically trivial. The pfaffian line bundle  $\text{Det } V \rightarrow T$  has order two.

The pfaffian and pfaffian line may be constructed for skew-adjoint Fredholm operators  $D$  (see [Qn], [S2]). In this case the pfaffian line depends on  $D$ , not just on the underlying topological vector space. The pfaffian line of a Fredholm operator does not carry a natural metric or connection.

A *geometric family of Dirac operators* parametrized by a manifold  $T$  is given by the following data: a fiber bundle  $\pi : X \rightarrow T$  with finite-dimensional fibers; a metric on the tangent bundle to the fibers of  $\pi$ ; a horizontal distribution on  $\pi$ , i.e., a distribution in  $TX$  complementary to the relative tangent bundle  $T(X/T) \subset TX$ ; an orientation and  $\text{spin}^{19}$  structure on  $T(X/T)$ ; and a vector bundle  $E \rightarrow X$  with connection. In addition we assume that the fibers of  $\pi$  are closed (compact without boundary). Then the Dirac operator has discrete spectrum and the eigenvalues grow in absolute value at a rate depending on the dimension. This leads to a well-defined heat kernel and ultimately to geometric invariants.

Consider first the case where the fibers of  $\pi$  have dimension  $n = 1$ . The Dirac operator  $D$  is a real skew-adjoint Fredholm operator, so there is a general construction of the pfaffian line bundle in a family. However, we need a *geometric* line bundle, so must examine  $D$  in more detail. Relative to appropriate trivializations we may identify  $D$  with the operator  $\frac{d}{dx} + A$ , where  $A$  is a constant skew-adjoint matrix and  $x$  a local coordinate. The spectrum is symmetric about the origin, and if the operator is invertible<sup>20</sup> the absolute value of the pfaffian, which is formally

$$(2.11) \quad |\text{pfaff } D| = \prod_{\substack{\lambda \in \text{spec } D \\ |\lambda| > 0}} |\lambda|,$$

is defined using the heat kernel, or equivalently by analytic continuation of an appropriate  $\zeta$ -function [RS]. But the sign of the pfaffian is not defined. As we move in the parameter space we encounter zeros in the spectrum, and as pairs of eigenvalues pass through zero the sign becomes ill-defined. More precisely, suppose

<sup>19</sup>To construct complex Dirac operators a  $\text{spin}^c$  structure suffices. In most cases we require a  $\text{spin}$  structure to construct the pfaffian, though variations are possible in some situations.

<sup>20</sup>There is a mod 2 index, which is locally constant on  $T$ , which if nonzero guarantees the existence of a zero eigenvalue. In that case the pfaffian is defined to be identically zero. This is analogous to the requirement in finite dimensions that the dimension of the underlying vector space of a skew-adjoint operator be even.

we look along a path in  $T$  which has an isolated zero, say at  $s = 0$  for a parameter  $s$  along the path. Then to leading order  $|\text{pfaff } D_s| \sim cs^k$ , where  $\dim \ker D(0) = 2k$ . A *smooth* choice of a real-valued function  $\text{pfaff } D$  must change sign at  $s = 0$  if  $k$  is odd and must not change sign at  $s = 0$  if  $k$  is even. Thus there is a smooth function  $\text{pfaff } D$  with absolute value (2.11) if and only if for each loop (with appropriate transversality) the total number of pairs of eigenvalues which cross zero is even. The function which counts such zeros is an element of  $H^1(T; \mathbb{Z}/2\mathbb{Z})$ .

A construction which yields a smooth line bundle  $\text{Pfaff } D \rightarrow T$  and a section  $\text{pfaff } D$  proceeds as follows. For each nonnegative real number  $a$  let  $U_a \subset T$  be the subset of parameter values  $t \in T$  for which  $\pm a \notin \text{spec } D_t$ . Over  $U_a$  there is a finite dimensional real vector bundle  $V_a \rightarrow U_a$  whose fiber at  $t \in T$  is the sum of eigenspaces of  $D_t$  for eigenvalues of absolute value less than  $a$ . Then  $D$  restricts to a family of skew-adjoint operators on  $V_a$ , so the finite dimensional construction yields a real line bundle  $L_a \rightarrow U_a$  with a section. There is a natural isomorphism  $L_a \cong L_b$  over  $U_a \cap U_b$ , which is the pfaffian of  $D$  restricted to the sum of eigenspaces for eigenvalues of absolute value between  $a$  and  $b$ , and the sections of  $L_a$  and  $L_b$  agree under this isomorphisms. These isomorphisms patch  $L_a \rightarrow U_a$  into a smooth line bundle  $\text{Pfaff } D \rightarrow T$  and the sections into a smooth section  $\text{pfaff } D$ . One uses the zeta function regularization of (2.11) to construct a metric on  $\text{Pfaff } D$ ; see [Q], [BF], [F4]. A real line bundle with a metric has a unique compatible connection. The topological equivalence class of  $\text{Pfaff } D$  in  $H^1(T; \mathbb{Z}/2\mathbb{Z})$  agrees with that in the previous paragraph.

Now consider the case  $n = 2$ , so a family of compact oriented spin Riemannian surfaces parametrized by  $T$ . Such surfaces have a complex structure and also a square root  $K^{1/2}$  of the canonical bundle. The (chiral) Dirac operator  $D$  is identified with the  $\bar{\partial}$  operator coupled to  $K^{1/2} \otimes E$ :

$$\begin{array}{ccc} D = \bar{\partial}(K^{1/2} \otimes E) : & \Omega^{0,0}(K^{1/2} \otimes E) & \longrightarrow & \Omega^{0,1}(K^{1/2} \otimes E) \\ & \parallel & & \parallel \\ & \Omega^{1/2,0}(E) & \longrightarrow & \Omega^{1/2,1}(E) \end{array}$$

The domain and codomain are naturally dual—the duality pairing is the integral of the product—and by Stokes' theorem the Dirac operator is skew-adjoint. This family of complex skew-adjoint Fredholm operators determines a complex Pfaffian line bundle over  $T$ , but without a metric and connection. For a geometric construction we use a covering  $\{U_a\}_{a>0}$  of  $T$  as above, where now

$$U_a = \{t \in T : a \text{ is not in } \text{spec } D_t^* D_t\}.$$

As before, on  $U_a$  each Dirac operator  $D$  restricts to a skew-adjoint operator on the sum of the eigenspaces of  $D^* D$  with eigenvalue less than  $a$ ; the codomain is the corresponding sum of eigenspaces of  $D^* D$ . These finite dimensional vector spaces are dual if the numerical index of the Dirac operator vanishes; if not, then the pfaffian is identically zero. There is a patching construction of  $\text{Pfaff } D$  with metric and connection. As before we use zeta functions to regularize infinite products and infinite sums. Since  $\text{Pfaff } D$  is complex, the connection does not follow automatically from the metric, and more linear analysis is needed [BF], [F4].

**2.4. Computation of the Topological Anomaly.** Topological invariants of families of Dirac operators were thoroughly investigated by Atiyah and Singer [AS1]. The papers of particular relevance are [AS2] and [AS3]. In even dimensions the

Atiyah-Singer index theorem gives a formula for the topological equivalence class of the line bundle  $L_{\text{fermi}}$  in  $H^2(T; \mathbb{Z})$ . In odd dimensions it gives a formula for the topological equivalence class in  $H^1(T; \mathbb{Z}/2\mathbb{Z})$  of  $L_{\text{fermi}}$  with its real structure. Both are formulas in  $K$ -theory, or perhaps in the more refined  $KO$ - and  $KSp$ -theories.<sup>21</sup> The formula depends on  $n \pmod{8}$ , where  $n$  is the dimension of the fibers of  $\pi : X \rightarrow T$ , though the story for  $i \pmod{8}$  is quite similar to the story for  $i+4 \pmod{8}$ . We simply summarize the formulas here. In each case the formula is expressed in terms of the equivalence class  $[E]$  of the vector bundle  $E \rightarrow X$  in the appropriate  $K$ -theory group and the pushforward  $\pi_!$  in the appropriate  $K$ -theory constructed from the orientation and spin structure on the fibers of  $\pi$ . Recall that  $E$  is real, complex, or quaternionic as indicated in Table 1.

$\mathbf{n} \cong 1 \pmod{8}$ . In this case  $E$  is real, so  $[E] \in KO^0(X)$ . Suppose  $n = 8r + 1$ . The equivalence class of the pfaffian line bundle with its real structure is the image of  $[E]$  under the sequence of maps

$$(2.12) \quad KO^0(X) \xrightarrow{\pi_!} KO^{-(8r+1)}(T) \rightarrow H^1(T; \mathbb{Z}/2\mathbb{Z}).$$

One model for an element of  $KO^{-1}(T)$  is a homotopy class of maps  $T \rightarrow O(\infty)$ , and there is a universal class in  $H^1(O(\infty); \mathbb{Z}/2\mathbb{Z})$  which defines the second map in (2.12).

$\mathbf{n} \cong 2 \pmod{8}$ . In this case there are two real bundles  $E_1, E_2$  over  $X$ , corresponding to the two types of spinor fields in Minkowski spacetime. Then  $[L_{\text{fermi}}]$  is the image of  $[E_1] - [E_2]$  under the sequence of maps

$$(2.13) \quad KO^0(X) \xrightarrow{\pi_!} KO^{-(8r+2)}(T) \rightarrow H^2(T; \mathbb{Z}).$$

This last map is properly called the pfaffian line bundle; it fits into a commutative diagram

$$\begin{array}{ccc} KO^{-2}(T) & \xrightarrow{\text{Pfaaf}} & H^2(T; \mathbb{Z}) \\ \otimes \mathbb{C} \downarrow & & \downarrow \times 2 \\ K^{-2}(T) & \xrightarrow{\text{Det}} & H^2(T; \mathbb{Z}) \end{array}$$

$\mathbf{n} \cong 3 \pmod{8}$ .  $[L_{\text{fermi}}]$  is the image of  $[E]$  under the sequence of maps

$$(2.14) \quad KO^0(X) \xrightarrow{\pi_!} KO^{-(8r+3)}(T) \rightarrow H^1(T; \mathbb{Z}) \rightarrow H^1(T; \mathbb{Z}/2\mathbb{Z}).$$

The second map may be understood analytically as the exponential of  $\pi i/2$  times the  $\eta$ -invariant; of course, there is a topological interpretation as well. The analytic interpretation enters into the definition of pfaff  $D$  (see [Si1]). The last map is simply reduction modulo two.

$\mathbf{n} \cong 4 \pmod{8}$ .  $[L_{\text{fermi}}]$  is the image of  $[E]$  under the sequence of maps

$$(2.15) \quad K^0(X) \xrightarrow{\pi_!} K^{-(8r+4)}(T) \rightarrow H^2(T; \mathbb{Z}).$$

<sup>21</sup>In low dimensions the  $K$ -theory formulas below may be expressed in cohomological terms, starting with a characteristic class of the vector bundle  $E$ . For example, if  $n = 1$  and we restrict to oriented bundles  $E$ , the  $K$ -theory formula reduces to the pushforward of the second Stiefel-Whitney class of  $E$  in cohomology [FW, §5]. (There does not appear to be a cohomology formula if  $E$  is not oriented.) For  $n = 2$  there is a theorem along these lines in [F4, §5] if  $E$  is a virtual bundle of rank zero and is oriented.

In this case  $\text{pfaff } D$  is simply the determinant of the Dirac operator coupled to  $E$ , and (2.15) computes this determinant.

