INFINITELY MANY SIGN-CHANGING SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$∗
YINBIN DENG†, SHUANGJIE PENG‡, AND JIXIU WANG§

Abstract. This paper is concerned with constructing radial solutions with arbitrarily many sign changes for quasilinear Schrödinger equations in $\mathbb{R}^N$ which have appeared as several models in mathematical physics. For any given integer $k \geq 0$, by using a minimization argument, we obtain a sign-changing minimizer with $k$ nodes of a minimization problem with double constraints, and by applying an energy comparison method we prove that the minimizer is indeed a solution of the quasilinear Schrödinger equation.

Key words. Minimization problem, quasilinear Schrödinger equations, radial solutions, sign-changing solutions.

AMS subject classifications. 35J10, 35P30, 35Q55, 58E40, 58J32.

1. Introduction

This paper has been motivated by the quasilinear Schrödinger equations

$$i \partial_t z = -\Delta z + W(x)z - f(|z|^2)z - \kappa \Delta h(|z|^2)h'(|z|^2)z,$$

(1.1)

where $z : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $W : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given potential, $\kappa$ is a real constant, and $f,h$ are real functions of essentially pure power forms.

The semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years; for example, see Berestycki and Lions [7], Floer and Weinstein [17], Rabinowitz [34], and Strauss [36]. Quasilinear equations of form (1.1) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of $h$. For instance, the case $h(s) = s$ models the time evolution of the condensate wave function in super-fluid film ([21, 22]). This equation has been called the superfluid film equation in fluid mechanics by Kurihara [21]. In the case $h(s) = (1+s)^{1/2}$, problem (1.1) models the self-channeling of a high-power ultra short laser in matter, the propagation of a high-irradiance laser in a plasma creates an optical index depending nonlinearly on the light intensity and this leads to an interesting new nonlinear wave equation (see [8, 12, 16, 35]). Problem (1.1) also appears in plasma physics and fluid mechanics [4, 20, 33], in mechanics [19], in the theory of Heisenberg ferromagnets and magnons [4] and in condensed matter theory [28]. For more physical motivations and more references dealing with applications, we can refer Brüll and Lange [10], Lange et al. [23], Poppenberg et al. [32], and references therein.

In the mathematical literature, very few results are known on equations of the form (1.1). In the case $h(s) = (1+s)^{1/2}$ the local well posedness is proved in [16] for the space dimension $N = 1,2,3$ where smallness assumptions on the initial value are needed if $N = 2,3$. The case $h(s) = s, N = 1$ is investigated in [23], and the case of

∗Received: September 21, 2010; accepted (in revised version): January 10, 2011. Communicated by Jack Xin.
†School of Mathematics and Statistics, Huazhong Normal University, Wuhan, 430079, P.R. China (ybdeng@mail.ccnu.edu.cn).
‡School of Mathematics and Statistics, Huazhong Normal University, Wuhan, 430079, P.R. China (sjpeng@mail.ccnu.edu.cn).
§Department of Mathematics, University of Xiangfan, Xiangfan, 441053, P.R. China (wangjixiu127@yahoo.com.cn).
general $h(s)$ is considered in [30] for $N = 1$ and in [31] for arbitrary space dimension $N \geq 1$.

Here we focus on the case $h(s) = s, \kappa = 1/2$. It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to this case, and numerical results on this equation have been given in [9]. Our special interest is the existence of standing waves, that is, solutions of type $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u$ is a real function. It is well known that $z(t, x)$ satisfies (1.1) if and only if the function $u(x)$ solves the following equation

$$
\begin{cases}
-\Delta u + V(x)u - \frac{1}{2}\Delta(u^2)u = f(u^2), & x \in \mathbb{R}^N, \\
u \to 0, & \text{as } |x| \to \infty,
\end{cases}
$$

(1.2)

where $V(x) = W(x) - E$ is a new potential.

The existence of a positive ground state solution of problem (1.2) has been proved in Poppenberg et al. [32] and Liu and Wang [24] by using a constrained minimization argument, which gives a solution with an unknown Lagrange multiplier $\lambda$ in front of the nonlinear term. In Liu et al. [25], by a change of variables the quasilinear problem was transformed to a semilinear one and an Orlitz space framework was used as the working space, and the existence of a positive solution of problem (1.2) for any prescribed $\lambda > 0$ was proved by using the Mountain-Pass theorem (e.g., Ambrosetti and Rabinowitz [1]). The same method of changing variables was also used recently to obtain positive solutions of problem (1.2) in [13] for the case of subcritical growth and in [5] for the case of critical growth. Along this line, one could also look for sign-changing solutions (as, for example, in [26]) by utilizing the Nehari method; Liu et al. treated more general quasilinear problems and obtained positive and sign-changing solutions. The main mathematical difficulties with problem (1.2) are caused by the nonlinearity involving second order space derivatives. In the variational formulation, these difficulties concern the nonlinear functional $\Psi(u) = \int_{\mathbb{R}^N} u^2|\nabla u|^2$ which is homogeneous of order 4 and non-convex. A further problem is caused by the usual lack of compactness since these problems are dealt with in the whole $\mathbb{R}^N$.

We remark that the solutions given in the above papers were obtained mainly by using a constrained minimization method or the Mountain-Pass theorem and hence are ground state solutions. Generally, these types of solutions are orbitally stable, since they have the least energy (i.e. the mountain-pass level $c$) among all solutions. However, concerning sign-changing solutions, we found no results except in [26] where the solutions change sign exactly one time. Physically speaking, a sign-changing solution $u$ of (1.2) corresponds an excited standing wave $z(t, x) = \exp(-iEt)u(x)$ of (1.1) (see [15]), which generally has less orbital stability since a sign-changing solution has at least double the least energy $c$. In this paper, we construct radial solutions of (1.2) with arbitrarily many sign changes. When $\kappa = 0$ in (1.1), the existence of radial sign-changing solutions has been explored thoroughly; we refer the readers to [2, 11, 14, 37] and the references therein. In dimensions $N = 4$ and $N \geq 6$, Bartsch and Willem [3] were able to construct sequences of nonradial sign-changing solutions of (1.1). Concerning nonradial positive solutions with higher energy of (1.2), we should point out that a recent paper [27] gives a very interesting result: on the annulus $\{x \in \mathbb{R}^N : a \leq |x| \leq a + 1\}$, (1.2) has a sequence of nonradial positive solutions with higher energy for large $a$.

We will construct infinitely many solutions to problem (1.2) by the Nehari method. To emphasize our main idea, we only concentrate on the case where $f(s)$ is purely in
power form; that is, we consider

\[
\begin{aligned}
-\Delta u + V(x)u - \frac{1}{2}\Delta(|u|^2)u &= \lambda|u|^{p-2}u, & x \in \mathbb{R}^N, \\
u \to 0, & \text{ as } |x| \to \infty,
\end{aligned}
\]

(1.3)

where \(\lambda > 0\), \(N \geq 2\), \(4 < p < 22^*\), \(2^*\) is the Sobolev critical exponent, that is, \(2^* = 2N/(N-2)\) if \(N \geq 3\) and \(2^* = +\infty\) if \(N = 2\).

We assume \(V(x)\) is a radially symmetric function and satisfies

(V): \(V \in C(\mathbb{R}^N, \mathbb{R})\), \(0 < V_0 := \inf_{\mathbb{R}^N} V(x)\).

Our main result for (1.3) is:

**Theorem 1.1.** Suppose that \(V(x)\) satisfies (V), \(4 < p < 22^*, \lambda > 0\). Then for any \(k \in \{0, 1, 2, \cdots\} \) there exists a pair of radial solutions \(u_k^\pm\) of (1.3) with the following properties:

(i) \(u_k^-\) have at least the energy \((k+1)c\) and hence belong to higher energy solutions if \(k \geq 1\), where \(c\) is the least energy corresponding to (1.3). Since \(|u|^p u\) is odd in \(u\), we see that \(-u_k^+\) and \(-u_k^-\) are also sign-changing solutions to (1.3). However, we can not claim that \(u_k^+ = -u_k^-\) since the nodes \(r_1, \cdots, r_k\) might not be unique. It is a very interesting problem to study the uniqueness of \(k\)-node solutions (up to a sign) of (1.3) for given \(k \in \mathbb{N}\) even \(\kappa = 0\).

