communications in
number theory and physics
Volume 4, Number 4, 623–657, 2010

K3 surfaces, $\mathcal{N} = 4$ dyons and the Mathieu group M_{24}

Miranda C. N. Cheng

A close relationship between $K3$ surfaces and the Mathieu groups has been established in the last century. Furthermore, it has been observed recently that the elliptic genus of $K3$ has a natural interpretation in terms of the dimensions of representations of the largest Mathieu group M_{24}. In this paper, we first give further evidence for this possibility by studying the elliptic genus of $K3$ surfaces twisted by some simple symplectic automorphisms. These partition functions with insertions of elements of M_{24} (the McKay–Thompson series) give further information about the relevant representation. We then point out that this new "moonshine" for the largest Mathieu group is connected to an earlier observation on a moonshine of M_{24} through the 1/4-BPS spectrum of $K3 \times T^2$-compactified type II string theory. This insight on the symmetry of the theory sheds new light on the generalized Kac–Moody algebra structure appearing in the spectrum, and leads to predictions for new elliptic genera of $K3$, perturbative spectrum of the toroidally compactified heterotic string, and the index for the 1/4-BPS dyons in the $d = 4$, $\mathcal{N} = 4$ string theory, twisted by elements of the group of stringy $K3$ isometries.

1. Introduction and summary

Recently there have been two new observations relating $K3$ surfaces and the largest Mathieu group M_{24}. They seem to suggest that the sporadic group M_{24} naturally acts on the spectrum of $K3$-compactified string theory. In this paper, we will further explore this relation by studying various spectra, perturbative and non-perturbative, of $K3$-compactified string theories.

Throughout the development of string theory, $K3$ surfaces have been playing a prominent role. Being the unique Calabi–Yau two-fold, $K3$ compactifications serve as an important playground for supersymmetric compactifications. It also features in one of the most important dualities in string theory that links heterotic strings to the rest of the string family.

It is well–known that there are 26 sporadic groups in the classification of finite simple groups. The first ones discovered are the five so-called Mathieu groups. The largest one in the family is M_{24}, which can be thought of as acting as permutations of 24 objects. It has an index 24 subgroup M_{23}, which can be thought of as acting as permutations of 24 objects with one of them held fixed.

The relation between sporadic groups and modular objects has been a fascinating topic of mathematical research in the past few decades, connecting different fields such as automorphic forms, Lie algebras, hyperbolic lattices and number theory. The most famous example is the so-called Monstrous Moonshine, which relates the largest sporadic (the Monster) group to modular functions of discrete subgroups of $PSL(2,\mathbb{R})$. On the physics side, this topic has been deeply connected to string theory from the start. For example, the "monstrous module" $V^\natural$ (an infinite-dimensional graded representation of the Monster group) and Borcherds' Monster Lie algebra naturally arise from compactifying bosonic string theory on an $\mathbb{Z}_2$-orbifolded Leech lattice. See [1], for example, for a review of the topic. A possible relation with three-dimensional gravity has also been suggested recently [2, 3]. In this paper, we would like to explore various aspects of a possible similar "moonshine" for the sporadic group M_{24}, which appears to arise from string theory probing $K3$ surfaces.

In the last century, S. Mukai [4] considered the finite groups which act on $K3$ surfaces and fix the holomorphic two-form, a condition tied to the physical condition of supersymmetry and defining the so-called symplectic automorphisms, and showed that all of them can be embedded inside the sporadic group M_{23}. This relation between the classical geometry of $K3$ and the Mathieu group was later further explained by S. Kondo [5] from a lattice theoretic view point.

Before discussing the recent observations relating $K3$ surfaces and the largest Mathieu group M_{24}, let us first comment on the possible origin of the appearance of M_{24} in the elliptic genus. Since it was shown that all symplectic automorphisms of $K3$ surfaces can be embedded in M_{23}, the idea that the full M_{24} acts on the spectrum of conformal field theory (CFT) on $K3$ seems simply excluded. The fallacy of such an argument lies in the fact that the moduli space of string theory on $K3$ surfaces includes more than just the classical geometry part. The B-field degrees of freedom for example, should definitely be taken into account. To be more specific, the analysis of Kondo [5] shows that the action of symplectic automorphisms on the lattice $H^2(K3,\mathbb{Z}) \simeq \Gamma^{3,19}$ of classical $K3$ geometry can be identified with the action of certain elements of M_{23} on Niemeier lattices (i.e., even, self-dual, negative-definite lattices of rank 24). But once the two-form field is taken into account, the relevant lattice for the CFT is in fact the larger quantum lattice $H^{2*}(K3, \mathbb{Z}) \simeq \Gamma^{4,20}$.

Moreover, there does not have to be a point in the moduli space where g generates an symmetry of the CFT for every element g of M_{24} in order for the elliptic genus of $K3$ to be a representation of M_{24}. This is because the elliptic genus is a moduli-independent object, which implies when different symmetries are enhanced at different points in the moduli space, the elliptic genus has to be invariant under the whole group these different symmetries generate. But this does not imply that all subgroups of the group in question are realised as actual symmetries of the CFT[1].

Finally, as we will see in details later, the Fourier coefficients of the $K3$ elliptic genus also have the interpretation as the root multiplicity of a generalized Kac–Moody algebra generating the 1/4-BPS spectrum of $K3 \times T^2$-compactified string theory, or equivalently T^6-compactified heterotic string theory. It is hence possible that the whole M_{24} should better be understood in this context, where the relevant charge lattice is the larger $\Gamma^{6,22} \oplus \Gamma^{6,22}$.

Now we are ready to discuss the recent observations relating $K3$ and M_{24}. In [6], Eguchi, Ooguri and Tachikawa studied the following decomposition of the elliptic genus of $K3$ in the characters of representations of an $\mathcal{N} = 4$ superconformal algebra

$$\begin{aligned}
\mathcal{Z}_{K3}(\tau, z) &= \mathrm{Tr}_{\mathrm{RR}}\left((-1)^{J_0+\bar{J}_0} q^{L_0} \bar{q}^{\bar{L}_0} \mathrm{e}^{2\pi \mathrm{i} z J_0}\right) \\
&= \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} (24\mu(\tau, z) + q^{-1/8}(-2 + T(\tau))).
\end{aligned}$$

They then observe that the first few Fourier coefficients of the q-series $T(\tau)$ are given by simple combinations of dimensions of irreducible representations of the sporadic group M_{24}. In other words, it appears as if there is an infinite-dimensional representation, which we will call $K^\natural$, of the largest Mathieu group, that has a grading related to the L_0-eigenvalues of the CFT, such that

$$K^\natural = \bigoplus_{n=1}^{\infty} K_n^\natural, \quad T(\tau) = \sum_{n=1}^{\infty} q^n \dim(K_n^\natural).$$

It would constitute very strong further evidence for the validity of the above assumption if one can show that the corresponding McKay–Thompson

[1] I thank A. Sen for useful discussions on this point.

series, or twisted partition functions

$$T_g(\tau) = \sum_{n=1}^{\infty} (\mathrm{Tr}_{K_n^\natural} g)\, q^n, \quad g \in M_{24},$$

as well play a role in the $K3$ CFT. This motivates us to look at the $K3$ elliptic genera twisted by generators of the simplest groups $\mathbb{Z}_{p=2,3,5,7}$ of symplectic automorphism groups of classical $K3$ geometry, which are all the cyclic $\mathbb{Z}_p$ groups in Nikulin's list of $K3$ quotients with prime p. These objects have been computed in [7] and we found that under similar decomposition they are indeed given by the McKay–Thompson series of the appropriate elements $g \in M_{24}$. Apart from further supporting the existence of such a M_{24} module, the twisted elliptic genera also give much finer information about the representation $K^\natural$. Furthermore, using the conjectural relation between M_{24} characters and twisted elliptic genera, we can now easily compute the elliptic genera twisted by various other elements of M_{24}. The reason for this is the following. Using standard CFT argument we expect the twisted elliptic genera to be weak Jacobi forms of congruence subgroups of $PSL(2,\mathbb{Z})$. This modularity leaves just a few unknown coefficients to be fixed in the expression for the elliptic genera. The knowledge about the few lowest-lying representations $K_n^\natural$ with small n together with the character table (table 2, appendix A) is sufficient to fix all unknown coefficients. The full expression can then in turn be used to determine the representation $K_n^\natural$ for higher n. We demonstrate this by giving explicit formulas for elliptic genera twisted by various other elements of M_{24}, including both those generating other geometric symplectic automorphisms and those that do not have known geometric interpretations.

Now we turn to the other recent observation regarding the perturbative spectrum of heterotic string theories. It is a familiar fact that the 1/2-BPS spectrum of the heterotic theory compactified on T^6 is given by nothing but the bosonic string partition function

$$\frac{1}{\eta^{24}(\tau)} = q^{-1} + \sum_{n=0}^{\infty} d_n q^n, \quad q = \mathrm{e}^{2\pi \mathrm{i}\tau},$$

where $\eta(\tau)$ denotes the Dedekind Eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n).$$

In [8, 9], at points with enhanced symmetries, the partition function of the above theory twisted by an $\mathbb{Z}_{p=2,3,5,7}$ symmetry has been computed to be

$$\frac{1}{\eta^{24/(p+1)}(\tau)\eta^{24/(p+1)}(p\tau)}.$$

Very recently, Govindarajan and Krishna [10] extended the above results to other $\mathbb{Z}_N$ orbifolds in Nikulin's list of $K3$ quotients. In their derivation, these authors have used the heterotic-type II duality and the fact that all $K3$ symplectic automorphisms can be embedded in $M_{23} \subset M_{24}$. They also observed that the resulting partition functions are given by the product of η-functions with the so-called "cycle shape" of the corresponding elements of M_{24}. See Section 3.1 and table 2 for details.

In fact, these η-products are known to be the central objects of a version of the "moonshine" relating M_{24} and cusp forms [11]. The first observation is

$$\eta^{24}(\tau) = \sum_{n=1}^{\infty} \tau_n q^n = \sum_{n=1}^{\infty} q^n \,\mathrm{sdim}\tilde{K}_n,$$

where τ_n are the Ramanunjan τ-functions and $\tilde{K} = \oplus_{n=1}^{\infty}\tilde{K}_n$ is an infinite-dimensional graded representation of M_{24}. The Ramanunjan numbers are integers of both signs, and this is reflected in the fact that the representation should be thought of as coming in both bosonic and fermionic varieties, with positive and negative super-dimensions. For $g \in M_{24}$, the corresponding η-product is then the corresponding McKay–Thompson series

$$\eta_g(\tau) = \sum_{n=1}^{\infty} q^n \,\mathrm{Tr}_{\tilde{K}_n}((-1)^F g),$$

which reduces to the above formula for the Ramanunjan numbers when g is taken to be the identity element. For those g that generate the geometric $\mathbb{Z}_p$ symplectic automorphisms, the corresponding η-products are exactly those that give the above twisted heterotic string spectrum.