$\mathbf{n} \cong 5 \pmod{8}$ . The next three cases may be related to the first three using Bott periodicity  $KSP^i \cong KO^{i+4}$ . Thus  $[L_{\text{fermi}}]$  is the image of  $[E]$  under the sequence of maps

$$(2.16) \quad KSp^0(X) \xrightarrow{\pi_1} KSp^{-(8r+5)}(T) \rightarrow H^1(T; \mathbb{Z}/2\mathbb{Z}).$$

$\mathbf{n} \cong 6 \pmod{8}$ .  $[L_{\text{fermi}}]$  is the image of  $[E_1] - [E_2]$  under the sequence of maps

$$(2.17) \quad KSp^0(X) \xrightarrow{\pi_1} KSp^{-(8r+6)}(T) \rightarrow H^2(T; \mathbb{Z}).$$

$\mathbf{n} \cong 7 \pmod{8}$ .  $[L_{\text{fermi}}]$  is the image of  $[E]$  under the sequence of maps

$$(2.18) \quad KSp^0(X) \xrightarrow{\pi_1} KSp^{-(8r+7)}(T) \rightarrow H^1(T; \mathbb{Z}) \rightarrow H^1(T; \mathbb{Z}/2\mathbb{Z}).$$

$\mathbf{n} \cong 8 \pmod{8}$ .  $[L_{\text{fermi}}]$  is the image of  $[E]$  under the sequence of maps

$$(2.19) \quad K^0(X) \xrightarrow{\pi_1} K^{-(8r+8)}(T) \rightarrow H^2(T; \mathbb{Z}).$$

**2.5. Computation of the Geometric Anomaly.** As mentioned several times, in odd dimensions the equivalence class of the anomaly  $L_{\text{fermi}}$  is topological, so the Atiyah-Singer theorem suffices. But in even dimensions there is geometric information, and one needs a geometric refinement of the Atiyah-Singer theorem to compute the class of  $L_{\text{fermi}}$  in  $\check{H}^2(T)$ . The class is determined by the holonomy of  $L_{\text{fermi}}$  about all loops in  $T$ . Witten's global anomaly formula [W3] expresses the holonomy in terms of exponentiated  $\eta$ -invariants. See [BF], [F4] for the interpretation (and proof) in terms of determinant and pfaffian line bundles, and [Si2], [C] for other interpretations and proofs. The curvature of  $L_{\text{fermi}}$ , or local anomaly, has a local formula in terms of the curvature of the metric on  $X \rightarrow T$  and the curvature of  $E \rightarrow T$ .

Just as there is a refinement of  $H^2(T; \mathbb{Z})$  to  $\check{H}^2(T)$ , so too there are refinements of topological  $K$ -theory groups  $K^i(T)$  to *differential  $K$ -theory groups*  $\check{K}^i(T)$ . (There are similar refinements for real and quaternionic  $K$ -theory, indeed for any generalized cohomology theory.) An ongoing project with Michael Hopkins and Isadore Singer is designed to interpret invariants of geometric families of Dirac operators as cocycles for differential  $K$ -theory. From this point of view a vector bundle with connection is a cocycle for a differential  $K$ -theory class, and the sequences of maps in the previous section have refinements in the differential version. The theory to be developed will imply that the geometric anomaly (in even dimensions) is computed by the refined sequences of maps.

### Part 3: Abelian Gauge Fields

In §1.2 we met a finite version of gauge fields, namely principal bundles with finite structure group. In this section we discuss principal bundles whose structure group is abelian. Furthermore, they are endowed with a connection, which is usually called the *gauge field* or *gauge potential*. In classical electromagnetism the gauge group is the translation group  $\mathbb{R}$ , and the connection is determined up to isomorphism by its curvature. As we recall this curvature, or *field strength*, encodes the electric and magnetic fields. The entire classical theory may be expressed in terms of the field strength; the gauge field plays only a formal role. On the other hand, in quantum theories the gauge field is essential. Furthermore, the gauge group is  $\mathbb{R}/q\mathbb{Z}$  for an appropriate  $q \in \mathbb{R}$ . This compactification of the gauge group from  $\mathbb{R}$  to  $\mathbb{R}/q\mathbb{Z}$  is equivalent to the quantization of electric charge. Dirac's argument for charge quantization depends on the existence of nonzero magnetic current, which again is a feature of the quantum theory not found in classical electromagnetism. This story can be told with some variations in arbitrary dimensions and with field strengths which are differential forms of arbitrary degree. One particularly illuminating case, in which the geometry is more apparent, is when the field strength has degree one. Then the "gauge field" is a (twisted) map to the circle. In this case we illustrate concretely an anomaly in the electric coupling, which occurs when there is simultaneous magnetic and electric current.

In higher degrees the gauge field is a differential geometric object—a generalized differential cocycle—whose precise nature depends on the quantization law for charges.

In superstring theories it turns out that the appropriate quantization law for electric and magnetic Ramond-Ramond charges, as well as fluxes, is some form of  $K$ -theory. The electric coupling anomaly in these cases is expressed in terms of differential  $K$ -theory. As explained in §2.5 fermion anomalies are also (conjecturally) expressed in differential  $K$ -theory, so can potentially cancel an electric coupling anomaly. Such a cancellation is termed the *Green-Schwarz mechanism*.

Different expositions of this material, and further information about the superstring examples, is contained in [FH] and [F5]. Here we emphasize the elementary pictures which motivate that work.

**3.1. Maxwell's Equations.** We work on a four-dimensional spacetime of the form  $M^4 = \mathbb{E}^1 \times N^3$ , where  $(N^3, g_N)$  is a Riemannian manifold. We endow  $M$  with the Lorentz metric  $dt^2 - g_N$ , where  $t$  is a (time) coordinate on  $\mathbb{E}^1$  and the speed of light is set to unity. Minkowski spacetime is the case  $N = \mathbb{E}^3$ .



Classical electromagnetism involves four fields:

$$\begin{array}{ll} E \in \Omega^1(N) & \text{electric field} \\ B \in \Omega^2(N) & \text{magnetic field} \\ \rho_E \in \Omega_c^3(N) & \text{electric charge density} \\ J_E \in \Omega_c^2(N) & \text{electric current} \end{array}$$

Here  $\Omega_c$  denotes differential forms of compact support. Traditional texts identify  $E, B, J_E$  with vector fields and  $\rho_E$  with a function.<sup>22</sup> But the differential form language is more convenient and leads to a better geometric picture. The classical Maxwell equations are

$$(3.1) \quad \begin{array}{ll} dB = 0 & \frac{\partial B}{\partial t} + dE = 0 \\ d*_N E = \rho_E & *_N \frac{\partial E}{\partial t} - d*_N B = J_E \end{array}$$

We reformulate these equations using differential forms on  $M$  with its Lorentz metric and corresponding Hodge  $*$  operator as follows. Set

$$\begin{array}{ll} F = B - dt \wedge E & \in \Omega^2(M) \\ j_E = \rho_E + dt \wedge J_E & \in \Omega^3(M). \end{array}$$

The *electric current*  $j_E$  has compact spatial support. Maxwell's equations (3.1) are equivalent to the pair of equations

$$(3.2) \quad \begin{array}{ll} dF = 0 \\ d*F = j_E. \end{array}$$

As a consequence of the second equation we have

$$(3.3) \quad dj_E = 0.$$

Equation (3.3) leads to conservation laws through the use of Stokes' theorem.

There is a global condition (3.8) which needs to be added to (3.2) in the classical theory; we discuss it below.

**REMARK 3.4.** *There is an asymmetry in (3.2) which would be corrected if we postulate a magnetic current  $j_B \in \Omega^3(M)$  of compact spatial support and replace the first Maxwell equation with*

$$(3.5) \quad dF = j_B.$$

*In the classical theory there is no magnetic current, but the quantum theory allows for it and, as we explain below, leads to the quantization of both electric and magnetic charge. If (3.5) were admitted in the classical theory, it would obey the symmetry of electromagnetic duality:*

$$\begin{array}{l} F \longleftrightarrow *F \\ j_B \longleftrightarrow j_E. \end{array}$$

*There is a quantum version of this symmetry, but for the classical equations (3.2) it only holds in a vacuum ( $j_E = 0$ ).*

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<sup>22</sup>This assumes that  $M$  is oriented. For simplicity we assume that spacetime  $M$ —and later in the Euclidean version the Riemannian “spacetimes”  $X$ —are oriented. If not, then  $\rho_E, J_E$  are forms twisted by the orientation bundle. The vector field which corresponds to  $B$  is also twisted.

Let  $\iota_t : N \hookrightarrow M$  be the inclusion at time  $t$ . Then (3.3) implies that the de Rham cohomology class

$$\overline{Q}_E = [\iota_t^* j_E] \in H_c^3(N; \mathbb{R})$$

is independent of  $t$ . It is called the total *electric charge*. If  $N$  is connected, then  $H_c^3(N; \mathbb{R}) \cong \mathbb{R}$ , and the total charge is a real number. Notice that the second Maxwell equation (3.2) implies

$$(3.6) \quad \overline{Q}_E \in \ker(H_c^3(N; \mathbb{R}) \longrightarrow H^3(N; \mathbb{R})).$$

In particular, it vanishes if  $N$  is compact.

A typical source for electric current is a collection of charged particles. They may be fixed background objects, too heavy to be affected by the electromagnetic field  $F$ , or may be dynamical objects whose equations of motion are coupled to Maxwell's equations. Let the worldlines of the particles be a submanifold  $i : W^1 \hookrightarrow M$ . We don't assume  $W$  is connected, so allow for several particles, but do assume that for each  $t$  the intersection  $W^1 \cap (\{t\} \times N)$  is compact. Physically, we should also assume that  $W$  is timelike, though this is not essential for the issues at hand. The electric charges are specified by a locally constant function  $q_E : W \rightarrow \mathbb{R}$ , that is  $q_E \in \Omega^0(W)$  with  $dq_E = 0$ . The induced electric current may be written<sup>23</sup>.

$$(3.7) \quad j_E = i_* q_E \in \Omega^3(M).$$

The most straightforward interpretation of (3.7) is as a distributional differential form—a de Rham current—but we prefer to use a smooth representative to avoid illegal products of distributions. Of course, this smoothing involves a choice. In any case the electric charge  $\overline{Q}_E \in H_c^3(N; \mathbb{R})$  is independent of the choice. Notice that the current  $j_E$ , which is a differential-geometric quantity, encodes the positions, velocities, and charges of the individual particles, whereas the charge  $\overline{Q}_E$ , which is a topological quantity, only encodes the total charge.

We can generalize this classical picture easily in two directions. First, we may replace  $N^3$  with a Riemannian manifold  $N^{n-1}$  of arbitrary dimension. Secondly, we may replace the electromagnetic field  $F$  with a differential form of arbitrary degree  $d$ . There are corresponding changes in the degrees of  $j_E$  (and of  $j_B$  in the quantum theory). We can go further and consider a collection of forms  $F = (F_1, \dots, F_k)$  of multidegree  $\mathbf{d} = (d_1, \dots, d_k)$ . But for the moment we continue with a single field of degree  $d = 2$  in  $n = 4$  dimensions.

**3.2. The Action Principle for Electromagnetism.** We treat Maxwell's equations (3.2) asymmetrically to write an action principle. First, we need to add to the first Maxwell equation the condition

$$(3.8) \quad [F] = 0 \in H^2(M; \mathbb{R}).$$

Of course, for  $N = \mathbb{E}^3$  this holds automatically, but for example if  $N = \mathbb{E}^3 \setminus \{0\}$  it is a nontrivial condition. It ensures that all periods of  $F$  around 2-cycles in  $N$  vanish, which physically means that there are no magnetic charges. As we already discussed magnetic charges are excluded in the classical theory. It follows that there exist 1-forms  $A \in \Omega^1(M)$  such that

$$(3.9) \quad F = dA.$$

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<sup>23</sup>Various “twistings”, due to orientations or put in by hand, may be present in the fields; see the end of [F5, §2]

Furthermore, the *gauge field* or *gauge potential*  $A$  is determined up to addition of closed forms. The space of classical fields is then the quotient space

$$(3.10) \quad \mathcal{F}_{\text{classical}} = \frac{\Omega^1(M)}{\Omega_{\text{cl}}^1(M)},$$

where  $\Omega_{\text{cl}}$  denotes closed differential forms. The differential  $d$  maps it isomorphically onto the space of exact 2-forms, in other words to the space of electromagnetic fields  $F$ .

REMARK 3.11. *We can reformulate the gauge field in the language of principal bundles and connections. Namely, take  $A$  to be a connection on a principal bundle over  $M$  whose structure group is the real numbers  $\mathbb{R}$  (under addition). The space of these connections up to equivalence is an affine space based on  $\Omega^1(M)/d\Omega^0(M)$ . The quotient by equivalence classes of flat connections is an affine space based on (3.10). This is the correct space for classical electromagnetism; no “classical experiment” distinguishes between fields which differ by a flat connection. In the quantum theory, however, there are such experiments (the Aharonov-Bohm effect).*

Equation (3.9) ensures that the first Maxwell equation (3.2) is satisfied. The second Maxwell equation is the Euler-Lagrange equation of the action

$$(3.12) \quad S(A) = \int_M -\frac{1}{2}dA \wedge *dA + A \wedge j_E, \quad A \in \Omega^1(M).$$

It is *not* the case that  $S$  is well-defined on the quotient (3.10). However, the integrand (lagrangian) is well-defined up to an exact term. Namely, it follows from (3.6) that  $j_E = dG$  for a global 2-form  $G$ . (We can take  $G = *F$  for any solution  $F$  to the classical equations, but we do not need  $G$  to solve any equations for this argument, which is topological in nature.) Then (3.12) may be rewritten as

$$(3.13) \quad S(A) = \int_M -\frac{1}{2}dA \wedge *dA + dA \wedge G - d(A \wedge G).$$

This shows that  $S$  depends only on  $dA$ , that is, the image of  $A$  in the quotient (3.10), up to an exact term. That is all we require of an action in classical physics, as the Euler-Lagrange equations are unaffected by exact terms.<sup>24</sup> Now a straightforward computation shows that the Euler-Lagrange equation of (3.13) is the second Maxwell equation (3.2).