In Theorem 1.1, we mainly deal with the nonlinear term \(|u|^p u\). But, the oddness assumption on nonlinear term is actually unnecessary. Our main result holds true for general nonlinearity \(f(|x|, u)\) with properties similar to those in [2] or [11]. Indeed, in this case we only need to extend \(f(|x|, u)\) as follows:

\[
\begin{aligned}
f^+(|x|, u) := \begin{cases} f(|x|, u), & \text{if } u \geq 0, \\
-f(|x|, -u), & \text{if } u < 0,
\end{cases}
\end{aligned}
\]

or

\[
\begin{aligned}
f^-(|x|, u) := \begin{cases} -f(|x|, -u), & \text{if } u \geq 0, \\
f(|x|, u), & \text{if } u < 0,
\end{cases}
\end{aligned}
\]

and define \(J^\pm(u), c_k^\pm = \inf_{u \in M_k^\pm} J^\pm(u)\) in the same way as those in [11]. By a similar argument, we can prove that \(c_k^\pm\) can be attained by \(u_k^\pm\), which must be the \(k\)-node solutions for the corresponding problem.

Particularly, we can prove the following Corollary.

**Corollary 1.1.** Suppose that \(V(x)\) satisfies (V), \(4 < p, q < 22^*, \lambda, \mu > 0\). Then for every integer \(k \geq 1\) there exists a pair \(u_k^\pm\) of radial solutions of

\[
\begin{aligned}
-\Delta u + V(x)u - \frac{1}{2}\Delta(|u|^2)u &= \lambda u_+^{p-1} - \mu u_-^{q-1}, & x \in \mathbb{R}^N, \\
u \to 0, & \text{ as } |x| \to \infty,
\end{aligned}
\]

with \(u_k^-\) having exactly \(k\) nodes.
Since $2^{*} = 4N/(N-2)$ behaves like a critical exponent for equation, by using a Pohozaev type variational identity, Liu et al. [26] proved that (1.3) has no positive solutions in $H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u^2 |\nabla u|^2 < \infty$ if $p \geq 2^{*}$. A natural problem is whether or not (1.3) has sign-changing solutions (including radial solutions) if $p \geq 2^{*}$. We will give some results on the problem in a future work.

In [5, 13, 24, 26, 32], to obtain ground state solutions of (1.3), $V(x)$ should satisfy

$$0 < V_0 = \inf_{x \in \mathbb{R}^N} V(x) \leq V_\infty = \lim_{|x| \to \infty} V(x). \quad (1.4)$$

Hence, another interesting problem is whether or not ground state solutions or higher energy solutions exist if (1.4) is not satisfied?

We will prove Theorem 1.1 by looking for a minimizer for a constrained minimization problem in a special space in which each function changes sign $k$ ($k \in \{0, 1, 2, \cdots\}$) times, and then verify that the minimizer is smooth and indeed a solution to (1.3) by analyzing the least energy related to the minimizer. Unlike those done in the above mentioned papers, we will work directly with the functional $I$ corresponding to (1.3) in spite of its lack of smoothness. We mention here that the main method to prove our theorem was introduced by Bartsch and Willem in [2] and Cao and Zhu in [11] independently, but, as we can see later, the appearance of the quasilinear operator $\Delta^{(\gamma^2)}$ may cause more difficulties.

The paper is organized as follows: Section 2 is devoted to preliminaries and some useful lemmas. Theorem 1.1 will be proved in Section 3.

2. Some preliminary lemmas

In this section, we give some definitions and lemmas. The proof of some lemmas mentioned here can be found in the corresponding references. Here we use $|u|_q$ to denote the $L^q(\mathbb{R}^N)$ norm. In the following, we set

$$H^1_1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u(x) = u(|x|)\},$$

and

$$X = \{u \in H^1_1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V|u|^2dx < +\infty, \int_{\mathbb{R}^N} |\nabla u|^2|u|^2dx < +\infty\}.$$ 

Set

$$||u|| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V|u|^2)dx\right)^{1/2}.$$ 

A function $u \in X$ is called a weak solution of problem (1.3) if for all $\phi \in C_0^{\infty}(\mathbb{R}^N)$ it holds

$$\int_{\mathbb{R}^N} (1 + u^2)\nabla u \nabla \phi dx + \int_{\mathbb{R}^N} |\nabla u|^2u \phi dx + \int_{\mathbb{R}^N} Vu \phi dx - \lambda \int_{\mathbb{R}^N} |u|^{p-2}u \phi dx = 0.$$ 

Define the energy functional $I$ on $X$ by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (1 + u^2)|\nabla u|^2dx + \frac{1}{2} \int_{\mathbb{R}^N} V|u|^2dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} |u|^pdx.$$ 

Formally, our problem has a variational structure. Given $u \in X$ and $\phi \in C_0^{\infty}(\mathbb{R}^N)$, the Gateaux derivative of $I$ in the direction $\phi$ at $u$, denoted by $\langle I'(u), \phi \rangle$, is defined as $\lim_{t \to 0^+} \frac{I(tu + \phi) - I(u)}{t}$. It is easy to check that

$$\langle I'(u), \phi \rangle = \int_{\mathbb{R}^N} \left[ (1 + u^2)\nabla u \nabla \phi + |\nabla u|^2u \phi + Vu \phi - \lambda |u|^{p-2}u \phi \right]dx.$$ 

Hence, \( u \) is a weak solution of problem (1.3) if this derivative is zero in every direction \( \phi \in C_0^\infty(\mathbb{R}^N) \). In particular, for \( u \in X \), we denote

\[
\gamma(u) = \langle I'(u), u \rangle = \int_{\mathbb{R}^N} (1+2u^2)|\nabla u|^2\,dx + \int_{\mathbb{R}^N} V|u|^2\,dx - \lambda \int_{\mathbb{R}^N} |u|^p\,dx.
\]

Note that we do not claim that \( I \) is well-defined nor of class \( C^1 \) in \( X \).

From [32], we have the following two lemmas.

**Lemma 2.1.** For \( N \geq 2 \), there is a constant \( C = C(N) > 0 \) such that

\[
|u(x)| \leq C|x|^\frac{2-N}{2} \|u\|_{H^1},
\]

for any \( |x| \geq 1 \) and \( u \in H^1_0(\mathbb{R}^N) \).

**Lemma 2.2.** Let \( \{u_n\} \subset H^1_0(\mathbb{R}^N) \) satisfy \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^N) \). Then

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2|u_n|^2\,dx \geq \int_{\mathbb{R}^N} |\nabla u|^2|u|^2\,dx.
\]

The following lemma was first proved by Strauss [36].

**Lemma 2.3.** Let \( N \geq 2 \) and \( 2 < q < 2^* \). Then the imbedding

\[
H^1_0(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)
\]

is compact.

**Lemma 2.4.** (Brézis-Lieb lemma [6]) Let \( \{u_n\} \subset L^q(\mathbb{R}^N) \) be a bounded sequence, where \( 1 \leq q < \infty \), such that \( u_n \to u \) almost everywhere in \( \mathbb{R}^N \). Then

\[
\lim_{n \to \infty} (|u_n|^q - |u_n - u|^q) = |u|^q.
\]

**Lemma 2.5.** ([26]) Let \( u \) be a weak solution of (1.3). Then \( u \) and \( \nabla u \) are bounded. Moreover, \( u \) satisfies the following exponential decay at infinity

\[
|u(x)| \leq Ce^{-\delta R}, \quad |x| = R, \quad \int_{\mathbb{R}^N \setminus B_R} (|\nabla u|^2 + |u|^2)\,dx \leq Ce^{-\delta R},
\]

for some positive constants \( C, \delta \).

