DECAY OF THE SOLUTION FOR THE BIPOLAR EULER-POISSON SYSTEM WITH DAMPING IN DIMENSION THREE

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Abstract. The global solution to Cauchy's problem of the bipolar Euler-Poisson equations with damping in dimension three are constructed when the initial data in $H^3$ norm is small. Moreover, by using a refined energy estimate together with the interpolation trick, we improve the decay estimate in [Y.P. Li and X.F. Yang, J. Diff. Eqs., 252(1), 768–791, 2012], and we need not the smallness assumption of the initial data in $L^1$ space in [Y.P. Li and X.F. Yang, J. Diff. Eqs., 252(1), 768–791, 2012].

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1. Introduction

The compressible bipolar Euler-Poisson equations with damping (BEP) takes the following form:

\[
\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \text{div}(\rho_1 u_1) &= 0, \\
\frac{\partial \rho_1 u_1}{\partial t} + \text{div}(\rho_1 u_1 \otimes u_1) + \nabla P(\rho_1) &= \rho_1 \nabla \phi - \rho_1 u_1, \\
\frac{\partial \rho_2}{\partial t} + \text{div}(\rho_2 u_2) &= 0, \\
\frac{\partial \rho_2 u_2}{\partial t} + \text{div}(\rho_2 u_2 \otimes u_2) + \nabla P(\rho_2) &= -\rho_2 \nabla \phi - \rho_2 u_2, \\
\Delta \phi &= \rho_1 - \rho_2, \quad x \in \mathbb{R}^3, \ t \geq 0,
\end{aligned}
\]

(1.1)

where the unknown functions $\rho_i(x,t), u_i(x,t) \ (i = 1, 2), \phi(x,t)$ represent the charge densities, current densities, velocities, and electrostatic potential, respectively, and the pressure $P = P(\rho)$ is a smooth function with $P'(\rho) > 0$ for $\rho > 0$. The system (1.1) usually describes charged particle fluids, for example, electrons and holes in semiconductor devices, and positively and negatively charged ions in a plasma. We refer to [5, 19] for the physical background of the system (1.1).

In this paper, we will study the global existence and large time behavior of the smooth solutions for the system (1.1) with the following initial data:

\[
\rho_i(x,0) = \rho_{i0}(x) > 0, \ u_i(x,0) = u_{i0}(x), \ i = 1, 2.
\]

(1.2)

A lot of important work has been done on the system (1.1). For the one-dimensional case, we refer to Zhou and Li [30] and Tsuge [24] for the unique existence of the stationary solutions, Natalini [18] and Hsiao and Zhang [8] for global entropy weak solutions in the framework of compensated compactness on the whole real line and bounded domain respectively, Natalini [18] and Hsiao and Zhang [9] for the relaxation-time limit, Gasser and Marcati [3] for the combined limit, Huang and Li [7] for the large-time behavior and quasi-neutral limit of $L^\infty$-solution, Zhu and Hattori [31] for the stability of steady-state solutions to a recombined one-dimensional
bipolar hydrodynamical model, and Gasser, Hsiao and Li [2] for large-time behavior of smooth small solutions.


Recently, Using the classical energy method together with the analysis of the Green’s function, Li and Yang [15] investigated the optimal decay rate of the classical solution of Cauchy’s problem of the system (1.1) when the initial data is small in the space $H^3 \cap L^1$. They deduced that the electric field (a nonlocal term in hyperbolic-parabolic system) slows down the decay rate of the velocity of the BEP system. For more background, see the relevant works on the unipolar Navier-Stokes-Poisson equations (NSP) and unipolar Euler-Poisson equations with damping [11, 12, 29, 25, 27, 28]. In fact, by the detailed analysis of the Green’s function, all of these works show that the presence of the electric field field slows down the decay rate in $L^2$-norm of the velocity of the unipolar NSP system with the factor $\frac{1}{2}$ comparing with the Navier-Stokes system (NS) when the initial perturbation $\rho_0 - \tilde{\rho}, u_0 \in L^p \cap H^3$ with $p \in [1, 2]$.

However, Wang [26] gave a different treatment of the effect of the electric field on the time decay rates of the solution of the unipolar NSP system. The key idea is to make an instead assumption on the initial perturbation $\rho_0 - \tilde{\rho} \in H^{-1}, u_0 \in L^2$. As a result, the electric field does not slow down but rather enhances the time decay rate of the density with the factor $\frac{1}{2}$. The method in [26] is initially established in Guo and Wang [4] for the estimates in the negative Sobolev’s space. The proof in [4] is based on a family of energy estimates with minimum derivative counts and interpolations among them without linear decay analysis. Very recently, using this kind of energy estimate, Tan and Wang [22] discussed the Euler equations with damping in $\mathbb{R}^3$, where they also gave the estimates in the negative Besov’s space.

The main purpose of this paper is to improve the $L^2$-norm decay estimates of the solutions in Li and Yang [15] by using this refined energy method together with the interpolation trick in [4, 26, 22]. Comparing with [4, 26, 22], the main additional difficulties are due to the presence of the electronic field and the coupling of two carriers by the Poisson equation. First, as Wang [26] pointed out, for the bipolar NSP system, there is one term $\partial_t u \nabla \phi$ which cannot be controlled by the dissipation terms when using this refined energy method; see the introduction in [26]. However, after a careful observation and an elaborate calculation, we can deal with this term for the BEP system (1.1); see the estimates (2.26)-(2.28) and (2.38)-(2.39) in Lemma 2.9 and Lemma 2.10. Second, the key point of this refined energy method to get the decay result is to prove the boundedness of the solution in the $\dot{H}^{-s}$ norm ($0 \leq s < 3/2$) (or the $\dot{B}^{s}_{2, \infty}$ norm ($0 < s \leq 3/2$)). Wang [26] separated $s$ into two parts: $s \in (0, \frac{3}{2}]$ and $s \in (\frac{1}{2}, \frac{3}{2})$. For the case $s \in (\frac{1}{2}, \frac{3}{2})$, in [26] it strongly depends on the derived decay result of the case $s = \frac{1}{2}$ and the fact that the electric field enhances the decay of the density for the unipolar case: $\|\rho - \tilde{\rho}\|_{L^2} = \|\nabla(\nabla \phi)\|_{L^2}$. While, for the bipolar case in the present paper, we know that the electric field does not enhance the decay of each density $\rho_1, \rho_2$, since the Poisson equation only suffices to prove $\|\nabla^k (\rho_1 - \rho_2)\|_{L^2} \leq \|\nabla^{k+1} \nabla \phi\|_{L^2}$. So, we have to find some new skills to deal with the case $s \in (\frac{1}{2}, \frac{3}{2})$. In fact, by separating the cases that $s \in [0, \frac{1}{2}], s \in (\frac{1}{2}, 1), s \in [1, \frac{3}{2})$, and $s \in [1, \frac{3}{2}]$. For the space $\dot{H}^{-s}$ and $s \in [0, \frac{3}{2}], s \in (\frac{1}{2}, 1), s \in [1, \frac{3}{2})$, and $s = \frac{3}{2}$ for the space $\dot{B}^{s}_{2, \infty}$, we obtain expected
estimates (see Lemma 2.12, Lemma 2.13, and Subsection 3.2).

Our main results are stated in the following theorems.

**Theorem 1.1.** Let $P'(\rho_i) > 0$ ($i = 1, 2$) for $\rho_i > 0$, and $\bar{\rho} > 0$. Assume that $(\rho_i - \bar{\rho}, u_{i0}, \nabla \phi_0) \in H^3(\mathbb{R}^3)$ for $i = 1, 2$, with $\epsilon_0 =: \|\rho_{i0} - \bar{\rho}, u_{i0}, \nabla \phi_0\|_{H^3(\mathbb{R}^3)}$ small. Then there exists a unique, global, classical solution $(\rho_1 - \bar{\rho}, u_1, \rho_2 - \bar{\rho}, u_2, \phi)$ such that for all $t \geq 0$,

$$
\|(\rho_1 - \bar{\rho}, u_1, \rho_2 - \bar{\rho}, u_2, \nabla \phi)\|^2_{H^3} + \int_0^t \|(u_1, u_2)\|^2_{H^3} + \|(\nabla \rho_1, \nabla \rho_2, \nabla (\nabla \phi))\|^2_{H^2} \, dt \\
\leq C\|(\rho_{i0} - \bar{\rho}, u_{i0}, \rho_{20} - \bar{\rho}, u_{20}, \nabla \phi_0)\|^2_{H^3}
$$

(1.3)

**Theorem 1.2.** Under the assumptions of Theorem 1.1, if $(\rho_{i0} - \bar{\rho}, u_{i0}, \nabla \phi_0) \in \dot{H}^{-s}$ ($i = 1, 2$) for some $s \in [0, 3/2)$ or $(\rho_{i0} - \bar{\rho}, u_{i0}, \nabla \phi_0) \in \dot{B}^{-s}_{2, \infty}$ for some $s \in (0, 3/2]$, then for all $t \geq 0$ there exists a positive constant $C_0$ such that

$$
\|(\rho_1 - \bar{\rho}, u_1, \nabla \phi)(t)\|_{\dot{H}^{-s}} \leq C_0
$$

(1.4)

or

$$
\|(\rho_1 - \bar{\rho}, u_1, \nabla \phi)(t)\|_{\dot{B}^{-s}_{2, \infty}} \leq C_0,
$$

(1.5)

and

$$
\|\nabla^l(\rho_1 - \bar{\rho}, u_1, \nabla \phi)(t)\|_{H^{3-l}} \leq C_0(1 + t)^{-\frac{l+s}{4}} \text{ for } l = 0, 1, 2, \ s \in [0, \frac{3}{2}] ;
$$

(1.6)

$$
\|\nabla^l(\rho_1 - \rho_2)(t)\|_{L^2} \leq C_0(1 + t)^{-\frac{l+s+1}{4}} \text{ for } l = 0, 1, \ s \in [0, \frac{3}{2}] .
$$

(1.7)

**Remark 1.1.** (1.7) is derived from (1.6) and the fact that

$$
\|\nabla^l(\rho_1 - \rho_2)\|_{L^2} = \|\nabla^l \Delta \phi\|_{L^2} \leq \|\nabla^{l+1} \nabla \phi\|_{L^2}, \ l \geq 0,
$$

which shows the presence of the electric field enhances the time decay rate of disparity between two species.