**3.3. Dirac Charge Quantization.** To write a quantum mechanical theory which incorporates electromagnetism—for example, the nonrelativistic Schrödinger equation for a charged particle moving in a background electromagnetic field—the gauge potential  $A$ , and not just the electromagnetic field  $F = dA$ , appears. This assertion has an experimental basis, due to Aharonov and Bohm. Furthermore, it is an empirical fact that nobody has written a quantum theory in terms of  $F$  alone; see [Fyn, §II-15-5] for a discussion. Accepting the necessity of the gauge potential, the quantization of charge is based on: (i) the existence of a system in which the magnetic current  $j_B$  and electric current  $j_E$  are both nonzero, and (ii) the particular coupling of  $A$  to the electric current in the quantum theory.

<sup>24</sup>The Euler-Lagrange equations are derived by considering *compactly supported* variations  $\dot{A}$  of  $A$ , and for these  $\int_M d(\dot{A} \wedge G) = 0$ .

We continue with standard electromagnetism on  $M^4 = \mathbb{E}^1 \times N^3$ , and assume that  $N = \mathbb{E}^3$  is standard Euclidean space. Suppose there is a static *magnetic monopole* of magnetic charge  $q_B$  at the origin of space. We represent it as a magnetic current

$$j_B = q_B \cdot \delta$$

localized at the origin. Here  $\delta$  is the distributional 3-form on  $M^4$  dual to the 1-manifold  $\mathbb{E}^1 \times \{0\} \subset \mathbb{E}^1 \times \mathbb{E}^3$ . The first Maxwell equation, modified as in (3.5) to incorporate magnetic current, is then

$$(3.14) \quad dF = q_B \cdot \delta.$$

We can no longer write  $F = dA$  globally on  $M$ ; the right-hand side of (3.14) is a local obstruction to writing  $F = dA$ . Although it is eventually important to us to define the gauge field  $A$  on all of spacetime, we first try to define it on subsets. For example, on  $\mathbb{E}^1 \times (\mathbb{E}^3 \setminus \{0\})$  there is no local obstruction, but now there is a global obstruction

$$(3.15) \quad \int_S F = q_B,$$

where, say,  $S$  is the unit 2-sphere in  $\{t\} \times (\mathbb{E}^3 \setminus \{0\})$  for any  $t$ . Dirac eliminates this global obstruction by introducing a ray  $R$  emanating from the origin  $0 \in \mathbb{E}^3$ , a so-called ‘‘Dirac string’’. (A physical model is a semi-infinite solenoid.) Then  $A$  can be defined on  $\mathbb{E}^1 \times (\mathbb{E}^3 \setminus R)$  and Dirac’s argument is based on the premise that the string be invisible. In essence, it is based on the global obstruction (3.15) due to  $H^2(\mathbb{E}^3 \setminus \{0\}) \neq 0$ . However one phrases the argument, there is a tension between the necessity of a local gauge potential  $A$  in quantum theory and the global topological obstruction to its existence as a 1-form.

Differential geometry provides a well-known relief of this tension. We emphasize, however, that there is a choice<sup>25</sup>, and in other theories different differential-geometric constructions may enter at this point. Namely, we take  $F$  to be the curvature of a connection  $A$  on a principal  $\mathbb{R}/q_B\mathbb{Z}$ -bundle  $\pi : P \rightarrow \mathbb{E}^1 \times (\mathbb{E}^3 \setminus \{0\})$ . Thus,  $A$  is a right-invariant form on  $P$  which satisfies  $dA = \pi^*F$ . The global condition (3.15) implies that  $P$  represents the generator of  $H^2(\mathbb{E}^1 \times (\mathbb{E}^3 \setminus \{0\}))$ . The classical space of fields (3.10) is now replaced by the quantum groupoid of fields

$$\mathcal{F}_{\text{quantum}} = \text{category of principal } \mathbb{R}/q_B\mathbb{Z}\text{-bundles with connection.}$$

Morphisms are connection-preserving bundle isomorphisms. The space of equivalence classes  $\overline{\mathcal{F}_{\text{quantum}}}$  fits into an exact sequence

$$0 \longrightarrow H^1(\cdot; \mathbb{R}/q_B\mathbb{Z}) \longrightarrow \overline{\mathcal{F}_{\text{quantum}}} \longrightarrow \mathcal{F}_{\text{classical}} \longrightarrow 0.$$

On the spacetime  $M = \mathbb{E}^1 \times (\mathbb{E}^3 \setminus \{0\})$  we have  $\overline{\mathcal{F}_{\text{quantum}}} = \mathcal{F}_{\text{classical}}$ , since the first cohomology vanishes, but in general  $\overline{\mathcal{F}_{\text{quantum}}}$  is an extension of  $\mathcal{F}_{\text{classical}}$  by the equivalence classes of flat connections. This fits the fact that flat connections can be detected in quantum mechanics, but not in classical physics.

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<sup>25</sup>The choice is based partly on the form of the electric coupling in the theory, and partly on other physical considerations (anomaly cancellation, for example). See §3.7 for a discussion about this choice for Ramond-Ramond charges in superstring theory.

The Dirac quantization law follows from the requirement that the exponentiated *electric coupling*

$$(3.16) \quad \exp\left(\frac{i}{\hbar} \int_M A \wedge j_E\right)$$

be well-defined. Here  $\hbar$  is Planck's constant, which appears in any quantum theory. This expression is a factor in the exponentiated action which enters the functional integral formulation of quantum mechanics and quantum field theory. The same expression appears in the exponentiated Euclidean action, except that  $M = \mathbb{E}^1 \times N$  is replaced by an arbitrary Riemannian 4-manifold  $X$ . In the Euclidean framework, to which we now turn,  $A$  is a connection in an  $\mathbb{R}/q_B\mathbb{Z}$ -bundle  $P \rightarrow X$ . For now we do not consider general electric currents  $j_E$ , but rather the current associated to a "Wilson loop" of electric charge  $q_E$ . That is, suppose as in (3.7) that  $j_E$  is  $q_E$  times the Poincaré dual to an oriented 1-manifold  $W^1 \subset X$ , which we now assume is closed. Then (3.16) reduces to

$$(3.17) \quad \exp\left(\frac{i}{\hbar} q_E \int_W A\right).$$

We interpret the integral as the holonomy of  $A$  around  $W$ , which takes values in  $\mathbb{R}/q_B\mathbb{Z}$ . Thus (3.17) is well-defined if

$$(3.18) \quad \frac{q_E q_B}{\hbar} \in 2\pi\mathbb{Z}.$$

Equation (3.18) is Dirac's quantization law.

The same conclusion may be derived in the nonrelativistic quantum mechanics of a particle of charge  $q_E$  on  $N = \mathbb{E}^3 \setminus \{0\}$ . The wave function of the charged particle is a section of the line bundle associated to the representation

$$\begin{aligned} \mathbb{R}/q_B\mathbb{Z} &\longrightarrow \mathbb{T} \\ x &\longmapsto \exp\left(\frac{i}{\hbar} q_E x\right), \end{aligned}$$

which is well-defined only if (3.18) is satisfied.

In either argument we see directly that the compactness of the gauge group immediately implies the quantization of charge.

Summarizing, the argument for charge quantization appears in two stages. First, a nonzero magnetic current  $j_B$  alters the global nature of the gauge potential  $A$ . Second, the electric coupling (3.17) forces a quantization law on the electric current  $j_E$ , and so on the electric charge. Observe that the global nature of  $A$  depends on the magnetic current  $j_B$ , a result of which is that the quantization law involves the product of magnetic and electric charges.

**3.4. The Electric Coupling Anomaly.** We begin to move away from the Maxwell theory toward more general abelian gauge fields. From now on we work in the Euclidean framework, so over an oriented Riemannian manifold  $X$ . (In fact, the Riemannian metric is irrelevant in this discussion, but still it is better viewed in the framework of Euclidean quantum field theory.) We assume  $X$  is compact to avoid convergence issues. Let  $X$  have arbitrary dimension  $n \geq 2$ . The geometry of the anomaly is clearest for a single field strength  $F$  of degree  $d = 1$ . Classically,

then,  $F$  is an exact 1-form on  $X$  and the classical gauge field

$$A_{\text{classical}} \in \frac{\Omega^0(X)}{\Omega_{\text{cl}}^0(X)}$$

is a function up to locally constant functions.

In the quantum theory we admit a nonzero magnetic current  $j_B$ , which is a closed 2-form on  $X$ . The case  $\dim X = 2$  is easiest to visualize. Then let  $i_B : W_B \hookrightarrow X$  to be the inclusion of a finite set of oriented points, and  $q_B : W_B \rightarrow \mathbb{R}$  the magnetic charges of these points. The induced magnetic current  $j_B = (i_B)_*(q_B)$  is  $q_B$  times a smooth Poincaré dual form to  $W_B$ . Normalize<sup>26</sup> the magnetic charges so that they lie in  $2\pi\mathbb{Z}$ :

$$(3.19) \quad q_B : W_B \longrightarrow 2\pi\mathbb{Z};$$

then  $j_B/2\pi$  has integral periods. Represent the magnetic current  $j_B$  geometrically<sup>27</sup> by a principal  $\mathbb{R}/2\pi\mathbb{Z}$ -bundle  $\mathcal{J}_B \rightarrow X$  with connection whose curvature is  $j_B$ . This is a refinement of the differential form current  $j_B$ , as  $j_B$  does *not* determine  $\mathcal{J}_B$  up to isomorphism if  $H^1(X; \mathbb{R}/2\pi\mathbb{Z}) \neq 0$ .

In the previous section we defined the gauge potential  $A$  outside the support of  $j_B$ , and here we begin similarly. Set  $X^0 = X \setminus \text{supp } j_B$ . Now the quantization condition (3.19) leads to the fact that  $[j_B/2\pi] \in H^2(X; \mathbb{R})$  is in the image of integer cohomology. It is natural to assume that the restriction of  $F$  to  $X^0$ , which is closed, also has a cohomology class which is in the image of integer cohomology. Then there exists a map

$$(3.20) \quad A^0 : X^0 \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$$

whose differential is  $F$  restricted to  $X^0$  and whose winding number about  $W_B$  is given by  $q_B/2\pi$ . (These conditions are related by Stokes' theorem.)

Now we go further and define a gauge potential  $A$  on all of  $X$ . Namely, we define a gauge potential to be a section

$$(3.21) \quad A : X \longrightarrow \mathcal{J}_B$$

of  $\mathcal{J}_B$ . In other words, *the gauge field is a trivialization of the magnetic current*. This is a geometric version of (3.5). In this picture the gauge field (3.21) is not a map to a fixed circle, as in (3.20), but rather a map to a variable circle defined by the magnetic current. The existence of  $A$  forces  $\mathcal{J}_B$  to be globally trivial, just as (3.20) forces  $[j_B]$  to vanish in absolute cohomology. Furthermore, if we trivialize  $\mathcal{J}_B$  on  $X^0$ , or even on  $X \setminus W_B$ , then the ratio of  $A$  to the trivialization is the gauge field (3.20).

We emphasize that it is important that the geometric magnetic current  $\mathcal{J}_B$  be a particular bundle with connection, not a class of bundles up to isomorphism. Otherwise, we could not talk about trivializations. Further, once the gauge field is defined as a trivialization of the magnetic current, then its geometric nature—in particular its integrality—is tied to that of the magnetic current. We remark that this story goes through for  $n = \dim X$  arbitrary, except that  $\dim W_B = n - 2$  in general.

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<sup>26</sup>There is a nontrivial restriction on the values of  $q_B$ , namely that they be integer multiples of a nonzero real number. If not, we would not be able to define the gauge potential, at least in this framework.

<sup>27</sup>This is a differential-geometric version of the construction of a holomorphic line bundle on a complex curve from a divisor.