Let \( \Omega \) be one of the following three types of domains:

\[
\begin{align*}
\{x \in \mathbb{R}^N & \mid |x| < R_1\}, \\
\{x \in \mathbb{R}^N & \mid 0 < R_2 \leq |x| < R_3 < +\infty\}, \\
\{x \in \mathbb{R}^N & \mid |x| \geq R_4 > 0\}.
\end{align*}
\]

Set

\[
H^1_{0,r}(\Omega) = \{u \in H^1_0(\Omega) \mid u(x) = u(|x|)\}
\]

and

\[
X(\Omega) = \{u \in H^1_{0,r}(\Omega) \mid \int_{\Omega} Vu^2\,dx < +\infty, \int_{\Omega} |\nabla u|^2u^2\,dx < +\infty\}.
\]
Now we consider the following equation on $\Omega$:

$$
\begin{cases}
-\Delta u + Vu - \frac{1}{2} \Delta |u|^2 u = \lambda |u|^{p-2} u, & x \in \Omega, \\
|u|_{\partial \Omega} = 0.
\end{cases}
$$

(2.2)

Corresponding to (2.2), we define the functional

$$
I_{\Omega}(u) = \frac{1}{2} \int_{\Omega} (1 + u^2) |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} V |u|^2 dx - \frac{\lambda}{p} \int_{\Omega} |u|^p dx, \quad u \in X(\Omega).
$$

Similarly we can define the Gateaux derivative of $I_{\Omega}$ at $u \in X(\Omega)$ and weak solutions of problem (2.2). Set

$$
\gamma_{\Omega}(u) = \langle I'_{\Omega}(u), u \rangle = \int_{\Omega} (1 + 2u^2) |\nabla u|^2 dx + \int_{\Omega} V |u|^2 dx - \lambda \int_{\Omega} |u|^p dx
$$

and define

$$
M(\Omega) = \{ u \in X(\Omega) \backslash \{0\} | \gamma_{\Omega}(u) = 0 \}.
$$

**Remark 2.1.** If $\Omega$ is of the second or the third shape of (2.1), then by Lemma 2.1 $X(\Omega)$ is a subspace of $H^1_{0, r}(\Omega)$ and $I_{\Omega}$ is well defined and $C^1$ smooth in $X(\Omega)$. Hence, in these two cases, we can obtain solutions of problem (2.2) much easier by using the variational formulation.

From [26] we have the following lemma.

**Lemma 2.6.** Suppose that $p > 4$ and $u \in X(\Omega)$. Then there is a unique $t > 0$ such that $tu \in M(\Omega)$. Moreover, if $\gamma_{\Omega}(u) < 0$, then $t \in (0, 1)$.

**Lemma 2.7.** Suppose that the domain $\Omega$ is one of the forms of (2.1). Then $c_1 = \inf_{M(\Omega)} I_{\Omega}(u)$ can be achieved by some positive function $u$ which is a solution of problem (2.2), i.e., $\forall \phi \in C^\infty_c(\Omega)$ it holds

$$
\int_{\Omega} \left[ (1 + u^2) \nabla u \nabla \phi + |\nabla u|^2 u \phi + Vu \phi - \lambda |u|^{p-2} u \phi \right] dx = 0.
$$

(2.3)

Moreover, the above equation holds for $\phi \in X(\Omega)$ with the property that

$$
\int_{\Omega} u^2 |\nabla \phi|^2 dx < \infty \quad \text{and} \quad \int_{\Omega} |\nabla u|^2 \phi^2 dx < \infty.
$$

**Proof.** Since the quasilinear operator $\Delta(\cdot^2)$ appears, we can not use the Mountain-Pass theorem here. Now we use a minimization method. The proof can be divided into three steps.

**Step 1.** $c_1$ is attained.

By the definition of $c_1$, there exists a sequence $\{u_n\} \subset M(\Omega)$ such that

$$
I_{\Omega}(u_n) = c_1 + o(1),
$$

i.e.,

$$
0 = \int_{\Omega} \left[ |\nabla u_n|^2 + V |u_n|^2 \right] dx + 2 \int_{\Omega} u_n^2 |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^p dx,
$$

$$
c_1 + o(1) = I_{\Omega}(u_n) = \frac{1}{2} \int_{\Omega} \left[ |\nabla u_n|^2 + V |u_n|^2 \right] dx + \frac{1}{2} \int_{\Omega} u_n^2 |\nabla u_n|^2 dx - \frac{\lambda}{p} \int_{\Omega} |u_n|^p dx.
$$
Thus we have
\[ c_1 + o(1) = \left( \frac{1}{2} - \frac{1}{p} \right) \|u_n\|^2 + \left( \frac{1}{2} - \frac{2}{p} \right) \int_{\Omega} u_n^2 |\nabla u_n|^2 \, dx. \] (2.4)

From \( p > 4 \), it is easy to verify that \( \{u_n\} \) is bounded in \( X(\Omega) \). Hence, by Lemma 2.3, we can extract a subsequence of \( \{u_n\} \) (still denoted by \( \{u_n\} \)), such that
\[ u_n \rightharpoonup u \quad \text{in} \quad X(\Omega), \]
\[ u_n \rightarrow u \quad \text{in} \quad L^q(\Omega), \quad 2 < q < 2^*. \]

Since \( \nabla (u_n^2) \) is uniformly bounded in \( L^2(\Omega) \) from (2.4), by Sobolev’s inequality we have
\[ |u_n|^2 \leq C, \]
which gives
\[ |u_n|^{2^*} \leq C. \]
By Hölder’s inequality we have
\[ u_n \rightarrow u \quad \text{in} \quad L^p(\Omega), \quad 2 < p < 2^*. \] (2.5)

Next we want to prove \( u_n \rightarrow u \) in \( X(\Omega) \). Let \( \nu_n = u_n - u, \ u_n \in M(\Omega) \). Then by Lemmas 2.2, 2.4 and 2.5,
\[ 0 = \lim_{n \rightarrow \infty} \gamma(\nu_n) = \lim_{n \rightarrow \infty} \|\nu_n\|^2 + 2|u_n \nabla u_n|_2^2 - \lambda |u_n|^p \]
\[ \geq \lim_{n \rightarrow \infty} \|\nu_n\|^2 + \|u\|^2 + 2|u \nabla u|_2^2 - \lambda |u|^p \]
\[ = \gamma(u) + \lim_{n \rightarrow \infty} \|\nu_n\|^2, \] (2.6)
so that \( \gamma(\nu) \leq 0 \). If \( \gamma(u) < 0 \), then by Lemma 2.6 there exists \( t \in (0,1) \) such that \( tu \in M(\Omega) \). From \( \gamma(u) = 0 \), we have
\[ 2|u-n \nabla u_n|_2^2 = -\|u_n\|^2 + \lambda |u_n|^p, \]
so that
\[ I(\nu_n) = \frac{1}{4} \|u_n\|^2 + \left( \frac{1}{4} - \frac{1}{p} \right) \lambda |u_n|^p. \]

Therefore we get
\[ c_1 = \lim_{n \rightarrow \infty} I(\nu_n) = \frac{1}{4} \|u\|^2 + \left( \frac{1}{4} - \frac{1}{p} \right) \lambda |u|^p + \frac{1}{4} \lim_{n \rightarrow \infty} \|\nu_n\|^2 \]
\[ \geq \frac{1}{4} \|u\|^2 + \left( \frac{1}{4} - \frac{1}{p} \right) \lambda |u|^p \]
\[ = \frac{1}{4} t^{-2} \|tu\|^2 + \left( \frac{1}{4} - \frac{1}{p} \right) \lambda t^{-p} |tu|^p. \] (2.7)

For \( t \in (0,1) \), we have
\[ c_1 > \frac{1}{4} \|tu\|^2 + \left( \frac{1}{4} - \frac{1}{p} \right) \lambda |tu|^p = I(tu), \]
which contradicts the definition of \( c_1 \). Thus we get
\[ \gamma(u) = 0 \quad \text{and} \quad u \in M(\Omega). \] (2.8)
Combining (2.6) and (2.8), we obtain
\[
\lim_{n \to \infty} \|\nu_n\|^2 = 0. \tag{2.9}
\]
Substituting (2.8) and (2.9) into (2.7), we see
\[
I_\Omega(u) = c_1,
\]
and \(c_1\) is attained by \(u\).