Note that Lemma 2.4 (the Hardy-Littlewood-Sobolev theorem) implies that for $p \in (1, 2]$, $L^p \subset \dot{H}^{-s}$ with $s = 3(\frac{1}{p} - \frac{1}{2})$ and Lemma 2.6 implies that for $p \in [1, 2)$, $L^p \subset \dot{B}^{-s}_{2, \infty}$ with $s = 3(\frac{1}{p} - \frac{1}{2})$. Then Theorem 1.2 yields the following usual optimal decay results of $L^p - L^2$ type.

**Corollary 1.1.** Under the assumptions of Theorem 1.2 except that we replace the $\dot{H}^{-s}$ or $\dot{B}^{-s}_{2, \infty}$ assumption by that $(\rho_{i0} - \bar{\rho}, u_{i0}, \nabla \phi_0) \in L^p$ for some $p \in [1, 2]$, the following decay results hold:

$$
\|\nabla^l(\rho_1 - \bar{\rho}, u_1, \nabla \phi)(t)\|_{H^{3-l}} \leq C_0(1 + t)^{-\frac{l}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{l}{2}}, \text{ for } l = 0, 1, 2;
$$

(1.8)

$$
\|\nabla^l(\rho_1 - \rho_2)(t)\|_{L^2} \leq C_0(1 + t)^{-\frac{l}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{l+1}{4}}, \text{ for } l = 0, 1.
$$

(1.9)
Remark 1.2. From Corollary 1.1, we know the each order derivative of the density \( \rho - \bar{\rho} \) and the velocity \( u \) has the same decay rate in the \( L^2 \) norm as the solution of the Navier-Stokes equations, while the velocity \( u \) in [15] decays at the rate \((1 + t)^{-4/3}\) in the \( L^2 \) norm, which is slower than the rate \((1 + t)^{-4}\) for the compressible Navier-Stokes equations. That is, we improve the decay result in [15], and what’s more, we need not the smallness of the initial data in \( L^1 \) space.

Remark 1.3. The energy method (close the energy estimates at each \( l \)-th level with respect to the spatial derivatives of the solutions) in this paper cannot be applied to the bipolar Navier-Stokes-Poisson equations. In fact, as Wang [26] pointed out, there is one term \( n_u \nabla \phi \) cannot be controlled by the dissipation terms; see the introduction in [26]. Hence, it is also interesting to consider the bipolar Navier-Stokes-Poisson equations by using this new energy method with some big modifications or a new method.

Notations. In this paper, \( \nabla^l \) with an integer \( l \geq 0 \) stands for the any spatial derivative of order \( l \). For \( 1 \leq p \leq \infty \) and an integer \( m \geq 0 \), we use \( L^p \) and \( W^{m,p} \) to denote the usual Lebesgue space \( L^p(\mathbb{R}^n) \) and Sobolev spaces \( W^{m,p}(\mathbb{R}^n) \) with norms \( \| \cdot \|_{L^p} \) and \( \| \cdot \|_{W^{m,p}} \), respectively, and set \( H^m = W^{m,2} \) with norm \( \| \cdot \|_{H^m} \) when \( p = 2 \). In addition, for \( s \in \mathbb{R} \), we define a pseudo-differential operator \( \Lambda^s \) by

\[
\Lambda^s g(x) = \int_{\mathbb{R}^n} |\xi|^s \hat{g}(\xi) e^{2\pi i x \cdot \xi} d\xi,
\]

where \( \hat{g} \) denotes the Fourier transform of \( g \). We define the homogeneous Sobolev space \( \dot{H}^s \) of all \( g \) for which \( \| g \|_{\dot{H}^s} \) is finite, where

\[
\| g \|_{\dot{H}^s} := \| \Lambda^s g \|_{L^2} = \| |\xi|^s \hat{g} \|_{L^2}.
\]

Let \( \eta \in C_0^\infty (\mathbb{R}^3) \) be such that \( \eta(\xi) = 1 \) when \( |\xi| \leq 1 \) and \( \eta(\xi) = 0 \) when \( \xi \geq 2 \). We define the homogeneous Besov’s space \( \dot{B}_{p,q}^{-s} (\mathbb{R}^3) \) with norm \( \| \cdot \|_{\dot{B}_{p,q}^{-s}} \) defined by

\[
\| f \|_{\dot{B}_{p,q}^{-s}} := \sup_{j \in \mathbb{Z}} \| 2^{sj} \Delta_j f \|_{L^p},
\]

where \( \Delta_j f := F^{-1}(\varphi_j * f) \), \( \varphi(\xi) = \eta(\xi) - \eta(2\xi) \) and \( \varphi_j(\xi) = \varphi(2^{-j} \xi) \).

Throughout this paper, we will use a non-positive index \( s \). For convenience, we will change the index to be “\(-s\)” with \( s \geq 0 \). \( C \) or \( C_i \) denotes a positive generic (generally large) constant that may vary at different places. For simplicity, we write \( \int f := \int_{\mathbb{R}^3} f dx \).

The rest of the paper is arranged as follows. In Section 2, we give some useful Sobolev’s inequalities and Besov’s inequalities, then we give an energy estimate in the \( H^3 \) norm and some estimates in \( H^{-s} \) and \( B_{2,\infty}^{-s} \). The proof of global existence and temporal decay results of the solutions will be derived in Section 3.

2. Nonlinear energy estimates

2.1. Preliminaries. In this subsection we give some Sobolev’s inequalities and Besov’s inequalities, which will be used in the next sections.

Lemma 2.1. (Gagliardo-Nirenberg’s inequality). If \( 0 \leq m, k \leq l \), then we have

\[
\| \nabla^k g \|_{L^p} \leq C \| \nabla^m g \|_{L^q}^{1-\theta} \| \nabla^l g \|_{L^r}^\theta,
\]

with

\[
\frac{1}{q} = \frac{m}{p} + \frac{k}{l} - \frac{l}{r}.
\]
where $k$ satisfies
\[
\frac{1}{p} - \frac{k}{n} = (1 - \theta) \left( \frac{1}{q} - \frac{m}{n} \right) + \theta \left( \frac{1}{r} - \frac{l}{n} \right).
\]

**Lemma 2.2.** (Moser-type calculus) (i) Let $k \geq 1$ be an integer and define the commutator
\[
[\nabla^k, g]h = \nabla^{k}(gh) - g\nabla^k h.
\]
Then we have
\[
||[\nabla^k, g]h||_{L^2} \leq C_k (||\nabla g||_{L^\infty} ||\nabla^{k-1} h||_{L^2} + ||\nabla^k g||_{L^2} ||h||_{L^\infty}).
\]
(ii) If $F(\cdot)$ is a smooth function, $f(x) \in H^k \cap L^\infty$, then we have
\[
||\nabla^k F(f)||_{L^2} \leq C(k, F, ||f||_{L^\infty}) ||\nabla^k f||_{L^2}.
\]

**Lemma 2.3.** ([4], Lemma A.5) If $s \geq 0$ and $l \geq 0$, then we have
\[
||\nabla^l g||_{L^2} \leq C ||\nabla^{l+1} g||_{L^2}^{1-\theta} ||g||_{H^{-s}}^{\theta}, \text{ where } \theta = \frac{1}{l+s+1}.
\]

**Lemma 2.4.** ([21], Chapter V, Theorem 1) If $0 < s < n$, $1 < p < q < \infty$, and $\frac{1}{q} + \frac{s}{n} = \frac{1}{p}$, then
\[
||\Lambda^{-s} g||_{L^p} \leq C ||g||_{L^p}.
\]

Next, we give some lemmas on Besov space $\dot{B}^{-s}_{2,\infty}$.

**Lemma 2.5.** ([20], Lemma 4.5) If $k \geq 0$ and $s > 0$, then we have
\[
||\nabla^k f||_{L^2} \leq C ||\nabla^{k+1} f||_{L^2}^{1-\theta} ||f||_{\dot{B}^{-s}_{2,\infty}}^{\theta}, \text{ where } \theta = \frac{1}{l+1+s}.
\]

**Lemma 2.6.** ([20], Lemma 4.6) Suppose that $s > 0$ and $1 \leq p < 2$. We have the embedding $L^p \subset \dot{B}^{-s}_{q,\infty}$ with $1/2 + s/3 = 1/p$. In particular we have the estimate
\[
||f||_{\dot{B}^{-s}_{2,\infty}} \leq C ||f||_{L^p}.
\]

**Lemma 2.7.** ([22], Lemma A.7) If $1 \leq r_1 \leq r_2 \leq \infty$, then
\[
\dot{B}^{-s}_{2,r_1} \subset \dot{B}^{-s}_{2,r_2}.
\]

**Lemma 2.8.** ([22], Lemma A.8) If $m > l \geq k$ and $1 < p < q < r \leq \infty$, then we have
\[
||g||_{\dot{B}^{l}_{2,q}} \leq C ||g||_{\dot{B}^{l}_{2,r}}^{\theta} ||g||_{\dot{B}^{l}_{2,p}}^{1-\theta},
\]
where $l = k\theta + m(1-\theta)$, $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{p}$. 