Suppose now we have electric charges described as the inclusion  $i_E : W_E \hookrightarrow X$  of a finite set of oriented points. The quantization law (3.18) asserts that the electric charge is  $\hbar$  times a function

$$(3.22) \quad q_E : W_E \longrightarrow \mathbb{Z}.$$

The exponentiated electric coupling (3.17) takes the form

$$(3.23) \quad \prod_{w \in W_E} \exp(iA)^{q_E(w)}.$$

If  $W_E \cap W_B = \emptyset$  we can use the trivialization to replace  $A$  by the  $\mathbb{R}/2\pi\mathbb{Z}$ -valued function  $A^0$ , defined on the complement of  $W_B$ , and so define (3.23) as an element of  $\mathbb{T} \subset \mathbb{C}$ . Without using the trivialization, we regard  $\exp(iA)$  as an element of the fiber of the hermitian line bundle  $L_B \rightarrow X$  associated to  $\mathcal{J}_B \rightarrow X$  via the character  $x \mapsto \exp(ix)$  of  $\mathbb{R}/2\pi\mathbb{Z}$ . Then (3.22) is an element of

$$(3.24) \quad \bigotimes_{w \in W_E} (L_B)^{\otimes q_E(w)}.$$

We can imagine a theory in which the electric and magnetic sources  $W_E$  and  $W_B$  vary, perhaps as functions of other fields. Then (3.24) determines a hermitian line bundle with connection over any parameter space  $T$  of fields. As explained in §2.1, this line bundle is the anomaly in the exponentiated electric coupling. Note from (3.24) that it is in some sense the integral of the product of the magnetic and electric currents, but where the currents are refined to differential-geometric objects  $\check{j}_B$  and  $\check{j}_E$ . We write the anomaly in a family  $X \rightarrow T$  as

$$(3.25) \quad L_{\text{electric}} = \exp \left( 2\pi i \int_{X/T} \check{j}_B \cdot \check{j}_E \right).$$

In this form it generalizes to more complicated situations.

**3.5. Generalized Differential Cocycles.** We return briefly to the Hamiltonian formalism, so to a spacetime  $M^n = \mathbb{E}^1 \times N^{n-1}$ . Consider now a general abelian field strength  $F \in \Omega^{\mathbf{d}}(M)$  of multi-degree  $\mathbf{d} = (d_1, \dots, d_k)$ . Then the currents  $j_B \in \Omega^{\mathbf{d}+1}(M)$  and  $j_E \in \Omega^{n-\mathbf{d}+1}(M)$  have corresponding charges  $\bar{Q}_B \in H_c^{\mathbf{d}+1}(N; \mathbb{R})$  and  $\bar{Q}_E \in H_c^{n-\mathbf{d}+1}(N; \mathbb{R})$ . We generalize Dirac's quantization law (3.18) to postulate full lattices  $\bar{\Gamma}^\bullet$  in real cohomology so that the charges  $\bar{Q}_B, \bar{Q}_E$  take values in  $\bar{\Gamma}^\bullet(N)$ . The argument for this follows the one presented in §3.3 for 2-form field strengths. Namely, the gauge field can only be defined in  $\bar{Q}_B$  is "quantized", and then the electric coupling forces  $\bar{Q}_E$  to be quantized as well. In fact, the magnetic current and gauge field should be quantized by one lattice, and the electric current by a lattice which is complementary in the sense that there is a duality pairing which generalizes (3.18). However, to keep things simple we use the same lattice and postulate a multiplication which implements the analog of (3.18).

The most natural lattice which comes to mind is

$$\bar{\Gamma}^\bullet(N) = \text{image}(H^\bullet(N; \mathbb{Z}) \longrightarrow H^\bullet(N; \mathbb{R})),$$

but this is not the only possible choice. Indeed, any generalized cohomology theory defines a full lattice in real cohomology.

REMARK 3.26. *As an example, consider complex  $K$ -theory. Recall that the Chern character is a ring homomorphism*

$$\text{ch} : K^0(X) \longrightarrow H^{\text{even}}(X; \mathbb{R}).$$

*The kernel is the torsion subgroup, and the image is a full lattice isomorphic to  $K^0(X)/\text{torsion}$ . A more canonical version maps to real cohomology with coefficients in  $K^\bullet(\text{pt}) \otimes \mathbb{R} \cong \mathbb{R}[[u, u^{-1}]]$ , where  $\deg u = 2$ :*

$$(3.27) \quad \text{ch} : K^\bullet(X) \longrightarrow H(X; \mathbb{R}[[u, u^{-1}]])^\bullet.$$

*This is a homomorphism of  $\mathbb{Z}$ -graded rings.*

It is natural, now, to refine the charges  $\overline{Q}_B, \overline{Q}_E \in \overline{\Gamma}_c^\bullet(N)$  to charges  $Q_B, Q_E \in \Gamma_c^\bullet(N)$  in the abelian groups, rather than in the groups mod torsion. This allows for torsion charges, and in any case  $\Gamma(\cdot)$  is better behaved than  $\Gamma(\cdot)/\text{torsion}$ .

REMARK 3.28. *In all examples of which we are aware the lattice  $\overline{\Gamma}^\bullet$  in real cohomology is defined by a generalized cohomology theory, which furthermore is multiplicative. Locality in quantum field theory suggests that  $\Gamma(X)$  should depend locally on  $X$ , and the Mayer-Vietoris property of generalized cohomology theories is certainly an expression of locality. However, it seems possible that  $\Gamma^\bullet$  may be defined as something other than a generalized cohomology theory.*

Switch to the Euclidean setting and drop the support condition. The refinement  $\check{j}$  of electric and magnetic currents that we seek must encode both the integral charge  $Q \in \Gamma^\bullet(X)$  and the local information of the differential form  $j \in \Omega_{\text{cl}}^\bullet(X)$ . A first guess is the fiber product<sup>28</sup>

$$\begin{array}{ccc} A_\Gamma^\bullet(X) & \longrightarrow & \Omega_{\text{cl}}^\bullet(X) \\ \downarrow & & \downarrow \\ \Gamma^\bullet(X) & \longrightarrow & H^\bullet(X; \mathbb{R}) \end{array}$$

Note that  $\overline{Q}$  is the common image of  $Q$  and  $j$  in  $H^\bullet(X; \mathbb{R})$ . But the naive fiber product of abelian groups is not sufficient here. Rather, we need to take the fiber product in the sense of cohomology theories. This gives a pullback diagram

$$(3.29) \quad \begin{array}{ccc} \check{\Gamma}^\bullet(X) & \longrightarrow & \Omega_{\text{cl}}^\bullet(X) \\ \downarrow & & \downarrow \\ \Gamma^\bullet(X) & \longrightarrow & H^\bullet(X; \mathbb{R}) \end{array}$$

in which the upper left-hand entry is the *generalized differential cohomology*. A generalized differential  $q$ -cocycle is a triple  $(c, h, \omega)$ , where  $c$  is a cocycle for  $\Gamma^q$ ,  $\omega$  is a closed  $q$ -form, and  $h$  is a  $(q-1)$ -cocycle for real singular cohomology whose differential is  $\omega - c$ . In other words,  $h$  is a homotopy from  $c$  to  $\omega$ . There are many possible cocycle models possible for topological (generalized) cohomology, and they lead to different cocycle models for the differential theory. For a heuristic exposition see [F5, §1]; for a detailed development see [HS].

Briefly, then, charge quantization is implemented by a choice of generalized cohomology theory. The electric and magnetic currents  $j_E, j_B$  are lifted to generalized differential cocycles  $\check{j}_E, \check{j}_B$ . The gauge field  $\check{A}$  is a trivialization of  $\check{j}_B$ . The electric coupling is written using the product in differential cohomology, and it has

<sup>28</sup>The coefficients should be  $\Gamma^\bullet(\text{pt}) \otimes \mathbb{R}$ , not simply  $\mathbb{R}$ , but for simplicity we do not incorporate that into the notation here.



an anomaly computed by (3.25). This picture of abelian gauge fields is explained in detail in [F5, §2].

**3.6. Self-Dual Fields.** Let  $\mathbb{M}^n$  denote  $n$ -dimensional Minkowski spacetime, an affine space whose underlying vector space  $V$  carries a metric of signature  $(1, n-1)$ . The *Poincaré group* is the double cover of the connected component of affine isometries of  $\mathbb{M}^n$ . The corresponding double cover group of the connected component of linear isometries of  $V$  is the *Lorentz group*. In relativistic quantum mechanics a *particle* is an irreducible unitary representation of the Poincaré group. We restrict to massless representations, which may be described as certain function spaces on the positive nullcone  $N^+$  in the dual space  $V^*$ . The Lorentz group acts on  $N^+$ , transitively if  $n \geq 3$ , and the reductive part of the stabilizer—the *massless little group*—is isomorphic to the compact group  $\text{Spin}(n-2)$ . Massless particles correspond to irreducible complex representations of  $\text{Spin}(n-2)$ .

Our interest is in the exterior power representations  $\bigwedge^p$ . They factor through  $SO(n-2)$ , so correspond to *bosons*. The Hodge  $*$  operator gives an isomorphism

$$(3.30) \quad \bigwedge^p \cong \bigwedge^{n-2-p},$$

so only the representations for  $p = 1, \dots, \lfloor \frac{n-2}{2} \rfloor$  are distinct. If  $n = 4\ell + 2$ ,  $n \geq 6$ , the representation  $\bigwedge^{2\ell}$  splits into the sum of two irreducible representations

$$(3.31) \quad \bigwedge^{2\ell} \cong \bigwedge_+^{2\ell} \oplus \bigwedge_-^{2\ell}$$

In quantum field theory there is a restriction on the total particle representation—the CPT theorem—which for  $n$  even asserts that the representation of the little group must carry a real structure, which is why we only consider the decomposition (3.31) for  $n \equiv 2 \pmod{4}$ . The irreducible representation  $\bigwedge_+^{2\ell}$  gives the *self-dual* particle;  $\bigwedge_-^{2\ell}$  the *anti-self-dual* particle. For  $n = 2$  the representation  $\bigwedge^0$  is of course irreducible, but the positive nullcone  $N^+$  breaks into two orbits  $N^+ = N_L^+ \cup N_R^+$  of the Lorentz group, and the massless representation of Poincaré corresponding to  $\bigwedge^0$  breaks up as a sum of two irreducible representations, one supported on  $N_L^+$  and one on  $N_R^+$ .

Free field theories on  $\mathbb{M}^n$  lead to particles by quantization: the space of solutions to the Poincaré-invariant free field equations is a real symplectic vector space with a symplectic action of Poincaré. Its quantization is the Hilbert space completion of the symmetric algebra of a unitary representation of Poincaré. This representation is the *particle content* of the theory. For example, the classical Maxwell equations (3.2) with  $j_E = 0$  lead to the massless irreducible representation  $\bigwedge^1$ . (See [F6, Lecture 3], for one account.) More generally, we may consider a *real*  $d$ -form field strength  $F$  on  $\mathbb{M}^n$  which satisfies the free field equations

$$(3.32) \quad dF = d * F = 0.$$

This free system has a lagrangian description in terms of a gauge potential  $A$  up to gauge transformations; the relevant quotient space of fields is (3.10) with 1-forms replaced by  $(d-1)$ -forms. The particle content of the corresponding quantum system is the massless representation  $\bigwedge^{d-1}$ . The symmetry (3.30) of particles corresponds to the *electromagnetic duality* symmetry

$$F \longleftrightarrow *F$$

of the classical field equations (3.32)—see Remark 3.4.

Specialize to  $n = 4\ell + 2$  and  $d = 2\ell + 1$ . Then the Hodge  $*$  fixes the space  $\Omega^{2\ell+1}(\mathbb{M})$  of  $d$ -forms on  $\mathbb{M}^n$ , and since  $*^2 = 1$  we have a decomposition

$$\Omega^{2\ell+1}(\mathbb{M}^n) \cong \Omega_+^{2\ell+1}(\mathbb{M}^n) \oplus \Omega_-^{2\ell+1}(\mathbb{M}^n)$$

This induces a decomposition of the space of solutions to (3.32) into a direct sum of two symplectic subspaces, each invariant under Poincaré, so each giving a free relativistic classical mechanical system in its own right. In fact, the quantization of these systems give the irreducible massless representations  $\bigwedge_{\pm}^{2\ell}$  of Poincaré. (For  $n = 2$  we obtain the representations supported on  $N_L^+, N_R^+$ .) In other words, the classical systems which correspond to (anti-)self-dual particles are the  $(2\ell + 1)$ -forms which satisfy (3.32) and

$$(3.33) \quad F = \pm * F.$$

In §3.2 we explained how the field equations (3.32) arise from an action principle on gauge potentials. Unfortunately, there is no straightforward Poincaré-invariant lagrangian theory which also gives the (anti-)self-duality equation (3.33). This is not a handicap for the classical theory—the field equations contain all the information—but leads to an extra complication in the *Euclidean* version of the quantum theory. Namely, to define Euclidean correlation functions on arbitrary Riemannian manifolds, there is an extra piece of data which is needed [W5]. It is best expressed as a quadratic refinement of the bilinear form (3.25) on electric and magnetic currents. Dirac quantization enters as for any gauge field, so the quadratic form is best described in the language of generalized differential cocycles. Note that for (anti-)self-dual fields the magnetic current determines the electric current, and the Euclidean partition function on  $X$  depends on this single current. A precise geometric formula for this partition function and its anomaly is still lacking in general.

