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Large Diffusivity Stability of Attractors in the $C(\bar{\Omega})-$ Topology for a Semilinear Reaction and Diffusion System of Equations.

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Abstract. A semilinear system of reaction and diffusion equations with nonlinear boundary conditions and discontinuous real valued data for which a mild solution is only known to exist is studied from the point of view of the Hölder continuity of the solutions. This regularity of the solutions furnishes the stability of attractors in an adequate notion via Arzela-Ascoli's Theorem uniformly on the domain in arbitrary space dimensions. The limit in question is of large diffusivity. In this case, the dynamics of the limit process are governed by a nonlinear coupled system of ordinary differential equations. This allows to assert that in chemical engineering or biochemical reactions models, the long time dynamics of systems of reaction and diffusion equations yielding the formation of spatially heterogeneous stable states of patterns of concentrations by certain chemical substances can only occur in the case of relatively small diffusions.

Contents

1. Introduction

Let $\Omega \subset I\!R^N$ be an open bounded regular domain with boundary $\partial\Omega = \Gamma$, $\varepsilon > 0$ be a given parameter and $\lambda > 0$ be a real number. In this paper, we consider

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the following weakly coupled system of reaction and diffusion equations

$$
\begin{array}{rcll}
u_t - \mathrm{Div}(d^\varepsilon(x)\nabla u) + V^\varepsilon(x)u + \lambda u &=& f(u) & \text{in } \Omega \\
d^\varepsilon(x)\frac{\partial u}{\partial \vec{n}} + b^\varepsilon(x)u &=& g(u) & \text{on } \Gamma, \quad t \in (0,T) \\
u(0) &=& u_0^\varepsilon & \text{in } \Omega,
\end{array} \tag{1}
$$

with nonlinear boundary conditions, where $u = (u_1, \ldots, u_m)^\top$ is a vector function, $\vec{n} = (n_1, \ldots, n_N)^\top$ denotes the unit external vector at the boundary Γ,

$$
\begin{array}{rcl}
d^\varepsilon &=& \mathrm{diag}[d_i^\varepsilon] \in L^\infty(\Omega; I\!M_{m\times m}), \\
V^\varepsilon &=& \mathrm{diag}[V_i^\varepsilon] \in L^{p_0}(\Omega; I\!M_{m\times m}), \quad \text{and} \\
b^\varepsilon &=& \mathrm{diag}[b_i^\varepsilon] \in L^{q_0}(\Omega; I\!M_{m\times m}) \quad \text{for } i = 1, \ldots, m
\end{array} \tag{2}
$$

are diagonal matrices with the first being positive definite and

$$
p_0 \left\{ \begin{array}{lll} \geq & 1 & \text{if } N = 1 \\ > & 1 & \text{if } N = 2 \\ \geq & \frac{N}{2} & \text{if } N \geq 3, \end{array} \right. \quad \text{and} \quad q_0 \left\{ \begin{array}{lll} \geq & 1 & \text{if } N = 1 \\ > & 1 & \text{if } N = 2 \\ \geq & N-1 & \text{if } N \geq 3 \end{array} \right.
$$

respectively. We assume in the last two matrices of (2) that the functions have at most their integrated values converging as $\varepsilon \to 0$. Also we suppose that $u_0^\varepsilon \in H^1(\Omega)$ and the nonlinearities of the system $N = f$, or $g : I\!R^m \to I\!R^m$ are such that

$$
\begin{array}{l}
|N(u) - N(v)| \leq \mu(u,v)|u - v| \quad \text{componentwise, where} \\
\qquad \mu(u,v) = C(|u|^{\rho-1} + |v|^{\rho-1} + 1) \qquad \text{and} \\
\rho_f \leq 1 + \dfrac{4}{N-2}, \quad \rho_g \leq 1 + \dfrac{2}{N-2} \text{ if } N \geq 3. \\
\text{If } N = 2 \text{ we assume } \forall \eta > 0, \exists C_\eta \geq 0 \text{ such that} \\
\qquad \mu(u,v) = C_\eta(e^{\eta|u|^2} + e^{\eta|v|^2} + 1). \\
\text{If } N = 1 \text{ no growth conditions are imposed.}
\end{array} \tag{3}
$$

Our interest in the above semilinear initial and boundary value parabolic problem, is to study the asymptotic global in time limit dynamics of the the system of equations in the presence of large diffusivity. This theme has been studied by many authors [**7, 8, 13, 16, 20, 22, 33, 36**]. However, to the best of our knowledge, none of these references has proved the stability of attractors in the $C(\Omega)-$ topology at infinite diffusivity in the system of equations when the domain of the equation is an open regular bounded subset in arbitrary space dimensions. Some possible reason for this is either a study of the problem via Sobolev space embedding theorems [**13, 20, 22**] or via elliptic regularity results [**33**]. These methods naturally yield restrictions on the space dimension of the domain in the equations. To bypass this type of rectrictions, a lot more work need be done. Our approach is a direct method based on Moser-Nash-De Giorgi type of iterations [**2, 31, 25, 26**]. The success of this technique has insofar been demonstrated mostly for quasilinear parabolic equations. We reobtain its non trivial applicability in our present situation of equations (1). The novel aspects of this application are exact general iterations schemes for both global boundedness in time uniform in the domain and Hölder continuity of solutions on the cylinder $\Omega \times (0,T)$.

The main hypothesis yielding large diffusivity in the system of equations (1) is the following

$$
\sigma_1(\varepsilon) \stackrel{def}{=} \inf_{\Omega}\{d_i^\varepsilon(x) \in d^\varepsilon\} \to \infty \quad \text{as } \varepsilon \to 0. \tag{4}
$$

Asserting precisely the sense in Banach spaces of the limit process of reaction and diffusion equations under assumption (4) is not always a trivial task, even in the case of a simple non homogeneous linear parabolic initial and boundary value problem. It is much more difficult when nonlinear boundary conditions are taken into account in the problems. For more details on these facts see for example in [**8, 13, 33, 36**].

The interest in the topic on asymptotic global dynamics in time of semilinear reaction and diffusion equations with large diffusivity, comes from the need to understand situations in pattern formation. These normally arise in models either of chemical engineering or biochemical processes, see [**13, 20, 40**]. A more recent study on pattern formation has been provided in [**24**]. It yields together with analytical results, a detailed numerical simulation of a 2D simple chemical system of reaction and diffusion equations with a general order of autocatalysis and decay. Another study has been furnished by [**42**], without numerical simulations. This reference analyses a situation that incorporates both mechanisms of small and large diffusivity in a one dimensional system of Gierer-Meinhardt equations. It also provides a large bibliography on the topic. There has been important results [**9, 21, 35**] on large diffusivity effects in semilinear wave equations.

In the sequel, we assume that the oscillation of the diffusion coefficients is bounded, that is

$$\operatorname{osc}_{\Omega}\{d_i^{\varepsilon}(x) \in d^{\varepsilon}\} < \infty, \quad \forall \varepsilon > 0. \tag{5}$$

This last assumption is not immediately necessary. However it is important when proving the Hölder regularity of solutions to equations (1).

The paper is organized as follows. In section 2 we give some preliminaries. These will review properties in standard Sobolev spaces of Hilbert type of the given elliptic system spatial differential operator of second order in (1). A study of the behaviour of the eigenvalue problem in large diffusivity is also given. The system elliptic differential operator, therefore turns out to be an infinitesimal generator of a linear semigroup in scales of Hilbert spaces and we can solve the homogeneous problem associated to our system (1). This semigroup admits a spectral representation in the mentioned scales of spaces. Consequently, the asymptotic behaviour of the principal eigenvalues and their corresponding eigenfunctions in large diffusivity furnishes the existence of a limit semigroup and the convergence is obtained in operator norm. In Section 3, we study the well posedness of the complete problem (1).We prove the existence of at most one mild solution in the space $C((0,T);H^1(\Omega))$. The remaining part of the section, studies the Hölder regularity of solutions. A necessary condition for the cited regularity of solutions is to prove uniform boundedness on $\overline{\Omega} \times [0,\infty)$ of the nonlinear semigroup generated by the system of equations. Consequently, in Section 4 we are able to prove the existence of a global compact attractor for the nonlinear evolution problem (1). This attractor is obtained in the space $H^1(\Omega) \cap C(\overline{\Omega})$, without imposing dissipative conditions on the nonlinearities of the problem. Since this attractor depends on the diffusion of the system of equations, we prove in the hypothesis (4) the stability in the $C(\overline{\Omega})$-topology of attractors. A review of the definition of an attractor and criteria for determining its existence for nonlinear semigroups defined by evolution problems, is provided at the beginning of the cited section. Concluding remarks, confirm that the dynamical properties of the system of equations (1) are completely determined by those of its natural limit processes in large diffusivity. This includes upto the case of finite escape times of the nonlinear semigroups.

2. Preliminaries.

Throughout this paper, we assume that the reader is familiar with standard notations of normed function spaces. For simplicity we do not make distinctions in these function space notations when vector or matrix functions are involved. Thus we use the same function spaces notation for single real valued functions defined on the domain. The inner product of the space $L^2(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$ and generic constants will be denoted as $C \geq 0$. Let X be a Banach space, then the norm notation $\|\cdot\|_X$ will be used.

2.1. The elliptic system problem. In this subsection, shall prove results on the behaviour of the principal elements of the associated eigenvalue problem of the elliptic system differential equations in (1). Thanks to the assumption that the system of equations (1) is weakly coupled. This implies that the coupling in the equations only occurs in the nonlinear terms of the system. Thus considerations for the elliptic part of the system of equations can be obtained by analyzing only the scalar $m = 1$ case, then adequate generalization concludes results for the system of elliptic second order differential operators.