2.2. Energy estimates in $H^3$-norm. We reformulate the nonlinear system (1.1) for $(ρ_1,u_1,ρ_2,u_2)$ around the equilibrium state $(\bar{ρ},0,\bar{ρ},0)$. Without loss of generality, we can assume $\bar{ρ} = 1$ and $P'(\bar{ρ}) = 1$. Denoting

$$n_i = ρ_i - 1, \ h(n_i) = \frac{P'(ρ_i)}{ρ_i} - 1,$$

the Cauchy problem for $(n_1,u_1,n_2,u_2,φ)$ is given by

$$\begin{aligned}
&\partial_t n_1 + \text{div} u_1 = -u_1 \cdot ∇ n_1 - n_1 \text{div} u_1, \\
&\partial_t u_1 + u_1 + ∇ n_1 - ∇ φ = -u_1 \cdot ∇ u_1 - h(n_1) ∇ n_1, \\
&\partial_t n_2 + \text{div} u_2 = -u_2 \cdot ∇ n_2 - n_2 \text{div} u_2, \\
&\partial_t u_2 + u_2 + ∇ n_2 + ∇ φ = -u_2 \cdot ∇ u_2 - h(n_2) ∇ n_2, \\
&Δ φ = n_1 - n_2,
\end{aligned}$$

(2.1)

In this section, we will derive a priori nonlinear energy estimates for the equivalent system (2.1). Hence we make the a priori assumption that for a sufficiently small constant $\delta > 0$,

$$\|n_i(t)\|_{H^3} + \|u_i(t)\|_{H^3} + \|∇ φ(t)\|_{H^3} ≤ \delta, \ i = 1,2,$$

(2.2)

which together with Sobolev’s inequality, yields

$$1/2 ≤ n_i ≤ 2, \ |h(n_i)| ≤ C|n_i|, \ |h^{(k)}(n_i)| ≤ C, \ i = 1,2, \ \text{for any } k ≥ 1.$$  

(2.3)

We first deduce the following energy estimates, which contain the dissipation estimate for $u_1,u_2$.

Lemma 2.9. Assume that $0 ≤ k ≤ 2$. Then we have

$$\frac{1}{2} \frac{d}{dt} \int \|\nabla^k(n_1,u_1,n_2,u_2,φ)\|^2 + \|\nabla^k(u_1,u_2)\|^2 \leq C(\|\nabla^{k+1}n_1\|^2_{L^2} + \|\nabla^k u_1\|^2_{L^2} + \|\nabla^{k+1}u_2\|^2_{L^2} + \|\nabla^k u_2\|^2_{L^2} + \|\nabla^{k+1}φ\|^2_{L^2}).$$

(2.4)

Proof. For $0 ≤ k ≤ 2$, applying $\nabla^k$ to (2.1)$_1$, (2.1)$_2$ and then multiplying the resulting equations by $\nabla^k n_1, \nabla^k u_1$ respectively, and sumning up and integrating over $\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int \|\nabla^k(n_1,u_1)|^2 + \|\nabla^k u_1\|^2_{L^2} - \int \nabla^k u_1 \nabla^k φ \\
= & - \int \nabla^k n_1 \nabla^k (u_1 \cdot ∇ n_1 + n_1 \text{div} u_1) + \nabla^k u_1 \nabla^k (u_1 \cdot ∇ u_1 + h(n_1) ∇ n_1) \\
= & - \int \nabla^k (u_1 \cdot ∇ n_1) \nabla^k n_1 - \nabla^k (u_1 \cdot ∇ u_1) \nabla^k u_1 - \nabla^k (n_1 \text{div} u_1) \nabla^k n_1 - \nabla^k (h(n_1) ∇ n_1) \nabla^k u_1 \\
:= & I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

(2.5)

We shall first estimate each term in the right hand side of (2.5). By Hölder’s inequalities and Lemma 2.1, we get

$$I_1 = - \int \sum_{0 ≤ l ≤ k} C_l^k \|\nabla^{k-l} u_1 \cdot ∇ n_1 \nabla^l n_1 \|_{L^6/5} \|\nabla^k n_1\|_{L^6} \leq \sum_{0 ≤ l ≤ k} \|\nabla^{k-l} u_1 \nabla^l n_1\|_{L^6/5} \|\nabla^k n_1\|_{L^6} \|\nabla^k u_1\|_{L^2}.$$  

(2.6)
When $0 \leq l \leq \lfloor \frac{k}{2} \rfloor$, by H"{o}lder’s inequality and Lemma 2.1, we have
\[
\| \nabla^{k-l} u_1 \nabla^{l} n_1 \|_{L^{6/5}} \leq \| \nabla^{k-l} u_1 \|_{L^2} \| \nabla^{l+1} n_1 \|_{L^2}
\leq \| u_1 \|_{L^2} \| \nabla^{l+1} n_1 \|_{L^2} \leq \delta(\| \nabla^{l+1} n_1 \|_{L^2} + \| \nabla^{k} u_1 \|_{L^2}),
\tag{2.7}
\]
where $\alpha$ satisfies
\[
l + \frac{3}{2} = \alpha \left( 1 - \frac{l}{k} \right) + (k+1) \frac{l}{k},
\]
which gives $\alpha = \frac{3k-2l}{2k+2} \in [\frac{3}{2}, 3)$ since $l \leq \frac{k}{2}$.

When $[\frac{k}{2}] + 1 \leq l \leq k$, by H"{o}lder’s inequality and Lemma 2.1 again, we obtain
\[
\| \nabla^{k-l} u_1 \nabla^{l} n_1 \|_{L^{6/5}} \leq \| \nabla^{k-l} u_1 \|_{L^2} \| \nabla^{l+1} n_1 \|_{L^2}
\leq \| u_1 \|_{L^2} \| \nabla^{l+1} n_1 \|_{L^2} \| \nabla^{l+1} u_1 \|_{L^2}
\leq \delta(\| \nabla^{l+1} n_1 \|_{L^2} + \| \nabla^{k} u_1 \|_{L^2}),
\tag{2.8}
\]
where $\alpha$ satisfies
\[
k - l + \frac{1}{2} = \alpha \frac{l+1}{k+1} + k \frac{k-l}{k+1},
\]
which implies $\alpha = \frac{3k-2l+1}{2k+2} \in [\frac{1}{2}, 3)$ since $l \geq \frac{k+1}{2}$.

From (2.6), (2.7), and (2.8), one has
\[
I_1 \leq \delta(\| \nabla^{l+1} n_1 \|_{L^2} + \| \nabla^{k} u_1 \|_{L^2}).
\tag{2.9}
\]

For $I_2$, using Lemma 2.1 and H"{o}lder’s inequality, we get
\[
I_2 = -\int \left( \nabla^{k} u_1 \nabla u_1 + u_1 \nabla \nabla^{k} u_1 \right) \nabla^{k} u_1 \leq \| \nabla u_1 \|_{L^\infty} \| \nabla^{k} u_1 \|_{L^2}^2 - \frac{1}{2} \int u_1 \nabla(\nabla^{k} u_1 \nabla^{k} u_1)
\leq \| \nabla u_1 \|_{L^\infty} \| \nabla^{k} u_1 \|_{L^2}^2 + \frac{1}{2} \int \nabla^{k} u_1 \cdot \nabla^{k} u_1 \leq \delta \| \nabla^{k} u_1 \|_{L^2}^2.
\tag{2.10}
\]

For $I_3$, we have
\[
I_3 = -\int \nabla^{k} (n_1 \nabla u_1) \nabla^{k} n_1
= -\int \sum_{0 \leq l \leq k-1} C^l_k \nabla^{k-l} n_1 \nabla^{l} \nabla u_1 \nabla^{k} n_1 - \int n_1 \nabla \nabla^{k} u_1 \nabla^{k} n_1
:= I_{31} + I_{32}.
\tag{2.11}
\]

First, we estimate $I_{31}$. By H"{o}lder’s inequality, Lemma 2.1, and Cauchy’s inequality, we obtain
\[
I_{31} = -\int \sum_{0 \leq l \leq k-1} C^l_k \nabla^{k-l} n_1 \nabla^{l} \nabla u_1 \nabla^{k} n_1
\leq C \sum_{0 \leq l \leq k-1} \| \nabla^{k-l} n_1 \|_{L^{6/5}} \| \nabla^{k+1} n_1 \|_{L^2}.
\tag{2.12}
\]
When \( 0 \leq l \leq \lfloor \frac{k}{2} \rfloor \), using Lemma 2.1 and H"older’s inequality, we have
\[
\| \nabla^{k-l} n_1 \nabla \text{div} u_1 \|_{L^{6/5}} \leq C \| \nabla^{k-l} n_1 \|_{L^3} \| \nabla^{l+1} u_1 \|_{L^2} \\
\leq C \| n_1 \|_{L^3} ^{\frac{l+1}{2}} \| \nabla^{k-l} n_1 \|_{L^2} \| \nabla^{n} u_1 \|_{L^2} \| \nabla^{k} u_1 \|_{L^2} ^{\frac{k-l-1}{2}} \\
\leq C \delta (\| \nabla^{k-l} n_1 \|_{L^2} + \| \nabla^{k} u_1 \|_{L^2}),
\tag{2.13}
\]
where \( \alpha \) satisfies \( l + \frac{3}{2} = \alpha - \frac{k-l+1}{k} \), which yields \( \alpha = \frac{k+2l+3}{2k-2l} \) since \( l \leq \frac{k}{2} \).

When \( \lfloor \frac{k}{2} \rfloor + 1 \leq l \leq k-1 \), using Lemma 2.1 and H"older’s inequality, we have
\[
\| \nabla^{k-l} n_1 \nabla \text{div} u_1 \|_{L^{6/5}} \leq C \| \nabla^{k-l} n_1 \|_{L^3}^{l+1} \| u_1 \|_{L^2} \| \nabla^{k-l} n_1 \|_{L^2} \| u_1 \|_{L^2} \| \nabla^{k} u_1 \|_{L^2} \| \nabla^{k} n_1 \|_{L^2} \| \nabla^{k} u_1 \|_{L^2} ^{l+1} \\
\leq C \delta (\| \nabla^{k-l} n_1 \|_{L^2} + \| \nabla^{k} u_1 \|_{L^2}),
\tag{2.14}
\]
where \( k-l+1 = \frac{l+1}{k} + (k+1) \frac{k-l-1}{k} \),

which yields \( \alpha = \frac{k+2l+3}{2k-2l} \) since \( l \geq \frac{k+1}{2} \).