**3.7. Ramond-Ramond Charge and  $K$ -Theory.** The physically correct choice of a generalized cohomology theory  $\Gamma$  for the quantization of charge is motivated by many different considerations in the quantum theory. We illustrate this in Type II superstring theory, where integer cohomology quantizes Neveu-Schwarz (NS) charge and complex  $K$ -theory quantizes Ramond-Ramond (RR) charge. Variations of these arguments occur in other superstring theories.

We work with the field theory approximation to the low energy limit of Type II superstring theory. The first step is the quantization of the free superstring [d'H], §7.8 on  $\mathbb{M}^n$ . The massless particles which occur include exterior power representations  $\bigwedge^p$  of the little group  $\text{Spin}(8)$ . As explained in §3.6 they are obtained by quantizing differential form fields. Specifically, there is a 3-form NS field strength and there are RR field strengths of various degrees. In Type IIA these have degrees 2 and 4; in Type IIB the degrees are 1, 3, and  $5_+$ , where the subscript indicates that the field is self-dual. (The self-dual field corresponds to the representation  $\bigwedge_+^4$  of  $\text{Spin}(8)$ .) We need to specify the appropriate Dirac quantization condition for these gauge fields. We work in the Euclidean theory, so on a Riemannian 10-manifold  $X$ .

We begin with the NS 2-form gauge potential, which we denote  $B$ . It turns out that the correct quantization condition is integer cohomology. In other words,

in the absence of magnetic current  $B$  is lifted to a cocycle  $\check{B}$  for the ordinary differential cohomology group  $\check{H}^3(X)$ . There are a few justifications for using integer cohomology here. First, the  $B$ -field has an electric coupling of the form (3.17). The fundamental string is represented by a map  $\phi : \Sigma \rightarrow X$  of a closed oriented surface  $\Sigma$  to  $X$ , and the exponentiated electric coupling is

$$(3.34) \quad \exp\left(\frac{i}{2\pi\alpha'\hbar} \int_{\Sigma} \phi^* B\right),$$

where  $\alpha'$  is the *Regge slope*. Comparison with (3.17) shows that the electric NS charge of the fundamental string is  $q_E = 1/(2\pi\alpha')$ . Now  $\Sigma$  carries a fundamental class in integer cohomology, and this allows us to lift (3.34) to an integral of an ordinary differential cocycle  $\check{B}$ :

$$(3.35) \quad \exp\left(\frac{i}{2\pi\alpha'\hbar} \int_{\Sigma} \phi^* \check{B}\right).$$

This simple form of the electric coupling is the first rationale for using integer cohomology to quantize the NS charge; it is not clear how to write (3.35) with other differential cohomology theories. This electric coupling occurs in the worldsheet theory, where  $\check{B}$  is a background field and  $\phi$  a dynamical field. In the strong coupling limit of Type II superstring theory there can be a fixed background source  $\phi : W \hookrightarrow X$  for the dynamical field  $\check{B}$ , which is more in keeping with the context of (3.17). In that situation the dual magnetically charged object is the NS 5-brane, whose magnetic charge is an integer multiple of  $(2\pi)^2\alpha'\hbar$ , in accordance with (3.18). This quantization of charge is consistent with the form of classical 5-brane solutions to supergravity [CHS, §6], [P1, §§14.1,14.4], and provides further confirmation that integer cohomology is the correct choice of quantization law for the  $B$ -field.<sup>29</sup> Another piece of evidence that integer cohomology is correct involves twistings of RR-fields, as explained below.

We turn now to the RR gauge fields, where the correct quantization is provided by complex  $K$ -theory if the NS  $\check{B}$ -field vanishes. Before reviewing physical arguments for this, we interpret the statement mathematically in terms of differential  $K$ -theory. (Compare [MW, (2.17)].) The first step is to introduce electromagnetic duals to the minimal set of differential form fields. So we postulate inhomogeneous RR field strengths

$$G' = \begin{cases} G_2 + G_4 + G_6 + G_8, & \text{Type IIA;} \\ G_1 + G_3 + G_5 + G_7 + G_9, & \text{Type IIB,} \end{cases}$$

where  $G_d$  is a differential form of degree  $d$ . In the classical theory on  $\mathbb{M}^{10}$  we impose the self-duality equation  $G' = *G'$ , but in the Euclidean quantum theory on a Riemannian 10-manifold the self-duality condition is manifest by using a certain quadratic form in the definition of the partition function (see §3.6). It is convenient to redefine the RR field strength to be homogeneous. Recalling from Remark 3.26

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<sup>29</sup>Contrast to the D-branes which couple to the RR gauge fields. In that case the charge is represented by a complex vector bundle on the D-brane, not by an integer, and this bundle appears in the electric coupling. The low energy fluctuations of the NS 5-brane include a gauge field living on the brane, but it is not involved in the electric coupling.

the inverse Bott element  $u \in K^2(pt)$ , we set

$$G = \begin{cases} u^{-1}G_2 + u^{-2}G_4 + u^{-3}G_6 + u^{-4}G_8, & \text{Type IIA;} \\ u^{-1}G_1 + u^{-2}G_3 + u^{-3}G_5 + u^{-4}G_7 + u^{-5}G_9, & \text{Type IIB,} \end{cases}$$

Then  $\deg G = 0$  in IIA and  $\deg G = -1$  in IIB. In the absence of current, then, the refined gauge field  $\check{C}$  is a cocycle for an element of  $\check{K}^\bullet(X)$ , where  $\bullet = 0$  in IIA and  $\bullet = -1$  in IIB. Note that the defining pullback square (3.29) for differential  $K$ -theory is

$$\begin{array}{ccc} \check{K}^\bullet(X) & \longrightarrow & \Omega_{\text{cl}}(X; \mathbb{R}[[u, u^{-1}]])^\bullet \\ \downarrow & & \downarrow \\ K^\bullet(X) & \longrightarrow & H(X; \mathbb{R}[[u, u^{-1}]])^\bullet \end{array}$$

The bottom map is the Chern character (3.27). The current  $\check{j}$  is a cocycle of one degree higher than  $\check{C}$ ; if it is nonzero, then  $\check{C}$  is a trivialization of  $\check{j}$ .

One indication that  $K$ -theory is the correct cohomology theory to quantize RR charge is the form of RR-charged objects. These are *D-branes*; see [P2], [P1, §13] for a review. Geometrically, a D-brane in the Euclidean theory is a submanifold  $i : W \hookrightarrow X$  of codimension  $r$  together with a complex vector bundle  $Q \rightarrow W$  and unitary connection. In Type IIA the codimension  $r$  is odd, in Type IIB it is even. The vector bundle  $Q$  is analogous to the function (3.22) in electromagnetism; it encodes the charge carried by the D-brane. Physicists refer to  $\text{rank } Q$  as the number of D-branes, so a single D-brane comes with a line bundle and unitary connection, that is, a 1-form gauge field quantized by integer cohomology. We allow virtual bundles as well; negative rank corresponds to D-antibranes. One motivation for the vector bundle  $Q \rightarrow W$  is the Chan-Paton construction in *open* string theory [d'H, §2.11]. The important point for us is the coupling of the D-brane to the RR gauge potential  $C$  involves the gauge field on the brane. This was discussed in several papers, for example [GHM], [MM], [CY]. In particular, motivated by the presence of the  $\hat{A}$ -genus in the differential form expression of the coupling, Minasian and Moore [MM] suggested the  $K$ -theory formula for RR charge. It is important to note that this coupling has the standard form (3.17) when the RR gauge potential is lifted to a differential  $K$ -theory cocycle  $\check{C}$ :

$$\exp\left(\frac{i}{2\hbar} \int_W u^{-[\frac{r}{2}]} \check{C} \cdot \overline{Q}\right).$$

Here the bundle  $Q \rightarrow W$  with its connection gives rise to a differential  $K$ -theory cocycle  $\check{Q}$  whose complex conjugate is  $\overline{\check{Q}}$ . The factor of  $1/2$  is inserted since  $\check{C}$  is self-dual, and the correct interpretation involves the quadratic form mentioned above.<sup>30</sup> The RR charge of the D-brane is  $u^{-[\frac{r}{2}]} i_*[\check{Q}]$ , where  $i_* : K^0(W) \rightarrow K^r(X)$  is the pushforward, which is defined if the normal bundle to  $W$  is oriented and  $\text{spin}^c$ . In fact, there is a fermion anomaly in the perturbative open string [FW] which shows that for  $\text{rank } Q = 1$  the line bundle  $Q$  should be interpreted in terms of  $\text{spin}^c$  structures. (In general one must interpret it in *twisted*  $K$ -theory.) This anomaly, which fits the orientation condition for  $K$ -theory, is yet another piece of evidence for the quantization condition.

<sup>30</sup>The notion of a lagrangian for self-dual fields is itself only a schematic for the definition of the partition function; see the end of §3.6.

A rather different piece of evidence was put forth by Witten [W4]. He exhibited objects in superstring theory with torsion RR charge which naturally live in  $K$ -theory (rather than integer cohomology, for example). His examples lie in Type I, but Type I may be constructed from Type II and so his examples are also relevant for understanding Type II. Furthermore, Witten gave several formal arguments for  $K$ -theory and also showed that Sen's brane-antibrane annihilation [Se] may be interpreted as the difference construction in  $K$ -theory.

The choices of integer cohomology to quantize the NS  $\check{B}$ -field and  $K$ -theory to quantize the RR  $\check{C}$ -field fit together if both are nonzero. Namely, the  $\check{B}$ -field provides a *twisting* of differential  $K$ -theory, and the  $\check{C}$ -field lives in this twisted theory. First, topologically a cocycle for  $H^3(X)$  defines a twisting of  $K$ -theory, and if  $[\check{B}] \in H^3(X)$  is nonzero then the RR charge lives in this twisted  $K$ -theory. There is a more refined statement on the level of differential cohomology:  $\check{C}$  is a  $\check{B}$ -twisted differential  $K$ -theory cocycle. We remark that the mathematical groundwork for these twisted theories has not yet been fully developed. Also, these assertions about twisted (differential)  $K$ -theory enter the anomaly considerations to which we now turn.

The Type II theories on a 10-manifold are believed to be anomaly-free. For Type IIA this is easily proved, since both chiralities of fermion occur and so the fermion anomalies cancel by an elementary argument; see the footnote preceding (A.29). In Type IIB the *local* fermion anomalies cancel the local anomaly in the self-dual RR gauge field, that is, the total partition function is a section of a line bundle with zero curvature. However, as mentioned at the end of §3.6, there is not yet a precise definition of the partition function of the self-dual field which demonstrates a cancellation of *global* anomalies, i.e., which demonstrates that the line bundle has no holonomy (and even better has a canonical flat section of unit norm). Nevertheless, that  $K$ -theory quantizes the RR gauge field must enter into the eventual argument. What has been worked out are the additional anomalies due to a D-brane [FH], [MW]. Here the electric coupling anomaly (3.25) cancels a fermion anomaly (2.8). The formulas for both live, at least conjecturally, in differential  $K$ -theory, and if the RR gauge field were lifted to a different generalized differential cohomology theory this argument would not work.

Similar reasoning indicates that in any situation where this *Green-Schwarz mechanism* cancels fermi anomalies against and electric coupling anomaly, the gauge field will be quantized by a cohomology theory which, if not some form of  $K$ -theory, at least maps to it.