To begin, we note that system of elliptic second order differential operators of the system of reaction and diffusion equations (1) can be defined (taking into account the regularity of the data) in a distributional sense as the operator $(A^\varepsilon, D(A^\varepsilon))$ where

$$D(A^\varepsilon) = \left\{ \begin{array}{lcl} u \in H^1(\Omega, I\!R^m) & : & -\mathrm{Div}(d^\varepsilon \nabla u) + (V^\varepsilon + \lambda)u \in L^2(\Omega) \\ & & d^\varepsilon \frac{\partial u}{\partial \vec{n}} + b^\varepsilon u = 0 \quad \text{on } \Gamma. \end{array} \right\} \tag{6}$$

Moreover, this operator $A^\varepsilon : D(A^\varepsilon) \subset L^2(\Omega) \to L^2(\Omega)$ is maximal monotone and self adjoint. Actually it is the restriction to the space $L^2(\Omega)$ of the isomorphism $L^\varepsilon(u) \in H^{-1}(\Omega)$ defined for every $u \in H^1(\Omega)$ by

$$\langle L^\varepsilon(u), \varphi\rangle = \int_\Omega d^\varepsilon \nabla u \nabla \varphi + \int_\Omega (V^\varepsilon + \lambda) u\varphi + \int_\Gamma b^\varepsilon u \varphi, \quad \forall \varphi \in H^1(\Omega). \tag{7}$$

Note that (7) is a bilinear form and by hypotheses Lax-Migram's Theorem [**6, 29**] is satisfied. This implies in particular we can solve for at most one solution $u^\varepsilon \in H^1(\Omega)$ the corresponding elliptic linear non homogeneous problem

$$\begin{array}{rcll} -\mathrm{Div}(d^\varepsilon \nabla u) + V^\varepsilon u + \lambda u & = & f^\varepsilon & \text{in } \Omega \\ d^\varepsilon \frac{\partial u}{\partial \vec{n}} + b^\varepsilon u & = & g^\varepsilon & \text{on } \Gamma, \end{array} \tag{8}$$

where in particular, if the Sobolev imbeddings

$$H^1(\Omega) \hookrightarrow L^{\Theta'}(\Omega) \quad \text{are satisfied} \tag{9}$$

we may consider

$$\begin{array}{l} f^\varepsilon = (f_1^\varepsilon, \ldots, f_m^\varepsilon)^\top \in L^p(\Omega) \quad \text{with } \frac{1}{p} + \frac{1}{\Theta'} = 1 \text{ and} \\ g^\varepsilon = (g_1^\varepsilon, \ldots, g_m^\varepsilon)^\top \in H^{-\frac{1}{2}}(\Gamma). \end{array} \tag{10}$$

Also, as consequence of Lax-Milgram's Theorem we have that the associated eigenvalue problem

$$\begin{array}{rcll} -\mathrm{Div}(d^\varepsilon \nabla u) + V^\varepsilon u + \lambda u & = & \mu u & \text{in } \Omega \\ d^\varepsilon \frac{\partial u}{\partial \vec{n}} + b^\varepsilon u & = & 0 & \text{on } \Gamma, \end{array} \tag{11}$$

has solutions. More precisely, there exists a sequence of eigenvalues

$$0<\mu_1^\varepsilon \le \dots \le \mu_n^\varepsilon \le \mu_{n+1}^\varepsilon \le \dots \nearrow \infty \quad \text{as } n\to\infty \tag{12}$$

with corresponding orthonormal in $L^2(\Omega)$ eigenfunctions $\{\varphi_n^\varepsilon : n\in I\!N^*\}\subset H^1(\Omega)$ that form a Hilbert basis of the space $L^2(\Omega)$. Thus from [**23, 30**] the power spaces $X_\varepsilon^\alpha \cong D((A^\varepsilon)^\alpha)$, $0\le\alpha<\infty$ defined by the operator (6) are well defined and endowed with the graph norm

$$\|u\|^2_{\alpha,\varepsilon}=\|(A^\varepsilon)^\alpha u\|^2_{L^2(\Omega)}=\sum_{n=1}^{\infty}|\mu_n^\varepsilon|^{2\alpha}|\langle u,\varphi_n^\varepsilon\rangle|^2 \quad \text{if } u\in X_\varepsilon^\alpha.$$

As usual the dual spaces are denoted by $X_\varepsilon^{-\alpha}=[X_\varepsilon^\alpha]^*$.

We now say a few words on the limit process (4) of large diffusivity in the system of equations (8). Following, from our results in [**36**] if we consider the elliptic system problem (8) and we assume given in (10) is that the integrated values of the functions converge as $\varepsilon\to 0$. Then the sequence of solutions $\{u^\varepsilon:\varepsilon>0\}\subset H^1(\Omega)$ associated with this problem (8) has a strong limit as $\varepsilon\to 0$ in the space $H^1(\Omega)$ which we denote as $u_\Omega=(u_\Omega^1,\dots u_\Omega^m)^\top\in I\!R^m$ and satisfy that

$$u_\Omega \;=\; L_0^{-1}h_\Omega \quad \text{where} \quad L_0=\text{diag}[\mu_i] \quad \text{for } i=1,\dots,m \text{ and} \tag{13}$$

$$\mu_i=\fint_\Omega V_i+\frac{|\Gamma|}{|\Omega|}\fint_\Gamma b_i+\lambda\neq 0, \tag{14}$$

$$h_\Omega=\left(\fint_\Omega f_1+\frac{|\Gamma|}{|\Omega|}\fint_\Gamma g_1,\dots,\fint_\Omega f_m+\frac{|\Gamma|}{|\Omega|}\fint_\Gamma g_m\right)^\top.$$

$$\text{Furthermore, we have} \quad \lim_{\varepsilon\to 0}\int_\Omega d^\varepsilon|\nabla u^\varepsilon|^2=0. \tag{15}$$

Note that the integral type define spatial average functions and also that in the subspace $W=\left\{\varphi\in H^1(\Omega):\int_\Omega\varphi=0\right\}$ of $H^1(\Omega)$ the limit function $u_\Omega=0$. If the spatial average limits of the reaction terms given in (10) are not bounded in $\varepsilon>0$ then $u_\Omega=\infty$.

In what follows we study the behaviour of the fundamental elements of the eigenvalue problem (11) in limit of large diffusivity in the equations.

2.2. Aysmptotic behaviour of eigenvalues. In subsection, we prove the asymptotic behaviour of the principal elements of the eigenvalue problem (11) in limit (4) of large diffusivity. First we recall the min-max characterization of the eigenvalues in (12) and for this we define

$$J_\varepsilon(w)=\frac{\langle L^\varepsilon(w),w\rangle}{\int_\Omega|w|^2}.$$

This functional is known as the Rayleigh quotient and following [**14**] we have for $n\ge 2$ that

$$\begin{aligned}\mu_n^\varepsilon &= \inf_{\substack{w\neq 0\\ w\perp span[\varphi_1^\varepsilon,\dots,\varphi_{n-1}^\varepsilon]}} J_\varepsilon(w)\\ &= \sup_{\substack{w\neq 0\\ w\in span[\varphi_1^\varepsilon,\dots,\varphi_n^\varepsilon]}} J_\varepsilon(w)\\ &= \inf_{\substack{W_\varepsilon\subset H^1(\Omega)\\ dimW_\varepsilon=n}}\ \sup_{w\in W_\varepsilon} J_\varepsilon(w),\end{aligned} \tag{16}$$

and the first eigenvalue is given by

$$\mu_1^\varepsilon \stackrel{def}{=} \inf_{w\in H^1(\Omega)\setminus\{0\}} J_\varepsilon(w) = J_\varepsilon(\varphi_1^\varepsilon). \tag{17}$$

It is easy see to that the eigenvalue (17) is simple. In fact, if $\varphi, \psi \in H^1(\Omega)\setminus\{0\}$ are any two eigenfuctions associated with it, then

$$0 = J_\varepsilon(\varphi) - J_\varepsilon(\psi) \geq \beta \int_\Omega (\varphi-\psi)^2 \Rightarrow \int_\Omega \varphi\psi = \frac{1}{2}\left(\int_\Omega \varphi^2 + \int_\Omega \psi^2\right) > 0$$

for some $\beta > 0$. Thus the eigenfunctions can not be orthogonal in $L^2(\Omega)$ and the simplicity of the first eigenvalue (17) is proved.

Concerning the behaviour of eigenvalues of the problem (11) when large diffusivity (4) is assumed, we have the following theorem.

Theorem 2.1. *Consider the eigenvalue problem (11) and assume (4) is satisfied. Then the first m-eigenvalues and associated eigenfunctions verify*

$$(\mu_n^\varepsilon, \varphi_n^\varepsilon) \to (\mu_n, |\Omega|^{-\frac{1}{2}}\vec{e}_n) \quad \textit{and} \quad \lim_{\varepsilon\to 0}\int_\Omega d_n^\varepsilon |\nabla\varphi_n^\varepsilon|^2 = 0 \tag{18}$$

for $n = 1,\ldots,m$ as $\varepsilon \to 0$ where the convergence of eigenfunctions is strong in $H^1(\Omega)$, $\{\vec{e}_n : 1 \leq n \leq m\}$ denotes the canonic basis of $I\!R^m$ and $\mu_n > 0$ is as in (14). Also there holds

$$\liminf_{\varepsilon\to 0} \frac{\mu_{m+1}^\varepsilon}{\sigma_1(\varepsilon)} \geq \lambda_2^N, \tag{19}$$

where $\lambda_2^N \neq 0$ is the first non zero eigenvalue of $-\Delta$ subject to Neumann zero boundary conditions.

Proof. We prove the results of the theorem from the case $m = 1$ of (11). First let us observe that from (16) we get that

$$\mu_2^\varepsilon = \inf_{\substack{w\in H^1(\Omega)\\ w\perp\varphi_1^\varepsilon, \int_\Omega w=0}} J_\varepsilon(w) \geq \beta_1^\varepsilon \inf_{\substack{w\in H^1(\Omega)\\ w\perp\varphi_1^\varepsilon, \int_\Omega w=0}} \left\{\frac{\int_\Omega |\nabla w|^2}{\int_\Omega |w|^2}\right\} + \beta_0 \to \infty \tag{20}$$

where $\beta_0 > 0$ and $\beta_1^\varepsilon \to \infty$ as $\varepsilon \to 0$. Thus the sequence $\{\varphi_1^\varepsilon : \varepsilon > 0\} \subset H^1(\Omega)$ of first eigenfuctions is uniformly bounded with respect to $\varepsilon > 0$. By compactness there exists a weak limit function $\varphi \in H^1(\Omega)$ and

$$\int_\Omega |\nabla\varphi|^2 \leq \liminf_{\varepsilon\to 0}\int_\Omega |\nabla\varphi_1^\varepsilon|^2 = 0. \tag{21}$$

This implies that $\varphi \notin W = \{w \in H^1(\Omega) : \int_\Omega w = 0\}$. On the other hand, the Sobolev embeddings $H^1(\Omega) \subset L^2(\Omega)$ are compact. Thus the convergence of the first eigenfuctions is strongly in $L^2(\Omega)$ and their normalization in this space yields that $\varphi = |\Omega|^{-\frac{1}{2}}$. Now in view of (21) we get that the convergence is stongly in $H^1(\Omega)$ as $\varepsilon \to 0$.