From (2.12), (2.13), and (2.14), we get
\[
I_{31} \leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2} ^{2} + \| \nabla^{k} u_1 \|_{L^2} ^{2}).
\tag{2.15}
\]

For \( I_{32} \), by H"older’s inequality, Lemma 2.1, and Cauchy’s inequality, we obtain
\[
I_{32} = - \int n_1 \text{div} \nabla v_1 \nabla^{k-1} n_1 = - \int n_1 \text{div} (\nabla v_1 \nabla^{k-1} n_1) + \int \nabla^{k+1} n_1 \nabla v_1 \\
\leq C \| n_1 \|_{L^3} \| \nabla v_1 \|_{L^2} \| \nabla^{k-1} n_1 \|_{L^2} + \| n_1 \|_{L^\infty} \| \nabla^{k+1} n_1 \|_{L^2} \| \nabla v_1 \|_{L^2} \\
\leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2} ^{2} + \| \nabla^{k} u_1 \|_{L^2} ^{2}).
\tag{2.16}
\]

Thus, (2.11), (2.15), and (2.16) imply
\[
I_3 \leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2} ^{2} + \| \nabla^{k} u_1 \|_{L^2} ^{2}).
\tag{2.17}
\]

Next, we will estimate \( I_4 \).
\[
I_4 = - \int \nabla^{k} (h(n_1) \nabla v_1) \nabla v_1 = - \int \sum_{0 \leq i \leq k-1} C_k^{i} \nabla^{k-i} h(n_1) \nabla^{i+1} n_1 \nabla^{k-i} u_1 + h(n_1) \nabla^{k} n_1 \nabla^{k} u_1 \\
:= I_{41} + I_{42}.
\tag{2.18}
\]

For \( I_{41} \), by H"older’s inequality and Lemma 2.1, we obtain
\[
I_{41} = - \int \sum_{0 \leq i \leq k-1} C_k^{i} \nabla^{k-i} h(n_1) \nabla^{i+1} n_1 \nabla^{k} u_1 \leq C \| \nabla^{k-l} n_1 \nabla^{l+1} n_1 \|_{L^2} \| \nabla^{k} u_1 \|_{L^2}.
\tag{2.19}
\]

When \( 0 \leq l \leq \lfloor \frac{k}{2} \rfloor \), by using H"older’s inequality and Lemma 2.1, we get
\[
\| \nabla^{k-l} h(n_1) \nabla^{l+1} n_1 \|_{L^2} \leq \| \nabla^{k-l} h(n_1) \|_{L^3} \| \nabla^{l+1} n_1 \|_{L^2} \\
\leq C \| \nabla^{k-l} h(n_1) \|_{L^3} \| \nabla^{l+1} n_1 \|_{L^2} \| \nabla^{k-l} h(n_1) \|_{L^6} \| \nabla^{l+1} n_1 \|_{L^2} \\
\leq C \| \nabla^{k-l} n_1 \|_{L^2} \| \nabla^{k+1} n_1 \|_{L^2} \| \nabla^{l+1} n_1 \|_{L^2} \| \nabla^{k+1} n_1 \|_{L^2} ^{\frac{1}{2}} \\
\leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2}),
\tag{2.20}
\]
where \( \alpha \) satisfies \( l + \frac{1}{2} = \alpha (1 - \frac{l}{k-1}) + l \), which implies \( \alpha = \frac{3k+3}{2k-2l+2} \in \left[ \frac{3}{2}, 3 \right) \), since \( l \leq k / 2 \).

When \( \left( \frac{k}{2} \right) + 1 \leq l \leq k - 1 \), by Hölder’s inequality and Lemma 2.1, we get

\[
\| \nabla^{k-l} h(n_1) \nabla^{l+1} n_1 \|_{L^2} \leq \| \nabla^{k-l} h(n_1) \|_{L^2} \| \nabla^{l+1} n_1 \|_{L^2} \\
\leq C \| \nabla^{k-l} h(n_1) \|_{L^2} \| \nabla^{k+1} h(n_1) \|_{L^2} \| \nabla^2 n_1 \|_{L^2} \| \nabla^{l+1} n_1 \|_{L^2} \\
\leq C \| \nabla^{k+l} n_1 \|_{L^2} \| \nabla^{k+1} n_1 \|_{L^2} \| \nabla^2 n_1 \|_{L^2} \| \nabla^{k+1} n_1 \|_{L^2},
\]

(2.21)

where \( \alpha \) satisfies

\[
k - l + \frac{1}{2} = \alpha \left( \frac{l}{k-1} + (k+1) \left( 1 - \frac{l}{k-1} \right) \right),
\]

which implies \( \alpha = 2 + \frac{k+1}{2l} \in \left[ \frac{3}{2}, 3 \right) \) since \( l \geq k / 2 \).

Thus, from (2.18), (2.19), (2.20), and (2.21), we deduce that

\[
I_4 \leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2}^2 + \| \nabla^k u_1 \|_{L^2}^2).
\]

(2.22)

Hence, for \( n_1 \) and \( u_1 \), we have

\[
\frac{1}{2} \frac{d}{dt} \int \| \nabla^k (n_1, u_1) \|^2 + \| \nabla^k u_1 \|_{L^2}^2 - \int \nabla^k u_1 \nabla^k \nabla \phi \leq C \delta \| \nabla^{k+1} n_1 \|_{L^2}^2.
\]

(2.23)

In the same way, we can get the following estimates for \( n_2 \) and \( u_2 \), that is,

\[
\frac{1}{2} \frac{d}{dt} \int \| \nabla^k (n_2, u_2) \|^2 + \| \nabla^k u_2 \|_{L^2}^2 + \int \nabla^k u_2 \nabla^k \nabla \phi \leq C \delta \| \nabla^{k+1} n_2 \|_{L^2}^2.
\]

(2.24)

Lastly, we will estimate the last term in left hand side of (2.23) and (2.24). By using the Poisson equation, we estimate them simultaneously as follows:

\[
- \int \nabla^k \nabla \phi \cdot \nabla u_1 + \int \nabla^k \nabla \phi \cdot \nabla u_2 = \int \nabla^k (\text{div} u_1) \nabla^k \phi - \int \nabla^k (\text{div} u_2) \nabla^k \phi \\
= - \int \nabla^{k} \partial_t n_1 + \text{div}(n_1 u_1) \nabla^k \phi + \int \nabla^{k} \partial_t n_2 + \text{div}(n_2 u_2) \nabla^k \phi \\
= - \int \nabla^k (\partial_t n_1 - n_2) \nabla^k \phi - \int \nabla^k (\text{div}(n_1 u_1)) \nabla^k \phi + \int \nabla^k (\text{div}(n_2 u_2)) \nabla^k \phi \\
= - \int \nabla^k \partial_t \Delta \phi \nabla^k \phi - \int \nabla^k (\text{div}(n_1 u_1)) \nabla^k \phi + \int \nabla^k (\text{div}(n_2 u_2)) \nabla^k \phi \\
= \frac{1}{2} \frac{d}{dt} \| \nabla^k \nabla \phi \|_{L^2}^2 + \int \nabla^k (n_1 u_1) \nabla^k \nabla \phi - \int \nabla^k (n_2 u_2) \nabla^k \nabla \phi \\
:= \frac{1}{2} \frac{d}{dt} \| \nabla^k \nabla \phi \|_{L^2}^2 + I_{51} + I_{52}.
\]

(2.25)

For \( I_{51} \), when \( k = 0 \), by Hölder’s inequality, Sobolev’s inequality, and Cauchy’s inequality, we have

\[
\int n_1 u_1 \nabla \phi \leq C \| \nabla \phi \|_{L^6} \| n_1 \|_{L^2} \| u_1 \|_{L^3} \leq C \delta (\| \nabla \nabla \phi \|_{L^2} + \| u_1 \|_{L^2}^2).
\]

(2.26)

Similarly, for \( k = 1 \) and \( k = 2 \), we get

\[
\int \nabla (n_1 u_1) \nabla \phi = - \int (n_1 u_1) \nabla^2 \nabla \phi \leq C \| \nabla^2 \nabla \phi \|_{L^2} \| n_1 \|_{L^6} \| u_1 \|_{L^6} \| n \|_{L^3} \\
\leq C \| \nabla^2 \nabla \phi \|_{L^2} \| n_1 \|_{L^2} \| u_1 \|_{L^6} \| n \|_{L^6} \leq C \delta (\| \nabla^2 \nabla \phi \|_{L^2} + \| \nabla u_1 \|_{L^2}^2);
\]

(2.27)
\[ \int \nabla^2(n_1 u_1) \nabla^2 \nabla \phi = - \int \nabla(n_1 u_1) \nabla^3 \nabla \phi \leq \| \nabla^3 \nabla \phi \|_{L^2} \| \sum_{0 \leq l \leq 1} \nabla^{1-l} n_1 \nabla^l u_1 \|_{L^2} \]
\[ \leq C \| \nabla^3 \nabla \phi \|_{L^2} \| \nabla^n n_1 \|_{L^2} \| \nabla^3 n_1 \|_{L^2} \| u_1 \|_{L^2} \| \nabla^2 u_1 \|_{L^2} \]
\[ \leq C \delta (\| \nabla^3 \nabla \phi \|_{L^2}^2 + \| \nabla^3 n_1 \|_{L^2}^2 + \| \nabla^2 u_1 \|_{L^2}^2), \quad (2.28) \]

where \( \alpha = \frac{t}{1+t} \), \( t = 0.1 \).

I_{52} can be estimated in the same way. Hence, from (2.25) to (2.28), we have
\[ I_{51} + I_{52} \geq -C \delta (\| \nabla^{k+1} n_1 \|_{L^2}^2 + \| \nabla^k u_1 \|_{L^2}^2 + \| \nabla^{k+1} u_2 \|_{L^2}^2 + \| \nabla^{k+1} \nabla \phi \|_{L^2}^2). \quad (2.29) \]

Combining (2.23), (2.24), (2.25), and (2.29), we deduce that
\[ \frac{1}{2} \frac{d}{dt} \int |\nabla^{k+1}(n_1, u_1, n_2, u_2, \nabla \phi)|^2 + \| \nabla^{k+1}(u_1, u_2) \|_{L^2}^2 \]
\[ \leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2}^2 + \| \nabla^k u_1 \|_{L^2}^2 + \| \nabla^{k+1} u_2 \|_{L^2}^2 + \| \nabla^{k+1} \nabla \phi \|_{L^2}^2). \quad (2.30) \]

This proves Lemma 2.9.