## Appendix: The Partition Function of Rarita-Schwinger Fields

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This appendix grew out of a concrete question: Why for the computation of anomalies is the Rarita-Schwinger field on an even dimensional manifold  $X$  treated as a spinor field coupled to the virtual bundle  $TX - 1$ , whereas in 11 dimensional M-theory it is treated as a spinor field coupled to  $TX - 3$ ? The perturbative anomaly computation in even dimensions is explained in [AgW], and the statement in 11 dimensions appears in [DMW, (2.7)]. Here we analyze general Rarita-Schwinger fields from first principles. First, we quantize the free classical Rarita-Schwinger field on Minkowski spacetime to obtain particle representations of the Poincaré group. There are three different classical theories we consider, and they lead to different particle content. The first of these generalizes neatly to Euclidean field theory, and using it we define the Rarita-Schwinger partition function. This leads to the anomaly computations cited above.

The material here is well-known to physicists. Our notation follows the treatments in [DF, §3] and [F6, Lecture 3], of scalar, spinor, and 1-form fields.

**A.1. Quantization of Spinor Fields: Review.** Let  $\mathbb{M}^n$  denote  $n$ -dimensional Minkowski spacetime. It is an affine space modeled on a vector space  $V$  equipped with a Lorentz metric  $g$  of signature  $(1, n - 1)$ . Fix a basis  $\{e_\mu\}$  of  $V$  and dual basis  $\{e^\mu\}$  of  $V^*$ . These give rise to coordinates on  $\mathbb{M}^n$ , so partial derivatives  $\partial_\mu$ . Let  $|d^n x|$  denote the density induced from the metric. When we take Fourier transforms below we choose a basepoint of  $\mathbb{M}^n$  to identify it with  $V$ ; then the Fourier transform is a function on  $V^*$ . Let  $N \subset V^*$  be the cone of null covectors, and  $N^+ \subset N$  a distinguished component of  $N \setminus \{0\}$ . (This gives a notion of positive energy.)

Let  $\text{Spin}(V)$  denote the connected Lorentz spin group. We develop the theory for an arbitrary finite-dimensional real spinor representation  $S$  of  $\text{Spin}(V)$ . It is a fact [De, Chapter 6] that for any  $S$  there exist symmetric pairings

$$\begin{aligned} \Gamma &: S^* \otimes S^* &\longrightarrow V \\ \tilde{\Gamma} &: S \otimes S &\longrightarrow V \end{aligned}$$

which satisfy a Clifford relation. Fix bases  $\{f^a\}$  of  $S$  and dual basis  $\{f_a\}$  of  $S^*$ . Then the Clifford relation reads

$$\Gamma_{ab}^\mu \tilde{\Gamma}^{\nu bc} + \Gamma_{ab}^\nu \tilde{\Gamma}^{\mu bc} = 2g^{\mu\nu} \delta_a^c,$$

where  $g^{\mu\nu}$  is the inverse metric  $g^{-1}$  on  $V^*$ . Put differently,  $\Gamma, \tilde{\Gamma}$  determine Clifford multiplications

$$(A.1) \quad \begin{aligned} c : V^* &\longrightarrow \text{Hom}(S, S^*) \\ c : V^* &\longrightarrow \text{Hom}(S^*, S) \end{aligned}$$

which satisfy

$$(A.2) \quad c(k)c(\ell) + c(\ell)c(k) = 2g^{-1}(k, \ell), \quad k, \ell \in V^*.$$

Thus  $S \oplus S^*$  is a real Clifford module for the Clifford algebra  $\text{Cliff}(V^*, g^{-1})$ .

Spinor fields and Rarita-Schwinger fields are odd in the sense of  $\mathbb{Z}/2\mathbb{Z}$ -graded geometry. Let  $\Pi S$  denote the odd vector space which is the parity-reversal of  $S$ .<sup>31</sup> Then a spinor field is a function

$$(A.3) \quad \psi : \mathbb{M}^n \longrightarrow \Pi S.$$

The lagrangian is the symmetric (in the graded sense) bilinear form based on the Dirac operator:

$$(A.4) \quad L = \frac{1}{2} \psi \not{D} \psi |d^n x| = \frac{1}{2} \tilde{\Gamma}^{\mu ab} \psi_a \partial_\mu \psi_b |d^n x|.$$

Now the Fourier transform of (A.3)

$$\hat{\psi} : V^* \longrightarrow \Pi S_{\mathbb{C}}$$

lands in the complexified spin space and satisfies the reality condition

$$(A.5) \quad \hat{\psi}(-k) = \overline{\hat{\psi}(k)}.$$

The equation of motion derived from (A.4) is the Dirac equation

$$(A.6) \quad c(k) \hat{\psi}(k) = \tilde{\Gamma}^{\mu ab} k_\mu \psi_a(k) f_b = 0.$$

If  $|k|^2 \neq 0$  we apply  $c(k)$  to (A.6) to deduce that  $\hat{\psi}$  is supported on the nullcone  $N$ . Define a complex vector bundle  $S' \rightarrow N \setminus \{0\}$  by

$$(A.7) \quad S'_k = \ker(c(k) : S_{\mathbb{C}} \rightarrow S^*_{\mathbb{C}}), \quad |k|^2 = 0, \quad k \neq 0$$

Typically this is a bundle of rank  $\frac{1}{2} \dim S$ , though there are exceptions in  $n = 2$  dimensions. Using (A.6) we identify<sup>32</sup> the space of classical solutions with the space of sections of  $\Pi S' \rightarrow N \setminus \{0\}$  which satisfy the reality condition (A.5). The lagrangian determines a symplectic structure on this vector space.

Quantization proceeds by complexifying the symplectic space of solutions—so dropping the reality condition—and then considering the lagrangian subspace of sections supported on  $N^+ \subset N$ . This is a unitary representation of the Poincaré group called the *one-particle Hilbert space*; the full quantum Hilbert space is the completion of its symmetric algebra (in the graded sense).

We describe the bundle  $S' \rightarrow N^+$  more explicitly as follows. Introduce the real vector bundle  $V' \rightarrow N^+$  of rank  $n - 2$  whose fiber at  $k \in N^+$  is

$$V'_k = \frac{k^\perp}{\mathbb{R} \cdot k}.$$

The Lorentz group  $\text{Spin}(V)$  acts on  $N^+$  and there is a lifted action on the bundles  $V', S'$ . The subgroup of  $\text{Spin}(V)$  which stabilizes  $k$  has a reductive part which

<sup>31</sup>More precisely,  $\Pi S = \mathbb{C}^{\text{odd}} \otimes S$ , where  $\mathbb{C}^{\text{odd}}$  is a fixed odd line.

<sup>32</sup>Throwing out  $k = 0$  does not cause problems if  $n \geq 3$ , but leads to special considerations in dimension  $n = 2$ .

is  $\text{Spin}(V'_k)$ , also known as the “little group”. It is isomorphic to the compact spin group  $\text{Spin}(n-2)$ . These groups fit together to form a bundle of groups  $\text{Spin}(V') \rightarrow N^+$ . Then  $S'_k$  is a complex spin representation of  $\text{Spin}(V'_k)$ , and the associated unitary representation of Poincaré is called a massless spin 1/2 particle. (See [De, Chapter 5] for more about the representation  $S'$ .) Depending on  $S$  this may or may not be irreducible, so is better called a collection of spin 1/2 particles. The complexification of  $V' \rightarrow N^+$  is also associated to a unitary representation of Poincaré, irreducible if  $n \geq 3$ , called the spin 1 particle.

**A.2. Quantization of Rarita-Schwinger Fields: First Approach.** Continuing the discussion of particles in relativistic quantum mechanics, we first describe a spin 3/2 particle. As in (A.7) define a complex vector bundle  $S'' \rightarrow N^+$  by

$$S''_k = \ker(c(k) : S_{\mathbb{C}}^* \rightarrow S_{\mathbb{C}}), \quad k \in N.$$

Then Clifford multiplication (A.1) induces a map

$$(A.8) \quad c' : V' \otimes S' \rightarrow S''$$

which is equivariant for  $\text{Spin}(V')$ . Set  $R' = \ker c'$ . The sections of the odd  $\text{Spin}(V)$ -equivariant complex vector bundle  $\Pi R' \rightarrow N^+$  form a unitary representation of the Poincaré group which represents a collection of spin 3/2 particles. (Again, depending on  $S$  it may or may not be irreducible.)

It is useful to note that there is a splitting of (A.8):

$$\begin{aligned} S'' &\rightarrow V' \otimes S' \\ s^a f_a &\mapsto \frac{1}{n} g_{\mu\nu} \Gamma_{ab}^{\mu} s^a e^{\nu} \otimes f^b. \end{aligned}$$

Thus we have a decomposition

$$(A.9) \quad V' \otimes S' \cong R' \oplus S''$$

as  $\text{Spin}(V)$ -equivariant complex vector bundles over  $N^+$ .

A *Rarita-Schwinger field* is a fermionic field

$$\chi : \mathbb{M}^n \rightarrow V^* \otimes \Pi S.$$

We write  $\chi = \chi_{\mu} e^{\mu} = \chi_{\mu a} e^{\mu} \otimes f^a$ , where  $\chi_{\mu} \in \Pi S$  and  $\chi_{\mu a} \in \Pi R$ . The first lagrangian we consider generalizes (A.4):

$$(A.10) \quad L = \frac{1}{2} \chi \not{D} \chi = \frac{1}{2} \tilde{\Gamma}^{\mu ab} g^{\nu\rho} \chi_{\nu a} \partial_{\mu} \chi_{\rho b}.$$

Note the factor of the inverse metric to absorb the vector index on  $\chi$ . The associated Euler-Lagrange equation, written on the Fourier transform, is

$$(A.11) \quad (id \otimes c(k)) \hat{\chi}(k) = \tilde{\Gamma}^{\mu ab} k_{\mu} \hat{\chi}_{\rho a}(k) f_b = 0.$$

As before we conclude that  $\hat{\chi}$  is supported on the nullcone  $N$ . Polarize—drop the reality condition and restrict to  $N^+ \subset N$ —to obtain the space of sections of  $V_{\mathbb{C}}^* \otimes \Pi S' \rightarrow N^+$ . Now the  $\text{Spin}(V)$ -equivariant vector bundle  $V^* \rightarrow N^+$  decomposes as the sum of  $V'$  and a rank 2 trivial bundle. Hence, using (A.9) we see that the one-particle Hilbert space of the theory is the space of sections of

$$(A.12) \quad \Pi(R' \oplus S'' \oplus S' \oplus S') \rightarrow N^+.$$



In other words, the particle content of the theory is a collection of spin 3/2 particles based on  $R'$ , a collection of spin 1/2 particles based on  $S''$ , and two collections of spin 1/2 particles based on  $S'$ .

REMARK A.13. *To obtain a collection of pure spin 3/2 particles we need to “subtract” off the spin 1/2 particles in (A.12). There are various mechanisms to do so. Our interest is in the Euclidean partition function, and it is in that context that we isolate the spin 3/2 particles.*

REMARK A.14. *There is an analogous quantization of a 1-form field on Minkowski spacetime which leads to a spin 1 particle and two spin 0 particles. The more usual quantization of 1-forms uses a gauge symmetry to avoid the spin 0 particles. We give the analogous treatment of Rarita-Schwinger fields in the next section.*

### A.3. Quantization of Rarita-Schwinger Fields: Second Approach.

We begin with some linear algebra. Recall from (A.1) that  $S \oplus S^*$  is a real Clifford module for  $\text{Cliff}(V^*)$ . As a real vector space  $\text{Cliff}(V^*)$  is isomorphic to  $\bigwedge^\bullet(V^*)$ , so there is an induced map

$$c^{(3)} : \bigwedge^3 V^* \longrightarrow \text{End}(S \oplus S^*)$$

which exchanges  $S$  and  $S^*$ . From the Clifford identity (A.2) it follows that for  $k_1, k_2, k_3 \in V^*$  we have

$$(A.15) \quad c^{(3)}(k_1 \wedge k_2 \wedge k_3) = c(k_1)c(k_2)c(k_3) - \langle k_1, k_2 \rangle c(k_3) + \langle k_1, k_3 \rangle c(k_2) - \langle k_2, k_3 \rangle c(k_1),$$

where we write  $\langle \cdot, \cdot \rangle$  for the inverse metric on  $V^*$ .