A quick glimpse at the character table (table 2) suggests that the two M_{24} modules $\tilde{K}$ and $K^{\natural}$ have really nothing to do with each other. Hence we have now two versions of moonshine for the largest Mathieu group, one related to the $K3$ elliptic genus and one to the heterotic string spectrum. But it turns out that they are not unrelated after all. Before explaining the physics, let us first switch gear and remind ourselves a few things about Borcherds' generalised Kac–Moody algebra. A generalized Kac–Moody algebra is an infinite-dimensional Lie algebra, which has a denominator identity equating an infinite sum to an infinite product. In the example of the Monstrous Moonshine, the denominator of the Monster Lie algebra is given by the famous expression

$$j(p) - j(q) = \left(\frac{1}{p} - \frac{1}{q}\right) \prod_{n,m=1}^{\infty} (1 - p^n q^m)^{d(nm)},$$

where the j-function

$$j(q) - 744 = q^{-1} + 196884q + \cdots = \sum_{n=-1}^{\infty} q^n \dim(V_n^\natural) = \sum_{n=-1}^{\infty} d(n)\, q^n$$

is related to an infinite-dimensional graded representation $V^\natural = \oplus_{n=-1}^{\infty} V_n^\natural$ of the Monster group. The multiplicity of the positive root corresponding to the factor $(1 - p^n q^m)$ inside the infinite product is given by the Fourier coefficients $d(nm) = \dim(V_{nm}^\natural)$ of the j-function. One can also twist the above expression by elements of the Monster group and obtain the so-called twisted denominators that have played an important role in Borcherds' proof of the moonshine conjecture [12]. More precisely, the product expression for the denominator twisted by an element g of the Monster is given by the Fourier coefficients of the McKay–Thompson series $\sum_{n=-1}^{\infty} q^n \mathrm{Tr}_{V_n^\natural}(g^k)$, where k runs from one to the order of g.

It turns out that a similar structure is present in our M_{24} case as well. It has been proposed that the 1/4-BPS dyon spectrum in the $\mathcal{N} = 4$, $d = 4$ theory obtained by compactifying type II string on $K3 \times T^2$ is generated by a generalized Kac–Moody superalgebra [13, 14]. The multiplicities of the roots of the algebra are given by the Fourier coefficients of the elliptic genus of $K3$

$$\mathcal{Z}_{K3}(\tau, z) = \sum_{n \geq 0, \ell \in \mathbb{Z}} c(4n - \ell^2) q^n y^\ell,$$

and are hence given by the representation $K^\natural$. These data give the denominator of the algebra

$$\Phi(\rho, \sigma, \nu) = \mathrm{e}^{2\pi \mathrm{i}(\rho+\sigma-\nu)} \prod_{\substack{n,m \geq 0, \ell \in \mathbb{Z} \\ \ell > 0 \text{ when } n=m=0}} (1 - \mathrm{e}^{2\pi \mathrm{i}(n\rho + m\sigma + \ell\nu)})^{c(4nm - \ell^2)},$$

which gives the 1/4-BPS spectrum of the theory. For consistency, the 1/2-BPS spectrum, in this case given by another representation $\tilde{K}$, should also be encoded in this 1/4-BPS spectrum. This is because the wall-crossing phenomenon in the $\mathcal{N}=4$, $d=4$ theory dictates that the 1/4-BPS spectrum jumps across the walls of marginal stability in the moduli space where 1/4-BPS bound states decay into a pair of 1/2-BPS particles. Indeed, at its poles

$$\frac{1}{\Phi(\rho,\sigma,\nu)} \xrightarrow{\nu\to 0} -\frac{1}{4\pi^2\nu^2}\frac{1}{\eta^{24}(\rho)}\frac{1}{\eta^{24}(\sigma)} + \text{reg.},$$

where 'reg.' denotes terms that are regular as $\nu \to 0$, the 1/4-BPS partition function factorises into two 1/2-BPS partition functions and correctly reproduces the expected jump in the indices [13, 15]. Hence, rather remarkably these two different versions of moonshine for M_{24} both make their appearance in the denominator of the dyon algebra. Turning it around, this fact can serve as further evidence that this sporadic group indeed acts on the algebra as a symmetry group.

If the idea described in the last paragraph is true, a mathematical consequence will be the following: the denominator twisted by an element $g \in M_{24}$ should be given by an infinite product involving the $K3$ elliptic genus twisted by g^k, or equivalently its Fourier coefficients $\mathrm{Tr}_{K_n^\natural}(g^k)$, with k again running from one to the order of g. Furthermore its poles should be given by a product of two copies of the twisted 1/2-BPS partition function $1/\eta_g$. One can show that this is indeed true for all elements $g \in M_{24}$. In fact, for the elements g generating the $\mathbb{Z}_{2,3,5,7}$ symplectic automorphisms of $K3$, we will show that the twisted denominator is the same object considered recently [7, 9] by Sen and his collaborators. In these references, they are computed as the generating function of twisted 1/4-BPS indices [16, 17], and are studied as a microscopic test for the quantum entropy function proposal for black holes with AdS_2 as part of the near horizon geometry [18].

These observations provide important new insights into the symmetries and structures of the $K3$-compactified string theories, or equivalently toroidally compactified heterotic strings. Furthermore, the assumption of an underlying M_{24} symmetry leads to new conjectural formulae for the twisted $K3$ elliptic genera, the twisted perturbative spectrum of heterotic strings, and the twisted indices for 1/4-BPS dyons in the $\mathcal{N}=4$, $d=4$ theories.

The rest of the paper is organized as follows. In Section 2, we study the twisted $K3$ elliptic genera and clarify their relations to the McKay–Thompson series of M_{24}. From this consideration, we give predictions for the twisted $K3$ elliptic genera that have not been computed before. In Section 3, we start with a review on some previously known mathematical and physical results. These include on the one hand the relation between η-products, M_{24}, and heterotic string spectrum, and the relationship between the

$K3 \times T^2$-compactified string theory and a certain generalised Kac–Moody superalgebra on the other hand. We then study the (twisted) denominators of this algebra, and show they relate the two sets of modular objects (elliptic genera and η-products) which are both attached to M_{24}. Finally, in the last section we summarize the lessons learned from this study. In particular, we summarize our conjectural formulae for the various (twisted) spectra of the theories arising from $K3$-compactified string theory, while some of the symmetries are yet to be understood explicitly.

2. Twisted $K3$ elliptic genera and the McKay–Thompson Series of M_{24}

In this section, we will study the twisted elliptic genera of $K3$ surfaces and discuss their relation to the characters of a certain infinite-dimensional graded representation of the largest Mathieu group M_{24}.

In [6], Eguchi, *et al.* studied the following decomposition of the elliptic genus of $K3$ in terms of characters of representations of an $\mathcal{N} = 4$ superconformal algebra

$$\begin{aligned} \mathcal{Z}_{K3}(\tau, z) &= \mathrm{Tr}_{\mathrm{RR}}\left((-1)^{J_0+\bar{J}_0} q^{L_0} \bar{q}^{\bar{L}_0} \mathrm{e}^{2\pi \mathrm{i} z J_0}\right) \\ &= \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \left(\chi\mu(\tau, z) + q^{-1/8}(-2 + T(\tau))\right), \end{aligned} \tag{2.1}$$

where the first term corresponds to the short multiplet of spin zero in the Ramond sector, $\chi = 24$ is the Euler number of $K3$, and we have written

$$q = \mathrm{e}^{2\pi \mathrm{i}\tau}, \quad y = \mathrm{e}^{2\pi \mathrm{i} z}.$$

Note that the elliptic genus is a weak Jacobi form of the modular group $SL(2, \mathbb{Z})$, as one can show from the standard CFT spectral flow argument.

On the other hand, the so-called Appel–Lerch sum

$$\mu(\tau, z) = \frac{-\mathrm{i} y^{1/2}}{\theta_1(\tau, z)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n y^n q^{n(n+1)/2}}{1 - y q^n}$$

is not such a simple modular object and is in fact a mock theta function [19, 20]. This implies that the q-series $T(\tau)$ does not have simple modular transformation either. The appearance of such a mock modular object in the moonshine for the sporadic group M_{24} is very interesting, although a detailed discussion is out of the scope of the present paper.

The fascinating observation made by the authors of [6] is about the object $T(\tau)$, which shows up in the above superconformal decomposition of the $K3$ elliptic genus (2.1). They observe that the first few Fourier coefficients, which read

$$T(\tau) = 2(45q + 231q^2 + 770q^3 + 2277q^4 \\ + 5796q^5 + (3520 + 10395)q^6 + \cdots), \tag{2.2}$$

are given by the (sums of) dimensions of irreducible representations of the sporadic group M_{24}.

In other words, it is as if there is an infinite-dimensional representation, which we will call $K^\natural$, of the largest Mathieu group, that has a $\mathbb{Z}$-grading related to the L_0 of the CFT, such that

$$K^\natural = \bigoplus_{n=1}^{\infty} K_n^\natural, \quad T(\tau) = \sum_{n=1}^{\infty} q^n \dim(K_n^\natural). \tag{2.3}$$

Furthermore, the Euler number $\chi = 24$ can similarly be written as

$$\chi = \dim(\rho_1 \oplus \rho_{23}) = 1 + 23,$$

where ρ_1 denotes the trivial representation and ρ_{23} the 23-dimensional representation of M_{24} [6].

To further test this proposal and to gain more information about the purported M_{24} module $K^\natural$, a natural set of objects to consider is the twisted version of $K3$ elliptic genus. Quite a few things about these twisted elliptic genera have been studied in recent years in the program to understand $\mathcal{N} = 4$, $d = 4$ string theory compactifications [21]–[47]. These include the so-called CHL models in heterotic string theory [48], or type II string theory compactified on $K3 \times T^2$ orbifolded by a $\mathbb{Z}_N$ symmetry, which acts on $K3$ as an order N symplectic automorphism and as a $\mathbb{Z}_N$ shift along one of the circles in T^2. In particular, the twisted elliptic genera has been computed for all $\mathbb{Z}_p$ symmetries with prime p in Nikulin's list of $K3$ quotients [7]: letting g_p be the order p generator of the discrete $\mathbb{Z}_p$ transformation on the geometry of $K3$, the following object in the $K3$ CFT was considered by the authors of [7]

$$\mathrm{Tr}_{\mathrm{RR},(g_p)^m}\left((-1)^{J_0+\bar{J}_0}(g_p)^n q^{L_0}\bar{q}^{\bar{L}_0} e^{2\pi i z J_0}\right), \; m, n = 0, \ldots, p-1, \tag{2.4}$$

where the trace is taken over the $(g_p)^m$-twisted sector. We are interested in the case $m = 0, n = 1$, the answer for which is given by

$$\begin{aligned}\mathcal{Z}_{g_p}(\tau,z) &= \mathrm{Tr}_{\mathrm{RR}}\left((-1)^{J_0+\bar{J}_0} g_p\, q^{L_0}\bar{q}^{\bar{L}_0}\mathrm{e}^{2\pi i z J_0}\right)\\ &= \frac{2}{p+1}\phi_{0,1}(\tau,z) + \frac{2p}{p+1}\phi_{-2,1}(\tau,z)\phi_2^{(p)}(\tau)\\ &\quad \text{for } (g_p)^p = \mathbb{1},\ \ p = 2,3,5,7.\end{aligned} \tag{2.5}$$

where $\phi_{-2,1}(\tau)$ is the weight -2, index one weak Jacobi form and $\phi_2^{(p)}(\tau)$ is a weight two modular form under the congruence subgroup $\Gamma_0(p) \subset PSL(2,\mathbb{Z})$. The total expression $\mathcal{Z}_{g_p}(\tau,z)$ hence transforms as a weak Jacobi form of weight zero, index one of $\Gamma_0(p)$. More details on these modular objects can be found in appendix B.