The second assertion in (18) is obvious. Otherwise we obtain a contradiction with the boundedness of (17). On the other hand, the continuity of the bilinear form in the left hand side of (7) yields passing to the limit as $\varepsilon \to 0$ the convergence of the first eigenvalue of (11) to the value $\mu_1 = (14) > 0$.

Assume that a similar estimate (20) also holds for the second eigenvalue using the third eigenvalue. However this is impossible as we find a contradiction with the ortogonality of the associated second eigenfunction in limit (4). Thus in (20) if we set

$$\lambda_{2,\varepsilon}^N \stackrel{def}{=} \inf_{\substack{w\in H^1(\Omega)\\ w\perp\varphi_1^\varepsilon,\int_\Omega w=0}} \left\{ \frac{\int_\Omega |\nabla w|^2}{\int_\Omega |w|^2} \right\}$$

then, since $w \perp \mathrm{span}[v^\varepsilon, |\Omega|^{-\frac{1}{2}}]$ with $v^\varepsilon \in W$, we get $\lambda_{2,\varepsilon}^N \geq \lambda_2^N$. Hence dividing throughout by $\sigma_1(\varepsilon)$ and noting that $\lim_{\varepsilon\to 0} \frac{\beta_1^\varepsilon}{\sigma(\varepsilon)} = 1$, leads to

$$\liminf_{\varepsilon\to 0} \frac{\mu_2^\varepsilon}{\sigma_1(\varepsilon)} \geq \lambda_2^N. \tag{22}$$

Consequently, generalizing this proof to the non coupled system eigenvalue problem (11) we obtain the conclusion of the theorem. □

2.3. The linear semigroup. We now conclude these preliminaries, by reviewing the well posedness of the homogeneous problem associated with the system of equations (1). We also will prove the asymptotic behaviour in large diffusivity (4) of the linear semigroup generated by the homogeneous equations of the system (1).

We note the homogeneous equations of (1) can written using the isomorphism operator in (7) as

$$u_t + L^\varepsilon(u) = 0, \quad u_0^\varepsilon \in X_\varepsilon^\alpha, \alpha \in I\!R \tag{23}$$

in $H^{-1}(\Omega)$ for almost $t \in (0,T)$. Relating to the well posedness of the system evolution equations (23) we have from [**23, 30**] the following theorem.

THEOREM 2.2. *The operator* $(L^\varepsilon, X_\varepsilon^{\alpha+\frac{1}{2}})$ *is an infinitesimal generator of an analytic semigroup*

$$\{S^\varepsilon(t); t\geq 0\} = \{exp(-L^\varepsilon t), t \geq 0\} \subset X_\varepsilon^\alpha, \alpha \in I\!R \tag{24}$$

such that the mapping

$$t \longmapsto u^\varepsilon(t) = exp(-L^\varepsilon t)u_0^\varepsilon \in X_\varepsilon^\alpha, \alpha \in I\!R$$

and the problem (23) has at most one solution of class in $\alpha = 0$,

$$u^\varepsilon \in C([0,\infty), X_\varepsilon^{-\frac{1}{2}}))) \cap C^\omega((0,\infty), X_\varepsilon^\alpha), \quad \forall \alpha \in I\!R.$$

In addition, for any $\alpha_0, \alpha_1 \in I\!R$ *the mapping* $S_\varepsilon(t) : X_\varepsilon^{\alpha_0} \to X_\varepsilon^{\alpha_1}$ *is bounded and there exists* $M > 0$ *such that for any* $0 < \omega \leq \inf\{\mu : \mu \in \sigma(L^\varepsilon)\}$ *we have*

$$\|S^\varepsilon(t)\|_{\alpha_0,\alpha_1} \leq \frac{M}{t^{\alpha_1-\alpha_0}} e^{-\omega t}, \quad \textit{for all } t > 0. \tag{25}$$

In other words, the linear semigroup has a smoothening effect and is aymptotically stable.

Observe that in view of the fact that the operator (6) is maximal monotone and self adjoint, we the semigroup $\{S^\varepsilon(t) : t \geq 0\}$ compact. This conclusion we obtain from [**27**] Proposition 5.2. Also the given semigroup does admit a spectral representation in the scales of Hilbert spaces X_ε^α,$\alpha \in I\!R$. This enables us to write

$$S^\varepsilon(t)w = \sum_{n=1}^\infty e^{-\mu_n^\varepsilon t} |\mu_n^\varepsilon|^\alpha w_n^\varepsilon \varphi_n^\varepsilon \in X_\varepsilon^\alpha, \quad t \geq 0 \tag{26}$$

for any $w \in X_\varepsilon^\alpha$, where the pair $(\mu_n^\varepsilon, \varphi_n^\varepsilon)$,$n \geq 1$ solves the system eigenvalues problem (11) and $w_n^\varepsilon = \langle w, \varphi_n^\varepsilon \rangle$. Concerning the asymptotic behaviour of this semigroup in view of hypothesis (4) we have the following lemma.

LEMMA 2.3. *Consider the semigroup* $T(t) = e^{-L_0 t}; t > 0$ *where* $L_0 = diag[\mu_n]$ *for* $n = 1, \ldots, _m]$ *is definite positive and is defined as in (13). Then*

$$\lim_{\varepsilon \to 0} \sup_{t>0} e^{\delta t} \| S^\varepsilon(t) - T(t) \|_{\alpha_1, \alpha_0} = 0, \quad \forall \alpha_0, \alpha_1 \in I\!R \tag{27}$$

where $S^\varepsilon(t)$,$t \geq 0$, *is the semigroup generated by (23) and* $\inf\{\mu_n; 1 \leq n \leq m\} > \delta$ *with* μ_n *satisfying (18).*

PROOF. Let $w \in X_\varepsilon^{\alpha_0}$,$\alpha_0 \in I\!R$ then write realization of the finite dimensional semigroup as

$$T(t)w = \sum_{n=1}^{m} e^{-\mu_n t} (\mu_n^\varepsilon)^{\alpha_1} w_n \varphi_n \in X_\varepsilon^{\alpha_1}, \quad t \geq 0 \tag{28}$$

where $\varphi_n = |\Omega|^{-1/2} \vec{e}_n$, $\{\vec{e}_n; 1 \leq n \leq m\}$ is the canonic basis of $I\!R^m$, $w_n = \langle w, \varphi_n \rangle$, $n = 1, \ldots, m$. Using (26) we find that

$$\begin{aligned} e^{2\delta t} \| (S^\varepsilon(t) - T(t))w \|_{\alpha_1, \varepsilon}^2 \leq & \sum_{n=m+1}^{\infty} e^{-2\mu_n^\varepsilon t} |\mu_n^\varepsilon|^{2(\alpha_1 - \alpha_0)} |\mu_n^\varepsilon|^{2\alpha_0} |w_n^\varepsilon|^2 \quad + \\ & + \quad \sum_{n=1}^{m} \Big(|e^{-\mu_n^\varepsilon t} - e^{-\mu_n t}|^2 |\mu_n^\varepsilon|^{2(\alpha_1 - \alpha_0} |\mu_n^\varepsilon|^{2\alpha_0} |w_n^\varepsilon|^2 \quad + \\ & \qquad + \quad 2\, e^{-2\mu_n t} |\mu_n^\varepsilon|^{2(\alpha_1 - \alpha_0)} |\mu_n^\varepsilon|^{2\alpha_0} |w_n^\varepsilon|^2 \| \varphi_n^\varepsilon - \varphi_n \|_0^2 \Big). \end{aligned}$$

This enables us to deduce the following

$$\begin{aligned} e^{2\delta t} \| (S^\varepsilon(t) - T(t))w \|_{\alpha_1, \varepsilon}^2 \quad \leq \quad & \Bigg(\sup_{n \geq m+1} \left\{ \frac{|\mu_n^\varepsilon|^{\alpha_1 - \alpha_0}}{e^{\mu_n^\varepsilon t_0}} \right\}^2 \quad + \\ & + \sup_{1 \leq n \leq m} \left\{ \sup_{t \geq t_0} |e^{-\mu_n^\varepsilon t} - e^{-\mu_n t}| |\mu_n^\varepsilon|^{\alpha_1 - \alpha_0} \right\}^2 + \\ & + 2 e^{2\mu_n t_0} \sup_{1 \leq n \leq m} \left\{ |\mu_n^\varepsilon|^{\alpha_1 - \alpha_0} \| \varphi_n^\varepsilon - \varphi_n \|_0^2 \right\}^2 \Bigg) \|w\|_{\alpha_0}^2, \end{aligned}$$

for any $0 < t_0 \leq t$. Thus implying the result (27) on taking into account (18)-(19) since $\sigma_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Our proof of the lemma is complete. □

3. Regularity of solutions.

In this section, we prove the well posedness of the system of equations (1) for at most one mild solution in the space $C((0,T), H^1(\Omega))$. The remaining part of the section, proves under additional hypotheses on the initial data, mild solution and nonlinearities of the system of equations, well posedness in the sense of Hölder continuity of the solutions. This regularity of the solutions to the system of equations (1) has a necessary condition the proof for uniform boundedness of solutions on $\overline{\Omega} \times (0, \infty)$.

Our first result on the regularity of solutions of the semilinear system of reaction and diffusion equations (1) states the following.

THEOREM 3.1. *Consider the semilinear initial and boundary value problem defined by the system of reaction and diffusion equations (1) in given hypotheses upto to conditions (3). Then, there exists at most one mild solution of class* $u^\varepsilon \in C([0,T), H^1(\Omega))$. *Furthermore,* $u^\varepsilon_t \in L^2(0,T,L^2(\Omega))$ *and for almost all* $t \in (0,T)$ *we have that*

$$\int_\Omega u^\varepsilon_t \varphi + \langle L^\varepsilon(u), \varphi\rangle_{H^{-1}(\Omega),H^1(\Omega)} = \int_\Omega f(u^\varepsilon)\varphi + \int_\Gamma g(u^\varepsilon)\varphi \tag{29}$$

for any $\varphi \in H^1(\Omega)$.