**Step 2.** \(u\) is a radial solution of (2.2).

We use an indirect argument which is similar to [26]. Suppose that \(u \in M(\Omega), I_\Omega(u) = c_1\), but the conclusion (2.3) of the lemma is not true. Then first we can find a function \(\phi \in X(\Omega)\) with the property that
\[
\int_\Omega u^2|\nabla \phi|^2 \, dx < \infty \quad \text{and} \quad \int_\Omega |\nabla u|^2 \phi^2 \, dx < \infty,
\]
but
\[
\langle I'_\Omega(u), \phi \rangle = \int_\Omega \left[ (1 + u^2)|\nabla u| \nabla \phi + |\nabla u|^2 u \phi + Vu \phi - \lambda|u|^{p-2}u \phi \right] \, dx \leq -1.
\]
Choose \(\varepsilon > 0\) small enough such that
\[
\langle I'_\Omega(tu + \sigma \phi), \phi \rangle \leq -\frac{1}{2}, \quad \forall \ |t-1| + |\sigma| \leq \varepsilon.
\]
Let \(\eta\) be a cut-off function such that
\[
\eta(t) = \begin{cases} 1, & |t-1| \leq \frac{1}{2} \varepsilon, \\ 0, & |t-1| \geq \varepsilon. \end{cases}
\]
We estimate \(\sup_t I_\Omega(tu + \varepsilon \eta(t) \phi)\). If \(|t-1| \leq \varepsilon\), then
\[
I_\Omega(tu + \varepsilon \eta(t) \phi) = I_\Omega(tu) + \int_0^1 \langle I'_\Omega(tu + \sigma \varepsilon \eta(t) \phi), \varepsilon \eta(t) \phi \rangle \, d\sigma \leq I_\Omega(tu) - \frac{1}{2} \varepsilon \eta(t). \tag{2.10}
\]
If \(|t-1| \geq \varepsilon\), then \(\eta(t) = 0\), and the above estimate is trivial. Now since \(u \in M(\Omega)\), for \(t \neq 1\) we get \(I_\Omega(tu) < I_\Omega(u)\). Hence it follows from (2.10) that
\[
I_\Omega(tu + \varepsilon \eta(t) \phi) \leq \begin{cases} I_\Omega(tu) < I_\Omega(u), & \text{if } t \neq 1, \\ I_\Omega(u) - \frac{1}{2} \varepsilon \eta(1) = I_\Omega(u) - \frac{1}{2} \varepsilon, & \text{if } t = 1. \end{cases}
\]
In any case we have \(I_\Omega(tu + \varepsilon \eta(t) \phi) < I_\Omega(u) = c_1\). In particular,
\[
\sup_{0 \leq t \leq 2} I_\Omega(tu + \varepsilon \eta(t) \phi) < c_1. \tag{2.11}
\]
Since \(u \in M(\Omega)\), we have
\[
\int_\Omega \left[ (|\nabla u|^2 + Vu^2) + 2u^2|\nabla u|^2 - \lambda|u|^p \right] \, dx = 0. \tag{2.12}
\]
Let
\[
h(t) = \int_{\Omega} \left[ |\nabla(tu + \varepsilon \eta(t) \phi)|^2 + V|tu + \varepsilon \eta(t) \phi|^2 + 2|tu + \varepsilon \eta(t) \phi|^2 |\nabla(tu + \varepsilon \eta(t) \phi)|^2 - \lambda |tu + \varepsilon \eta(t) \phi|^p \right] dx.
\]

Without loss of generality, we assume \( \varepsilon < \frac{1}{4} \). For \( t = 2 \), we have \( \eta(2) = 0 \), and thus from (2.12),
\[
h(2) = \int_{\Omega} \left[ 4(|\nabla u|^2 + V|u|^2) + 32|u|^2 |\nabla u|^2 - 2^p \lambda |u|^p \right] dx
\]
\[
= (4 - 2^p) \int_{\Omega} (|\nabla u|^2 + V|u|^2) dx + (32 - 2^{p+1}) \int_{\Omega} |u|^2 |\nabla u|^2 dx < 0.
\]

For \( t = \frac{1}{2} \), we see
\[
h(\frac{1}{2}) = \int_{\Omega} \left[ \frac{1}{4}(|\nabla u|^2 + V|u|^2) + \frac{1}{8} |u|^2 |\nabla u|^2 - \frac{1}{2^p} \lambda |u|^p \right] dx
\]
\[
= \left( \frac{1}{4} - \frac{1}{2^p} \right) \int_{\Omega} (|\nabla u|^2 + V|u|^2) dx + \left( \frac{1}{8} - \frac{1}{2^p} \right) \int_{\Omega} |u|^2 |\nabla u|^2 dx > 0.
\]

As a result, we can find \( t \in (\frac{1}{2}, 2) \) such that \( h(t) = 0 \), which implies that \( \bar{t}u + \varepsilon \eta(\bar{t}) \phi \in M(\Omega) \). However, it follows from (2.11) that \( I_{\Omega}(\bar{t}u + \varepsilon \eta(\bar{t}) \phi) < c_1 \). Hence, we get a contradiction.

**Step 3.** \( u > 0 \).

Firstly, by Lemma 2.5, we have \( u, |\nabla u| \in L^\infty(\Omega) \). Moreover, by the condition \( V(x) \in C(\mathbb{R}^N, \mathbb{R}) \) and the \( L^p \) estimate, we know that \( u \in W_{loc}^{2,p}(\Omega) \) for any \( p < +\infty \). Hence \( u \in C_{loc}^{1,\alpha}(\Omega), \alpha \in (0, 1). \) Since \( u \) satisfies the equation
\[
-u_{rr} - \frac{N-1}{r} u_r = \frac{\lambda |u|^{p-2} - V + |u_r|^2}{1 + u^2} u, \quad (2.13)
\]
we know that \( u_{rr} \) is continuous, except possibly at 0. Set \( G(r) = \frac{\lambda |u|^{p-2} - V + |u_r|^2}{1 + u^2} u \), and note that \( G(r) \) is continuous on \([0, +\infty)\). Rewriting (2.13) as \( -\frac{d}{dr} (r^{N-1} u_r) = r^{N-1} G(r) \) and integrating from 0 to \( r \), we find
\[
r^{N-1} u_r = -\int_0^r s^{N-1} G(s) ds.
\]

With a change of variable, we get
\[
r^{N-1} u_r = -r \int_0^1 s^{N-1} G(sr) ds \quad \text{or} \quad \frac{u_r}{r} = - \int_0^1 s^{N-1} G(s) ds.
\]

Since
\[
\int_0^1 s^{N-1} G(s) ds \to \frac{G(0)}{N} \quad \text{as} \quad r \to 0,
\]
Hence we can assume \( u(0) \) exists and \( u_r(0) = \frac{G(0)}{N} \). Furthermore, from Equation (2.13) we see that
\[
  u_r \to -\frac{G(0)}{N} \quad \text{as} \quad r \to 0,
\]
and thus \( u \in C^2(\Omega) \).