For $\mathbb{Z}_N$ orbifolds with N not prime, ideas for how to compute the twisted elliptic genus using geometric data have been put forth in [10, 17], as well as an explicit formula for the case $N = 4$ in [10].

As was mentioned earlier, it is known that all the groups in Nikulin's list of $K3$ quotients can be embedded in M_{24} in a natural way. Provided that this "classical geometric" part of M_{24} is indeed realised on the purported representation $K^\natural$ according to the embedding of Mukai and Kondo, it is a mere consequence of the conjecture (2.3) that, for an element g generating a certain symplectic automorphism, the twisted elliptic genera

$$\mathcal{Z}_g(\tau,z) = \mathrm{Tr}_{\mathrm{RR}}\left((-1)^{J_0+\bar{J}_0}\, g\, q^{L_0}\bar{q}^{\bar{L}_0}\mathrm{e}^{2\pi \mathrm{i} z J_0}\right) \tag{2.6}$$

are related in the following way to the so-called McKay–Thompson series $T_g(\tau)$, or twisted partition functions, of M_{24}

$$\mathcal{Z}_g(\tau,z) = \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}\left(\chi_g\mu(\tau,z) + q^{-1/8}(-2+T_g(\tau))\right), \tag{2.7}$$

$$T_g(\tau) = \sum_{n=1}^{\infty}(\mathrm{Tr}_{K_n^\natural} g)\, q^n, \quad g \in M_{24}. \tag{2.8}$$

In the above formula, $\chi_g = \mathcal{Z}_g(\tau, z=0)$ is the twisted Euler number (trace of g on $K3$ cohomologies), which is expected to be

$$\chi_g = \mathrm{Tr}_{\rho_1\oplus\rho_{23}} g. \tag{2.9}$$

A few comments are in order here. First, taking the identity element $g = \mathbb{1}$, we recover the series $\sum_{n=1}^{\infty}\dim(K_n^\natural)q^n = T(\tau)$ encoding the dimensions of the representation. Second, it is clear from their definition that the McKay–Thompson series depend only on the conjugacy class of the group element. Namely, we have

$$T_g(\tau) = T_{hgh^{-1}}(\tau).$$

Despite of the large number

$$|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \sim 10^8$$

of elements, M_{24} has only 26 conjugacy classes, as listed in table 2. Hence we expect at most 26 distinct McKay–Thompson series $T_g(\tau)$. Third, the modularity properties of the above McKay–Thompson series $T_g(\tau)$ are somewhat obscure due to the subtraction of the Mock theta function $\mu(\tau, z)$ and are most easily seen in the combined CFT object $\mathcal{Z}_g(\tau, z)$. This is to be contrasted with the more familiar case of the Monstrous Moonshine, where the corresponding McKay–Thompson series are all modular functions of discrete subgroups of $PSL(2, \mathbb{R})$.

Finally, following the same logic we also expect the more general twisted elliptic genus which can be calculated when the CFT has a symmetry group $G \subset M_{24}$

$$\mathcal{Z}_{h,g}(\tau, z) = \mathrm{Tr}_{\mathrm{RR},h}\left((-1)^{J_0+\bar{J}_0}\, g\, q^{L_0}\bar{q}^{\bar{L}_0}\mathrm{e}^{2\pi i z J_0}\right), \quad h, g \in G, \tag{2.10}$$

to be related to the so-called generalized McKay–Thompson series for commuting pairs of elements (g, h) [49]

$$T_{h,g}(\tau) = \sum_{n=1}^{\infty}(\mathrm{Tr}_{K^\natural_n,h}\, g)\, q^n, \quad g, h \in M_{24} \tag{2.11}$$

of the group M_{24}, where the element h denotes the twisted sectors in the Hilbert space.

Now we are ready to test the proposal (2.7) for the known cases when g is taken to be the element generating the $\mathbb{Z}_{p=2,3,5,7}$ symplectic automorphisms of $K3$. Rewriting the known answer (2.5) we get the results that the twisted elliptic genus $\mathcal{Z}_{g_p}(\tau)$ can indeed be written in the form (2.7). The corresponding elements $g_p \in M_{24}$ in their ATLAS names and the first seven Fourier coefficients of the q-series $T_{g_p}(\tau)$ are given in table 1.

Comparing this coefficient table with the character table of M_{24} (table 2), there are quite a few things to be learned. First of all, the first six columns

Table 1: Here we list the first seven coefficients of the q-series $T_g(\tau)$ that show up in the elliptic genera of $K3$ twisted by $\mathbb{Z}_p$ symplectic automorphisms (2.5,2.7), together with the twisted Euler number. In the first column, we have adopted the same ATLAS naming for the conjugacy classes as in table 2. In particular, in the first row are the coefficients of the non-twisted $K3$ elliptic genus as given in [6].

g	χ_g	q^1	q^2	q^3	q^4	q^5	q^6	q^7
$1A$	24	2×45	2×231	2×770	2×2277	2×5796	2×13915	2×30843
$2A$	8	2×-3	2×7	2×-14	2×21	2×-28	2×43	2×-69
$3A$	6	0	2×-3	2×5	0	2×-9	2×10	0
$5A$	4	0	2×1	0	2×-3	2×1	0	2×7
$7A$	3	-1	0	0	4	0	-2	2

are consistent with the hypothesis that $K^\natural_i, i=1,\ldots,5$ are given by the irreducible representations of M_{24} with the corresponding dimensions. Also the twisted Euler number $\chi_{g_p}=\frac{24}{p+1}$ is consistent with the formula (2.9). This gives very strong new evidence for the possibility that M_{24} does act on the elliptic genus of $K3$. Furthermore, this also shows that the part of M_{24} that does have interpretation as automorphisms of classical $K3$ geometry indeed acts on $K3$ in a way that we expect it to.

Second, we see that the McKay–Thompson series give useful extra information about the M_{24} module $K^\natural$. Note that the dimensions of irrep's of M_{24} are highly degenerate. For $n>5$, $K^\natural_n$ are not simply given by irreducible representations anymore but rather by positive integral combinations of them. The decompositions are in fact never unique and one needs more information than the dimension to determine the representation $K^\natural_n$. For example, for $n=7$ the authors of [6] propose

$$30843 = 10395 + 2\times 5796 + 5544 + 3312,$$

which is unfortunately inconsistent with the Fourier coefficients of the McKay–Thompson series listed in table 1.

Instead, the following decomposition for example, is consistent with the limited character table we have so far and also the examples we are going to consider later[2]

$$30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771\ .$$

[2]It is in fact the only possibility if one assumes that the largest irrep appears once, irrep's with dimension smaller than 500 do not appear, and each irrep appears at most twice. Similar computer scans can be extended to more general assumptions.

This is a concrete example of how the twisted q-series contain much more information about the appropriate representation. We expect them to give important hints about the representation before we eventually obtain other ways to "naturally" define $K^\natural$ from physical considerations.

Finally, it is a familiar fact that all Fourier coefficients of $K3$ elliptic genus are even integers. As can be seen in the table and can be shown in general [23], this remains true for the $K3$ elliptic genus twisted by element generating the $\mathbb{Z}_{2,3,5}$ automorphisms. A directly related statement about the generalized Kac–Moody algebra that generates the 1/4-BPS spectrum in the corresponding $\mathcal{N}=4$, $d=4$ string theory will be discussed in section 3.2. Note though that in the last row in table 1, the Fourier coefficients are no longer even for the $\mathbb{Z}_7$ twisted character. Furthermore, by comparing with the character table we conclude that all representations in $K^\natural$ come in conjugate pairs[3]. In other words, a better way to write the numbers in Table 1 is $\mathbf{c}+\bar{\mathbf{c}}$ instead of $2\times\mathbf{c}$. Such a distinction is invisible unless one studies the twisted version of the partition functions of the conformal field theory. Later we will see how this sheds new light on the structure of 1/4-BPS spectrum of the $K3\times T^2$-compactified string theory.

After this simple exercise we are now not only more confident about the existence of the M_{24} module $K^\natural=\oplus_n K^\natural_n$, but also how the few lowest-lying representations $K^\natural_n$ look like. Combined with modularity properties, the proposal (2.7) can be used to derive explicit expressions for the $K3$ elliptic genera twisted by various elements of M_{24}. This derivation is purely arithmetic and does not depend on whether the element $g\in M_{24}$ has a classical geometrical meaning or not. To illustrate this, let us first derive the $K3$ elliptic genera twisted by elements that generate the $\mathbb{Z}_{4,6,8}$ symplectic automorphisms of $K3$. Together with the $\mathbb{Z}_{p=2,3,5,7}$ we just discussed, they exhaust the $\mathbb{Z}_N$ groups in the list of Nikulin's $K3$ involutions. An alternative proposal to compute these $\mathbb{Z}_{4,6,8}$-twisted elliptic genera from geometric consideration can be found in [17].

From the standard CFT arguments about the change of boundary conditions under modular transformations, we expect $\mathcal{Z}_g(\tau,z)$ to be invariant up to a phase under transformations

$$\tau\to\frac{a\tau+b}{c\tau+d},\quad z\to\frac{z}{c\tau+d},\quad \begin{bmatrix}a & b\\ c & d\end{bmatrix}\in SL(2,\mathbb{Z}),\ c=0 \bmod \mathrm{ord}(g).$$

[3]We say that an irreducible representation is its own conjugate if it does not come in conjugate pairs.

This condition leaves just a few unknown coefficients to determine in the expression for $\mathcal{Z}_g(\tau, z)$. See appendix B for more information on the relevant modular objects.