Moreover, this solution is given by the variation of constants formula

$$u^\varepsilon(t) = e^{-L^\varepsilon t} u^\varepsilon_0 + \int_0^t e^{-L^\varepsilon(t-s)} h(u^\varepsilon(s)) ds \tag{30}$$

where the generator of the linear semigroup is the system isomorphic differential operator defined in (7) and $h(u) \stackrel{def}{=} f_\Omega(u) \oplus g_\Gamma(u) \in H^{-1}(\Omega)$ *is a nonlinear form.*

PROOF. First we note that the system of equations (1) can be read using the isomorphic operator defined in (7) as

$$\begin{cases} u_t + L^\varepsilon u &= h(u) \\ u(t_0) &= u^\varepsilon_0 \in H^1(\Omega), \end{cases} \tag{31}$$

for almost all $t \in (0,T)$, where the nonlinearity $h(u) \in H^{-1}(\Omega)$ is defined by

$$\langle h(u), \varphi\rangle_{-1,1} \stackrel{def}{=} \int_\Omega f(u)\varphi + \int_\Gamma g(u)\varphi, \quad \forall \varphi \in H^1(\Omega).$$

In abbreviation, we write this as

$$h(u) \stackrel{def}{=} f_\Omega(u) + g_\Gamma(u) \in L^{\sigma_1}(\Omega) \oplus X^\beta_\varepsilon \subset H^{-1}(\Omega)$$

where $\frac{1}{\sigma_1} + \frac{1}{\rho} = 1$ and $-1/4 \geq \beta \geq -1/2$. In this case, the Nemytskii operators

$$f_\Omega : H^1(\Omega) \longmapsto L^{\sigma_1}(\Omega) \quad \text{and} \quad g_\Gamma : H^1(\Omega) \longmapsto X^\beta_\varepsilon \tag{32}$$

are well defined and the Sobolev embeddings of the space $H^1(\Omega) \subset L^\rho$ are continuous and compact. Technically, we seek solutions to the evolution problem (31) that are continuous functions on $(0,T)$ with values $u^\varepsilon(t) \in H^1(\Omega)$ and are fixed points of the nonlinear mapping $\mathcal{F}(u)(t)$ defined by the right-hand side of(30) on some subspace $V \subset C([t_0, t_0+h], H^1(\Omega))$ such that $\|u(t) - u_0\| \leq \delta$ with $h, \delta > 0$ sufficiently small. Then use the extension theorems to get a solution on the maximum interval of existence $(0,T)$. This yields existence and uniqueness results in [**23, 30**]. Thus it is sufficient to prove local Lipschitz continuity of the nonlinear terms in (1) to obtain our theorem.

Consider the nonlinear operators (32), denote by X either Ω or Γ and without loss of generality assume $|X| \geq 1$. Also recall the following inequality

$$(a+b)^p \leq 2^p(a^p + b^p), \quad a, b \geq 0, p > 0. \tag{33}$$

Now from (3) a repeated application of (33) and Hölder's inequality yields

$$\begin{aligned}\int_X |N(u)-N(v)|^{\sigma_i} &\le 2^{2\sigma_i} L_j^{\sigma_i}|X|\left(\int_X |u|^{(\rho-1)\sigma_i}|u-v|^{\sigma_i} \quad + \right.\\ &\qquad \left. + \quad \int_X |v|^{(\rho-1)\sigma_i}|u-v|^{\sigma_i} + \int_X |u-v|^{\sigma_i}\right)\\ &\le 2^{2\sigma_i} L_j^{\sigma_i}(|X|^{\frac{1}{\sigma_i}})^{\sigma_i}\left[\left(\int_X |u|^{\rho+1}\right)^{\frac{\rho-1}{\rho+1}\sigma_i} + \left(\int_X |v|^{\rho+1}\right)^{\frac{\rho-1}{\rho+1}\sigma_i} + 1\right]\times\\ &\qquad \times\left(\int_X |u-v|^{\rho+1}\right)^{\frac{\sigma_i}{\rho+1}},\end{aligned}$$

where $\frac{1}{\sigma_i}+\frac{1}{\rho}=1$ for $i=1,2$ with $\rho>1$ as in (3). Therefore

$$\begin{aligned}\|N(u)-N(v)\|_{L^{\sigma_i}(X)} &\le (2^{2\sigma_i+2})^{\frac{1}{\sigma_i}} L_j|X|^{\sigma_i}\times\\ &\times \left(\|u\|^{\rho-1}_{L^{\rho+1}(X)} + \|v\|^{\rho-1}_{L^{\rho+1}(X)}+1\right)\|u-v\|_{L^{\rho+1}(X)},\end{aligned}$$

which is also valid in the case of $|X|<1$. Moreover, the embeddings of the space $H^1(\Omega)\subset L^{\rho+1}$ are continuous and compact (see [**23, 30**]). Thus the Nemytskii operators in (32) are locally Lipschitz continuous. Now results in [**23, 29**], more precisely Proposition 7.4 and Theorem 7.8 of [**29**], give the conclusion of the theorem for case $N\ge 3$.

The same is true for the case $N=2$. In fact, if we set $\varphi=\frac{|u|}{\|u\|_{H^1(\Omega)}}$ for any $u\in H^1(\Omega)$ then $\|\varphi\|_{H^1(\Omega)}=1$ and there exists η_0 such that

$$\int_\Omega e^{(\eta_0\varphi)^2}\le M \tag{34}$$

where the constant $M<\infty$. Indeed suppose not, that is for all $\eta_0>0$ there is a sequence $\{\varphi_n, n\ge 1\}\subset H^1(\Omega)$ verifying $\|\varphi_n\|_{H^1(\Omega)}=1$ and $D>0$ such that

$$\int_\Omega e^{(\eta_0\varphi_n)^2}dx > D, \quad \forall n. \tag{35}$$

Since the sequence $\{\varphi_n, n\ge 1\}$ is bounded in $H^1(\Omega)$, a passage to subsequences if necessary implies that $\varphi_{n_k}\to\varphi$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ as $k\to\infty$. On the other hand, if we let $\tau(\varphi_{n_k})=\frac{1}{2\eta_0^2\varphi_{n_k}\varphi'_{n_k}}e^{(\eta_0\varphi_{n_k})^2}$ then $\langle\tau(\varphi_{n_k}),\psi\rangle\to\langle\tau(\varphi),\psi\rangle$, for any $\psi\in L^2(\Omega)$ as $k\to\infty$. In particular, we find that $\tau(\varphi_{n_k})$ is bounded in $L^2(\Omega)$ norm, which contradicts (35).

Next we notice that

$$L(\varphi\|u\|_{H^1(\Omega)})^\rho \le \frac{\rho}{2}|\varphi|^2 + \frac{(2-\rho)L^{\frac{2\rho}{2-\rho}}}{\rho}\|u\|^{\frac{2\rho}{2-\rho}}_{H^1(\Omega)}$$

where $L>0$ and we have used the standard Young's inequality with $s=\frac{2}{\rho}$. Therefore, if we let $\eta_0=\rho/2$ and $C=(2-\rho)L^{\frac{2\rho}{2-\rho}}/\rho$ then from (34) we get that

$$\int_\Omega e^{L|u|^\rho}\le M|\Omega|e^{C\|u\|^{\frac{2\rho}{2-\rho}}_{H^1(\Omega)}}. \tag{36}$$

Consequently, upon taking $C=2\eta$ we have $H^1(\Omega)\ni u\longmapsto f(u)\in L^\rho(\Omega)$ is well defined for any $\rho\ge 1$. Finally, if $\rho=2$ then the conditions (3), Schwartz's inequality and repeated use of (33) conclude

$$\|f(u)-f(v)\|^2_{L^2(\Omega)} \leq 16|\Omega|^{1/2}C^2_\eta\|u-v\|^2_{L^4(\Omega)}\left(\int_\Omega e^{4\eta|u|^\rho}+\int_\Omega e^{4\eta|v|^\rho}+1\right)^{1/2},$$

again provided $|\Omega|\geq 1$ and with obvious changes in the otherwise case. This yields the desired results including the case of $\rho<1$, since the inclusions $H^1(\Omega)\subset L^4(\Omega)$ are satisfied.

Now we prove that $g: H^1(\Omega)\longmapsto X^\beta_\varepsilon$ for $-1/4\geq\beta\geq -1/2$ is locally Lipschitz. But this is even much simpler, since the trace space $H^{\frac{1}{2}}(\Gamma)\subset L^\infty(\Gamma)$ continuously. Thus the conditions (3) implies

$$\|g(u)-g(v)\|_{\beta,\varepsilon}\leq(\mu^\varepsilon_1)^{\beta-1/4}C_\eta(2M|\Gamma|+1)\|u-v\|_{H^1(\Omega)}.$$

The case $N=1$ is trivial, thanks to the Sobolev embeddings theorem $H^1(\Omega)\subset C^\theta(\overline{\Omega})$, $0<\theta<1$ continuously. It now follows from results in [**29**] pp.50 and pp. 52 that the theorem is proved. □

The delicate regularity result of this paper is related to the Hölder continuity of the mild solution in Theorem 3.1. This is stated in the following main theorem of the section.

THEOREM 3.2. *Consider the system of equations (1) and suppose that $u^\varepsilon_0\in L^\infty(\Omega)$ for all $\varepsilon>0$. Let $r>1$ be such that conditions (3) for the nonlinearities are verified with*

$$\rho_f<1+\frac{2r}{N},\quad \rho_g<1+\frac{r}{N-1} \tag{37}$$

respectively. Assume that an a priori bound of the mild solution $u^\varepsilon\in L^r(\overline{\Omega})$ for all $t\in(0,T)$ and for all $\varepsilon>0$ is given. Suppose the condition (5) on the oscillation of the diffusion coefficients of the system of equations is satisfied. Then, the mild solution of the system of equations (1) is Hölder continuous and bounded for any $\varepsilon>0$. In other words,

$$u^\varepsilon\in C^{\alpha,\frac{\alpha}{2}}(\Omega\times(0,T)), 0<\alpha<1$$

for any $\varepsilon>0$.

To render a proof of this Theorem 3.2 we need first prove the following theorem on uniform boundedness of solutions to (1) on $\overline{\Omega}\times[0,\infty)$.

THEOREM 3.3. *Assume that all other hypotheses of the Theorem* 3.2 *except for the hypothesis (5) are satisfied. Then, the mild solution to the system of equations (1) is bounded uniformly on $\Omega\times[0,\infty)$. More precisely,*

$$\sup_\Omega|u^\varepsilon(t,u^\varepsilon_0)|\leq C,\quad \forall t\geq 0, \forall\varepsilon>0. \tag{38}$$

Thus the nonlinear semigroup generated by the system of evolution equations associated with the problem (1) is globally defined and bounded in $L^\infty(\Omega)$ independently of $\varepsilon>0$.