Next, we derive the second type of energy estimates excluding \( n_1, u_1 \) and \( n_2, u_2 \) themselves.

**Lemma 2.10.** If \( 0 \leq k \leq 2 \), then we have
\[ \frac{1}{2} \frac{d}{dt} \int |\nabla^{k+1}(n_1, u_1, n_2, u_2, \nabla \phi)|^2 + \| \nabla^{k+1}(u_1, u_2) \|_{L^2}^2 \]
\[ \leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2}^2 + \| \nabla^{k+1} u_1 \|_{L^2}^2 + \| \nabla^{k+1} u_2 \|_{L^2}^2 + \| \nabla^{k+1} \nabla \phi \|_{L^2}^2). \quad (2.31) \]

**Proof.** Applying \( \nabla^{k+1} \) to (2.1)₁, (2.1)₂ and then multiplying the resulting equations by \( \nabla^{k+1} n_1, \nabla^{k+1} u_1 \) respectively, summing up, and integrating over \( \mathbb{R}^3 \), one has
\[ \frac{1}{2} \frac{d}{dt} \int |\nabla^{k+1}(n_1, u_1)|^2 + \| \nabla^{k+1} u_1 \|_{L^2}^2 = \int \nabla^{k+1} u_1 \cdot \nabla^{k+1} \nabla \phi \]
\[ = - \int \nabla^{k+1} n_1 \nabla^{k+1}(u_1 \cdot \nabla n_1 + n_1 \div u_1) + \nabla^{k+1} u_1 \nabla^{k+1}(u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1) \]
\[ = - \int \nabla^{k+1}(u_1 \cdot \nabla n_1) \nabla^{k+1} n_1 + \nabla^{k+1}(u_1 \cdot \nabla u_1) \nabla^{k+1} u_1 \]
\[ - \int \nabla^{k+1}(n_1 \div u_1) \nabla^{k+1} n_1 + \nabla^{k+1}(h(n_1) \nabla n_1) \nabla^{k+1} u_1 \]
\[ := J_1 + J_2, \quad 0 \leq k \leq 2. \quad (2.32) \]

Now we shall estimate \( J_1 \) and \( J_2 \). By Lemma 2.2, Hölder’s inequality, and Cauchy’s inequality, we get
\[ J_1 = - \int \nabla^{k+1}(u_1 \cdot \nabla n_1) \nabla^{k+1} n_1 + \nabla^{k+1}(u_1 \cdot \nabla u_1) \nabla^{k+1} u_1 \]
\[ = - \int \nabla^{k+1}(u_1) \cdot \nabla n_1 \nabla^{k+1} n_1 + (\nabla^{k+1} u_1) \cdot \nabla^{k+1} u_1 \]
\[ - \int u_1 \cdot \nabla \nabla^{k+1} n_1 \nabla^{k+1} n_1 + (u_1 \cdot \nabla^{k+1} u_1) \cdot \nabla^{k+1} u_1 \]
\[ \leq C(\| \nabla u_1 \|_{L^\infty} \| \nabla^{k+1} n_1 \|_{L^2} + \| \nabla^{k+1} u_1 \|_{L^2} \| n_1 \|_{L^\infty}) \| \nabla^{k+1} n_1 \|_{L^2} \]
\[ + \| \nabla u_1 \|_{L^\infty} \| \nabla^{k+1} u_1 \|_{L^2} - \frac{1}{2} \int \nabla (\nabla^{k+1} n_1 \nabla^{k+1} n_1 + \nabla^{k+1} u_1 \cdot \nabla^{k+1} u_1) \leq C \| \nabla (n_1, u_1) \|_{L^\infty} \| \nabla^{k+1} (n_1, u_1) \|_{L^2}^2 + \frac{1}{2} \text{div} u_1 \nabla^{k+1} n_1 \nabla^{k+1} n_1 + \text{div} u_1 \nabla^{k+1} u_1 \cdot \nabla^{k+1} u_1 \leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2}^2 + \| \nabla^{k+1} u_1 \|_{L^2}^2). \] (2.33)

In the same way, one can deduce that
\[ J_2 \leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2}^2 + \| \nabla^{k+1} u_1 \|_{L^2}^2). \] (2.34)

Thus we have
\[ \frac{1}{2} \frac{d}{dt} \int \| \nabla^{k+1} (n_1, u_1) \|^2 + \frac{d}{dt} \| \nabla^{k+1} u_1 \|^2 - \int \nabla^{k+1} u_1 \nabla^{k+1} \phi \leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2}^2 + \| \nabla^{k+1} u_1 \|_{L^2}^2). \] (2.35)

The similar estimate of \( n_2, u_2 \) is
\[ \frac{1}{2} \frac{d}{dt} \int \| \nabla^{k+1} (n_2, u_2) \|^2 + \frac{d}{dt} \| \nabla^{k+1} u_2 \|^2 + \int \nabla^{k+1} u_2 \nabla^{k+1} \phi \leq C \delta (\| \nabla^{k+1} n_2 \|_{L^2}^2 + \| \nabla^{k+1} u_2 \|_{L^2}^2). \] (2.36)

Finally, we give the estimates of the last terms in the left hand side of (2.35) and (2.36) as follows:
\[
- \int \nabla^{k+1} \nabla \phi \cdot \nabla^{k+1} u_1 + \int \nabla^{k+1} \nabla \phi \cdot \nabla^{k+1} u_2
= \int \nabla^{k+1} (\text{div} u_1) \nabla^{k+1} \phi - \int \nabla^{k+1} (\text{div} u_2) \nabla^{k+1} \phi
= - \int \nabla^{k+1} \partial_t n_1 + \text{div}(n_1 u_1) \nabla^{k+1} \phi + \int \nabla^{k+1} \partial_t n_2 + \text{div}(n_2 u_2) \nabla^{k+1} \phi
= - \int \nabla^{k+1} \partial_t (n_1 - n_2) \nabla^{k+1} \phi - \int \nabla^{k+1} (\text{div}(n_1 u_1)) \nabla^{k+1} \phi + \int \nabla^{k+1} (\text{div}(n_2 u_2)) \nabla^{k+1} \phi
= - \int \nabla^{k+1} \partial_t \Delta \phi \nabla^{k+1} \phi - \int \nabla^{k+1} (\text{div}(n_1 u_1)) \nabla^{k+1} \phi + \int \nabla^{k+1} (\text{div}(n_2 u_2)) \nabla^{k+1} \phi
= \frac{1}{2} \frac{d}{dt} \| \nabla^{k+1} \nabla \phi \|_{L^2}^2 + \int \nabla^{k+1} (n_1 u_1) \nabla^{k+1} \nabla \phi - \int \nabla^{k+1} (n_2 u_2) \nabla^{k+1} \nabla \phi
= \frac{1}{2} \frac{d}{dt} \| \nabla^{k+1} \nabla \phi \|_{L^2}^2 + J_3 + J_4. \] (2.37)

Using Hölder’s inequality, Lemma 2.2, and Cauchy’s inequality, we obtain
\[ J_3 = \int \nabla^{k+1} (n_1 u_1) \cdot \nabla^{k+1} \nabla \phi \leq C \| \nabla^{k+1} \nabla \phi \|_{L^2} \| \nabla^{k+1} (n_1 u_1) \|_{L^2} \leq C \| \nabla^{k+1} \nabla \phi \|_{L^2} (\| n_1 \|_{L^\infty} \| \nabla^{k+1} u_1 \|_{L^2} + \| u_1 \|_{L^\infty} \| \nabla^{k+1} n_1 \|_{L^2}) \leq C \delta (\| \nabla^{k+1} u_1 \|_{L^2}^2 + \| \nabla^{k+1} n_1 \|_{L^2}^2 + \| \nabla^{k+1} \nabla \phi \|_{L^2}^2). \] (2.38)

Similarly, we have
\[ J_4 = \int \nabla^{k+1} (n_2 u_2) \cdot \nabla^{k+1} \nabla \phi \leq C \| \nabla^{k+1} \nabla \phi \|_{L^2} \| \nabla^{k+1} (n_2 u_2) \|_{L^2} \leq C \delta (\| \nabla^{k+1} u_2 \|_{L^2}^2 + \| \nabla^{k+1} n_2 \|_{L^2}^2 + \| \nabla^{k+1} \nabla \phi \|_{L^2}^2). \] (2.39)

Hence, plugging (2.33), (2.34), (2.37), (2.38), and (2.39) into (2.32), we deduce (2.31). This proves Lemma 2.10. \( \blacksquare \)
Now, we shall recover the dissipation estimate for $n_1, n_2$.