From this consideration, we obtain

$$
\begin{aligned}
\mathcal{Z}_{4B}(\tau,z) &= \frac{1}{3}\phi_{0,1}(\tau,z) + \phi_{-2,1}(\tau,z)\left(-\frac{1}{3}\phi_2^{(2)}(\tau) + 2\phi_2^{(4)}(\tau)\right),\\
\mathcal{Z}_{6A}(\tau,z) &= \frac{1}{6}\phi_{0,1}(\tau,z) + \phi_{-2,1}(\tau,z)\\
&\quad \times \left(-\frac{1}{6}\phi_2^{(2)}(\tau) - \frac{1}{2}\phi_2^{(3)}(\tau) + \frac{5}{2}\phi_2^{(6)}(\tau)\right),\\
\mathcal{Z}_{8A}(\tau,z) &= \frac{1}{6}\phi_{0,1}(\tau,z) + \phi_{-2,1}(\tau,z)\left(-\frac{1}{2}\phi_2^{(4)}(\tau) + \frac{7}{3}\phi_2^{(8)}(\tau)\right).
\end{aligned}
\tag{2.12}
$$

To make our discussion unambiguous, in the above equation we have adopted the ATLAS naming of the conjugacy classes, as listed in table 2. Comparing the answers to the known result, $\mathcal{Z}_{4B}(\tau, z)$ is identical to the expression given in [10] for $\mathbb{Z}_4$-twisting. For twisting by $\mathbb{Z}_6$ and $\mathbb{Z}_8$ geometric isometries, there are no explicit formulas available in the literature, and our answer is consistent with the topological constraints in [10]. Furthermore, for $N = 2, 3, 4, 6$, we observe that our results can be rewritten as

$$
\chi_{g_N} \frac{\theta_1(\tau, z + (1/N))\theta_1(\tau, -z + (1/N))}{\theta_1^2(\tau, (1/N))},
\tag{2.13}
$$

which can be interpreted geometrically as suggested in [17]. In the above formula, $\chi_{g_N} = \mathrm{Tr}_{\rho_1 \oplus \rho_{23}} g_N$ denotes the Euler number twisted by the order N elements $2A, 3A, 4B$ and $6A$ ($N = 2, 3, 4, 6$), respectively.

This procedure can be extended to various other elements of M_{24}, even when they do not have an interpretation as holonomy-preserving automorphisms of $K3$. We expect these twisted elliptic genera to provide valuable information about the action of the corresponding elements on stringy $K3$ geometry. For the brevity of the text, the formulas of these other twisted elliptic genera are placed in appendix C.

3. The two versions of M_{24} moonshine and the algebra for BPS dyons

In this section, we will first review the known relationship between products of η-functions, the perturbative spectra of heterotic string compactifications, and the sporadic group M_{24}. Putting different pieces of known results together, we observe how an old and the new versions of "M_{24} moonshine" are related in a generalised Kac–Moody algebra, which is known to have the physical interpretation as the spectrum-generating algebra for 1/4-BPS dyons in the $K3 \times T^2$-compactified type II string theory.

3.1. Review: η-products, M_{24} and heterotic strings

It is a familiar fact that the 1/2-BPS partition function of heterotic string compactified on T^6 is nothing but the bosonic string partition function

$$\frac{1}{\eta^{24}(\tau)} = q^{-1} + \sum_{n=0}^{\infty} d_n q^n, \quad q = e^{2\pi i\tau}.$$

In [8,9], the 1/2-BPS partition functions of the above theory twisted by $\mathbb{Z}_{p=2,3,5,7}$ symmetries have been computed to be

$$\frac{1}{\eta^{24/(p+1)}(\tau)\eta^{24/(p+1)}(p\tau)}. \tag{3.1}$$

Very recently, Govindarajan and Krishna [10] extended the above results to other $\mathbb{Z}_N$ orbifolds, with $N = 4, 6, 8$ being the non-prime numbers in the list of $\mathbb{Z}_N$ quotients in Nikulin's list of $K3$ quotients. The fact that all $K3$ symplectic automorphisms can be embedded in M_{23} and hence in M_{24} was crucial for their derivation. To understand the answer we have to know a few more things about M_{24}. The largest Mathieu group can be thought of as a subgroup of the group S_{24} of the permutation of 24 objects. In this way, one can assign to each conjugacy class of M_{24} a corresponding "cycle shape" of the form

$$1^{i_1}2^{i_2}\cdots r^{i_r}, \quad \sum_{\ell=1}^{r} \ell\, i_\ell = 24.$$

See table 2 for the cycle shapes of all conjugacy classes. Using Mukai's embedding of $K3$ symplectic automorphisms into $M_{23} \subset M_{24} \subset S_{24}$, one can also associate to a generator of an $K3$ symplectic automorphism a cycle shape. For example, the cycle shape of the identity element is 1^{24}, which denotes the collection of 24 cycles of length one. Similarly, the generator of the geometric $\mathbb{Z}_2$ symplectic automorphism is associated with $1^8 2^8$, the collection of 8 cycles of length 1 and 8 cycles of length 2, and $\mathbb{Z}_3$ is given by $1^6 3^6$ and so on. In the simple cases, the explicit meaning of the cycle shapes in terms of Narain lattices with enhanced symmetries on the heterotic side can be found in [50].

Using this mathematical relation to the Mathieu group and the asymptotic growth of the degeneracies in the physical system, it was argued in [10] that the result (3.1) can be extended to $\mathbb{Z}_N$ orbifolds with composite N, and the answer is given by the corresponding η-products. To be more specific, through the embedding of $K3$ automorphisms into M_{24}, the $\mathbb{Z}_4$ automorphism corresponds to the cycle shape $1^4 2^2 4^4$ and the corresponding twisted generating function is given by $(\eta^4(\tau)\eta^2(2\tau)\eta^4(4\tau))^{-1}$, and the $\mathbb{Z}_6$ theory has $(\eta^2(\tau)\eta^2(2\tau)\eta^2(3\tau)\eta^2(6\tau))^{-1}$, the $\mathbb{Z}_8$ theory $(\eta^2(\tau)\eta(2\tau)\eta(4\tau)\eta^2(8\tau))^{-1}$. For $\mathbb{Z}_p$ with p being prime, this procedure reproduces the known result (3.1).

In the same way, one can associate a so-called η-product to every conjugacy class of M_{24}, whether it has a known interpretation as a classical $K3$ automorphism or not. In other words, we have a map between all elements $g \in M_{24}$ and cusp forms $\eta_g(\tau)$

$$g \text{ with cycle shape } 1^{i_1} 2^{i_2} \cdots r^{i_r} \mapsto \eta_g(\tau) = \prod_{\ell=1}^{r} \eta(\ell\tau)^{i_\ell} = \sum_{m=1}^{\infty} a_{g,m} q^m \tag{3.2}$$

These η-products are long known to be the central objects of a version of "moonshine" of the group M_{24} [11,51] (later generalized to a "generalized moonshine" of M_{24} [52]). In other words, the Fourier coefficients of the of these η-products given by the cycle shapes of M_{24} are known to be the virtual, or generalized, characters of M_{24}. That is, the Fourier coefficients of $\eta_g(\tau)$ are (not necessarily positive) integral linear combinations of irreducible characters of M_{24}. For example, we have

$$\begin{aligned} \eta^{24}(\tau) &= q + (-1-23)q^2 + (-1+253)q^3 + \cdots, \\ \eta^8(\tau)\eta^8(2\tau) &= q + (-1-7)q^2 + (-1+13)q^3 + \cdots, \\ \eta^6(\tau)\eta^6(3\tau) &= q + (-1-5)q^2 + (-1+10)q^3 + \cdots. \\ &\cdots \end{aligned} \tag{3.3}$$

Interested readers can see [53] for a list of decompositions in terms of irrep's of M_{24} from q^1 to q^{30}.

For completeness we also mention here the second formulation of the relationship between η-products and M_{24}. Using the remarkable fact that all the η-products obtained from (3.2) are multiplicative (a Fourier series $F(\tau) = \sum_{m=1}^{\infty} a_m q^m$ is called multiplicative if $a_1 a_{mn} = a_m a_n$ for $(m,n) = 1$), one can show that its Mellin transform

$$F_g(s) = \sum_{m=1}^{\infty} a_{g,m} m^{-s}$$

is the Euler product $\prod_p (1 - a_{g,p} p^{-s} + b_{g,p} p^{-2s})$, where $b_{g,p} = \epsilon_{g,p} p^{k-1}$ with $\epsilon_{g,p} = 0, \pm 1$ certain Dirichlet character and k is the so-called "level" of the M_{24} element g, defined as half of the total number of cycles or equivalently the weight of the η-product $\eta_g(\tau)$. Then the statement is, for all prime p with $p \neq 3$, $b_{g,p}$ is a character of M_{24}. For example, for $p = 2$ we have

$$\begin{aligned}
1 \times 2^{12-1} &= 1 + 23 + 253 + 1771, \\
0 \times 2^{8-1} &= 1 + 7 + 13 - 21, \\
1 \times 2^{6-1} &= 1 + 5 + 10 + 16, \\
&\cdots
\end{aligned}$$

for $g = 1A, 2A, 3A, \ldots$.

3.2. The BPS dyon algebra and the two moonshines

It is a rather curious fact that we seem to have now two sets of modular objects, $\mathcal{Z}_g(\tau, z)$ and $\eta_g(\tau)$, each giving a version of moonshine for the group M_{24}. Note also that the underlying M_{24} module for these two sets of objects seem to be completely unrelated to each other. But in this subsection we would like to point out that they are in fact united in the denominator of the characters of a generalised Kac–Moody superalgebra, which has the physical interpretation as the spectrum-generating algebra for the 1/4-BPS states in the $K3 \times T^2$-compactified type II string theory, or equivalently the T^6-compactified heterotic string theory.

To explain this, let us remind ourselves a few things about generalized Kac–Moody superalgebras. See, for example, [13, 54, 55] for a bit more or much more information. A generalized Kac–Moody superalgebra, or sometimes called a Borcherds–Kac-Moody algebra, is an infinite dimensional algebra with non-positive definite Cartan matrix (Kac–Moody). Furthermore, it contains the so-called imaginary simple roots (generalized), which are simple roots which are time- or light-like (having positive or null norm squared in our convention). Finally, the roots come in bosonic or fermionic varieties (super). A very important object to consider in these algebras is the denominator of their characters, which satisfies the so-called denominator identity that relates an infinite sum to an infinite product:

$$e^{\varrho} \prod_{\alpha \in \Delta_+} (1 - e^{\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w \in W} \epsilon(w)\, w(e^{\varrho}\Sigma). \tag{3.4}$$

Here Δ_+ denotes the set of positive roots and $\mathrm{mult}(\alpha)$ the root multiplicity, ϱ the Weyl vector of the algebra satisfying the condition $(\varrho, \alpha_i) = (\alpha_i, \alpha_i)/2$ for all real simple roots (simple roots that are space-like) α_i, W the Weyl group which is typically a hyperbolic reflection group and $\epsilon(w) = \pm 1$. Finally, Σ is some combination of simple imaginary roots, which encodes the information of the degeneracies of these time-or light-like simple roots.