3.1. Proof of Theorem 3.3. We carry out this proof in several stages. Our first lemma is the following.

LEMMA 3.4. *Under the hypotheses of Theorem 3.3 the system of evolution equations (1) admits a differential inequality of the form*

$$\frac{d}{dt}\int_\Omega |u|^{r+1} + \beta\|(u)^{\frac{r+1}{2}}\|^2_{H^1(\Omega)} \le (rC)^{\sigma_1}\int_\Omega |u|^{r+1} + rC \tag{39}$$

where $r \ge 1$, $\beta = \beta(\eta) > 0$ and $\sigma_1 = \sigma(N, \rho_f, \rho_g, \vartheta, \theta) > 0$ for some $\vartheta, \theta > 0$.

PROOF. Consider equations (1). Then there holds

$$\frac{d}{dt}\int_\Omega |u|^{r+1} + \beta\|(u)^{\frac{r+1}{2}}\|^2_{H^1(\Omega)} \le rC\int_X (|u|^{\rho+r} + |u|^{r+1} + |u|^r) \tag{40}$$

for some $\beta > 0$, where $X = \Omega\cup\Gamma$, by taking $\varphi = |u|^{r-1}u \in H^1(\Omega)$ as a test function in (29). We estimate from above term by term the right hand side of the inequality (40). The first term yields that

$$\begin{aligned} rC\int_\Omega |u|^{\rho_f+r} &\le rC\left(\int_\Omega |u|^{\frac{N(r+1)}{N-2}}\right)^{\Theta_1}\left(\int_\Omega |u|^r\right)^{\Theta_2}\left(\int_\Omega |u|^{r+1}\right)^{\Theta_3} \\ &\le rC\left(\|(u)^{\frac{r+1}{2}}\|^2_{H^1(\Omega)}\right)^{\frac{N\Theta_1}{N-2}}\left(\int_\Omega |u|^r\right)^{\frac{2\Theta_1}{N-2}}\left(\int_\Omega |u|^{r+1}\right)^{\Theta_3} \\ &\le \eta\|(u)^{\frac{r+1}{2}}\|^2_{H^1(\Omega)} + r^{\frac{1}{\Theta_3}}C^{2(\rho-1)/2}\int_\Omega |u|^{r+1} \end{aligned} \tag{41}$$

where we have used a combination of Hölder's inequality and Sobolev embeddings [**1, 6**] in the first estimate. Also

$$\Theta_1 = \frac{(N-2)(\rho_f-1)}{N(\rho_f-1)+\vartheta}, \quad \Theta_2 = \frac{2(\rho_f-1)}{N(\rho_f-1)+\vartheta}, \quad \text{and} \quad \Theta_3 = \frac{\vartheta}{N(\rho_f-1)+\vartheta},$$

for some $\vartheta > 0$. On the other hand

$$\frac{N\Theta_1}{N-2} = \frac{\rho_f-1}{N(\rho_f-1)+\vartheta}, \qquad \frac{N\Theta_1}{N-2} + \Theta_3 = 1$$

where we have applied the following Young's inequality

$$ab \le \eta a^s + C_\eta b^{s'} \quad \text{with} \quad 0 < \eta < 1, \quad C_\eta = \eta^{-\frac{1}{s-1}} \tag{42}$$

to get the last estimate.

Similarly we estimate the same term on Γ to obtain

$$\begin{aligned} rC\int_\Gamma |u|^{\rho_g+r} &\le \eta\|(u)^{\frac{r+1}{2}}\|^2_{H^1(\Omega)} + r^{\frac{1}{\Theta_6}}C^{\frac{2(\rho_g-1)}{\theta}}\int_\Gamma |u|^{r+1}, \\ &\le 2\eta\|(u)^{\frac{r+1}{2}}\|^2_{H^1(\Omega)} + r^{\frac{2}{\Theta_6}}C^{\frac{2(\rho_g-1)}{\theta}}\int_\Omega |u|^{r+1} \end{aligned} \tag{43}$$

where $\theta > 0$, $\Theta_6 = \frac{\theta}{(N-1)(\rho_g-1)+\theta}$, and where we have used the Sobolev inequality

$$\int_\Gamma |w|^2 \le \delta\int_\Omega |\nabla w|^2 + C_\delta\int_\Omega |w|^2$$

to find the last estimate with $C_\delta = \frac{C}{\delta} > 0$ and $\delta = \left(r^{\frac{1}{\Theta_6}}C^{\frac{2(\rho_g-1)}{\theta}}\right)^{-1}\eta$.

Analogously this inequality, with $\delta = \frac{\eta}{rC}$, implies that

$$rC\int_{\Gamma}|u|^{r+1} \leq \eta\|(u)^{\frac{r+1}{2}}\|^2_{H^1(\Omega)} + rC\int_{\Omega}|u|^{r+1}. \tag{44}$$

Note that the other terms need not be estimated. Thus (41),(43) (44) and the hypotheses yield the desired estimate (39) from (40). □

Lemma 3.5. *Let* $y_i(t) = \int_\Omega |u|^{r_i+1}$ *with* $r_i = 2^i$,$i \in I\!N^*$. *Then (39) can be written in the form*

$$\frac{dy_i}{dt} + \beta y_i \leq (r_iC)^\sigma (y_{i-1}+1)^{s_i} \tag{45}$$

for some $\beta = \beta(\eta) > 0$, *where* $s_i = \frac{r_i+1}{r_{i-1}+1}$, $\sigma = \sigma_1(N+2)/2$ *and* $C \geq 1$*(without loss of generality).*

Proof. Let $r_i = 2^i$ for $i \geq 1$ and define

$$\Theta_i = \frac{2(r_i+1)}{N(r_i+1)-(N-2)(r_{i-1}+1)}, \qquad \Theta_i' = 1-\Theta_i.$$

It follows from Hölder's inequality and Sobolev's embeddings that

$$\begin{aligned}(r_iC)^{\sigma_1}\int_\Omega |u|^{r_i+1} &\leq \left(\int_\Omega |u|^{\frac{N(r_i+1)}{N-2}}\right)^{\Theta_i'}\left(\int_\Omega |u|^{r_i+1}\right)^{\Theta_i}\\ &\leq (r_iC)^{\sigma_1}\left(\|(u)^{\frac{r_i+1}{2}}\|^2_{H^1(\Omega)}\right)^{\frac{N\Theta_i'}{N-2}}\left(\int_\Omega |u|^{r_i+1}\right)^{\Theta_i}\\ &\leq \eta\|(u)^{\frac{r_i+1}{2}}\|^2_{H^1(\Omega)} + (r_iC)^\sigma\left(\int_\Omega |u|^{r_{i-1}+1}\right)^{s_i},\end{aligned}$$

where $\frac{N\Theta_i'}{N-2} < 1$ (we have used Young's inequality (42) to obtain the last estimate) and $s_i = \frac{r_i+1}{r_{i-1}+1}$ and $\sigma = \sigma_1(N+2)/2 \geq \sigma_1(Nr_{i-1}+r_i+2)/(r_i+2)$.

Consequently we obtain (45) from (39) and the proof of the lemma is complete. □

To complete our proof of the Theorem 3.3 we have the following additional lines.

Proof. Solving (45) using the above hypotheses, we find that there exists $K = K(\|u_0^\varepsilon\|_\infty) > 0$ such that

$$y_i(t) \leq C(r_iC)^\sigma\left(K^{(r_{i-1}+1)s_i} + (\sup_{t\in(0,T)} y_{i-1}(t))^{s_i}\right). \tag{46}$$

Iterating (46) for arbitrary $i=k\geq 1$ furnishes

$$\begin{aligned}
y_k(t) &\leq (2C)^{1+2s_k+2s_{k-1}s_k+\ldots+2s_2s_3\ldots s_k}\times\\
&\times (2C)^{k\sigma+(k-1)\sigma s_k+\ldots+\sigma s_2s_3\ldots s_k}K^{2s_1s_2\ldots s_k} +\\
&+ (2C)^{1+2s_k+2s_{k-1}s_k+\ldots+2s_2s_3\ldots s_k}\times\\
&\times (2C)^{k\sigma+(k-1)\sigma s_k+\ldots+\sigma s_2s_3\ldots s_k}\left(\sup_{t\in(0,T)}\int_\Omega |u|^2\right)^{s_1s_2s_3\ldots s_k}\\
&\leq (2C)^{2A_k}(2C)^{\sigma B_k}K^{2\chi_k}+(2C)^{2A_k}(2C)^{\sigma B_k}\left(\sup_{t\in(0,T)}\int_\Omega |u|^2\right)^{\chi_k}
\end{aligned}$$

where $\chi_k = s_k\ldots s_1 \leq \frac{r_k+1}{2}$ and

$$\begin{aligned}
A_k &= 1+s_k+s_ks_{k-1}+\ldots+s_ks_{k-1}\ldots s_1 \leq (r_k+1)\sum_{i=0}^{\infty}\frac{1}{r_i+1},\\
B_k &= k+(k-1)s_k+(k-2)s_ks_{k-1}+\ldots+s_ks_{k-1}\ldots s_1 \leq (r_k+1)\sum_{i=1}^{\infty}\frac{i}{r_i+1}.
\end{aligned}$$

Since $r_i=2^i$ the above series do converge. Therefore

$$\begin{aligned}
y_k(t) &\leq \left((2C)^{2\omega_1}(2C)^{\sigma\omega_2}K+(2C)^{2\omega_1}(2C)^{\sigma\omega_2}\sup_{t\in(0,T)}\left(\int_\Omega |u|^2\right)^{1/2}\right)^{r_k+1}\\
&\leq \left((2C)^{2\omega_1}(2C)^{\sigma\omega_2}K\left(\sup_{t\in(0,T)}\left(\int_\Omega |u|^2\right)^{1/2}+1\right)\right)^{r_k+1}
\end{aligned}$$

where $\omega_1=\sum_{i=1}^{\infty}\frac{1}{r_i+1}$ and $\omega_2=\sum_{i=1}^{\infty}\frac{i}{r_i+1}$.

Passing the power index r_k+1 to the left-hand side of the inequality, taking up limits as $k\to\infty$, gives (38). □

Theorem 3.3 has interesting consequences on the global dynamics generated by the nonlinear solutions semigroup of the evolution equations (1), see Section 4.

3.2. Proof of Theorem 3.2. We shall carry out this proof in several lemmas.

Proof. Let $w=\text{In}((\omega_4+R)/l(u))$ where ω_4 denote the oscillation of $|u|$ on

$$Q_{4R}=B_{4R}\times[t_0,t_0+4R^2]\quad\text{for some } 1/2\leq R\leq 1,$$

$l(u)=2(M_4-|u|)+R$ and $M_4=\sup_{Q_{4R}}|u|$. Then the following lemma holds.