**Lemma 2.11.** Assume that $0 \leq k \leq 2$, then we have

$$
\frac{d}{dt} \left\{ \int \nabla^k u_1 \cdot \nabla \nabla^k n_1 + \nabla^k u_2 \cdot \nabla \nabla^k n_2 \right\} + C \| \nabla^{k+1} (n_1, n_2, \nabla \phi) \|^2_{L^2} \\
\leq C(\| \nabla^k u_1 \|^2_{L^2} + \| \nabla^{k+1} u_1 \|^2_{L^2} + \| \nabla^k u_2 \|^2_{L^2} + \| \nabla^{k+1} u_2 \|^2_{L^2}).
$$
(2.40)

**Proof.** Let $0 \leq k \leq 2$. Applying $\nabla^k$ to (2.1) and then multiplying the resulting equality by $\nabla \nabla^k n_1$, we have

$$
\| \nabla^{k+1} n_1 \|^2_{L^2} - \int \nabla \nabla^k n_1 \nabla \phi \leq - \int \nabla^k \partial_t u_1 \cdot \nabla \nabla^k n_1 + C \| \nabla^k u_1 \|_{L^2} \| \nabla^{k+1} n_1 \|_{L^2} + \| \nabla^k (u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1) \|_{L^2} \| \nabla^{k+1} n_1 \|_{L^2}.
$$
(2.41)

First, we estimate the first term in the right hand side of (2.39):

$$
- \int \nabla^k u_1 \partial_t u_1 \cdot \nabla \nabla^k n_1 = - \frac{d}{dt} \int \nabla^k u_1 \cdot \nabla \nabla^k n_1 - \int \nabla^k \text{div} u_1 \nabla^k \partial_t n_1 \\
= - \frac{d}{dt} \int \nabla^k u_1 \cdot \nabla \nabla^k n_1 + \| \nabla^k \text{div} u_1 \|^2_{L^2} + \int \nabla^k \text{div} u_1 \nabla^k (u_1 \cdot \nabla n_1 + n_1 \text{div} u_1). 
$$
(2.42)

Next, we shall estimate the last two terms in (2.40) by

$$
\int \nabla^k \text{div} u_1 \cdot \nabla^k (u_1 \cdot \nabla n_1) = \int \sum_{0 \leq l \leq k} C^l_k \nabla^l u_1 \cdot \nabla \nabla^k-n_1 \cdot \nabla^k \text{div} u_1 \\
\leq C \sum_{0 \leq l \leq k} \| \nabla^l u_1 \cdot \nabla \nabla^k-n_1 \|_{L^2} \| \nabla^{k+1} u_1 \|_{L^2}.
$$
(2.43)

If $l = 0$, then

$$
\| u_1 \cdot \nabla \nabla^k n_1 \|_{L^2} \| \nabla^{k+1} u_1 \|_{L^2} \leq C \| \nabla^k u_1 \|_{L^2} \| \nabla^{k+1} u_1 \|_{L^2} \| \nabla^{k+1} u_1 \|_{L^2} \\
\leq C \delta (\| \nabla^{k+1} n_1 \|^2_{L^2} + \| \nabla^{k+1} u_1 \|^2_{L^2}).
$$
(2.44)

If $1 \leq l \leq [k/2]$, using Hölder’s inequality and Lemma 2.1, we get

$$
\| \nabla^l u_1 \cdot \nabla \nabla^{k-l} n_1 \|_{L^2} \leq C \| \nabla^{k+1-l} \|_{L^6} \| \nabla^l u_1 \|_{L^3} \\
\leq C \| n_1 \|_{L^\frac{3}{2}} \| \nabla^{k+1} n_1 \|_{L^2} \| \nabla^{k+1} u_1 \|_{L^2} \\
\leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2} + \| \nabla^{k+1} u_1 \|_{L^2}),
$$
(2.45)

where $\alpha = \frac{3k+3}{2k-2\alpha+4} \in [3/2, 3)$, since $l \leq k/2$.

If $[k/2] + 1 \leq l \leq k$, using Hölder’s inequality and Lemma 2.1 again, we obtain

$$
\| \nabla^l u_1 \cdot \nabla \nabla^{k-l} n_1 \|_{L^2} \leq C \| \nabla^{k+1-l} \|_{L^3} \| \nabla^l u_1 \|_{L^6} \\
\leq C \| \nabla^\alpha n_1 \|_{L^\frac{3}{2}} \| \nabla^{k+1} n_1 \|_{L^2} \| u_1 \|_{L^2} \| \nabla^{k+1} u_1 \|_{L^2} \\
\leq C \delta (\| \nabla^{k+1} n_1 \|_{L^2} + \| \nabla^{k+1} u_1 \|_{L^2}),
$$
(2.46)

where $\alpha = \frac{3k+3}{2k+2} \in [3/2, 3)$, since $l \geq k/2$. 

Thus, from (2.44), (2.45), and (2.46), we obtain
\[
\int \nabla^k \text{div} u_1 \cdot \nabla^k (u_1 \cdot \nabla n_1) \leq C \delta \left( \| \nabla^{k+1} n_1 \|_{L^2} + \| \nabla^{k+1} u_1 \|_{L^2} \right).
\] (2.47)

Similarly, we also get
\[
\int \nabla^k \text{div} u_1 \cdot \nabla^k (n_1 \text{div} u_1) \leq C \delta \left( \| \nabla^{k+1} n_1 \|_{L^2} + \| \nabla^{k+1} u_1 \|_{L^2} \right),
\] (2.48)

and
\[
\| \nabla^k (u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1) \|_{L^2} \leq C \delta \left( \| \nabla^{k+1} n_1 \|_{L^2} + \| \nabla^{k+1} u_1 \|_{L^2} \right).
\] (2.49)

Hence, by (2.40)-(2.49), we have
\[
\frac{d}{dt} \int \nabla^k u_1 \cdot \nabla \nabla^k n_1 + C \| \nabla^{k+1} n_1 \|_{L^2} \leq C \left( \| \nabla^k u_1 \|_{L^2}^2 + \| \nabla^{k+1} u_1 \|_{L^2}^2 \right).
\] (2.50)

On the other hand, by a method similar to the above, we have
\[
\frac{d}{dt} \int \nabla^k u_2 \cdot \nabla \nabla^k n_2 + C \| \nabla^{k+1} n_2 \|_{L^2} \leq C \left( \| \nabla^k u_2 \|_{L^2}^2 + \| \nabla^{k+1} u_2 \|_{L^2}^2 \right).
\] (2.51)

Finally, using the Poisson equation in (2.1), the second term on the left hand side of (2.50) and (2.51) can be estimated as
\[
- \int \nabla \nabla^k n_1 \nabla^k \phi + \int \nabla \nabla^k n_2 \nabla^k \phi = \frac{1}{2} \| \nabla^{k+1} \phi \|_{L^2}^2.
\] (2.52)

Summing (2.50) and (2.51), and using (2.52), one has
\[
\frac{d}{dt} \left\{ \int \nabla^k u_2 \cdot \nabla \nabla^k n_2 + \nabla^k u_1 \cdot \nabla \nabla^k n_1 \right\} + C \| \nabla^{k+1} (n_1, n_2, \nabla \phi) \|_{L^2} \leq C \left( \| \nabla^k u_1 \|_{L^2}^2 + \| \nabla^{k+1} u_1 \|_{L^2}^2 + \| \nabla^k u_2 \|_{L^2}^2 + \| \nabla^{k+1} u_2 \|_{L^2}^2 \right).
\] (2.53)

This proves (2.40). \qed

### 2.3. Estimates in $\dot{H}^{-s}(\mathbb{R}^3)$

The following lemma plays a key role in the proof of Theorem 1.2. It shows an energy estimate of the solutions in the negative Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$.

**Lemma 2.12.** If $\| n_{i0}, u_{i0}, \nabla \phi_0 \|_{\dot{H}^s} \ll 1$ with $i = 1, 2$, for $s \in (0, \frac{1}{2}]$, we have
\[
\frac{d}{dt} \left\{ \| n_i, u_i, \nabla \phi \|_{\dot{H}^{-s}}^2 \right\} \leq C (\| n_i \|^2_{\dot{H}^1} + \| u_i \|^2_{\dot{H}^2}) \| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}}, \quad i = 1, 2,
\] (2.54)

and for $s \in (\frac{1}{2}, \frac{3}{2})$ we have, for $i = 1, 2$,
\[
\frac{d}{dt} \left\{ \| n_i, u_i, \nabla \phi \|_{\dot{H}^{-s}}^2 \right\} \leq C \left\{ \| n_i, u_i \|_{L^2}^{\frac{1}{2}} \| \nabla (n_i, u_i) \|_{L^1}^{\frac{3}{2} - s} + \| u_i \|_{L^2} \| n_i \|_{L^2}^{\frac{1}{2}} \| \nabla n_i \|_{L^2}^{\frac{1}{2} - s} \right\} \| (n_i, u_i, \nabla \phi) \|_{\dot{H}^{-s}},
\] (2.55)
**Proof.** Applying $\Lambda^{-s}$ to (2.2)_1, (2.2)_2 and multiplying the resulting identity by $\Lambda^{-s}u_i$, $\Lambda^{-s}u_i$, respectively, and integrating over $\mathbb{R}^3$ by parts, we get

$$
\frac{d}{dt} \left( |\Lambda^{-s}u_i|^2 + |\Lambda^{-s}u_i|^2 \right) + \int |\nabla \Lambda^{-s}u_i|^2 + (-1)^i \int \Lambda^{-s} \nabla \phi \cdot \Lambda^{-s}u_i
$$

$$
= \int \Lambda^{-s}(-n_i \text{div}u_i - u_i \cdot \nabla n_i) \Lambda^{-s}u_i - \Lambda^{-s}(u_i \cdot \nabla u_i + h(n_i) \nabla n_i) \cdot \Lambda^{-s}u_i
$$

$$
\leq C|n_i \text{div}u_i + u_i \cdot \nabla n_i||_{\dot{H}^{-s}}||n_i||_{\dot{H}^{-s}} + ||u_i \cdot \nabla u_i + h(n_i) \nabla n_i||_{\dot{H}^{-s}}||u_i||_{\dot{H}^{-s}}. \quad (2.56)
$$

If $s \in (0,1/2]$, then by Lemma 2.1, Lemma 2.3, and Young’s inequality, the right hand side of (2.56) can be estimated as follows:

$$
||n_i \text{div}u_i||_{\dot{H}^{-s}} \leq C||n_i \text{div}u_i||_{L^{2+s/3}} \leq C||n_i||_{L^{3/s}} ||\nabla u_i||_{L^2}
$$

$$
\leq C||\nabla n_i||_{L^2}^{1/2+s} ||\nabla^2 n_i||_{L^2}^{1/2-s} ||\nabla u_i||_{L^2}
$$

$$
\leq C(||\nabla n_i||_{H^1}^2 + ||\nabla u_i||_{L^2}^2), \quad (2.57)
$$

where we have used the facts $\frac{1}{2} + \frac{s}{3} < 1$ and $\frac{3}{s} \geq 6$.