More than a decade ago a generating function has been proposed for the 1/4-BPS spectrum of the $\mathcal{N} = 4$, $d = 4$ string theory obtained by compactifying type II strings on $K3 \times T^2$ [14]. There it was also observed that the generating function has the form of the square of the denominator of a generalized Kac–Moody superalgebra. This algebra was first constructed by Gritsenko and Nikulin in [54] and has a (2,1)-dimensional Cartan matrix

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix},$$

for the real roots. It is also the Gram matrix with entries $-2(\alpha_i, \alpha_j)$ in our metric convention. Later in [13], a more physical interpretation is given to the presence of this algebra. In particular, its Weyl group is identified with the group controlling the wall-crossing phenomenon of the physical spectrum.

As is often the case for automorphic forms arising as the denominator of generalised Kac–Moody algebras, both the Fourier coefficients in the sum side and the exponents on the product side of the formula (3.4) are given by the Fourier coefficients of some (typically different) modular objects. These phenomena are often referred to as the "arithmetic lift" and the "Borcherds lift," respectively. For the algebra of interest here, the two sides of the (square of the) denominator formula (3.4) is given by

$$\begin{aligned} \Phi(\Omega) &= \mathrm{e}^{4\pi \mathrm{i}(\varrho, \Omega)} \prod_{\alpha} (1 - \mathrm{e}^{2\pi \mathrm{i}(\alpha, \Omega)})^{c(|\alpha|^2)} \\ &= \sum_{m > 0} \mathrm{e}^{2\pi \mathrm{i} m \sigma}\, T_m \phi_{10,1}(\rho, \nu), \end{aligned} \tag{3.5}$$

where T_m denotes the mth Hecke operator and we write the complexified vector in the (2,1)-dimensional weight space in a matrix form as

$$\Omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix},$$

with the norm $|X|^2 = (X,X) = \det X$. The Weyl vector is given in terms of the three real simple roots by $\varrho = \frac{1}{2}\sum_{i=1}^{3} \alpha_i$, and a choice for the set of real simple roots is

$$\alpha_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

The product is then taken over α's that are positive semi-definite integral linear combinations of the $\alpha_{1,2,3}$:

$$\alpha \in \{\mathbb{Z}_+\alpha_1 + \mathbb{Z}_+\alpha_2 + \mathbb{Z}_+\alpha_3\}.$$

Note that the denominator $\Phi(\Omega)$ is independent of the choice of real simple roots, or equivalently a choice of the fundamental Weyl chamber, as can be seen explicitly from the sum side.

The exponents (root multiplicities) $c(k)$ on the product side and the Fourier coefficients on the sum side are then given by the weak Jacobi forms

$$\begin{aligned} Z_{K3}(\tau,z) = 2\phi_{0,1}(\tau,z) &= \sum_{n,\ell\in\mathbb{Z},n>0} c(4n-\ell^2)q^n y^\ell, \\ \phi_{10,1}(\tau,z) &= -\frac{\theta_1^2(\tau,z)}{\eta^6(\tau)}\,\eta^{24}(\tau), \end{aligned} \tag{3.6}$$

and $\Phi(\Omega)$ is an automorphic form of the genus two modular group $Sp(2,\mathbb{Z})$ of weight 10. In the last interpretation, Ω should really be thought of as the period matrix of the genus two surface.

The 1/4-BPS index with a given charge $(P,Q) \in \Gamma^{22,6} \oplus \Gamma^{22,6}$ and at a given point in the moduli space is then determined in the following way. Consider the following two vectors in the future light-cone of $\mathbb{R}^{2,1}$:

$$\Lambda_{(P,Q)} = \begin{pmatrix} Q\cdot Q & Q\cdot P \\ Q\cdot P & P\cdot P \end{pmatrix}$$

and

$$\tilde{Z} = \frac{1}{\lambda_2}\begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} + \frac{1}{\sqrt{Q_{\rm R}^2 P_{\rm R}^2 - (Q_{\rm R}\cdot P_{\rm R})^2}}\begin{pmatrix} Q_{\rm R}\cdot Q_{\rm R} & Q_{\rm R}\cdot P_{\rm R} \\ Q_R\cdot P_{\rm R} & P_{\rm R}\cdot P_{\rm R}\end{pmatrix},$$

where $\lambda = \lambda_1 + {\rm i}\lambda_2$ is the heterotic axion-dilaton and $Q_{\rm R}$, $P_{\rm R}$ are the right-moving part of the charges. See [13] for the details and motivations for these formulae. There are unique elements w_1 and w_2 of the Weyl group such that both $w_1(\Lambda_{(P,Q)})$ and $w_1 w_2(\tilde{Z})$ lie inside the fundamental Weyl chamber, which is defined as the region bounded by the planes of orthogonality to the real simple roots $\alpha_{1,2,3}$. Choosing the set of real simple roots $\alpha_{1,2,3}$ such that $w_1 = \mathbb{1}$, the corresponding 1/4-BPS index is then given by $D(w_2(\Lambda_{P,Q}))$, defined by

$$\text{(3.7)} \qquad \frac{1}{\Phi(\Omega)} = \sum_{\Lambda\in\mathcal{W}, w\in W} D(w(\Lambda)){\rm e}^{2\pi{\rm i}(w(\Lambda),\Omega)}.$$

In the above formula, an expansion in ${\rm e}^{2\pi{\rm i}(\alpha_i,\Omega)}, i = 1,2,3$ on the left-hand side should be understood. See also [25] for an equivalent formula in terms of a contour integral with moduli-dependent contours.

As mentioned earlier, the Weyl group of the algebra plays an important role in the counting of dyons due to the so-called wall-crossing phenomenon in this theory. When the moduli-vector $\tilde{Z}$ hits the wall of a Weyl chamber, some 1/4-BPS bound states corresponding to bound states of 1/2-BPS particles appear or disappear from the spectrum and as a result the 1/4-BPS index jumps. This physical phenomenon is reflected in the following property of the partition function [32]. The denominator $1/\Phi(\Omega)$ has double poles at the location where the vector Ω lies on wall of the Weyl chamber and factorizes into two parts, each corresponds to the 1/2-BPS partition function. For instance,

$$\text{(3.8)} \qquad \frac{1}{\Phi(\Omega)} \xrightarrow{\nu\to 0} -\frac{1}{4\pi^2\nu^2}\frac{1}{\eta^{24}(\rho)}\frac{1}{\eta^{24}(\sigma)} + \text{reg.}$$

and similarly for all other poles, where 'reg.' denotes terms that are regular as $\nu \to 0$.

Apart from the 1/4-BPS index, or the sixth helicity supertrace to be more precise, finer information about the spectrum can be obtained when extra isometries are present. Recently, such "twisted 1/4-BPS indices"

$$B_g(Q,P) = \frac{1}{6!}{\rm Tr}_{\mathcal{H}_{Q,P}}(g(-1)^{2J_3}(2J_3)^6),$$

of the above theory have been studied [16, 17] when there is an enhanced geometric $\mathbb{Z}_{p=2,3,5,7}$ symmetry in the internal manifold $K3\times T^2$. Here J_3 denotes the third component of the spacetime angular momentum quantum number. Denoting the generator of the $\mathbb{Z}_{p=2,3,5,7}$ isometry by g_p, the corresponding generating function is given by $1/\Phi_{g_p}(\Omega)$, where $\Phi_{g_p}(\Omega)$ is an automorphic form of a subgroup of $Sp(2,\mathbb{Z})$ which also has an infinite-product expression[4] [7, 16, 17]

$$\Phi_{g_p}(\Omega) = \mathrm{e}^{4\pi\mathrm{i}(\varrho,\Omega)} \prod_{r=0}^{p-1}\prod_{\alpha} \times \left(1-\mathrm{e}^{2\pi\mathrm{i}(r/p)}\mathrm{e}^{2\pi\mathrm{i}(\alpha,\Omega)}\right)^{\frac{1}{p}\sum_{s=0}^{p-1}\mathrm{e}^{-2\pi\mathrm{i}rs/p}c_{(g_p)^s}(|\alpha|^2)} \tag{3.9}$$

In this formula, the product is again taken over positive roots α as before, and the exponents $c_{(g_p)^s}(|\alpha|^2)$ are now the Fourier coefficients of the twisted $K3$ elliptic genus $\mathcal{Z}_{(g_p)^s}(\tau,z)$ (2.4).

Now, notice that (3.9) can be written as

$$\Phi_g(\Omega) = \mathrm{e}^{4\pi\mathrm{i}(\varrho,\Omega)}\prod_{\alpha}\exp\left(\sum_{k=1}^{\infty}-\frac{1}{k}c_{(g)^k}(|\alpha|^2)\,\mathrm{e}^{2\pi\mathrm{i}(k\alpha,\Omega)}\right), \tag{3.10}$$

when $g=g_p$. We recognize this is nothing but the twisted denominator of the original generalized Kac–Moody algebra (3.5). More precisely, this is the

[4] In fact they are also known to have the following infinite sum expression [9]:

$$\Phi_{g_p}(\Omega) = \sum_{m>0}\mathrm{e}^{2\pi\mathrm{i}m\sigma}\,T_m\left(-\frac{\theta_1^2(\rho,\nu)}{\eta^6(\rho)}\eta_{g_p}(\rho)\right),\quad p=2,3,5,7,$$

just as in the untwisted case (3.5). Unlike in the Monster case, the assumption that the sporadic group M_{24} is a symmetry of the positive root system does not directly imply that it acts on the system of simple roots in a simple way, and hence it is not straightforward to twist the sum side of the denominator identity. From the form of the above equation it is nevertheless tempting to see whether it is possible to generalize it to other elements of M_{24}. As to be expected probably, it fails for all conjugacy classes of M_{24} for which the object $-(\theta_1^2(\rho,\nu)/\eta^6(\rho))\eta_g(\rho)$ has zero or negative weight, and furthermore does not seem to hold for the conjugacy class $3B$. For all other cases it appears to hold up to the first few dozens of coefficients that we have checked.

expression one obtains if we regard the subalgebra E spanned by positive roots as an infinite-dimensional representation of M_{24} graded by a (2,1) lattice, and the denominator as calculating the alternating sum of exterior powers

$$\Lambda(E) = \Lambda^0(E) \oplus \Lambda^1(E) \oplus \cdots,$$

of E [12]. Furthermore, once accepting this interpretation, the above formula defines a twisted denominator for all elements of $g \in M_{24}$, whether or not they are known to generate the geometric $\mathbb{Z}_N$ actions.

To see the appearance of the η-products in the twisted denominators, let us again look at the wall-crossing poles of the (twisted) partition function which should relate the jump in 1/4-BPS index to 1/2-BPS indices. Using

$$\sum_\ell c_g(4n-\ell^2) = \mathrm{Tr}_{\rho_1\oplus\rho_{23}}(g)\delta_{n,0}, \quad c_g(-1) = -2, \quad c_g(-n) = 0, \text{ for } n > 1,$$

one indeed obtains

$$\frac{\nu^2}{\Phi_g(\Omega)} \xrightarrow{\nu\to 0} -\frac{1}{4\pi^2\nu^2}\frac{1}{\eta_g(\rho)}\frac{1}{\eta_g(\sigma)} + \text{reg.} \tag{3.11}$$

for all $g \in M_{24}$.