Lemma 3.6. *w is a super (sub) solution to some parabolic equation.*

Proof. Let $\varphi/l(u)$ with $\varphi\in H^1(\Omega)$ non negative be a test function in (29). Since

$$\nabla w=\begin{cases}\frac{2\nabla u}{l(u)} & \text{if}\quad u\geq 0\\ -\frac{2\nabla u}{l(u)} & \text{if}\quad u<0,\end{cases}\qquad w_t=\begin{cases}\frac{2u_t}{l(u)} & \text{if}\quad u\geq 0\\ -\frac{2u_t}{l(u)} & \text{if}\quad u<0,\end{cases}\tag{47}$$

we obtain in the positive case that

$$\begin{aligned} &\frac{1}{2}\int_\Omega w_t\varphi + \frac{1}{2}\int_\Omega d^\varepsilon(x)\nabla w\nabla\varphi \quad + \\ &\quad + \quad \frac{1}{2}\left(\frac{2M_4+R}{\omega_4+R}\right)\left(\int_\Omega (V^\varepsilon+\lambda)e^w\varphi + \int_\Gamma b^\varepsilon e^w\varphi\right) \\ &\quad\quad \le \int_\Omega f(w)\varphi + \int_\Gamma g(w)\varphi + \frac{1}{2}\int_\Omega (V^\varepsilon+\lambda)\varphi + \frac{1}{2}\int_\Gamma b^\varepsilon\varphi, \end{aligned} \tag{48}$$

where the nonlinear terms are in the form

$$h(w) = \frac{h(u)e^w}{\omega_4+R}, \quad |h(w)| \le \frac{C}{R^2} \tag{49}$$

for $C>0$ independent of R and ω_4.

Thus w is a super-solution to some parabolic problem. In the case $u<0$, w satisfy an inverse differential inequality to (48) and is a sub-solution to some parabolic equation. □

In what follows, we search for a local integral inequality in view of a Moser type iteration process. Let

$$Q^+_{4R} = B^+_{4R}\times[t_0, t_0+4R^2] \quad \text{where} \quad B^+_{4R} = \{x\in B_{4R} : u \ge M_4+m_4\}$$

with $m_4 = \inf_{Q_{4R}} u$, denote a cylinder and its base respectively. Also let $x_\varepsilon = \sigma_1(\varepsilon)^{-1/2}x - x_0$ where $\sigma_1(\varepsilon)$ is defined as in (4).

Next, consider a smooth function $\varphi : Q^+_{4R} \longmapsto I\!R$ such that

$$\left\{\begin{array}{l} 0\le\varphi\le 1, \quad \forall (x_\varepsilon,t)\in Q^+_{4R}, \\ \varphi \equiv \left\{\begin{array}{ll} 0 & \text{if } t\le t_0 \text{ or } x_\varepsilon\in\Omega\setminus B^+_{4R}, \\ 1 & \text{if } t_0\ge t_0+4R^2 \text{ and } x_\varepsilon\in B^+_{2R} \end{array}\right. \\ |\nabla(\varphi)| \le (\sigma R)^{-1}, \quad |\partial_t\varphi| \le (\sigma R^2)^{-1} \quad \text{for some } 0<\sigma<1, \end{array}\right. \tag{50}$$

independent of $\varepsilon>0$.

Lemma 3.7. *There exists a differential inequality in terms of w of the form*

$$\begin{aligned} &\frac{d}{dt}\int_{B^+_{4R}} (w^{\frac{r+2}{2}}\varphi)^2 + \beta\|w^{\frac{r+2}{2}}\varphi\|^2_{H^1(B^+_{4R})} \\ &\quad\quad \le \quad C_r(\varepsilon)\left(\frac{1}{(\sigma R)^2}\int_{B^+_{4R}} w^{r+2}\right) + C_r(\varepsilon) \end{aligned} \tag{51}$$

where $\beta>0$, $C_r(\varepsilon)\ge 0$ depends on $r>0$ and is uniformly bounded in $\varepsilon>0$.

Proof. Let $|w|^r w\varphi^2 = w^{r+1}\varphi^2$ for $r>0$ be a test function in (48) to obtain

$$\begin{aligned} &\frac{1}{2(r+2)}\frac{d}{dt}\int_{B^+_{4R}} (w^{\frac{r+2}{2}}\varphi)^2 + \frac{\sigma_1(\varepsilon)}{r+2}\int_{B^+_{4R}} |\nabla(w^{\frac{r+2}{2}}\varphi)|^2 + \frac{\lambda}{2}\int_{B^+_{4R}} (w^{\frac{r+2}{2}}\varphi)^2 \\ &\quad \le \quad \int_{B^+_{4R}} f(w)w^{r+1}\varphi^2 + \int_{B^+_{4R}\cap\Gamma} g(w)w^{r+1}\varphi^2 \quad + \\ &\quad + \quad \frac{1}{2}\left(\frac{M+R}{R}\right)\left(\int_{B^+_{4R}} (V^\varepsilon_+ +\lambda)w^{r+1}\varphi^2 + \int_{B^+_{4R}\cap\Gamma} b^\varepsilon_+ w^{r+1}\varphi^2\right) \\ &\quad + \quad \frac{\sigma_2(\varepsilon)}{2}\int_{B^+_{4R}} (w^{\frac{r+2}{2}}|\nabla(\varphi)|)^2 + \frac{1}{2}\int_{B^+_{4R}} w^{r+2}\varphi\partial_t\varphi, \end{aligned} \tag{52}$$

where $q_+(x)$ denotes positive part and $\sigma_2(\varepsilon)$ is the supremum of the diffusion coefficients.

Now we observe that (50), (49) and the following Young's inequality

$$ab \le \frac{1}{s}\delta^s a^s + \frac{1}{s'}\delta^{-s'} b^{s'} \quad \text{for} \quad a, b \ge 0, \delta > 0, \tag{53}$$

upon taking $s = 2(r+1)/(r+2) > 1$ and $\delta < R^2$, imply that

$$|\int_{B_{4R}^+} f(w) w^{r+1}\varphi^2| \le \int_{B_{4R}^+} (w^{\frac{r+2}{2}}\varphi)^2 + (r+2)^{-1} C_\delta^{r+2}, \tag{54}$$

since $\delta^{-(r+2)} > R^{-2(r+2)}$. Similarly (49), (50) (using Sobolev's embeddings [**1, 6**] and (53)) give an estimate of

$$\begin{aligned} |\int_{B_{4R}^+\cap\Gamma} g(w) w^{r+1}\varphi^2| &\le \left(\int_{B_{4R}^+} (w^{\frac{r+2}{2}}\varphi)^2 + \int_{B_{4R}^+} |\nabla (w^{\frac{r+2}{2}}\varphi)|^2 \right) \\ &+ (r+2)^{-1} C_\delta^{r+2}. \end{aligned}$$

It is easy to see for the next sum of terms in (52) that in virtue of (50), using Hölder's inequality and the Sobolev's embeddings $H^1(\Omega) \subset L^\vartheta$ where

$$\vartheta(p_0) = 2p_0'(r+1)/(r+2) \quad \text{or} \quad \vartheta(q_0) = 2q_0'(r+1)/(r+2),$$

respectively and (53) yield a bound from above of the form

$$\begin{aligned} &\le \left(\int_{B_{4R}^+} (w^{\frac{r+2}{2}}\varphi)^2 + \int_{B_{4R}^+} |\nabla (w^{\frac{r+2}{2}}\varphi)|^2 \right) \quad + \\ &+ \quad (r+2)^{-1} \left(C_\delta \|q_+^\varepsilon(x)\|_{r_0} \right)^{r+2}, \end{aligned}$$

for each term in the summed expression. It only remains to consider the last two integrated terms in the right-hand side of (52). These can be estimated as

$$\le \frac{C}{(\sigma R)^2} \int_{B_{4R}^+} w^{r+2}. \tag{55}$$

Thus there exist $\beta > 0$ and a strictly positive linear expression $\gamma(\varepsilon) > 0$ consisting of the sum of the constant $C_\delta > 0$ and the values of the coefficients $V_+^\varepsilon, b_+^\varepsilon$ in norms of the respective L^{r_0} spaces, for which there holds

$$\begin{aligned} &\frac{d}{dt} \int_{B_{4R}^+} (w^{\frac{r+2}{2}}\varphi)^2 + \beta \|w^{\frac{r+2}{2}}\varphi\|^2_{H^1(B_{4R}^+)} \\ &\le \quad (r+2)\gamma(\varepsilon)^{r+2} C \left(\frac{1}{(\sigma R)^2} \int_{B_{4R}^+} w^{r+2} \right) + \gamma(\varepsilon)^{r+2} C, \end{aligned} \tag{56}$$

after multiplying by $2(r+2)$. This proves (51). □

LEMMA 3.8. *Assume that Lemma* 3.7 *holds. Then, there exists an integral inequality of the form*

$$\int_{Q_{i+1}^+} w^{2s^{i+1}} \le C_i^s(\varepsilon) \left(\frac{1}{(\sigma R)^2} \int_{Q_i^+} w^{2s^i} \right)^s + C_i^s(\varepsilon) \tag{57}$$

where $C_i^s(\varepsilon) > 0$ *is bounded for all* $\varepsilon > 0$, $s = 1 + N/2 > 0$, $i \in I\!N$, *and* $Q_i^+ = B_{4R_i}^+ \times [\tau_i, t_0 + 4R^2]$ *for* $R_i = R_{i-1} - 3(2^{-i})R, \quad R_0 = 4R, \quad \tau_i = \tau_{i-1} + 2^{2-i}R^2, \quad \tau_0 = t_0.$

PROOF. Consider (51) given an in (56). Then integrate in time to find

$$\sup_{t_0\le t\le t_0+4R^2}\int_{B_{4R}^+}(w^{\frac{r+2}{2}}\varphi)^2+\beta\int_{t_0}^{t_0+4R^2}\|w^{\frac{r+2}{2}}\varphi\|^2_{H^1(B_{4R}^+)}$$
$$\le\quad (r+2)\gamma(\varepsilon)^{r+2}C(2R)^2\left(\frac{1}{(\sigma R)^2}\int_{Q_{4R}^+}w^{r+2}\right)+\gamma(\varepsilon)^{r+2}C(2R)^2, \tag{58}$$

because of the second property in (50) and since $R\ge 1/2$.