We use  $c^{(3)}$  in the following lagrangian for a Rarita-Schwinger field  $\chi : \mathbb{M}^n \rightarrow V^* \otimes \Pi S$ :

$$(A.16) \quad L = \frac{1}{2} \chi c^{(3)} \partial \chi |d^n x| = \frac{1}{2} \chi_\mu c^{(3)\mu\nu\rho} \partial_\nu \chi_\rho |d^n x|.$$

This lagrangian has a gauge invariance. Namely, if  $\psi : \mathbb{M}^n \rightarrow \Pi S$  is a spinor field, then  $\partial \psi$  is a Rarita-Schwinger field and  $L(\partial \psi) = 0$ . So in this model we consider the space of fields to be the quotient space

$$(A.17) \quad \frac{\{\chi : \mathbb{M}^n \rightarrow V^* \otimes \Pi S\}}{\partial\{\psi : \mathbb{M}^n \rightarrow \Pi S\}}.$$

The solutions to the equation of motion on this quotient form a symplectic vector space, and this is what we quantize to obtain the free quantum theory.

Now the equation of motion derived from (A.16) is  $c^{(3)} \partial \chi = 0$ . The maps

$$S_{\mathbb{C}} \xrightarrow{A_k} V_{\mathbb{C}}^* \otimes S_{\mathbb{C}} \xrightarrow{B_k} V_{\mathbb{C}} \otimes S_{\mathbb{C}}^*$$

defined by

$$A_k(s) = k \otimes s$$

$$B_k(\ell \otimes s) = e_\mu \otimes c^{(3)}(e^\mu \wedge k \wedge \ell) s$$

satisfy  $B_k \circ A_k = 0$ . The equation of motion on the Fourier transform asserts  $\hat{\chi}(k) \in \ker B_k$ .

LEMMA A.18. *Lemma (i) If  $|k|^2 \neq 0$  then  $\ker B_k = \text{im } A_k$ . (ii) If  $|k|^2 = 0$  and  $k \neq 0$ , then*

$$\frac{\ker B_k}{\text{im } A_k} \cong R_k'.$$

It follows that we can identify the space of solutions on the quotient space (A.17), after complexifying and polarizing, with the space of sections of  $R' \rightarrow N^+$ . In other words, in this approach we obtain exactly the desired collection of spin 3/2 particles.

PROOF. For (i) choose the basis  $e_1, \dots, e_n$  of  $V$  to be orthogonal and have  $e_1 = k$ . Assume that  $\sum_{i>1} e^i \otimes s^i$ ,  $s^i \in S_{\mathbb{C}}$ , is in  $\ker B_k$ ; we must show that  $s^i = 0$ . The hypothesis implies

$$\sum_{i>1} c^{(3)}(e^\mu \wedge e^1 \wedge e^i) s^i = 0, \quad \mu = 1, \dots, n.$$

For  $\mu = 1$  we learn nothing. If  $\mu > 1$ , then from (A.15) we have

$$\begin{aligned} &= \sum_{i>1} [c(e^\mu)c(e^1)c(e^i) + g^{\mu i}c(e^1)] s^i \\ (A.19) \quad &= \sum_{i>1} -c(e^\mu)c(e^i)c(e^1) s^i + g^{\mu\mu}c(e^1) s^\mu \\ &= \sum_{\substack{i \neq \mu \\ i>1}} c(e^\mu)c(e^1)c(e^i) s^i. \end{aligned}$$

Apply  $c(e^1)c(e^\mu)$  and set  $t^i = c(e^i)s^i$ ; then

$$(A.20) \quad \sum_{\substack{i \neq \mu \\ i>1}} t^i = 0, \quad \mu = 2, \dots, n.$$

Adding these equations we deduce  $\sum_{i>1} t^i = 0$ , and combining with (A.20) we find  $t^\mu = c(e^\mu)s^\mu = 0$  for all  $\mu$ . Apply  $c(e^\mu)$  to conclude  $s^\mu = 0$ .

For (ii) choose the basis of  $V$  to have  $e_1 = k$  and  $g_{11} = g_{22} = 0$ ,  $g_{12} \neq 0$ ,  $g_{1i} = g_{2i} = g_{ij} = 0$ , and  $g_{ii} \neq 0$ , where  $i, j > 2$ ,  $i \neq j$ . First, we show that  $B_k(e^2 \otimes s) = 0$ ,  $s \in S_{\mathbb{C}}$ , implies  $s = 0$ . For then if  $\mu > 2$  we apply (A.15) to find

$$(A.21) \quad 0 = c^{(3)}(e^\mu \wedge e^1 \wedge e^2) s = [c(e^\mu)c(e^1)c(e^2) - g^{12}c(e^\mu)] s,$$

from which  $c(e^1)c(e^2)s = g^{12}s$ , since  $c(e^\mu)^2 = g^{\mu\mu} \neq 0$ . Apply  $c(e^1)$  to deduce  $s \in \ker c(e^1)$ . Apply  $c(e^2)$  to deduce  $s \in \ker c(e^2)$ . Then since  $c(e^1)c(e^2) + c(e^2)c(e^1) = 2g^{12} \neq 0$  we find  $s = 0$ .

Finally, suppose  $\sum_{i>2} e^i \otimes s^i$ ,  $s^i \in S_{\mathbb{C}}$ , is in  $\ker B_k$ . Then

$$(A.22) \quad \sum_{i>2} c^{(3)}(e^\mu \wedge e^1 \wedge e^i) s^i = 0, \quad i = 1, \dots, n.$$

For  $\mu = 1$  we learn nothing. Set  $\mu = 2$  and apply (A.15) to find

$$c(e^2)c(e^1) \sum_{i>2} c(e^i) s^i = g^{12} \sum_{i>2} c(e^i) s^i.$$

As in the argument following (A.21) we conclude

$$(A.23) \quad \sum_{i>2} c(e^i) s^i = 0.$$

For  $\mu > 2$  we find from (A.22), analogously to (A.19), that

$$c(e^1) \sum_{\substack{i \neq \mu \\ i>2}} c(e^i) s^i = 0.$$

Together with (A.23) this implies  $c(e^\mu)s^\mu \in \ker c(e^1)$ , whence

$$(A.24) \quad s^\mu \in \ker c(e^1).$$

Then (A.23) and (A.24) together imply  $\sum_{i>2} e^i \otimes s^i \in R'$ , as desired.  $\square$

**A.4. Quantization of Rarita-Schwinger Fields: Third Approach.** Define the real representation  $R$  of  $\text{Spin}(V)$  as the kernel of Clifford multiplication  $V^* \otimes S \rightarrow S^*$ . In this approach we take a Rarita-Schwinger field to be a map

$$\tau : \mathbb{M}^n \longrightarrow \Pi R.$$

Then it is easy to verify using (A.15) that the lagrangians (A.10) and (A.16) agree when  $\tau$  replaces  $\chi$ . There is no gauge symmetry in this approach. The classical equations may be analyzed as in the text following (A.11). Thus the relevant equivariant vector bundle over  $N^+$  is the parity-reversed kernel of Clifford multiplication  $V_{\mathbb{C}}^* \otimes S' \rightarrow S_{\mathbb{C}}^*$ . This is isomorphic to  $\Pi(R' \oplus S') \rightarrow N^+$ . So in this approach we obtain a collection of spin 3/2 particles and a collection of spin 1/2 particles.

**A.5. The Euclidean Partition Function of a Rarita-Schwinger Field.** Here the setting is a Riemannian manifold  $X$ . One defines fields, a lagrangian, and an action as on Minkowski spacetime, but the classical equations of motion no longer carry physical meaning. Rather, these data are inputs into the functional integral; see (2.2). The fields and action are functorial in  $X$ . The correlation functions on Euclidean space are meant to have analytic continuations to Minkowski spacetime, and in this way one produces a relativistic quantum mechanical theory. Our goal, then, is to produce Euclidean theories—fields and an action—which analytically continue to the quantum theory of the spin 3/2 particle based on  $R'$ . We require that  $X$  be compact and spin (the compactness may be relaxed if conditions at the ends are imposed), and focus on the partition function rather than on general correlation functions.

In §2 we indicated that the choice of a real spin representation  $S$  of the Lorentz spin group  $\text{Spin}(V)$ —which we use to construct spinor fields  $\mathbb{M}^n \rightarrow \Pi S$  on Minkowski spacetime and then a collection of spin 1/2 particles based on  $S'$ —corresponds in Euclidean field theory on a spin manifold  $X$  to a section of a complex spin bundle over  $X$ . The details depend on the dimension  $n$ ; see the text before (2.5). For simplicity we denote this bundle as  $\Pi S_{\mathbb{C}} \otimes E \rightarrow X$ , where  $S_{\mathbb{C}} \rightarrow X$  is a complex spin bundle and  $E \rightarrow X$  a complex vector bundle.<sup>33</sup> We can take  $E$  to be trivial or we can take it to be an arbitrary bundle with connection, in which case the connection is a bosonic field in the theory. The Euclidean action<sup>34</sup> is the bilinear form (2.6) based on the Riemannian Dirac operator on  $X$  coupled to  $E$ . Then the functional integral over the spinor field is

$$(A.25) \quad \text{pfaff } \not{D}_E,$$

the pfaffian of the Dirac operator on spinors coupled to  $E$ , as in (2.7).

For Rarita-Schwinger fields we proceed similarly using our first approach to quantization in Minkowski spacetime. Thus the Euclidean Rarita-Schwinger field

<sup>33</sup>As explained in §2 this is correct for example if  $n \equiv 3 \pmod{8}$ , but must be modified for example if  $n \equiv 6 \pmod{8}$ .

<sup>34</sup>See [DF], §7.4 for a derivation, though what is written there should be modified in some dimensions, such as  $n \equiv 6 \pmod{8}$ .

is a section of  $\Pi\mathcal{S}_{\mathbb{C}} \otimes E \otimes T_{\mathbb{C}}X \rightarrow X$ . Since the Euclidean version of the lagrangian (A.10) is the action of a spinor field coupled to  $E \otimes T_{\mathbb{C}}X$ , the partition function is

$$(A.26) \quad \text{pfaff } \mathcal{D}_{E \otimes T_{\mathbb{C}}X}.$$

However, this corresponds to the physical theory with too many particles: the desired spin 3/2 particles plus three extraneous collections of spin 1/2 particles; see (A.12). Thus (A.26) represents the product of the desired partition function with the partition functions (A.25) of Euclidean spinor fields. The partition function we seek, which corresponds to the desired collection of spin 3/2 particles, is obtained by dividing out the spinor partition functions. Let  $\tilde{\mathcal{D}}_E$  denote the Dirac operator on the Euclidean spinor bundle  $\tilde{\mathcal{S}}_{\mathbb{C}} \otimes \tilde{E}$  which corresponds to the Lorentz spin representation  $S^*$ ; its relationship to  $\mathcal{D}_E$  depends on the dimension  $n$ . Then we define the Rarita-Schwinger partition function to be

$$(A.27) \quad \frac{\text{pfaff } \mathcal{D}_{E \otimes T_{\mathbb{C}}X}}{(\text{pfaff } \mathcal{D}_E)^2 (\text{pfaff } \tilde{\mathcal{D}}_E)}.$$

Finally, the Rarita-Schwinger anomaly is a tensor product of pfaffian line bundles

$$(A.28) \quad L_{RS} = (\text{Pfaff } \mathcal{D}_{E \otimes T_{\mathbb{C}}X}) \otimes (\text{Pfaff } \mathcal{D}_E)^{\otimes(-2)} \otimes (\text{Pfaff } \tilde{\mathcal{D}}_E)^{\otimes(-1)}.$$

#### A.6. Two Illustrative Examples.

The minimal supergravity in  $n = 10$  dimensions occurs in the low energy approximation to superstring theories with 16 supersymmetries, and it contains a Rarita-Schwinger field. The appropriate real spinor representation  $S$  of the Lorentz spin group is the minimal one of dimension 16; its dual  $S^*$  is an inequivalent representation.<sup>35</sup> In the corresponding Euclidean version  $\mathcal{S}_{\mathbb{C}} \rightarrow X$  is a half-spinor bundle of the spin 10-manifold  $X$  and  $\tilde{\mathcal{S}}_{\mathbb{C}} \rightarrow X$  is the opposite chirality half-spinor bundle. The vector bundle  $E$  is a trivial line bundle, so plays no role. The Dirac operator  $\mathcal{D}: \Gamma(\mathcal{S}_{\mathbb{C}}) \rightarrow \Gamma(\tilde{\mathcal{S}}_{\mathbb{C}})$  is complex skew-adjoint relative to the duality pairing between  $\mathcal{S}_{\mathbb{C}}$  and  $\tilde{\mathcal{S}}_{\mathbb{C}}$ . The Dirac operator  $\tilde{\mathcal{D}}$  in the denominator of (A.27) is the adjoint of  $\mathcal{D}$  relative to the hermitian structures on  $\Gamma(\mathcal{S}_{\mathbb{C}})$  and  $\Gamma(\tilde{\mathcal{S}}_{\mathbb{C}})$ . Its pfaffian line bundle is the canonically the inverse<sup>36</sup> of the pfaffian line bundle of  $\mathcal{D}$ , so that the anomaly (A.28) in this case simplifies to

$$(A.29) \quad (\text{Pfaff } \mathcal{D}_{T_{\mathbb{C}}X}) \otimes (\text{Pfaff } \mathcal{D})^{\otimes(-1)}.$$

Formally, this is the pfaffian line bundle of Dirac coupled to  $T_{\mathbb{C}}X - 1$ .