Now we see that the two versions of moonshine discussed in Sections 2 and 3.1, which involve two seemingly unrelated M_{24} modules, indeed co-exist and get related in the denominator of the generalized Kac–Moody algebra, which has been studied in the context of dyon counting in the $\mathcal{N} = 4$, $d = 4$ theory. From the physics point of view, this is consistent with the tentative interpretation we give to $1/\Phi_g(\Omega)$ as the generating function of the twisted 1/4-BPS index, and $1/\eta_g(\tau)$ as the twisted 1/2-BPS partition function, and the wall-crossing formulae that we already know [13, 25, 32].

This insight about the sporadic symmetry of the dyon algebra does not only shed light on the structure of the non-perturbative structure in the $d = 4$ theories, they also help us to predict various (twisted) spectra. We now proceed to discuss these consequences in our last section.

4. Conclusions and discussions

In this paper, by studying the twisted elliptic genera of $K3$ we first provided further evidence for the idea that associated with the elliptic genus of $K3$ is an infinite-dimensional graded representation of the sporadic group M_{24}. After that we pointed out that this M_{24} module and the previously observed

M_{24} moonshine given by the η-products are in fact united in the denominator of the generalized Kac–Moody algebra that is conjectured to generate the spectrum of $K3 \times T^2$-compactified type II string theory.

From these observations and again using the embedding of the symplectic automorphisms of $K3$ into M_{24}, we have the following conservative version of a conjecture: for all the symplectic automorphisms of $K3$ generated by $g \in M_{24}$

1. At a point in the moduli space where there is an enhanced symmetry generated by g, let us consider the $K3$ elliptic genus twisted by g. Then this object is given by (2.7), where $K^\natural$ is an infinite-dimensional graded representation of M_{24} with the first five $K^\natural_{n=1,\ldots,5}$ given by conjugate pairs of irreducible representations of M_{24} with the dimensions indicated in table 1.

2. Consider the $\mathcal{N} = 4, d = 4$ theory obtained by compactifying type II string theory on $K3 \times T^2$ with enhanced symmetry generated by $\tilde{g}$, which acts as a symplectic automorphism generated by g on $K3$ accompanied by a shift action on T^2. This shift is uniquely determined up to isomorphisms and we refer to [50] for their explicit forms. The 1/2-BPS partition function twisted by such a symmetry is given by $1/\eta_g(\tau)$ (3.2). See also [10].

3. The 1/4-BPS spectrum twisted by the same symmetry is given by the automorphic form $1/\Phi_g(\Omega)$, where $\Phi_g(\Omega)$ is the twisted denominator defined in the form of an infinite product as in (3.10).

A bolder, and perhaps more natural, version of the conjecture will state that the full M_{24} acts on moduli-invariant objects in $K3$-compactified string theories, including the $K3$ elliptic genus and the dyon algebra of the $K3 \times T^2$-compactified type II string theory. Note that, as we discussed in the introduction, this does not necessarily mean all subgroups of M_{24} have to be realized as an actual symmetry of the string theory at a certain point in the moduli space. But in the case when $g \in M_{24}$ does generate a symmetry group realized on some submanifold of the moduli space, we expect the above three properties to extend to it, once we drop the requirement that the action be fully geometric. A construction of the new string theories obtained by orbifolding with these non-geometric symmetries, perhaps from the heterotic side by associating cycle shapes to heterotic lattices at enhanced symmetry points in a way similar to [50], will be extremely interesting. Hence, here we are in one of the interesting occasions when we know quite a lot about the answers (given by the above three conjectural formulas) before quite knowing what exactly the question is (what the new symmetries are that we have twisted the partition functions with).

Finally, we finish with some discussions. First, a geometric interpretation of the above conjectures might be interesting in its own right. Recall that the 1/2-BPS spectrum $1/\eta^{24}(\tau)$ has an interpretation as counting nodal curves in $K3$ [56], and the 1/4-BPS index can be thought of as an appropriate counting of non-factorizable special Lagrangian cycles in $K3 \times T^2$. The twisted objects give finer geometric information: while the η-products $1/\eta_g(\tau)$ are now related to a certain counting of nodal curves in the corresponding orbifolded $K3$, the twisted denominator formula should give a version of the "twisted counting" of special Lagrangian cycles in $K3 \times T^2$. Of course such a geometric interpretation of the partition functions is only applicable when g generates a geometric symmetry.

Second, following the long string derivation in [57] we obtain the following generating function for the twisted elliptic genera for the symmetric product of $K3$:

$$\sum_{N\geq 0} p^N \mathcal{Z}_g(S^N K3; \tau, z) = \prod_{n>0, m\geq 0, \ell} \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} c_{(g)^k}(4nm - \ell^2)\,(p^n q^m y^\ell)^k\right).$$

This differs from the twisted 1/4-BPS partition function $1/\Phi_g$ by a factor

$$\frac{y}{(1-y)^2} \prod_{m\geq 1} \frac{(1-q^m)^4}{(1-q^m y)^2 (1-q^m y^{-1})^2} \times \frac{1}{\eta_g(\tau)},$$

which also has a physical interpretation in terms of the KK monopole and the center-of-mass degrees of freedom in the D1–D5-KK duality frame.

Third, as mentioned in the first section, by studying the twisted elliptic genera and relating them to the representations of M_{24} we have learned that the representation should really be thought of as coming in conjugate pairs, rather than two duplicate copies as one might have concluded by just looking at the untwisted elliptic genus. A direct parallel of this in the dyon counting algebra is, instead of thinking of the spectrum as being generated by two copies of the same algebra and the generating function as being related to the denominator of the algebra as $Z \sim (1/\text{den})^2$, it should really be thought of as $Z \sim (1/\text{den}) \times (1/\overline{\text{den}})$. In particular, if we use the first interpretation, an interpretation of the twisted 1/4-BPS indices as given by the denominator identity twisted by elements of M_{24} (3.10) of the original algebra will not be possible. Note that this insight is not available, as far as the author can see, without making the connection between the $\mathcal{N}=4, d=4$ string theory with the sporadic group M_{24}.

Finally, as alert readers might have already noted, apart from the twisted 1/4-BPS indices of the un-orbifolded theory, we can also ask what the twisted and untwisted 1/4-BPS indices of the theories orbifolded by the corresponding symmetries are. For the untwisted indices for the $\mathbb{Z}_N$-orbifolded theories, the partition function have been proposed in [9, 27] for $N = 2, 3, 5, 7$. Not surprisingly, these objects $1/\tilde{\Phi}_g(\Omega)$ are related to the twisted partition functions $1/\Phi_g(\Omega)$ by an automorphic lift of the S-transformation $\tau \to -\frac{1}{\tau}$ from the genus one modular group $SL(2,\mathbb{Z})$ into the genus two modular group $Sp(2,\mathbb{Z})$. In [23], an interpretation in terms of a spectrum-generating function similar to the one reviewed in Section 3.2 was proposed for $N = 2, 3$ and in [10] for the case $N = 4$. These algebras are not directly related to the original generalized Kac–Moody algebra (3.5)[5]. As discussed above, from their relations with the sporadic group M_{24}, we now know that the dyon partition function should not be of the form $Z \sim (1/\mathrm{den})^2$. We suspect that this previous prejudice might be related to the difficulty met in constructing the spectrum-generating algebra for $\mathbb{Z}_N$-orbifold with $N > 4$. It might be worthwhile to re-examine this issue, now that we have much more information about the expected structures thanks to the insight the largest Mathieu group has brought into the structure of $\mathcal{N}=4$, $d=4$ string theories.

Acknowledgments

I am deeply grateful to Clay Cordova, Atish Dabholkar, John Duncan, Jeff Harvey, Albrecht Klemm, Boris Pioline, Yuji Tachikawa, Cumrun Vafa, Erik Verlinde and especially Ashoke Sen for extremely helpful discussions during the course of this work. I also thank Eyjafjallajökull for making them possible. I would also like to thank LPTHE Jussieu, Universität Bonn, and Universität Heidelberg, where a part of this research was conducted, for their hospitality. This work was supported by the DOE grant number DE-FG02-91ER40654.

[5]For interested readers, they should be thought of as the analogues of the algebras $\mathfrak{m}_g, g \in \mathbf{M}$ constructed by S. Carnahan [58] as a part of the effort to prove Norton's conjecture on the modularity properties of the generalized McKay–Thompson series (2.11) in the context of Monstrous Moonshine.

Appendix A. Character tables

Table 2: Character table of M_{24}. See [59, 60]. We adopt the naming system of [59] and use the notation $e_n = \frac{1}{2}(-1 + \mathrm{i}\sqrt{n})$.

Cycle	1^{24}	$1^8 2^8$	$1^6 3^6$	$1^4 5^4$	$1^4 2^2 4^4$	$1^3 7^3$	$1^3 7^3$	$1^2 2\, 4\, 8^2$	$1^2 2^2 3^2 6^2$	$1^2\, 11^2$	$1\ 23$	$1\ 23$
Order	$1A$	$2A$	$3A$	$5A$	$4B$	$7A$	$\overline{7A}$	$8A$	$6A$	$11A$	$23A$	$\overline{23A}$
	1	1	1	1	1	1	1	1	1	1	1	1
	23	7	5	3	3	2	2	1	1	1	0	0
	45	−3	0	0	1	e_7	$\bar{e}_7$	−1	0	1	−1	−1
	$\overline{45}$	−3	0	0	1	$\bar{e}_7$	e_7	−1	0	1	−1	−1
	231	7	−3	1	−1	0	0	−1	1	0	1	1
	$\overline{231}$	7	−3	1	−1	0	0	−1	1	0	1	1
	252	28	9	2	4	0	0	0	1	−1	−1	−1
	253	13	10	3	1	1	1	−1	−2	0	0	0
	483	35	6	−2	3	0	0	−1	2	−1	0	0
	770	−14	5	0	−2	0	0	0	1	0	e_{23}	$\bar{e}_{23}$
	$\overline{770}$	−14	5	0	−2	0	0	0	1	0	$\bar{e}_{23}$	e_{23}
	990	−18	0	0	2	e_7	$\bar{e}_7$	0	0	0	1	1
	$\overline{990}$	−18	0	0	2	$\bar{e}_7$	e_7	0	0	0	1	1
	1035	−21	0	0	3	$2e_7$	$2\bar{e}_7$	−1	0	1	0	0
	$\overline{1035}$	−21	0	0	3	$2\bar{e}_7$	$2e_7$	−1	0	1	0	0
	1035’	27	0	0	−1	−1	−1	1	0	1	0	0
	1265	49	5	0	1	−2	−2	1	1	0	0	0
	1771	−21	16	1	−5	0	0	−1	0	0	0	0
	2024	8	−1	−1	0	1	1	0	−1	0	0	0
	2277	21	0	−3	1	2	2	−1	0	0	0	0
	3312	48	0	−3	0	1	1	0	0	1	0	0
	3520	64	10	0	0	−1	−1	0	−2	0	1	1
	5313	49	−15	3	−3	0	0	−1	1	0	0	0
	5544	−56	9	−1	0	0	0	0	1	0	1	1