Let $q=2(1+2/N)$. Then, from [**25**], the embeddings of the spaces of functions with values in Banach spaces

$$L^\infty(0,T,L^2(\Omega))\cap L^2(0,T,H^1(\Omega))\subset L^q(\Omega\times(0,T))$$

are verified. These embeddings restricted to the set Q_{4R}^+ in (58) with $s=1+2/N$ give

$$\int_{Q_{4R}^+}w^{(r+2)s}\quad\le\quad\big((r+2)\gamma(\varepsilon)^{r+2}(2C)(2R)^2\big)^s\times$$
$$\times\quad\left(\frac{1}{(\sigma R)^2}\int_{Q_{4R}^+}w^{r+2}\right)^s+\big(\gamma(\varepsilon)^{r+2}(2C)(2R)^2\big)^s \tag{59}$$

where we have used

$$(a+b)^p\le 2^p(a^p+b^p)\quad\text{for}\quad a,b\ge 0,\quad p>0. \tag{60}$$

It follows, upon defining nested cylinders of the form as in the lemma and taking $r_i=2(s^i-1)$ which imply $r_i+2=2s^i$, that the integral estimate (59) has the form

$$\int_{Q_{i+1}^+}w^{2s^{i+1}}\quad\le\quad\Big(2s^i\gamma(\varepsilon)^{2s^i}(2C)(2R)^2\Big)^s\times$$
$$\times\quad\left(\frac{1}{(\sigma R)^2}\int_{Q_i^+}w^{2s^i}\right)^s+\Big(\gamma(\varepsilon)^{2s^i}(2C)(2R)^2\Big)^s. \tag{61}$$

The desired integral inequality is obtained and (57) is proved. □

LEMMA 3.9. *Assume that Lemma* 3.8 *holds. Then, there exists a positive constant $C>0$ independent of R such that*

$$\sup_{Q_R}|w|\le C\left(\left(\frac{1}{R^{N+2}}\int_{Q_{4R}}|w|^2\right)^{\frac12}+1\right).$$

PROOF. The proof of this lemma follows from general inductive process of inequality (61). Since the process is recursive using repeatedly (60), we obtain for arbitrary $k\ge 1$ that

$$\int_{Q_k^+}w^{2s^k}\quad\le\quad 2^{2(k-1)s+2(k-2)s^2+\ldots+2s^k}\gamma(\varepsilon)^{2ks^k}\times$$
$$\times\quad\left(\frac{2C}{(\sigma R)}\right)^{2s+2s^2+\ldots+2s^k}\left(\frac{1}{(\sigma R)^2}\int_{Q_0^+}w^2\right)^{s^k}$$
$$+\quad\left(2^{2(k-1)s+2(k-2)s^2+\ldots+2s^k}(2\gamma(\varepsilon))^{2ks^k}\left(\frac{2}{\sigma}\right)^{2s+2s^2+\ldots+2s^k}\right), \tag{62}$$

since $R \geq 1$, $C \geq 1$, $(\sigma R)^{-1}$ is monotonic increasing with increasing exponents and since $1 < s \leq 2$ imply $k2s^k \leq 2^{2ks^k}$.

Next we note that

$$2(k-1)s + 2(k-2)s^2 + \ldots + 2s^k \leq 2s^k \sum_{i=0}^{\infty} \frac{i}{s^i} = 2s^k A.$$

Since $s = 1 + 2/N > 1$, the quotient rule yields convergence of the series. In addition, the sum to infinite of geometric series gives

$$2s + 2s^2 + \ldots + 2s^{k-1} \leq 2s^k \sum_{i=1}^{\infty} \frac{1}{s^i} = 2s^k \left(\frac{N}{2}\right).$$

Therefore, in (62), we arrive at

$$\begin{aligned}
\int_{Q_k^+} w^{2s^k} &\leq \left((2\gamma(\varepsilon))^A (2C)^{N/2} \frac{1}{(\sigma R)^{\frac{N}{2}}} \left(\frac{1}{\sigma R} \left(\int_{Q_0^+} w^2 \right)^{\frac{1}{2}} \right) \right)^{2s^k} \\
&+ \left((4\gamma(\varepsilon))^A \left(\frac{2}{\sigma} \right)^{\frac{N}{2}} \right)^{2s^k} \\
&\leq \left((2\gamma(\varepsilon))^A (2C)^{\frac{N}{2}} \left(\frac{1}{R^{N+2}} \int_{Q_0^+} w^2 \right)^{\frac{1}{2}} + (4\gamma(\varepsilon))^A \left(\frac{2}{\sigma} \right)^{\frac{N}{2}} \right)^{2s^k}.
\end{aligned}$$

Now passing the power $2s^k$ in the left-hand side of the inequality we get

$$\begin{aligned}
\left(\int_{Q_k^+} w^{2s^k} \right)^{\frac{1}{2s^k}} &\leq (4\gamma(\varepsilon))^A (2C)^{\frac{N}{2}} \left(\frac{2}{\sigma} \right)^{\frac{N}{2}} \left(\left(\frac{1}{R^{N+2}} \int_{Q_0^+} w^2 \right)^{\frac{1}{2}} + 1 \right) \\
&= cte \left(\left(\frac{1}{R^{N+2}} \int_{Q_0^+} w^2 \right)^{\frac{1}{2}} + 1 \right),
\end{aligned}$$

where the cte is independent of R. As a result, taking limits as $k \to \infty$, we conclude that

$$\sup_{Q_R} |w| \leq \lim_{k \to \infty} \left(\int_{Q_k^+} w^{2s^k} \right)^{\frac{1}{2s^k}} \leq cte \left(\left(\frac{1}{R^{N+2}} \int_{Q_0^+} w^2 \right)^{\frac{1}{2}} + 1 \right).$$

The rest is obvious (since the above two last estimates are true for the positive part w_+ of w on the cylinders $Q_k = B_{4R_k} \times [\tau_k, t_0 + 4R^2]$, $k \geq 0$) and our lemma is proved. □

Now, we assert the following lemma.

Lemma 3.10. *Assume that Lemma 3.9 holds. Then, the integral expression*

$$(63) \qquad \frac{1}{R^{N+2}} \int_{Q_{4R}} |w|^2 dxdt \quad \textit{is bounded above from independently of } R.$$

Let us first observe that if $\varphi^2/l(u)$ is taken as a test function in (29), then

$$\begin{aligned}\frac{d}{2dt}\int_\Omega w\varphi^2 + \frac{1}{2}\int_\Omega d^\varepsilon(x)|\nabla w|^2\varphi^2 + \frac{\lambda}{2}\int_\Omega w\varphi^2 \\ \le |\int_\Omega d^\varepsilon(x)\varphi\nabla w\nabla\varphi| + \int_\Omega f(w)\varphi^2 + \int_\Gamma g(w)\varphi^2 \\ +\frac{1}{2}\left(1+\frac{M}{R}\right)\int_\Omega (V_+^\varepsilon+\lambda)\varphi^2 + \frac{1}{2}\left(1+\frac{M}{R}\right)\int_\Gamma b_+^\varepsilon\varphi^2.\end{aligned} \tag{64}$$

We now estimate (64) from above. For the first term, we use the following Young's inequality

$$ab \le \frac{1}{2}\eta^2 a^2 + \frac{1}{2\eta^2}b^2, \quad a,b\ge 0, \quad 0<\eta<1, \tag{65}$$

with $\eta = (\delta\sigma_1(\varepsilon))^{\frac{1}{2}}/\sigma_2(\varepsilon)$ for some $\delta>0$ sufficiently small. Consequently

$$|\int_\Omega d^\varepsilon(x)\varphi\nabla w\nabla\varphi| \le \frac{\delta\sigma_1(\varepsilon)}{2}\int_\Omega |\nabla w\varphi|^2 + \frac{C}{R^2}.$$

The integrals involving the nonlinearities can be quite easily estimated as

$$\le CR^{-2}\int_\Omega \varphi^2 + CR^{N-2}. \tag{66}$$

A bound from above similiar to (66) is verified by the last sum of integrals. As a result, one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w\varphi^2 + \beta\int_\Omega |\nabla w|^2\varphi^2 + \frac{\lambda}{2}\int_\Omega w\varphi^2 \le \frac{C}{R^2}\int_\Omega \varphi^2 + CR^{N-2} \tag{67}$$

for some $\beta=\beta(\delta)>0$ and $C>0$ independent of R.

Now we can carry out the proof of Lemma 3.10.

Proof. Consider the closed time interval $I_2=[t_0,t_0+4R^2]$ and define the cylinder $Q=B_{2R}\times I_2$. Let $Q_u=\{(x,t)\in Q : u\le m_4+\frac{\omega_4}{2}\}$.

Then $w\le 0$ on Q_u if $u\ge 0$ and on $Q\setminus Q_u$ if $u<0$. Furthermore, w_+ must vanish on some $Q_0\subset Q$ with $|Q_0|\ge\frac{1}{2}|Q|$ depending on the case. Denote such a function by v and define

$$\Omega_t^0=\{x\in B_{4R} : v_+(x,t)=0\}, \quad m(t)=|\Omega_t^0| \quad \text{for} \quad t\in I_2.$$

Also define the function

$$W(t)=\left(\int_\Omega v\varphi^2\right)\left(\int_\Omega\varphi^2\right)^{-1}.$$

Since $W(t)\ge -\mathrm{In}2$ for all $t\in I_2$, the Poincaré inequality (due to Moser)

$$\int_\Omega \varphi^2(v-W)^2 \le 16R^2\int_\Omega |\varphi\nabla v|^2 \tag{68}$$

is satisfied. Hence we have an estimate from below in (67), for v in place of w in the complete inequality. This process yields, if we take the second and third terms in the lower estimate on a smaller set Ω_t^0 and if we let $\beta_R=\beta|\Omega_t^0|(8R^2)^{-1}$,

$$\frac{d}{dt}W\times\int_\Omega\varphi^2 + \beta_R W \le CR^{N-2}. \tag{69}$$

Since $\int_{I_2} m(t)dt = |Q_0| \ge CR^{N+2}$, we have (69) and [**26**] Lemma 2.5 implies $W(t_1)\le C$, with $t_1\in I_2$ independent of R. This fact, together with (67) in v and the third integrated term in the left-hand expression taken on Ω_t^0, then integrating the resultant inequality on subinterval $[t_1, t_2] \subset I_2$ yield

$$-\text{In}2 \leq W(t) \leq C, \quad \forall t \in I_2 \quad \text{and} \quad \int_{t_0}^{t_0+4R^2} \int_\Omega \varphi^2 (v - W)^2 \leq CR^{N+2},$$

upon using (68). This concludes (63) and the proof of the lemma is complete. □

To complete the proof of Theorem 3.2, we have that there exists a $C > 0$ independent of R such that $w = \text{In}((\omega_4 + R)/l(u))$ verifies $w(x,t) \leq C$ for all $(x,t) \in Q_{4R}$. This implies $\omega_1 \leq \delta\omega_4 + CR$ for $0 < \delta = (C-1)/C$ and, using [**19**] Lemma 8.23 yields the proof of Theorem 3.2 complete. □

4. Stability of attractors

We are in a position to study the stability in large diffusivity of attractors for the nonlinear dynamical system generated by the evolution equations (1). The basic theorem from which we obtain the existence of the attractor for the nonlinear semigroup to our equations is Theorem 3.3. Several notions of attractors and criteria for proving their existence for infinite dimensional dynamical systems arising in evolution equations have been extensively studied [**21, 38, 37, 39**]. The basic notion of a global compact attractor can be defined as follows.