Similarly, it holds that

$$
||u_i \cdot \nabla n_i||_{\dot{H}^{-s}} \leq C(||\nabla u_i||_{H^1}^2 + ||\nabla n_i||_{L^2}^2), \quad (2.58)
$$

$$
||u_i \cdot \nabla u_i||_{\dot{H}^{-s}} \leq C(||\nabla u_i||_{H^1}^2 + ||\nabla n_i||_{L^2}^2), \quad (2.59)
$$

$$
||h(n_i) \cdot \nabla n_i||_{\dot{H}^{-s}} \leq C(||\nabla n_i||_{H^1}^2 + ||\nabla n_i||_{L^2}^2). \quad (2.60)
$$

Now if $s \in (1/2,3/2)$, then $1/2 + s/3 < 1$ and $2 < 3/s < 6$. We shall estimate the right hand side of (2.55) in a different way. Using Sobolev’s inequality, we have

$$
||n_i \text{div}u_i||_{\dot{H}^{-s}} \leq C||n_i \text{div}u_i||_{L^{1/2+1/2s}} \leq C||n_i||_{L^{3/s}} ||\nabla u_i||_{L^2}
$$

$$
\leq C||n_i||_{L^2}^{s-1/2} ||\nabla n_i||_{L^2}^{3/2-s} ||\nabla u_i||_{L^2}, \quad (2.61)
$$

where we have used the facts $\frac{1}{2} + \frac{s}{3} < 1$ and $\frac{3}{s} \geq 6$.

Similarly, it holds for $s \in (1/2,3/2)$ that

$$
||u_i \cdot \nabla n_i||_{\dot{H}^{-s}} \leq C||u_i||_{L^2}^{s-1/2} ||\nabla u_i||_{L^2}^{3/2-s} ||\nabla n_i||_{L^2}, \quad (2.62)
$$

$$
||u_i \cdot \nabla u_i||_{\dot{H}^{-s}} \leq C||u_i||_{L^2}^{s-1/2} ||\nabla u_i||_{L^2}^{3/2-s} ||\nabla u_i||_{L^2}, \quad (2.63)
$$

$$
||h(n_i) \cdot \nabla n_i||_{\dot{H}^{-s}} \leq C||n_i||_{L^2}^{s-1/2} ||\nabla n_i||_{L^2}^{3/2-s} ||\nabla n_i||_{L^2}. \quad (2.64)
$$

Finally, we turn to the last term in the left hand side of (2.56) with $i = 1,2$. We have

$$
\frac{d}{dt} \left( |\Lambda^{-s}n_1|^2 + |\Lambda^{-s}u_1|^2 \right) + \int |\nabla \Lambda^{-s}n_1|^2 + (-1)^i \int \Lambda^{-s} \nabla \phi \cdot \Lambda^{-s}u_2
$$

$$
= \int \Lambda^{-s} \phi \Lambda^{-s} \text{div}u_1 - \int \Lambda^{-s} \phi \Lambda^{-s} \text{div}u_2
$$

$$
= -\int \Lambda^{-s} \phi \Lambda^{-s} \partial_1(n_1 - n_2) + \int \Lambda^{-s} \phi \Lambda^{-s} \text{div}(n_1u_1 - n_2u_2)
$$

$$
= \frac{1}{2} \frac{d}{dt} \int |\Lambda^{-s} \phi|^2 - \int \Lambda^{-s} \phi \Lambda^{-s}(n_1u_1 - n_2u_2). \quad (2.65)
$$
If \( s \in (0,1/2] \), we use Lemma 2.1 and Lemma 2.4 to obtain
\[
\| A^{-s}(n_i u_i) \|_{L^2} \leq C \| u_i \|_{L^2} \| n_i \|_{L^{3/s}} \cdot C \| u_i \|_{L^2} \| \nabla n_i \|_{L^2}^{1/2-s} \| \nabla^2 n_i \|_{L^2}^{1/2+s} \\
\leq C(\| u_i \|_{L^2}^2 + \| \nabla n_i \|_{H^1}^2),
\]
and if \( s \in (1/2,3/2) \), we have
\[
\| A^{-s}(n_i u_i) \|_{L^2} \leq C \| u_i \|_{L^2} \| n_i \|_{L^{3/s}} \leq C \| u_i \|_{L^2} \| \nabla n_i \|_{L^2}^{5/2} \| \nabla^2 n_i \|_{L^2}^{3/2-s}.
\]

Consequently, in light of (2.56)-(2.67), and using Young's inequality, we deduce (2.54) and (2.55).

### 2.4. Estimates in \( \dot{B}^{-s}_{2,\infty}(\mathbb{R}^3) \)

In this subsection, we will derive the evolution of the negative Besov norms of the solutions. The argument is similar to the previous subsection.

**Lemma 2.13.** If \( \| n_{i0}, u_{i0}, \nabla \phi_0 \|_{H^s} \ll 1 \) with \( i = 1,2 \), for \( s \in (0, \frac{1}{2}] \), we have
\[
\frac{d}{dt}(\| n_i u_i, \nabla \phi \|_{\dot{B}^{-s}_{2,\infty}}) \leq C(\| \nabla n_i \|_{L^1}^2 + \| u_i \|_{L^2}^2)(\| n_i u_i, \nabla \phi \|_{\dot{B}^{-s}_{2,\infty}}), \quad i = 1,2,
\]
and for \( s \in (\frac{1}{2}, \frac{3}{2}] \) we have, for \( i = 1,2 \),
\[
\frac{d}{dt}(\| n_i u_i, \nabla \phi \|_{\dot{B}^{-s}_{2,\infty}}) \leq C\left(\| (n_i u_i)^{s/2} \|_{L^2}^2 \| \nabla(n_i u_i) \|_{L^2}^{s/2} \| n_i \|_{L^2}^{s/2} \| \nabla n_i \|_{L^2}^{s/2} \right)(\| n_i u_i, \nabla \phi \|_{\dot{B}^{-s}_{2,\infty}}).
\]

**Proof.** Applying \( \Delta_j \) to (2.2)\(_1\), (2.2)\(_2\) and multiplying the resulting identity by \( \Delta_j n_1, \Delta_j u_1 \), respectively, and integrating over \( \mathbb{R}^3 \) by parts, we get
\[
\frac{d}{dt}(\| \Delta_j n_1 \|_{L^2}^2 + \| \Delta_j u_1 \|_{L^2}^2) + \int|\nabla \Delta_j u|^2 - \int\Delta_j \nabla \phi \cdot \Delta_j u_1 \\
= \int\Delta_j(-n_1 \nabla u_1 + u_1 \nabla n_1) \Delta_j n_1 - \Delta_j(u_1 \cdot \nabla u_1 + h(n_1) \nabla n_1) \cdot \Delta_j u_1 \\
\leq C\| n_1 \nabla u_1 + u_1 \nabla n_1 \|_{L^2} \| n_1 \|_{L^{2,\infty}_s} + \| u_1 \nabla u_1 + h(n_1) \nabla n_1 \|_{L^{2,\infty}_s} \| n_1 \|_{L^2}^{5/2}.
\]

Then, as the proof of Lemma 2.12, applying Lemma 2.6 instead to estimate the \( \dot{B}^{-s}_{2,\infty} \) norm, we complete the proof of Lemma 2.13.

### 3. Proof of Theorems

#### 3.1. Proof of Theorem 1.1

In this subsection, we shall use the energy estimates in Subsection 2.2 to prove the global existence in the \( H^3 \) norm.

We first close the energy estimates at each \( l \)-th level to prove (1.3). Let \( 0 \leq l \leq 2 \). Summing up the estimates (2.4) from \( k = l \) to \( k = 2 \), and then adding the resulting estimates to (2.31) for \( k = 2 \), by changing the index and since \( \delta \ll 1 \), we have
\[
\frac{d}{dt} \sum_{l \leq k \leq 3} \| \nabla^k(n_1, u_1, n_2, u_2, \nabla \phi) \|_{L^2}^2 + C_1 \sum_{l \leq k \leq 3} \| \nabla^k(u_1, u_2) \|_{L^2}^2 \\
\leq C_2 \delta \sum_{l+1 \leq k \leq 3} \| \nabla^k(n_1, n_2, \nabla \phi) \|_{L^2}^2.
\]
Summing up (2.40) of Lemma 2.11 from $k=l$ to $2$, we have
\[
\frac{d}{dt} \sum_{l \leq k \leq 3} \left( \text{some integral terms} \right) + C_3 \sum_{l+1 \leq k \leq 3} \left( \text{some integral terms} \right) \leq C_4 \sum_{l \leq k \leq 3} \left( \text{some integral terms} \right) \leq 0.
\] (3.2)

Computing $2C_2 \delta/C_3 \times (3.2) + (3.1)$, and by using the fact $\delta \ll 1$, we can conclude that there exists a constant $C_5 > 0$ such that for $0 \leq l \leq 2$,
\[
\frac{d}{dt} \left\{ \sum_{l \leq k \leq 3} \left( \text{some integral terms} \right) \right\} + C_5 \left( \sum_{l \leq k \leq 3} \left( \text{some integral terms} \right) \right) \leq 0.
\] (3.3)

By the smallness of $\delta$ and using Cauchy’s inequality, we deduce that
\[
C_6^{-1} \| \nabla^l (n_1, u_1, n_2, u_2, \nabla \phi) \|^2_{H^{3-l}} \leq \sum_{l \leq k \leq 3} \left( \text{some integral terms} \right) + \frac{2C_2 \delta}{C_3} \sum_{l \leq k \leq 3} \left( \text{some integral terms} \right) \leq C_5 \| \nabla^l (n_1, u_1, n_2, u_2, \nabla \phi) \|^2_{H^{3-l}}, 0 \leq l \leq 2.
\] (3.4)

As a result, we have the following estimate in Sobolev’s space for $0 \leq l \leq 2$:
\[
\frac{d}{dt} \| \nabla^l (n_1, u_1, n_2, u_2, \nabla \phi) \|^2_{H^{3-l}} + \left\{ \| \nabla^l (u_1, u_2) \|^2_{H^{3-l}} + \| \nabla^{l+1} (n_1, n_2, \nabla \phi) \|^2_{H^{2-l}} \right\} \leq 0.
\] (3.5)

Taking $l = 0$ in (3.5), and integrating directly in time, we have
\[
\| (n_1, u_1, n_2, u_2, \nabla \phi) \|^2_{H^3} \leq C_6 \| (n_{10}, u_{10}, n_{20}, u_{20}, \nabla \phi_0) \|^2_{H^3}.
\] (3.6)

By a standard continuity argument, since $\| (n_{10}, u_{10}, n_{20}, u_{20}, \nabla \phi_0) \|_{H^3}$ is sufficiently small, this closes the a priori estimates (2.2). Thus we obtain the global existence in Theorem 1.1.