5796	-28	-9	1	4	0	0	0	-1	-1	0	0		
10395	-21	0	0	-1	0	0	1	0	0	-1	-1		
1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	0	$-e_7$	$-\bar{e}_7$	1	-1	1	3	5	0	$\bar{e}_7$	e_7	-3	0
0	0	$-\bar{e}_7$	$-e_7$	1	-1	1	3	5	0	e_7	$\bar{e}_7$	-3	0
e_{15}	$\bar{e}_{15}$	0	0	0	0	3	0	-9	1	0	0	-1	-1
$\bar{e}_{15}$	e_{15}	0	0	0	0	3	0	-9	1	0	0	-1	-1
-1	-1	0	0	0	0	0	0	12	2	0	0	4	1
0	0	-1	-1	1	1	1	1	-11	-1	1	1	-3	0
1	1	0	0	0	0	3	0	3	-2	0	0	3	0
0	0	0	0	1	1	-2	-7	10	0	0	0	2	-1
0	0	0	0	1	1	-2	-7	10	0	0	0	2	-1
0	0	e_7	$\bar{e}_7$	1	-1	-2	3	-10	0	$\bar{e}_7$	e_7	6	0
0	0	$\bar{e}_7$	e_7	1	-1	-2	3	-10	0	e_7	$\bar{e}_7$	6	0
0	0	0	0	-1	1	-1	-3	-5	0	$-\bar{e}_7$	$-e_7$	3	0
0	0	0	0	-1	1	-1	-3	-5	0	$-e_7$	$-\bar{e}_7$	3	0
0	0	-1	-1	0	2	3	6	35	0	-1	-1	3	0
0	0	0	0	0	0	-3	8	-15	0	1	1	-7	-1
1	1	0	0	-1	-1	-1	7	11	1	0	0	3	0
-1	-1	1	1	0	0	0	8	24	-1	1	1	8	-1
0	0	0	0	0	2	-3	6	-19	1	-1	-1	-3	0
0	0	-1	-1	0	-2	0	-6	16	1	1	1	0	0
0	0	1	1	0	0	0	-8	0	0	-1	-1	0	0
0	0	0	0	0	0	-3	0	9	-1	0	0	1	1
-1	-1	0	0	0	0	0	0	24	-1	0	0	-8	1
1	1	0	0	0	0	0	0	36	1	0	0	-4	-1
0	0	0	0	0	0	3	0	-45	0	0	0	3	0

Appendix B. Modular Stuff

Theta functions

$$
\begin{aligned}
\text{(B.1)}\qquad \theta_1(\tau,z) &= -\mathrm{i}q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1}),\\
\theta_2(\tau,z) &= q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^{n-1}),\\
\theta_3(\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1+y\,q^{n-1/2})(1+y^{-1}q^{n-1/2}),\\
\theta_4(\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1-y\,q^{n-1/2})(1-y^{-1}q^{n-1/2}).
\end{aligned}
$$

Weak Jacobi forms

$$
\begin{aligned}
\phi_{0,1}(\tau,z) &= 4\left(\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2+\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\right),\\
\phi_{-2,1}(\tau,z) &= -\frac{\theta_1^2(\tau,z)}{\eta^6(\tau)}.
\end{aligned}
$$

They are weak Jacobi forms of index one and weight 0 and -2, respectively, and they generate the ring of weak Jacobi forms of even weight (Theorem 9.3 [61]).

Higher level

The family of congruence subgroups of the modular group $PSL(2,\mathbb{Z})$ that are most relevant for this paper are

$$
\Gamma_0(N)=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in PSL(2,\mathbb{Z}),\quad c=0 \bmod N\right\}.
$$

A weight two modular form of $\Gamma_0(N)$ is

$$
\phi_2^{(N)}(\tau)=\frac{24}{N-1}\,q\partial_q\log\left(\frac{\eta(N\tau)}{\eta(\tau)}\right)=1+\frac{24}{N-1}\sum_{k>0}\sigma(k)(q^k-Nq^{Nk}),
$$

where $\sigma(k)$ is the divisor function $\sigma(k)=\sum_{d|k}d$. This is the combination of weight two, non-holomorphic Eisenstein series $G_2^*(\tau)-NG_2^*(N\tau)$, normalized such that the constant term in one.

Apart from these so-called "old-forms," for some N there are also "new-forms," which are weight two modular forms not of the above kind. Together they span the space of weight two modular forms of congruence subgroup $\Gamma_0(N)$.

For $N = 11$, there is one such new form

$$f_{11}(\tau) = \eta^2(\tau)\eta^2(11\tau). \tag{B.2}$$

A similar thing happens for $N = 14, 15$. The respective new forms are

$$f_{14}(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau), \tag{B.3}$$

$$f_{15}(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau). \tag{B.4}$$

For $N = 23$, there are two such new forms, one with coefficients in $\mathbb{Z} + \mathbb{Z}\frac{1-\sqrt{5}}{2}$ and the other obtained by replacing $\sqrt{5}$ by $-\sqrt{5}$

$$\tilde{f}_{23,1}(\tau) = q - \frac{1-\sqrt{5}}{2}q^2 - \sqrt{5}q^3 - \frac{1+\sqrt{5}}{2}q^4 - (1-\sqrt{5})q^5 - \frac{5-\sqrt{5}}{2}q^6 + \cdots,$$
$$\tilde{f}_{23,2}(\tau) = q - \frac{1+\sqrt{5}}{2}q^2 + \sqrt{5}q^3 - \frac{1-\sqrt{5}}{2}q^4 - (1+\sqrt{5})q^5 - \frac{5+\sqrt{5}}{2}q^6 + \cdots.$$

Here we use the basis such that they are Hecke eigenforms. For the actual computation it is more convenient to use the basis

$$f_{23,1}(\tau) = \tilde{f}_{23,1}(\tau) + \tilde{f}_{23,2}(\tau), \quad f_{23,2}(\tau) = \frac{1}{\sqrt{5}}\left(\tilde{f}_{23,1}(\tau) - \tilde{f}_{23,2}(\tau)\right). \tag{B.5}$$

See [62] and Chapter 4D of [63] for more details on these modular forms.

A discussion about the ring of weak Jacobi forms of higher level can be found in [64]. From its Proposition 6.1, we conclude that weight zero, index one weak Jacobi forms of congruence subgroup $\Gamma_0(N)$ are linear combinations of $\phi_{0,1}(\tau, z)$ and $\phi_{-2,1}(\tau, z) \times f_N(\tau)$, where $f_N(\tau)$ is a weight two modular form of $\Gamma_0(N)$ and is hence linear combinations of the above "old" and "new" forms.

Appendix C. Formulae for twisted elliptic genera

In this appendix, we collect examples of elliptic genera of the $K3$ CFT twisted by elements of M_{24}. Unlike the examples discussed in Section 2, in these examples the elements $g \in M_{24}$ do not have known interpretation as acting on the classical geometry of $K3$ surfaces.

(C.1)

$$
\begin{aligned}
\mathcal{Z}_{11A}(\tau,z) &= \frac{1}{6}\phi_{0,1}(\tau,z) + \phi_{-2,1}(\tau,z)\left(\frac{11}{6}\phi_2^{(11)}(\tau) - \frac{22}{5}f_{11}(\tau)\right),\\
\mathcal{Z}_{23A}(\tau,z) &= \frac{1}{12}\phi_{0,1}(\tau,z) + \phi_{-2,1}(\tau,z)\\
&\quad\times\left(\frac{23}{12}\phi_2^{(23)}(\tau) - \frac{23}{22}f_{23,1}(\tau) - \frac{161}{22}f_{23,2}(\tau)\right),\\
\mathcal{Z}_{14A}(\tau,z) &= \frac{1}{12}\phi_{0,1}(\tau,z) + \phi_{-2,1}(\tau,z)\\
&\quad\times\left(-\frac{1}{36}\phi_2^{(2)}(\tau) - \frac{7}{12}\phi_2^{(7)}(\tau) + \frac{91}{36}\phi_2^{(14)}(\tau) - \frac{14}{3}f_{14}(\tau)\right),\\
\mathcal{Z}_{15A}(\tau,z) &= \frac{1}{12}\phi_{0,1}(\tau,z) + \phi_{-2,1}(\tau,z)\\
&\quad\times\left(-\frac{1}{16}\phi_2^{(3)}(\tau) - \frac{5}{24}\phi_2^{(5)}(\tau) + \frac{35}{16}\phi_2^{(15)}(\tau) - \frac{15}{4}f_{15}(\tau)\right),\\
\mathcal{Z}_{2B}(\tau,z) &= \phi_{-2,1}(\tau,z)\left(-2\phi_2^{(2)}(\tau) + 4\phi_2^{(4)}(\tau)\right)\\
&= 16\,\phi_{-2,1}(\tau,z)\,q\partial_q \log\left(\frac{\eta(\tau)\,\eta^2(4\tau)}{\eta^3(2\tau)}\right),\\
\mathcal{Z}_{4A}(\tau,z) &= \phi_{-2,1}(\tau,z)\left(\frac{1}{3}\phi_2^{(2)}(\tau) - 3\phi_2^{(4)}(\tau) + \frac{14}{3}\phi_2^{(8)}(\tau)\right)\\
&= 8\phi_{-2,1}(\tau,z)q\partial_q \log\left(\frac{\eta(2\tau)\,\eta^2(8\tau)}{\eta^3(4\tau)}\right).
\end{aligned}
$$

The weight two modular forms in the above formulae are discussed in the last appendix. As checks of these formulae, one can easily verify that all the Fourier coefficients are integers.

One curious observation is, one can show that the McKay–Thompson series of the form in (2.7) is a weak Jacobi form of the group $\Gamma_0(\mathrm{ord}(g))$ if and only the element g is not only an element of M_{24} but also an element of M_{23}. This fact supports the possibility that for all conjugacy class of M_{23}, there is a point in the moduli space where the symmetry is realized as a symmetry of the CFT. For the two examples ($g = 2B, 4A$) of elements that are in M_{24} but not in M_{23}, slightly more subtle modular properties of the corresponding twisted elliptic genus $\mathcal{Z}_g(\tau,z)$ are discussed in [65].[6]

[6]Paragraph rewritten in the second version. We thank M. Gaberdiel for correspondence on this point.

References

[1] T. Gannon, *Moonshine beyond the monster. The bridge connecting algebra, modular forms and physics*. Cambridge University Press, Cambridge, 2006.

[2] E. Witten, *Three-dimensional gravity revisited*, `arXiv:0706.3359`.

[3] J. F. R. Duncan and I. B. Frenkel, *Rademacher sums, moonshine and gravity*, `arXiv:0907.4529`.