The second example is  $n = 11$  dimensional supergravity, which is the low energy approximation to the (putative) M-theory. Again  $S$  is the minimal real spinor representation of the Lorentz group, which in this case is unique of dimension 32. So  $S^* \cong S$ . In the Euclidean version  $\mathcal{S}_{\mathbb{C}} \rightarrow X$  is the rank 32 spinor bundle of the spin 11-manifold  $X$ , and the Dirac operators  $\mathcal{D}$  and  $\tilde{\mathcal{D}}$  agree. The anomaly is then

$$(\text{Pfaff } \mathcal{D}_{T_{\mathbb{C}}X}) \otimes (\text{Pfaff } \mathcal{D})^{\otimes(-3)}.$$

Formally, this is the pfaffian line bundle of Dirac coupled to  $T_{\mathbb{C}}X - 3$ .

<sup>35</sup>In dimensions  $n \equiv 2, 6 \pmod{8}$  there are two inequivalent minimal real spinor representations, whereas in other dimensions there is only one.

<sup>36</sup>In the finite dimensional model (2.10) we are asserting that if  $V \rightarrow T$  is a hermitian vector bundle with connection, then  $(\text{Det } V^*) \otimes (\text{Det } V)$  is canonically isomorphic to the trivial line bundle with trivial metric and connection.

## References

- [Ab] L. Abrams, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications, **5** (1996), pp. 569–587.
- [AP] J. F. Adams, S. B. Priddy, *Uniqueness of BSO*, Math. Proc. Cambridge Philos. Soc., **80** (1976), pp. 475–509.
- [AgW] L. Alvarez-Gaumé, E. Witten, *Gravitational anomalies*, Nucl. Phys., **B234** (1983), pp. 269.
- [A1] M. F. Atiyah, *Topological quantum field theory*, Publ. Math. Inst. Hautes Etudes Sci. (Paris) **68** (1989), pp. 175–186.
- [A2] M. F. Atiyah, *K-Theory Past and Present*, math.KT/001221.
- [AH] M. F. Atiyah, F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Symp. Pure Math. **3** (1961), pp. 7–38.
- [AS1] M. F. Atiyah, I. M. Singer, *The index of elliptic operators. I*, Ann. Math. **87** (1968), pp. 484–530.
- [AS2] M. F. Atiyah, I. M. Singer, *The index of elliptic operators. IV*, Ann. Math. **93** (1971), pp. 119–138.
- [AS3] M. F. Atiyah, I. M. Singer, *The index of elliptic operators V*, Ann. of Math. **93** (1971), pp. 139–149.
- [AS4] M. F. Atiyah, I. M. Singer, *Dirac operators coupled to vector potentials*, Proc. Nat. Acad. Sci. **81** (1984), p. 2597.
- [ASe] M. F. Atiyah, G. B. Segal, *Exponential isomorphisms for  $\lambda$ -rings*, Quart. J. Math. Oxford Ser. **22** (1971), pp. 371–378.
- [BW] W. A. Bardeen, A. R. White, *Symposium on Anomalies, Geometry, Topology* (March, 1985), Singapore: World Scientific, (1985).
- [BCMMS] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray, D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, hep-th/0106194.
- [BF] J. M. Bismut, D. S. Freed, *The analysis of elliptic families I: Metrics and connections on determinant bundles*, Commun. Math. Phys. **106** (1986), pp. 159–176.  
 ———, *The analysis of elliptic families II: Dirac operators, eta invariants, and the holonomy theorem of Witten*, Commun. Math. Phys. **107** (1986), pp. 103–163.
- [BS] A. Borel, J.-P. Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France, **86** (1958), pp. 97–136.
- [B] Brylinski, Jean-Luc, *Loop spaces, characteristic classes and geometric quantization*, Boston, MA: Birkhäuser Boston Inc.
- [CHS] C. G. Callan, Jr., J. A. Harvey, A. Strominger, *Supersymmetric string solitons*, String theory and quantum gravity (Trieste, 1991) River Edge, NJ, World Sci. Publishing, (1992), pp. 208–244; hep-th/9112030.
- [CY] Y.-K. E. Cheung, Z. Yin, *Anomalies, branes, and currents*, Nuclear Phys. B **517** (1998), pp. 69–91; hep-th/9710206.
- [C] J. Cheeger,  *$\eta$ -invariants, the adiabatic approximation, and conical singularities*, Jour. Diff. Geo. **26** (1987).
- [De] P. Deligne, *Notes on spinors*, Quantum Fields and Strings: A Course for Mathematicians (P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten, eds.), 2 volumes, Providence, RI, American Mathematical Society, (1999), pp. 99–135.
- [DF] P. Deligne, D. S. Freed, *Classical field theory*, Quantum Fields and Strings: A Course for Mathematicians (P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten, eds.), 2 volumes, Providence, RI American Mathematical Society, (1999), pp. 137–225.
- [DMW] D.-E. Diaconescu, G. Moore, E. Witten, *E8 Gauge Theory, and a Derivation of K-Theory from M-Theory*, hep-th/0005090.
- [D] R. Dijkgraaf, *A geometrical approach to two-dimensional conformal field theory*, Ph.D. thesis (1989).
- [DK] P. Donovan, M. Karoubi, *Graded Brauer groups and K-theory with local coefficients*, Inst. Hautes Études Sci. Publ. Math. **38** (1970), pp. 5–25.
- [Fyn] R. P. Feynman, R. B. Leighton, M. Sands, *The Feynman Lectures on Physics*, Volume 2, Reading, Massachusetts, Addison-Wesley, (1964).

- [F1] D. S. Freed, *Higher algebraic structures and quantization*, Commun. Math. Phys. **159**, (1994), pp. 343–398; [hep-th/9212115](#).
- [F2] D. S. Freed, *Extended structures in topological quantum field theory*, Quantum Topology, (L. H. Kauffman, R. A. Baadhio, eds.), pp. 162–173, [hep-th/9306045](#).
- [F3] D. S. Freed, *The Verlinde algebra is twisted equivariant K-theory*, Turkish J. Math., **25** (2001), pp. 159–167; [math.RI/0101038](#).
- [F4] D. S. Freed, *On determinant line bundles* Mathematical Aspects of String Theory (ed. S. T. Yau) World Scientific Publishing, (1987), pp. 189–238
- [F5] D. S. Freed, *Dirac charge quantization and generalized differential cohomology*, Papers Dedicated to Atiyah, Bott, Hirzebruch, Singer International Press, (to appear.) [hep-th/0011220](#).
- [F6] D. S. Freed, Five lectures on supersymmetry, Providence, RI American Mathematical Society, (1999.)
- [FH] D. S. Freed, M. J. Hopkins, *On Ramond-Ramond fields and K-theory*, J. High Energy Phys. (2000), Paper 44; [hep-th/0002027](#).
- [FQ] D. S. Freed, F. Quinn, *Chern-Simons theory with finite gauge group*, Commun. Math. Phys. **156** (1993); [hep-th/9111004](#).
- [FW] D. S. Freed, E. Witten, *Anomalies in String Theory with D-Branes*, Asian J. Math **3** (1999), pp. 819–851; [hep-th/9907189](#).
- [GHM] M. B. Green, J. A. Harvey, G. Moore, *I-Brane Inflow and Anomalous Couplings on D-Branes*, Class. Quantum Grav., **14** (1997), pp. 47–52; [hep-th/9605033](#).
- [GS] M. B. Green, J. H. Schwarz, *Anomaly cancellations in supersymmetric D = 10 gauge theory and superstring theory*, Phys. Lett. B (1984.)
- [H] N. J. Hitchin, *Lectures on special lagrangian submanifolds*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA: International Press) pp. 151–182 (Providence, RI: Amer. Math. Soc., 2001);, [math.DG/9907034](#).
- [d'H] E. d'Hoker, *String Theory*, Quantum Fields and Strings: A Course for Mathematicians, (P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten, eds.), Providence, RI: American Mathematical Society, (1999), 2 volumes pp. 807–1011.
- [HS] M. J. Hopkins, I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, (in preparation).
- [L] R. Lawrence, *Triangulations, categories and extended topological field theories*, Quantum Topology, (L. H. Kauffman, R. A. Baadhio, eds.), World Scientific, pp. 191–208.
- [Lo] Lott, J.,  *$\mathbb{R}/\mathbb{Z}$  index theory*, Comm. Anal. Geom. **2** (1994), pp. 279–311.
- [MM] R. Minasian, G. Moore, *K-theory and Ramond-Ramond charge*, J. High Energy Phys., (1998), no. 11, Paper 2, 7 pp, [hep-th/9710230](#)
- [M] G. Moore, *Some comments on branes, G-flux, and K-theory*, Internat. J. Modern Phys. A, **16**, (2001.), pp. 936–944, [hep-th/0012007](#).
- [MW] G. Moore, E. Witten, *Self-duality, Ramond-Ramond fields, and K-theory*, J. High Energy Phys. (2000), 5 Paper 32 [hep-th/9912279](#).
- [P1] J. Polchinski, String theory. Vol. II, Cambridge University Press,, Cambridge (1998.)
- [P2] J. Polchinski, *Lectures on D-branes*, Fields, strings and duality, (Boulder, CO, 1996) pp. 293–356 River Edge, NJ: World Sci. Publishing, (1997), [hep-th/9611050](#).
- [Qn] D. Quillen, *Determinants of Cauchy-Riemann operators over a Riemann surface*, Funk. Anal. iprilozen, **19** (1985), pp. 37.
- [Q] F. Quinn, *Lectures on axiomatic topological quantum field theory*, Geometry and quantum field theory (Park City, UT, 1991), IAS/Park City Math. Ser., 1 Providence, RI: Amer. Math. Soc., (1995), pp. 323–453.
- [RS] D. B. Ray, I. M. Singer, *R-torsion and the laplacian on Riemannian manifolds*, Adv. Math. **7** (1971), pp. 145–210.
- [R] J. Rosenberg, *Continuous trace algebras from the bundle theoretic point of view*, Jour. Austr. Math. Soc. **47** (1989), pp. 368–381.
- [Sa] S. Sawin, *Direct sum decompositions and indecomposable TQFTs*, J. Math. Phys. **36** (1995), pp. 6673–6680, [q-alg/9505026](#).

- [S1] G. Segal, *Categories and cohomology theories*, *Topology*, **13** (1974), pp. 293–312.
- [S2] G. Segal, *The definition of conformal field theory*, (preprint).
- [Se] A. Sen, *Tachyon Condensation on the Brane Antibrane System*, *JHEP* 9808 (1998) 012 [hep-th/9805170](#).
- [Si1] I. M. Singer, *Families of Dirac operators with applications to physics*, The mathematical heritage of Élie Cartan, (Lyon, 1984) *Astisque*, (1985), pp. 323–340.
- [Si2] I. M. Singer, *The  $\eta$ -invariant and the index*, *Mathematical Aspects of String Theory*, ed. S. T. Yau, World Scientific Publishing, (1987).
- [T1] V. G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, Walter de Gruyter, Berlin (1994.)
- [T2] V. Turaev, *Homotopy field theory in dimension 3 and crossed group-categories*, [math.GT/0005291](#).
- [Wa] K. Walker, *On Witten's 3-manifold invariants*, preprint, 1991.
- [W1] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.*, **121** (1989).
- [W2] E. Witten, *World-sheet corrections via D-instantons*, *J. High Energy Phys.*, **2000** 2 Paper 30 [hep-th/9907041](#).
- [W3] E. Witten *Global gravitational anomalies*, *Commun. Math. Phys.*, **100** (1985), pp. 197–229.
- [W4] E. Witten, *D-branes and K-theory*, *J. High Energy Phys.*, **1998** 12, Paper 19. [hep-th/9810188](#).
- [W5] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.*, **22** (1997), pp. 103–133, [hep-th/9610234](#).

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