Secondly, we try to verify that the minimizer of \( c_1 \) will not change sign. If the attained function \( u \) changes sign in \( \Omega \), then \( u^+, u^- \in M(\Omega) \), where \( u^+ = \max\{u, 0\} \), \( u^- = -\min\{u, 0\} \). Thus
\[
  I_\Omega(u^+) < I_\Omega(u) = \inf_{M(\Omega)} I_\Omega(u) \leq I_\Omega(u^-),
\]
which is a contradiction. Therefore either \( u \geq 0 \) or \( u \leq 0 \). Without loss of generality, we can assume \( u \geq 0 \). Now we show that \( u > 0 \). If there exists \( x_0 \) such that \( u(x_0) = 0 \), then \( u'(x_0) = 0 \) for \( u \geq 0 \). By the Strong Maximum Principle (e.g., Gilbarg and Trudinger [18]), \( u = 0 \) near \( x_0 \) and \( u \) will vanish identically, which is impossible since \( u \in M(\Omega) \). Hence \( u > 0 \) and we complete the proof. \( \square \)

3. The proof of Theorem 1.1

In this section we will consider the existence of the nodal solutions of (1.3). For any given \( k \) numbers \( r_j \) \((j = 0, 1, \cdots, k+1)\) such that \( 0 = r_0 < r_1 < r_2 < \cdots < r_k < r_{k+1} = +\infty \), denote
\[
  \begin{align*}
    \Omega^1 &= \{x \in \mathbb{R}^N; |x| < r_1\}, \\
    \Omega^j &= \{x \in \mathbb{R}^N; r_{j-1} < |x| < r_j\}.
  \end{align*}
\]

We will always extend \( u_j \in X(\Omega^j) \) to \( X \) by setting \( u \equiv 0 \) on \( x \in \mathbb{R}^N \setminus \Omega^j \) for every \( u_j \in X(\Omega^j), j = 1, 2, \cdots, k+1 \). In this sense, we use \( I(u_j) \) to replace \( I_{\Omega^j}(u_j) \) and \( \gamma(u_j) \) to replace \( \gamma_{\Omega^j}(u_j) \) in the sequel. Define
\[
  \begin{align*}
    Y_k^\pm(r_1,r_2,\cdots,r_{k+1}) &= \left\{ u \in X \mid u = \pm \sum_{j=1}^{k+1} (-1)^{j-1} u_j, \ u_j \geq 0, \ u_j \neq 0, u_j \in X(\Omega^j), j = 1, 2, \cdots, k+1 \right\}, \\
    M_k^\pm &= \{u \in X \mid \exists 0 < r_1 < r_2 < \cdots < r_k < r_{k+1} = +\infty, \text{such that } u \in Y_k^\pm(r_1,r_2,\cdots,r_{k+1}) \text{ and } u_j \in M(\Omega^j), j = 1, 2, \cdots, k+1 \}. 
  \end{align*}
\]

Note that \( M_k^+ \neq \emptyset, k = 1, 2, \cdots \). In the following we will always refer to \( M_k \) and we will drop the “+”. For \( M_k^- \), everything could be done exactly in the same way. By the arguments of the standard Nehari method [29], it is easy to verify that
\[
  \forall \ u = \sum_{j=1}^{k+1} (-1)^{j-1} u_j \in M_k \iff I(u) = \max_{\alpha_j \geq 0} I \left( \sum_{j=1}^{k+1} \alpha_j \tilde{u}_j \right), \quad (3.1)
\]
where \( \tilde{u}_j = (-1)^{j-1} u_j \).

Set
\[
  c_k = \inf_{M_k} I(u), \ k = 1, 2, \cdots.
\]
Lemma 3.1. $c_k$ is attained provided that $4 < p < 22^*$, $k = 0, 1, \cdots$.

Proof. We will prove by induction that for each $k$ there exists $u_k \in M_k$ such that

$$I(u_k) = c_k.$$ 

The case that $k = 0$ can be deduced by setting $\Omega = \mathbb{R}^N$ in Lemma 2.7. We discuss the case $k \geq 1$ in the following.

Firstly, we prove $I$ is bounded from below on $M_k$ by a positive constant. Let $u \in M_k$; then $u = \sum_{j=1}^{k+1} (-1)^{j-1} u_j$ and $u_j \in M(\Omega^j)$, $j = 1, 2, \cdots, k + 1$. Denote

$$\eta_j^2 = \int_{\Omega_j} (1 + u_j^2)|\nabla u_j|^2 dx + \int_{\Omega_j} V |u_j|^2 dx.$$

By Hölder’s inequality and Sobolev’s inequality we have, with $\theta = (p-2)(N-2)/2(N+2)$,

$$\int_{\Omega_j} |u_j|^p dx \leq \left( \int_{\Omega_j} |u_j|^2 dx \right)^{1-\theta} \left( \int_{\Omega_j} |u_j|^{\frac{2N}{N-2}} dx \right)^{\theta} \leq C \left( \int_{\Omega_j} |u_j|^2 dx \right)^{1-\theta} \left( \int_{\Omega_j} u_j^2 |\nabla u_j|^2 dx \right)^{\frac{\theta N}{N-2}} \leq C \eta_j^{2(1-\theta)} \eta_j^{\frac{2N}{N-2}} = C \eta_j^{2+\frac{2(p-2)}{N+2}}.$$

Then

$$0 = \int_{\Omega_j} (1 + 2u_j^2)|\nabla u_j|^2 dx + \int_{\Omega_j} V |u_j|^2 dx - \int_{\Omega_j} |u_j|^p dx \geq \eta_j^2 - C \eta_j^{2+\frac{2(p-2)}{N+2}},$$

from which

$$\eta_j^2 \geq C_j > 0. \quad (3.2)$$

Then from (3.2) and $p > 4$,

$$I(u) = I \left( \sum_{j=1}^{k+1} (-1)^{j-1} u_j \right) = \sum_{j=1}^{k+1} I(u_j) \geq \sum_{j=1}^{k+1} \left\{ \frac{1}{2} \int_{\Omega_j} (1 + u_j^2)|\nabla u_j|^2 dx + \frac{1}{2} \int_{\Omega_j} V |u_j|^2 dx - \frac{\lambda}{p} \int_{\Omega_j} |u_j|^p dx \right\} \geq \sum_{j=1}^{k+1} \left\{ \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega_j} (|\nabla u_j|^2 + V |u_j|^2) dx + \left( \frac{1}{2} - \frac{2}{p} \right) \int_{\Omega_j} |\nabla u_j|^2 |u_j|^2 dx \right\} \geq \left( \frac{1}{2} - \frac{2}{p} \right) \sum_{j=1}^{k+1} C_j > 0.$$
Hence \( I \) is bounded from below on \( M_k \) by a positive constant.

Secondly, we suppose the claim is true for \( k-1 \) and let \( \{u_m\}_{m \geq 1} \) be a minimizing sequence of \( c_k \) in \( M_k \), that is,

\[
\lim_{m \to \infty} I(u_m) = c_k, \quad u_m \in M_k, \quad m = 1, 2, \cdots.
\]

\( u_m \) corresponds to \( k \) nodes, \( r_1^m, r_2^m, \cdots, r_k^m \), with \( 0 < r_1^m < r_2^m < \cdots < r_k^m < +\infty \). Set

\[
\Omega_k^m = \{ x \in \mathbb{R}^N \mid r_1^m < |x| < r_k^m \},
\]

and

\[
u_m^i = \begin{cases} u_m, & x \in \Omega_k^m, \\ 0, & x \notin \Omega_k^m. \end{cases}
\]

By selecting a subsequence, we may assume that \( \lim_{m \to \infty} r_i^m = r_i \), and clearly \( 0 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq +\infty \). Now we divide the rest of the proof into three steps.

**Step 1.** \( r^i \neq r^{i-1}, \; i = 1, 2, \cdots, k \). Here we denote \( r^0 = 0 \).

If there exists some \( i \in \{1, 2, \cdots, k\} \) such that \( r^i = r^{i-1} \), then \( \lim_{m \to \infty} r_i^m = \lim_{m \to \infty} r_{i-1}^m \). We denote the measure of \( \Omega_k^i \) by \( |\Omega_k^i| \), so that \( |\Omega_k^i| \to 0 \) as \( m \to \infty \).