[4] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94**(1) (1988), 183–221.

[5] S. Kondo, Shigeyuki Mukai, *Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of K3 surfaces. With an appendix by Shigeru Mukai.*, Duke Math. J. **92**(3) (1998), 593–603.

[6] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 surface and the Mathieu group M_{24}*, `arXiv:1004.0956`.

[7] J. R. David, D. P. Jatkar and A. Sen, *Product representation of dyon partition function in CHL models*, J. High Energy Phys. **0606** (2006) 064, `arXiv:hep-th/0602254`.

[8] A. Sen, *Black holes and the spectrum of Half-BPS states in $N = 4$ supersymmetric string theory*, Adv. Theor. Math. Phys. **9** (2005) 527–558, `arXiv:hep-th/0504005`.

[9] D. P. Jatkar and A. Sen, *Dyon spectrum in CHL Models*, J. High Energy Phys. **0604** (2006) 018, `arXiv:hep-th/0510147`.

[10] S. Govindarajan and K. G. Krishna, *BKM Lie superalgebras from dyon spectra in Z_N-CHL orbifolds for composite N*, `arXiv:0907.1410`.

[11] G. Mason, *M_{24} and certain automorphic forms*, Contemp. Math. **45** (1985), 223–244.

[12] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109**(2) (1992), 405–444.

[13] M. C. N. Cheng and E. P. Verlinde, *Wall crossing, discrete attractor flow, and Borcherds algebra*, `arXiv:0806.2337 [hep-th]`.

[14] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Counting dyons in $N = 4$ String Theory*, Nucl. Phys. B **484** (1997) 543–561 (Nucl. Phys. B 484:543-561,1997), `arXiv:hep-th/9607026`.

[15] A. Sen, *Wall crossing formula for $N = 4$ dyons: a macroscopic derivation*, J. High Energy Phys. **0807** (2008) 078, arXiv:0803.3857.

[16] A. Sen, *A twist in the dyon partition function*, arXiv:0911.1563.

[17] A. Sen, *Discrete information from CHL black Holes*, arXiv:1002.3857.

[18] A. Sen, *Quantum entropy function from AdS(2)/CFT(1) correspondence*, Int. J. Mod. Phys. A **24** (2009), 4225–4244 (Sept. 2008), arXiv:0809.3304.

[19] S. Zwegers, *Mock Theta Functions*, PhD thesis, Utrecht University, 2002.

[20] D. Zagier, *Ramanujan's Mock Theta Functions and Their Applications.* Sèminaire Bourbaki, 2007–2008, no. 986.

[21] A. Sen, *Two Centered Black Holes and $N = 4$ Dyon Spectrum*, J. High Energy Phys. **0709** (2007) 045 (May, 2007), arXiv:0705.3874.

[22] A. Sen, *Rare decay modes of quarter BPS dyons*, J. High Energy Phys. **0710** (2007) 059 (July, 2007), arXiv:0707.1563.

[23] M. C. N. Cheng and A. Dabholkar, *Borcherds–Kac–Moody symmetry of $N = 4$ dyons*, arXiv:0809.4258.

[24] M. C. N. Cheng and L. Hollands, *A geometric derivation of the dyon wall-crossing group*, J. High Energy Phys. **04** (2009) 067, arXiv:0901.1758 [hep-th].

[25] M. C. N. Cheng and E. Verlinde, *Dying dyons don't count*, J. High Energy Phys. **09** (2007) 070, arXiv:0706.2363 [hep-th].

[26] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, J. High Energy Phys. **01** (2008) 023, arXiv:hep-th/0702150.

[27] J. R. David, D. P. Jatkar and A. Sen, *Dyon spectrum in generic $N = 4$ supersymmetric Z_N orbifolds*, J. High Energy Phys. **0701** (2007) 016 (JHEP 0701:016,2007), arXiv:hep-th/0609109.

[28] J. R. David, D. P. Jatkar and A. Sen *Dyon spectrum in $N = 4$ supersymmetric type II string theories*, J. High Energy Phys. **0611** (2006) 073 (JHEP 0611:073,2006), arXiv:hep-th/0607155.

[29] J. R. David and A. Sen, *CHL dyons and statistical entropy function from D1-D5 system*, J. High Energy Phys. **0611** (2006) 072 (JHEP 0611:072,2006), `arXiv.hep-th/0605210`.

[30] A. Dabholkar and S. Nampuri, *Spectrum of dyons and Black Holes in CHL orbifolds using Borcherds Lift*, J. High Energy Phys. **0711** (2007) 077, `arXiv:hep-th/0603066`.

[31] A. Sen, *Three string junction and $N = 4$ dyon spectrum*, J. High Energy Phys. **0712** (2007) 019 (Aug., 2007), `arXiv:0708.3715`.

[32] A. Sen, *Walls of marginal stability and dyon spectrum in $N = 4$ supersymmetric string theories*, J. High Energy Phys. **0705** (2007) 039 (JHEP 0705:039,2007), `arXiv:hep-th/0702141`.

[33] S. Govindarajan and K. G. Krishna, *Generalized Kac–Moody Algebras from CHL dyons*, J. High Energy Phys. **0904** (2009) 032 (July, 2008), `arXiv:0807.4451`.

[34] G. L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Black hole partition functions and duality*, J. High Energy Phys. **0603** (2006) 074 (JHEP0603:074,2006), Preprint, `arXiv:hep-th/0601108`.

[35] A. Dabholkar, K. Narayan and S. Nampuri, *Degeneracy of decadent dyons*, J. High Energy Phys. **0803** (2008) 026 (Feb., 2008), `arXiv:0802.0761`.

[36] A. Dabholkar, J. Gomes and S. Murthy, *Counting all dyons in $N = 4$ string theory*, Preprint, `arXiv:0803.2692 [hep-th]`.

[37] A. Dabholkar, M. Guica, S. Murthy and S. Nampuri, *No entropy enigmas for $N = 4$ dyons*, `arXiv:0903.2481 [hep-th]`.

[38] A. Dabholkar, K. Narayan and S. Nampuri, *Degeneracy of decadent dyons*, J. High Energy Phys. **03** (2008) 026, Preprint, `arXiv:0802.0761 [hep-th]`.

[39] N. Banerjee, D. P. Jatkar and A. Sen, *Asymptotic expansion of the $N = 4$ dyon degeneracy*, J. High Energy Phys. **0905** (2009) 121 (Oct., 2008), `arXiv:0810.3472`.

[40] S. Banerjee and A. Sen, *Duality orbits, dyon spectrum and gauge theory limit of heterotic string theory on T^6*, J. High Energy Phys. **0803** (2008) 022 (Dec., 2007), `arXiv:0712.0043`.

[41] S. Banerjee, A. Sen and Y. K. Srivastava, *Genus two surface and quarter BPS dyons: the contour prescription*, J. High Energy Phys. **0903** (2009) 151 (Aug., 2008), `arXiv:0808.1746`.

[42] S. Banerjee, A. Sen and Y. K. Srivastava, *Partition functions of torsion >1 dyons in heterotic string theory on T^6*, J. High Energy Phys. **0805** (2008), 098 (Feb., 2008), `arXiv:0802.1556`.

[43] A. Mukherjee, S. Mukhi and R. Nigam, *Dyon death eaters*, J. High Energy Phys. **10** (2007) 037, `arXiv:0707.3035 [hep-th]`.

[44] A. Dabholkar and D. Gaiotto, *Spectrum of CHL dyons from genus-two partition function*, J. High Energy Phys. **0712** (2007), 087 (JHEP0712:087,2007), `hep-th/0612011`.

[45] S. Mukhi and R. Nigam, *Constraints on 'rare' dyon decays*, J. High. Energy Phys. **12** (2008), 056, `arXiv:0809.1157 [hep-th]`.

[46] D. Shih, A. Strominger and X. Yin, *Recounting dyons in $N = 4$ string theory*, J. High. Energy Phys. **0610** (2006), 087 (JHEP 0610:087,2006), `arXiv:hep-th/0505094`.

[47] D. Gaiotto, *Re-recounting dyons in $N = 4$ string theory*, `arXiv:hep-th/0506249`.

[48] S. Chaudhuri, G. Hockney and J. D. Lykken, *Maximally supersymmetric string theories in $D < 10$*, Phys. Rev. Lett. **75** (1995), 2264–2267, `arXiv:hep-th/9505054`.

[49] S. Norton, *Generalized Moonshine*, Proc. Symp. Pure Math. **47** (1987), 208–209.

[50] S. Chaudhuri and D. A. Lowe, *Type IIA-heterotic duals with maximal supersymmetry*, Nucl. Phys. B **459** (1996), 113–124, `arXiv:hep-th/9508144`.

[51] G. Mason, *On a system of elliptic modular forms attached to the large Mathieu group*, Nagoya Math. J. **118** (1990), 177–193.

[52] C. Y. Dong and G. Mason, *An Orbifold theory of genus zero associated to the sporadic group M(24)*, Commun. Math. Phys. **164** (1994), 87–104.

[53] D. Ford and J. McKay, *Ramifications of Ramanunjan's work on η-products*, Proc. Indian Acad. Sci. **99** (3) (1989), 221–229.

[54] V. A. Gritsenko and V. V. Nikulin, *Siegel automorphic form corrections of some Lorentz Kac-Moody algebras*, Am. J. Math. **119** (1991), 181.

[55] U. Ray, *Automorphic forms and Lie superalgebras.* Algebra and Applications 5. Dordrecht: Springer, 2006.

[56] S.-T. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on K3*, Nucl. Phys. B **471** (1996), 503–512, `arXiv:hep-th/9512121`.

[57] R. Dijkgraaf, G. W. Moore, E. P. Verlinde and H. L. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Commun. Math. Phys. **185** (1997), 197–209, `arXiv:hep-th/9608096`.

[58] S. Carnahan, *Monstrous Lie algebras and generalized moonshine.* PhD thesis, University of California, Berkeley, 2007.

[59] J. Conway, R. Curtis, S. Norton, R. Parker and R. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.* Oxford: Clarendon Press, 1985.

[60] G. D. James, *The modular characters of the mathieu groups*, J. Algebra **27** (1973), 57–111.

[61] M. Eichler and D. Zagier, *The theory of Jacobi forms.* Progress in Mathematics, Vol. 55. Boston-Basel-Stuttgart" Birkhäuser, 1985.

[62] `http://modi.countnumber.de/`.

[63] C. Itzykson, ed., *From number theory to Physics.* Berlin: Springer-Verlag, 1992.

[64] H. Aoki and T. Ibukiyama, *Simple graded rings of Siegel modular forms, differential operators and Borcherds products*, Int. J. Math. **16** (3) (2005), 249–279.

[65] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu twining characters for K3*, `arXiv:1006.0221`.

Department of Mathematics, Harvard University,
Cambridge, MA 02138, USA
E-mail address: `mcheng@math.harvard.edu`

Received June 1, 2010