### 3.2. Proof of Theorem 1.2.
In this subsection, we will prove the optimal time decay rates of the unique global solution to system (2.1) in Theorem 1.2.

First, from Lemma 2.12, we must use different arguments for different values of $s$. For $s \in [0, 1/2]$, integrating (2.54) in time, and by using the energy estimate (1.3), we have
\[
\| (n_i, u_i, \nabla \phi) \|^2_{H^{3-s}} \leq \sup_{0 \leq \tau \leq t} \{ \| (n_i, u_i, \nabla \phi) \|^2_{H^{3-s}} \}. \]
(3.7)

This yields
\[
\| (n_1, u_1, n_2, u_2, \nabla \phi) \|^2_{H^{3-s}} \leq C_0 \text{ for } s \in [0, 1/2].
\] (3.8)
Using Lemma 2.13, we similarly have
\[
\| (n_1,u_1,n_2,u_2, \nabla \phi) \|_{H_{s-\epsilon}^2} \leq C_0 \text{ for } s \in [0,1/2]. \tag{3.9}
\]
If \(0 \leq l \leq 2\), we may use Lemma 2.3 to have
\[
\| \nabla^{l+1} f \|_{L^2} \geq C \| f \|_{H^2}^{\frac{-1}{l+2}} \| \nabla f \|_{L^2}^{1+\frac{1}{l+2}}. \tag{3.10}
\]
By this fact and (3.9), we find
\[
\| \nabla^{l+1} (n_1,n_2, \nabla \phi) \|_{L^2} \geq C_0 \left( \| \nabla (n_1,n_2, \nabla \phi) \|_{L^2}^2 \right)^{1+\frac{1}{l+2}}. \tag{3.11}
\]
This together with (1.3) yields for \(l = 0,1,2\),
\[
\| \nabla^l (u_1,u_2), \nabla^{l+1} (n_1,n_2, \nabla \phi) \|_{H^{3-l}}^2 \geq C_0 \left( \| \nabla^l (u_1,u_2,n_1,n_2, \nabla \phi) \|_{H^{3-l}}^2 \right)^{1+\frac{1}{l+2}}. \tag{3.12}
\]
Hence, from (3.5), we have the following time differential inequality for \(l = 0,1,2\):
\[
\frac{d}{dt} \| \nabla^l (u_1,u_2,n_1,n_2, \nabla \phi) \|_{H^{3-l}}^2 + C_0 \left( \| \nabla^l (u_1,u_2,n_1,n_2, \nabla \phi) \|_{H^{3-l}}^2 \right)^{1+\frac{1}{l+2}} \leq 0, \tag{3.13}
\]
which gives
\[
\| \nabla^l (u_1,u_2,n_1,n_2, \nabla \phi) \|_{H^{3-l}} \leq C_0 (1+t)^{-\frac{(l+1)}{2}}, \text{ } l = 0,1,2; \text{ } s \in \left[ 0, \frac{1}{2} \right]. \tag{3.14}
\]
We now consider \(s \in (1/2,3/2)\). Notice that the arguments for the case \(s \in [0,1/2]\) cannot be applied to this case (see Lemma 2.12). Observing that we have \(n_{i0},u_{i0},n_{20},u_{20}, \nabla \phi_0 \in H^{-1/2}\) since \(H^{-s} \cap L^2 \subset H^{-s'}\) for any \(s' < 0\), we then deduce from what we have proved for (1.6) with \(s = 1/2\) that the following decay result holds:
\[
\| \nabla^l (n_1,u_1,n_2,u_2, \nabla \phi) \|_{H^{3-l}} \leq C_0 (1+t)^{-\frac{(l+1)}{2}} \text{ for } l = 0,1,2. \tag{3.15}
\]
Integrating (2.55) in time, for \(s \in (1/2,3/2)\) we have
\[
\| (n_1,u_i, \nabla \phi) \|_{H^{-s}} \leq \| (n_{i0},u_{i0},n_{20}, \nabla \phi_0) \|_{H^{-s}} + C \int_0^t \left\{ \| (n_1,u_i, \nabla \phi) \|_{H^{-s}} \right\} d\tau \leq \| (n_{i0},u_{i0},n_{20}, \nabla \phi_0) \|_{H^{-s}} + C \sup_{0 \leq \tau \leq t} \left\{ \| (n_i,u_i, \nabla \phi) \|_{H^{-s}} \right\} \tag{3.16}
\]
\[
	imes \int_0^t \left\{ \| (n_1,u_i, \nabla \phi) \|_{H^{-s}} \right\} d\tau 
\]
\[
= \| (n_{i0},u_{i0},n_{20}, \nabla \phi_0) \|_{H^{-s}} + C \sup_{0 \leq \tau \leq t} \left\{ \| (n_i,u_i, \nabla \phi) \|_{H^{-s}} \right\} \cdot (K_1 + K_2).
\]
For \(K_1\), by using (3.15), we deduce that for the case \(s \in (\frac{1}{2}, \frac{3}{2})\),
\[
K_1 = C \int_0^t \left\{ \| (n_1,u_i, \nabla \phi) \|_{H^{-s}} \right\} d\tau \leq C_0 + C_0 \int_0^t (1+\tau)^{-7/4-s/2} d\tau \sup_{0 \leq \tau \leq t} \left\{ \| (n_i,u_i, \nabla \phi) \|_{H^{-s}} \right\} \leq C_0 \left\{ 1 + \sup_{0 \leq \tau \leq t} \left\{ \| (n_1,u_i, \nabla \phi) \|_{H^{-s}} \right\} \right\}, \text{ } i = 1,2; \text{ } s \in \left( \frac{1}{2}, \frac{3}{2} \right). \tag{3.17}
\]
For $K_2$, we must vary the arguments for $s \in (\frac{1}{2}, 1)$ and $s \in [1, \frac{3}{2})$. When $s \in (\frac{1}{2}, 1)$,

$$
K_2 = C \int_0^t \{ \| u_i \|_{L^2} \| n_i \|_{L^2}^{\frac{s}{2}} \| \nabla n_i \|_{L^2}^{\frac{3}{2} - s} \} \langle (n_i, u_i, \nabla \phi) \rangle_{H^{-s}}, dt
$$

$$
\leq C \left\{ \int_0^t \| u_i \|_{L^2}^2 dt + \int_0^t \| n_i \|_{L^2}^{2s - 1} \| \nabla n_i \|_{L^2}^{3 - 2s} dt \right\}
$$

$$
\leq CC_0 + CC_0 \int_0^1 (1 + \tau)^{-\frac{1}{2}} (1 + \tau)^{\frac{1}{2}(s - 3)} d\tau
$$

$$
\leq CC_0 + CC_0 \int_0^1 (1 + \tau)^{-\frac{1}{2}} (s - 3) d\tau \leq CC_0, \ s \in \left( \frac{1}{2}, 1 \right).
$$

(3.18)

Thus, (3.16)-(3.18) imply that

$$
\| (n_i, u_i, \nabla \phi) \|_{H^{-s}} \leq CC_0, \ s \in [0, 1).
$$

(3.19)

Combining (3.19) together with a similar argument as for the case $s \in [0, \frac{1}{2}]$, we know that the decay result (1.6) is established for any $s \in [0, 1)$:

$$
\| \nabla^l (u_1, u_2, n_1, n_2, \nabla \phi) \|_{H^{s-l}} \leq C_0 (1 + t)^{-(l+s)}, \ l = 0, 1, 2, \ s \in [0, 1).
$$

(3.20)

Choosing a constant $s_1 = \frac{5}{8} + \frac{s}{4}$ with $s \in [1, \frac{3}{2})$, then $s_1 < 1$. Then, (3.20) gives

$$
\| \nabla^l (u_1, u_2, n_1, n_2, \nabla \phi) \|_{H^{s-l}} \leq C_0 (1 + t)^{-(l+s_1)}, \ l = 0, 1, 2, \ s_1 \in [0, 1).
$$

(3.21)

By (3.21), we can prove the decay result for $s \in [1, \frac{3}{2})$. In fact,

$$
K_2 = C \int_0^t \{ \| u_i \|_{L^2} \| n_i \|_{L^2}^{\frac{s}{2}} \| \nabla n_i \|_{L^2}^{\frac{3}{2} - s} \} \langle (n_i, u_i, \nabla \phi) \rangle_{H^{-s}}, dt
$$

$$
\leq CC_0 \int_0^t (1 + \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{1}{2}(s - 3)} (1 + \tau)^{-\frac{1}{2}(s - 3)} d\tau
$$

$$
= CC_0 \int_0^t (1 + \tau)^{s_1 - \frac{1}{2}} d\tau = CC_0 \int_0^1 (1 + \tau)^{s_1 - \frac{1}{2}} d\tau \leq CC_0, \ s \in \left[ 1, \frac{3}{2} \right).
$$

(3.22)

Hence, (3.16), (3.17), and (3.22) suffice to show that

$$
\| (n_i, u_i, \nabla \phi) \|_{H^{-s}} \leq CC_0, \ s \in [0, \frac{3}{2}).
$$

(3.23)

With (3.23) in hand, we repeat the arguments leading to (1.6) for $s \in [0, 1/2]$ to prove that it hold also for $s \in (1/2, 3/2)$.

Lastly, by using Lemma 2.5, Lemma 2.6, Lemma 2.7, Lemma 2.8, and Lemma 2.13, a similar argument as that leading to the estimate (3.23) for the negative Sobolev space can immediately yield that in the negative Besov’s space,

$$
\| (n_i, u_i, \nabla \phi) \|_{B^{-s}_{2, \infty}} \leq CC_0, \ s \in \left( 0, \frac{3}{2} \right].
$$

(3.24)

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