Since \( u_m^i \in M(\Omega_k^i) \),

\[
I(u_m^i) = \frac{1}{2} \int_{\Omega_k^i} (|\nabla (u_m^i)|^2 + V|u_m^i|^2)dx + \frac{1}{2} \int_{\Omega_k^i} |\nabla u_m^i|^2 |u_m^i|^2 dx - \frac{\lambda}{p} \int_{\Omega_k^i} |u_m^i|^p dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega_k^i} (|\nabla (u_m^i)|^2 + V|u_m^i|^2)dx + \left( \frac{1}{2} - \frac{2}{p} \right) \int_{\Omega_k^i} |\nabla u_m^i|^2 |u_m^i|^2 dx
\]

\[
\geq C \eta^2(u_m^i),
\]

where

\[
\eta^2(u_m^i) = \int_{\Omega_k^i} (1 + (u_m^i)^2)|\nabla (u_m^i)|^2 dx + \int_{\Omega_k^i} V|u_m^i|^2 dx.
\]

On the other hand, it follows from Hölder’s inequality, Sobolev’s inequality, and the fact \( u_m^i \in M(\Omega_k^i) \) that

\[
\eta^2(u_m^i) \leq \int_{\Omega_k^i} (|\nabla (u_m^i)|^2 + V|u_m^i|^2)dx + 2 \int_{\Omega_k^i} |\nabla u_m^i|^2 |u_m^i|^2 dx
\]

\[
= \lambda \int_{\Omega_k^i} |u_m^i|^p dx
\]

\[
\leq \lambda \left( \int_{\Omega_k^i} |u_m^i|^{22^{*}_p} dx \right)^{\frac{p}{22^{*}_p}} |\Omega_k^i|^{1 - \frac{p}{22^{*}_p}}
\]

\[
\leq C \left( \int_{\Omega_k^i} |\nabla u_m^i|^2 |u_m^i|^2 dx \right)^{\frac{p}{4}} |\Omega_k^i|^{1 - \frac{p}{22^{*}_p}}
\]

\[
\leq C \left( \eta^2(u_m^i) \right)^{\frac{p}{4}} |\Omega_k^i|^{1 - \frac{p}{22^{*}_p}}.
\]
Thus

\[
\left( \eta^2(u_m) \right)^{\frac{p-1}{2}} \geq C|\Omega_m|^{-1+\frac{p}{2p'}}. \tag{3.4}
\]

Since \(4 < p < 22^*\), we have from (3.4) that

\[
\eta^2(u_m) \to \infty, \quad \text{as} \quad m \to \infty.
\]

So (3.3) implies

\[
I(u_m) \to \infty, \quad \text{as} \quad m \to \infty. \tag{3.5}
\]

By the inductive assumption and (3.5), for \(\varepsilon > 0\) fixed we can choose \(M > 0\) such that

\[
I(u_m) > c_k - c_{k-1} + \varepsilon, \quad |I(u_m) - c_k| < \varepsilon, \quad \text{as} \quad m \geq M.
\]

Then we may define \(\tilde{u}(x) \in M_{k-1}\) by

\[
\tilde{u}(x) = \begin{cases} 
  u_l(x), & x \in \Omega^l_m \quad \text{as} \quad l < i, \\
  0, & x \in \Omega^i_m, \\
  -u_l(x), & x \in \Omega_l^m \quad \text{as} \quad l > i.
\end{cases}
\]

Hence

\[
I(\tilde{u}) = I(u_m) - I(u_m) < c_k + \varepsilon - (c_k - c_{k-1} + \varepsilon) = c_{k-1}, \quad \text{as} \quad m \geq M,
\]

which contradicts the fact that \(c_{k-1} = \inf_{M_{k-1}} I(u)\). Thus \(r^i \neq r^{i-1}, \quad i = 1, 2, \ldots, k\).

**Step 2.** \(r^k < +\infty\).

If \(r^k = +\infty\), then \(\lim_{m \to \infty} r^k_m = +\infty\). It follows from Lemma 2.1 and \(u^k_m \in M(u^k_m)\) that

\[
\eta^2(u^k_m) \leq \int_{\Omega^k_m} (|\nabla(u^k_m)|^2 + V|u^k_m|^2) dx + 2\int_{\Omega^k_m} |\nabla u^k_m|^2 |u^k_m|^2 dx - \lambda \int_{\Omega^k_m} |u^k_m|^p dx.
\]

\[
\leq \int_{\Omega^k_m} |u^k_m|^2 |u^k_m|^{p-2} dx
\]

\[
\leq \|u^k_m\|^{p-2} \int_{\Omega^k_m} |u^k_m|^2 |x|^{- (1-N)(p-2)/2} dx
\]

\[
\leq C \left( \eta^2(u^k_m) \right)^{\frac{p}{2}} |r^k_m|^{(1-N)(p-2)/2}.
\]

Thus

\[
\eta^2(u^k_m) \geq C|r^k_m|^{-1}. \tag{3.6}
\]

From (3.6) we have

\[
\eta^2(u^k_m) \to \infty, \quad \text{as} \quad m \to \infty.
\]
So by (3.3) we find
\[
I(u_k^m) \to \infty, \quad \text{as} \quad m \to \infty.
\] (3.7)

Repeating Step 1, we can obtain \(r^k < +\infty\).

**Step 3.** \(c_k\) is attained.

Using an argument similar to that in the proof of Lemma 2.7, we can find a subsequence (still denoted by \(\{u_m\}\)) such that
\[
u_m \rightharpoonup u \quad \text{in} \quad X,
\]
\[
u_m \to u \quad \text{in} \quad L^p(\mathbb{R}^N).
\]

Set \(\Omega^i = \{x \in \mathbb{R}^N \mid r^{i-1} < |x| < r^i\}\), for all \(i = 1, 2, \ldots, k+1, r^0 = 0, \) and \(r^{k+1} = +\infty\).

Lemma 2.7 implies that \(\bar{c} = \inf_{M(\Omega^i)} I(u)\) is attained by some positive function \(\hat{u}^i\) which satisfies the following boundary value problem
\[
\begin{aligned}
-\Delta u + Vu - \frac{1}{2} \Delta (|u|^2) u &= \lambda |u|^{p-2} u, \quad x \in \Omega^i, \\
|u|_{\partial \Omega} &= 0.
\end{aligned}
\]

Define \(u_k = \sum_{i=1}^{k+1} (-1)^{i-1} \hat{u}^i(x), (\hat{u}^i(x) = 0, x \notin \Omega^i)\). Then, clearly, \(u_k \in M_k\). Consider the coordinate transformations
\[
\Phi_m: \mathbb{R}^N \to \mathbb{R}^N, \quad m = 1, 2, \ldots,
\]
by
\[
\Phi_m(x) = \varphi_m(|x|) \frac{x}{|x|}, \quad x \in \mathbb{R}^N,
\]
where \(\varphi_m(r) = \left(\frac{r^i - r^{i-1}}{r_m^i - r_m^{i-1}} + r^{i-1}\right)\). For any \(r \in \mathbb{R}\), clearly, \(\Phi_m(\Omega^i) = \Omega^i\).

Let \(y = \Phi_m(x) \in \Omega^i\), if \(x \in \Omega^i_m\). It is easy to show that
\[
|\nabla u(y)| = (R_m^i)^{-1} |\nabla u(x)|,
\] (3.8)
\[
dy = |J_m^i| \, dx, \quad (3.9)
\]
and
\[
a_m^i \leq \left(\frac{\Phi_m(r)}{r}\right)^{N-1} \leq A_m^i, \quad (3.10)
\]
where
\[
R_m^i = \frac{r^i - r^{i-1}}{r_m^i - r_m^{i-1}}, \quad J_m^i = \left[\varphi_m(|x|)\right]^{N-1} (\varphi_m(|x|))' |x|^{1-N},
\]
\[
a_m^i = \left(\min\{r^i/r_m^i, r^{i-1}/r_m^{i-1}\}\right)^{N-1},
\]
We can prove that there exists a constant\( M > 0 \) such that
\[
\lim_{m \to \infty} \left| J_m^i \right| = A_m^i R_m^i.
\]

Clearly,\( a_m^i R_m^i \leq \left| J_m^i \right| \leq A_m^i R_m^i \)

and
\[
R_m^i \to 1, \quad a_m^i \to 1, \quad A_m^i \to 1, \quad J_m^i \to 1, \quad \text{as} \quad m \to \infty.
\]

Let
\[
f(t) = t^2 \int_{\Omega} (|\nabla (u_m^i)|^2 + V |u_m^i|^2) \, dy + 2t^4 \int_{\Omega} |\nabla u_m^i|^2 |u_m^i|^2 \, dy - \lambda t^p \int_{\Omega} |u_m^i|^p \, dy.
\]

Since \( p > 4 \), there exist some \( t_m^i > 0 \), such that \( f(t_m^i) = 0 \), thus \( t_m^i u_m^i \in M(\Omega^i) \).