Definition 4.1. *Let (X, d) be a metric space, and $S(t) : X \longmapsto X$ defined for $t \geq 0$ be a semigroup generated by the solutions to a certain nonlinear evolution system of equations. The set $\mathcal{A} \subset X$ is called a global attractor for the trajectories defined by this semigroup if*

(i). $\mathcal{A}$ is nonempty and compact

(ii). $\mathcal{A}$ is invariant i.e $S(t)\mathcal{A} = \mathcal{A} \quad \forall t \geq 0$

(iii). For any bounded set $B \subset X$, we have $\lim_{t\to\infty} dist(S(t)B, \mathcal{A}) = 0$,

where, for any pair of sets $A, B \subset X$,

$$dist(A, B) = \sup_{x\in A} \inf_{y\in B} dist(x, y)$$

denotes the Hausdorff semidistance.

We shall say that $B \subset X$ is an absorbing for the nonlinear semigroup if

$$\forall B_0 \subset X, \quad B_0 \quad \text{bounded}, \quad \exists t_0(B_0) : S(t)B_0 \subset B \quad \forall t \geq t_0(B_0.)$$

The semigroup is said to be uniformly compact if, for any bounded set $B \subset X$, there exists $t_0(B) > 0$ such that the positive trajectory $\gamma^+(B) = \cup_{t\geq t_0} S(t)B$ is relatively compact. If the nonlinear semigroup generated by a system of nonlinear evolution equations admits an absorbing set B and this semigroup is uniformly compact, then

$$\omega(B) = \cap_{\tau>0} \overline{\cup_{t\geq\tau} S(t)B}$$

is a global compact attractor and is unique for the semigroup. There are other criteria for asserting the existence of an attractor, for example in [**10, 11, 12, 21**], if $S(t), t \geq 0$, is a compact C^0- semigroup, that is in addition point dissipative, then, there exists a global compact attractor for semigroup $\{S(t) : t \geq 0\}$ in X.

We obtain from Theorem 3.3 the following corollary on the existence of a global compact attractor for the evolution equations generated by the semilinear system of reaction and diffusion equations (1).

COROLLARY 4.2. *Suppose that the hypotheses of Theorem 3.3 are satisfied. Then the nonlinear semigroup generated by the evolution equations (1) has a global compact attractor* $\mathcal{A}^\varepsilon \subset H^1(\Omega) \cap L^\infty(\Omega)$.

PROOF. Consider the variation of constants formula (30). Since the nonlinear term $h(u) \in H^{-1}(\Omega)$ satisfies $\|h(u)\|_{H^{-1}(\Omega)} \le C$ for all $t > 0$, we have

$$\begin{aligned} \|u^\varepsilon(t, u_0^\varepsilon)\|_{H^1(\Omega)} &\le Mt^{-\frac{1}{2}} e^{-\omega t} \|u_0^\varepsilon\|_{L^2(\Omega)} + C\int_0^t e^{-\omega(t-s)}(t-s)^{-\frac{1}{2}} ds \\ &\le Ct^{-\frac{1}{2}} e^{-\omega t} \|u_0^\varepsilon\|_{L^\infty(\Omega)} + C\int_0^\infty e^{-\omega t} t^{-\frac{1}{2}} dt. \end{aligned}$$

This together with Theorem 3.3 imply that the nonlinear solution semigroup has an absorbing property in $X = H^1(\Omega) \cap L^\infty(\Omega)$. More precisely

$$\limsup_{t\to\infty} \|u^\varepsilon(t, u_0^\varepsilon)\|_X \le C.$$

Thus, from [**27**] Proposition 5.4, the set $\{u^\varepsilon(t, u_0^\varepsilon) : t \ge 0\}$ is relatively compact. Let $D = \{w \in X : \|w\|_X \le 2C\}$, then the $\omega-$ limit set

$$\omega(u_0^\varepsilon) = \cap_{\tau>0} \overline{\cup_{t\ge\tau} u^\varepsilon(t, u_0^\varepsilon)} \subset D$$

is the smallest nonempty subset that is compact, connected, invariant and satisfies

$$\lim_{t\to\infty} \inf_{w\in\omega(u_0^\varepsilon)} \|u^\varepsilon(t, u_0) - w\|_X = 0.$$

Thus the semilinear evolution equations defined by (1) admits a global compact attractor $\mathcal{A}^\varepsilon \cong \omega(D)$. Alternatively, it is easy to see that the nonlinear semigroup generated $S_\varepsilon(t); t \ge 0$ using the variation of constants formula (30) is point dissipative and the conclusion follows from [**21**] Sec. 4.2. and our proof is complete. □

It is now relatively easier to deduce the limit process of the equations (1) in large diffusivity (5).

THEOREM 4.3. *Assume that Theorem 3.3 holds and consider the system of ordinary differential equations*

$$\dot{u}_\Omega + L_0(u_\Omega) = h_\Omega(u), \quad u_\Omega(0) = u_\Omega^0 \in I\!R^m \tag{70}$$

where

$$h_\Omega(u) = f(u_\Omega) + \frac{|\Gamma|}{|\Omega|} g(u_\Omega) \in I\!R^m \quad \text{and} \quad L_0 = diag[\mu_n]$$

for $n = 1, \ldots, m$ *as in Lemma 2.3 is definite positive. Then (70) is the limit process in (4) of the system of equations (1). This limit is given by convergence of solutions strongly in* $H^1(\Omega)$ *uniformly on compacts of* $I\!R^+$.

PROOF. The proof follows from the variations of constants formula (30), using Theorem 2.1 and Lemma 2.3. Since (38) is valid, we have by Lebesgue dominated convergence theorem,

$$\int_0^t e^{-L^\varepsilon(t-s)} h(u^\varepsilon(s)) ds \to \int_0^t e^{-L_0(t-s)} h_\Omega(u(s)) ds$$

as $\varepsilon \to 0$, and $h_\Omega(u) \in I\!R^m$ as in the theorem. Thus (70) holds in the limit since the solutions converge strongly in $H^1(\Omega)$ uniformly on compact sets of $(0, \infty)$. □

It follows from Theorem 4.3 that the system of ordinary differential equations (70) has a global compact attractor $\mathcal{A} \subset I\!R^m$ and the family of attractors $\{\mathcal{A}^\varepsilon \cup \mathcal{A}\}_{\varepsilon > 0}$ verifies

$$\lim_{\varepsilon \to 0} \delta_{H^1(\Omega)}(\mathcal{A}^\varepsilon, \mathcal{A}) = \lim_{\varepsilon \to 0} \sup_{u^\varepsilon \in \mathcal{A}^\varepsilon} \inf_{u_\Omega \in \mathcal{A}} \|u^\varepsilon - u_\Omega\|_{H^1(\Omega)} = 0, \tag{71}$$

so we have stability in large diffusivity of the attractors in the topology of $H^1(\Omega)$. This naturally leads, on the basis of Theorem 3.2 to a much stronger convergence of attractors in the following corollary.

Corollary 4.4. *Assume that the hypotheses in Theorem* 3.2 *are satisfied. Then*

$$\lim_{\varepsilon \to 0} \delta_{C(\Omega)}(\mathcal{A}^\varepsilon, \mathcal{A}) = 0,$$

for any open bounded regular domain $\Omega \subset I\!R^N$ *of arbitrary space dimensions.*

Proof. The proof is straightforward. Actually one uses the compact embeddings of the spaces

$$C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times (0, T)) \hookrightarrow C(\overline{\Omega} \times (0, T))$$

for $\alpha > 0$. This yields, via Arzelá - Ascoli's Theorem, the conclusion, since the limit of equations (1) in large diffusivity is unique. □

4.1. Finite escape time. In this subsection, we make some conclusion remarks about the dynamics of the system of equations (1) in view of those of its limit process (70) in large diffusivity. The remaining case from the discussion above of this section can be understood from the following simplified problem.

Consider the following modified system of equations

$$(PDE)\left\{\begin{array}{rcll} u_t - D\Delta u &=& |u|^{p-1}u & \text{in } \Omega \\ D\frac{\partial u}{\partial \vec{n}} &=& 0 & \text{on } \Gamma, \quad t \in (0, T) \\ u(0) &=& u_D^0 & \text{in } \Omega, \end{array}\right.$$

of the semilinear problem (1). This system of equations (PDE) is well posed for a mild solution in the sense of Theorem 3.1. Also we expect that its limit process as $D \to \infty$ be given by the finite dimensional equations

$$(ODE)\left\{\begin{array}{l} \dot{u}_\Omega = |u_\Omega|^{p-1}u_\Omega \\ \quad u_\Omega = u_0 \in I\!R^m \quad \text{for } t \in (0, T). \end{array}\right.$$

Now, without loss of generality, let us assume that $u_0 \neq 0$. Then the solution of the equations (ODE) have a finite escape time, namely $t_* = \frac{1}{(p-1)|u_0|^{p-1}}$. Thus, it is not difficult to see that, the local nonlinear semigroup generated by (PDE) blows up in norm of the space $H^1(\Omega)$ at the time $T = t_*$. Since $T = T(N, u_0)$ it is possible for small initial data of the system of equations and large but fixed N that solutions of (PDE) are almost globally defined and unbounded at time T. Thus from Theorem 3.1 we can further conclude that

either the nonlinear semigroup is globally defined,

OR

it blows up in a finite interval of time.

Furthermore, in view of the results of this section, we conclude that the dynamical systems of the infinite dimensional equations are completely determined by those of their limit processes in large diffusivity. More to it, the complicated spatial dynamics defining heterogeneous stable patterns of concentrations of reaction and diffusion problems modelling processes in chemical engineering or biochemical sciences [**13, 20, 24, 40, 42**], if large diffusion is introduced in the models, then the production of these patterns is made less effective to eventually being eliminated with time advances in the relevant experiments.

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