Now, we claim that
\[
t_m^i \to 1 \quad \text{as} \quad m \to \infty, \quad i = 1, 2, \ldots, k.
\]

Indeed, since \( f(t_m^i) = 0 \), we have
\[
\int_{\Omega} (|\nabla (u_m^i)|^2 + V |u_m^i|^2) \, dy + 2(t_m^i)^4 \int_{\Omega} |\nabla u_m^i|^2 |u_m^i|^2 \, dy
\]
\[
- \lambda(t_m^i)^{p-2} \int_{\Omega} |u_m^i|^p \, dy = 0.
\]

We can prove that there exists a constant \( M > 0 \) such that
\[
0 < t_m^i \leq M < \infty.
\]

By selecting a subsequence, we may assume that \( \lim_{m \to \infty} t_m^i = t^*_i \). Using (3.8)-(3.12), we have that
\[
\lim_{m \to \infty} \int_{\Omega_m} |\nabla u_m^i(y)|^2 \, dy = \lim_{m \to \infty} \int_{\Omega_m} |\nabla u_m^i(x)|^2 \, dx,
\]
\[
\lim_{m \to \infty} \int_{\Omega_m} V(y) |u_m^i(y)|^2 \, dy = \lim_{m \to \infty} \int_{\Omega_m} V(x) |u_m^i(x)|^2 \, dx,
\]
\[
\lim_{m \to \infty} \int_{\Omega_m} |\nabla u_m^i(y)|^2 |u_m^i(y)|^2 \, dy = \lim_{m \to \infty} \int_{\Omega_m} |\nabla u_m^i(x)|^2 |u_m^i(x)|^2 \, dx,
\]
\[
\lim_{m \to \infty} \int_{\Omega_m} |u_m^i|^p \, dy = \lim_{m \to \infty} \int_{\Omega_m} |u_m^i|^p \, dx.
\]

Substituting (3.15)-(3.18) into (3.14), we find that
\[
\lim_{m \to \infty} \int_{\Omega_m} (|\nabla (u_m^i)|^2 + V |u_m^i|^2) \, dx + 2(t^*_i)^2 \int_{\Omega_m} |\nabla u_m^i|^2 |u_m^i|^2 \, dx
\]
\[
- \lambda(t^*_i)^{p-2} \int_{\Omega_m} |u_m^i|^p \, dx = 0.
\]
Let
\[
\lim_{m \to \infty} \int_{\Omega_m} (|\nabla (u_m^i)|^2 + |u_m^i|^2) \, dx = a^i,
\]
\[
\lim_{m \to \infty} \int_{\Omega_m} |\nabla u_m^i|^2 |u_m^i|^2 \, dx = b^i,
\]
\[
\lim_{m \to \infty} \lambda \int_{\Omega_m} |u_m^i|^p \, dx = c^i.
\]
Then (3.19) reads
\[
a^i + 2b^i (t^*_i)^2 - c^i (t^*_i)^{p-2} = 0.
\] (3.20)

But from \(u_m^i(x) \in M(\Omega_m^i)\) we know that
\[
a^i + 2b^i - c^i = 0.
\] (3.21)

Set
\[
h(s) = a^i + 2b^i s^2 - c^i s^{p-2}.
\]
It is easy to verify that \(h(s)\) has only one zero point in \((0, +\infty)\). Taking (3.20) and (3.21) into account, we have \(t^*_i = 1\). So (3.13) holds. Moreover, by (3.13), (3.15)-(3.18), we deduce that
\[
\lim_{m \to \infty} I(t_i^m u_m^i(y)) = \lim_{m \to \infty} I(u_m^i(x)).
\] (3.22)

On the other hand, since \(I(\hat{u}^i) = \inf_{M(\Omega^i)} I(u)\) and \(t_m^i u_m^i(y) \in M(\Omega^i)\), we get
\[
I(\hat{u}^i) \leq I(t_m^i u_m^i(y)),
\]
and hence
\[
\lim_{m \to \infty} I(u_m^i(x)) \geq I(\hat{u}^i), \quad i = 1, 2, \ldots, k+1.
\]

Thus
\[
c_k = \lim_{m \to \infty} I(u_m) = \lim_{m \to \infty} \sum_{i=1}^{k+1} I(u_m^i) \geq \sum_{i=1}^{k+1} I(\hat{u}^i) = I(u_k).
\]

Since \(u_k \in M_k\), we have that \(c_k = I(u_k)\), which means that \(c_k\) is attained. \(\Box\)

Now, we are ready to prove the main result.

**Proof of Theorem 1.1.** By Lemma 3.1, there exists \(u_k \in M_k\) which attains \(c_k\). We will prove that \(u_k\) is indeed a solution to problem (1.3). For convenience, we denote \(u := u_k\). Thus we get \(k\) nodes: \(r_1, r_2, \ldots, r_k\), \(0 < r_1 < r_2 < \cdots < r_k < +\infty\). Clearly, \(u\) satisfies (1.3) in \(\{x \in \mathbb{R}^N : |x| \neq r_j, j = 1, 2, \ldots, k+1\}\). We set \(r := |x|\) and treat (1.3) as an ordinary differential equation. To simplify notation we write \(u(r)\) instead of \(u(|x|)\). We know already that \(u\) is of class \(C^2\) on
\[
E = \{r \in (0, +\infty) : r \neq r_j, j = 1, 2, \ldots, k\}.
\]
Clearly, according to (3.1), there exist $h$ and satisfies, for $r \in E$,
\[ -(1 + u^2)(pN - 1)u' = rN - 1(\lambda |u|^{p-2} - V + |u'|^2)u, \tag{3.23} \]
where $'$ denotes $\frac{d}{dr}$. To complete the proof, it suffices to show that $u$ satisfies (3.23) for all $r > 0$. This is the case if and only if
\[ u'_+ = \lim_{r \to r_j} u'(r) = \lim_{r \to r_j} u'(r) = u'_-, \quad j = 1, 2, \cdots, k. \tag{3.24} \]
In order to get (3.24), we use an indirect argument. Assume that $u'_+ \neq u'_-$ and set $\rho = r_{j-1}, \sigma = r_j, \tau = r_{j+1}$. We may assume that $u \geq 0$ on $[\rho, \sigma], u \leq 0$ on $[\sigma, \tau]$. Now fix $\delta > 0 (0 < \min\{\sigma - \rho, \tau - \sigma\})$ and define $v : [\rho, \tau] \to \mathbb{R}$ by
\[ v(r) = \begin{cases} u(r), & |r - \sigma| \geq \delta, \\ u(\sigma + \delta) + \frac{(r - \sigma + \delta)|u(\sigma + \delta) - u(\sigma - \delta)|}{2\delta}, & |r - \sigma| < \delta. \end{cases} \]
Clearly, $v$ is continuous on $[\rho, \tau]$. Let $s_0 = s_0(\delta) \in (\sigma - \delta, \sigma + \delta)$ be defined by $v(s_0) = 0$. According to (3.1), there exist $\alpha = \alpha(\delta) > 0, \beta = \beta(\delta) > 0$ such that
\[
\int_\rho^{s_0} (|\alpha v'|^2 + V|\alpha v|^2 + 2|\alpha v|^2|\alpha v|^2)rN - 1dr = \lambda \int_\rho^{s_0} |\alpha v|^prN - 1dr,
\]
\[
\int_{s_0}^{\tau} (|\beta v'|^2 + V|\beta v|^2 + 2|\beta v|^2|\beta v|^2)rN - 1dr = \lambda \int_{s_0}^{\tau} |\beta v|^prN - 1dr.
\]
Next we define
\[ w(r) = \begin{cases} \alpha v(r), & \rho \leq r \leq s_0, \\ \beta v(r), & s_0 \leq r \leq \tau, \\ u(r), & \text{otherwise}, \end{cases} \]
hence $w \in M_k$. By the definition of $u$, we have
\[ \psi(u) \leq \psi(w), \]
where
\[ \psi(h) = \int_\rho^{\tau} \left( \frac{1}{2}(h'^2 + Vh^2 + h^2h'^2) - \frac{\lambda}{p}|h|^p \right) rN - 1dr. \tag{3.25} \]
Since $|\sqrt{h}|^p = h^\frac{N}{2}$ is convex for $h > 0$, we have
\[
\frac{1}{p}|w|^p > \frac{1}{p}|u|^p + \frac{w^2 - u^2}{2}|u|^p - 2, \quad \text{if } u, w > 0.
\]
It follows that
\[
\begin{aligned}
& \left\{ \int_\rho^{\sigma - \delta} + \int_{\sigma + \delta}^{\tau} \right\} \left\{ \frac{1}{2}(|w'|^2 + Vw^2 + w^2|w'|^2) - \frac{\lambda}{p}|w|^p \right\} rN - 1dr \\
\leq & \left\{ \int_\rho^{\sigma - \delta} + \int_{\sigma + \delta}^{\tau} \right\} \left\{ \frac{1}{2}(|w'|^2 + Vw^2 + w^2|w'|^2) - \frac{\lambda}{p}|w|^p \\
- & \lambda \left( w^2 - u^2 \right)|w|^{p - 2} \right\} rN - 1dr. \tag{3.26} \end{aligned}
\]
On the other hand,
\[
\int_{\sigma-\delta}^{\sigma+\delta} \left( \frac{1}{2} (|u'|^2 + V w^2 + w^2 |u'|^2) - \frac{\lambda}{p} |w|^p \right) r^{N-1} dr
\]
\[
\leq \int_{\sigma-\delta}^{\sigma+\delta} \left( \frac{1}{2} (|u'|^2 + V w^2 + w^2 |u'|^2) - \frac{\lambda}{p} |w|^p + \frac{\alpha}{2} |u|^p \right) r^{N-1} dr.
\]

By the definition of \( u \), we have
\[
\int_{\rho}^{\tau} (|u'|^2 + V u^2 + 2u^2 |u'|^2) r^{N-1} dr = \lambda \int_{\rho}^{\tau} |u|^p r^{N-1} dr.
\]

Thus, combining (3.25)-(3.28) we obtain
\[
\psi(w) \leq \psi(u) + \int_{\rho}^{\tau} \left\{ \left( \frac{1}{2} (|u'|^2 + V w^2 + w^2 |u'|^2) - \frac{\lambda}{p} |w|^p \right) r^{N-1} dr \right\}
\]
\[
+ \int_{\sigma-\delta}^{\sigma+\delta} \left( \frac{1}{2} (|u|^2 + V u^2 + w^2 |u'|^2) - \frac{\lambda}{p} |w|^p + \frac{\alpha}{2} |u|^p \right) r^{N-1} dr.
\]

Using (3.23), we see that
\[
\int_{\rho}^{\sigma-\delta} \left( \frac{1}{2} (|u'|^2 + V u^2 + w^2 |u'|^2) - \frac{\lambda}{p} |w|^p \right) r^{N-1} dr
\]
\[
= \frac{\alpha^2}{2} \int_{\rho}^{\sigma-\delta} (|u|^2 + V u^2 + \alpha^2 u^2 |u'|^2 - \lambda |u|^p) r^{N-1} dr + \frac{1}{2} \int_{\rho}^{\sigma-\delta} u^2 |u'|^2 r^{N-1} dr
\]
\[
= \frac{\alpha^2}{2} \left[ u(\sigma-\delta) + u^2(\sigma-\delta) \right] (\sigma-\delta)^{N-1} u'(\sigma-\delta) + \frac{(\alpha^2-1)^2}{2} \int_{\rho}^{\sigma-\delta} u^2 |u'|^2 r^{N-1} dr.
\]

Since \( u(\sigma) = 0, (r^{N-1} u')|_{r=\sigma} = 0 \) by (3.23), we obtain
\[
u(\sigma-\delta) = -\delta u'_- + o(\delta),
\]
\[(\sigma-\delta)^{N-1} u'(\sigma-\delta) = \sigma^{N-1} u'_- + o(\delta).
\]

By (3.1), it is easy to verify that
\[
\lim_{\delta \to 0} \alpha(\delta) = \lim_{\delta \to 0} \beta(\delta) = 1.
\]

It follows from (3.30)-(3.32) that
\[
\int_{\rho}^{\sigma-\delta} \left( \frac{1}{2} (|u'|^2 + V u^2 + w^2 |u'|^2) - \frac{\lambda}{p} |w|^p \right) r^{N-1} dr
\]
\[
= \frac{\alpha^2}{2} (\sigma-\delta)^{N-1} u'(\sigma-\delta) u(\sigma-\delta) + \frac{(\alpha^2-1)^2}{2} \int_{\rho}^{\sigma-\delta} u^2 |u'|^2 r^{N-1} dr
\]
\[
= -\frac{\sigma^{N-1}}{2} (u'_-)^2 \delta + o(\delta).
\]
Similarly, we can prove that
\[
\int_{\sigma + \delta}^{\tau} \left( \frac{1}{2} (|w'|^2 + Vw^2 + w^2|w'|^2 + u^2|u'|^2) - \frac{\lambda}{2} w^2 |u|^p \right) r N^{-1} dr = -\frac{\sigma^{N-1}}{2} (u'_+)^2 \delta + o(\delta).
\]

On the other hand, it is not difficult to check that
\[
\int_{\sigma - \delta}^{\sigma + \delta} \left( \frac{1}{2} (V w^2 + w^2 |w'|^2 + u^2 |u'|^2) - \frac{\lambda}{p} w^p + \frac{\lambda}{p} |u|^p \right) r N^{-1} dr = o(\delta).
\]

\[
\frac{1}{2} \int_{\sigma - \delta}^{\sigma + \delta} |w'|^2 r N^{-1} dr = \frac{1}{2} \int_{\sigma - \delta}^{\sigma + \delta} |u'|^2 r N^{-1} dr + o(\delta)
\]
\[
= \left[ u(\sigma + \delta) - u(\sigma - \delta) \right]^2 \left( \frac{(\sigma + \delta)^N}{N} - \frac{(\sigma - \delta)^N}{N} \right) + o(\delta)
\]
\[
= \frac{\sigma^{N-1}}{4} (u'_+ + u'_-)^2 \delta + o(\delta).
\]

Now combining (3.29) and (3.33)-(3.36), we deduce that
\[
\psi(w) \leq \psi(u) - \frac{\sigma^{N-1}}{4} (u'_+ - u'_-)^2 \delta + o(\delta).
\]

This implies that \( \psi(w) < \psi(u) \) for \( \delta > 0 \) small enough, which contradicts the fact that \( \psi(u) \leq \psi(w) \).

As a result, we complete the proof.

Acknowledgement. The authors would like to thank the referees for careful reading of the details, valuable comments and suggestions, including pointing out the reference [27]. Deng and Peng were supported by the funds from NSFC(11071092, 11071094). Peng was also supported by NCET(07-0350) and the PhD specialized grant of the Ministry of Education of China (20100144110001).

REFERENCES

ON QUASILINEAR SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$


