Blow up of the solutions of the nonlinear parabolic equation

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Abstract. In this paper the Cauchy problem for nonlinear parabolic equation is investigated. We prove that the Cauchy problem has one nontrivial solution \(u(t, r)\) in the form
\[u(t, r) = v(t) \omega(r) \in C([0, 1] L^2([r_0, \infty])))\] for which
\[
\lim_{t \to 1} ||u||_{L^2([r_0, \infty)))} = \infty,
\] where \(r = |x|, r_0 \geq 1\) is arbitrary chosen and fixed. Also, we prove that the solution map is not uniformly continuous.

Contents

1. Introduction 1
2. Proof of Main Result 2
3. Appendix 10
References 14

1. Introduction

In this paper we consider the Cauchy problem
\[
\begin{align*}
(1.1) \quad &u_t - \Delta u = f(t, |x|, u), \quad t \in [0, 1], \quad |x| \geq r_0, \quad n \geq 2, \\
(1.2) \quad &u(0, x) = u_0(x) \in L^2(\mathbb{R}^n \setminus \{|x| < r_0\}),
\end{align*}
\]
where \(r_0 \geq 1\) is arbitrary chosen and fixed, \(f(t, |x|, u) \in C^1([0, 1]) \times C^1([r_0, \infty)) \times C^1(\mathbb{R}^1)\), \(a|u| \leq f'_u(t, |x|, u) \leq b|u|\) for every \(t \in [0, 1]\), for every \(|x| \geq r_0\), \(a\) and \(b\) are fixed positive constants, \(f(t, |x|, 0) = 0\) for every \(t \in [0, 1]\), \(\forall |x| \geq r_0\).

We will prove that the Cauchy problem (1.1), (1.2) has a nontrivial solution \(u(t, r)\) in the form \(u(t, r) = v(t) \omega(r) \in L^2([r_0, \infty))\) for every \(t \in [0, 1]\), for which
\[
\lim_{t \to 1} ||u||_{L^2([r_0, \infty))} = \infty.
\] Also we will prove that the solution map is not uniformly continuous. When we say that the solution map \(u_0 \longrightarrow u(t, r)\) is uniformly

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continuous we mean: for every positive constant $\varepsilon$ there exists positive constant $\delta$
such that for any two solutions $u, v$ of the Cauchy problem (1.1), (1.2), so that
$E(0, u - v) \leq \delta$,
the following inequality holds
$E(t, u - v) \leq \varepsilon$ for all $t \in [0, 1]$,
where
$E(t, u) := \|u(t, \cdot)\|_{L^1([r_0, \infty))}^2 + \|\partial_r u(\cdot, r)\|_{L^1([r_0, \infty))}^2$.

Here we use the approach which is used in [1], [2], [3], [4]. In the accessible
literature there are too many methods for investigations of this problem which are
different than the method which we propose in this paper.

Our main result is:

**Theorem 1.1.** Let $n \geq 2$ is fixed, $r_0 \geq 1$ is fixed, $f(t, |x|, u) \in C^4([0, 1]) \times
C^4([r_0, \infty)) \times C^4(R^1)$, $a|u| \leq f_a(t, |x|, u) \leq b|u|$ for every $t \in [0, 1]$, for every $|x| \geq r_0$, $a$ and $b$
are fixed positive constants, $f(t, |x|, 0) = 0$. Then the problem of Cauchy
(1.1), (1.2) has one nontrivial solution $u = u(t, r) \in C([0, 1] L^2([r_0, \infty)))$ for which
$\lim_{t \to 1} \|u\|_{L^2([r_0, \infty))} = \infty$. Also the solution map is not uniformly continuous.

The paper is organized as follows. In section 2 we will prove our main result.
In the appendix we will prove a result which we will use for the proof of our main result.

## 2. Proof of Main Result

Here $r_0 \geq 1$ is fixed, $n \geq 2$ is fixed.

Since we will search a positive solution $u(t, r)$ in the form $u(t, r) = v(t)\omega(r)$ we
rewrite the problem (1.1), (1.2) as follows
\begin{align*}
(2.1) \quad u_t - u_{rr} - \frac{n-1}{r} u_r &= f(t, r, u), \quad t \in [0, 1], \quad r \geq r_0, \\
(2.2) \quad u(0, r) &= u_0(r) \in L^2([r_0, \infty)).
\end{align*}

For fixed positive constants $n \geq 1$, $r_0 \geq 1$, $a, b$ we suppose that the constants
$A_1, A_2, A, B, c_1, d_1$ satisfy the following conditions
\begin{align*}
(i1) \quad \begin{cases}
1 \leq r_0 \leq c_1 \leq d_1, & 0 < A_1 \leq A_2, 0 \leq \frac{1}{A_1} \leq \frac{1}{A_2}, \\
A_1 > 2, & (A_1 - 1) \frac{d_1^2}{(1 + d_1)^2} \geq 1.
\end{cases}
\end{align*}

**Example.** Let $n = 14, r_0 \geq 2$. Let also
\begin{align*}
& a = 2r_0^{11}, b = 4r_0^{11}, A = r_0^n, B = \frac{1}{2} r_0^n, \\
& A_1 = r_0^{10}, A_2 = 2r_0^{10}, c_1 = r_0 + 1, d_1 = r_0 + 2.
\end{align*}

We note that $\frac{d_1}{2A_1} = A_1$.

For fixed $n \geq 1, r_0 \geq 1, a, b$ bellow we suppose that the constants $A_1, A_2, A, B, c_1, d_1$
satisfy the conditions (i1). Also we will suppose that the function $v(t)$ is
fixed function which satisfies the following hypotheses\[
\begin{aligned}
(H1) \quad & v(t) \in C^3([0, 1]), \quad v(t) > 0 \quad \forall t \in [0, 1], \quad \frac{v'(t)}{v(t)} > 0 \quad \forall t \in [0, 1], \\
(H2) \quad & A_1 \leq \frac{v'(t)}{v(t)} \leq A_2 \quad \forall t \in [0, 1], \quad \lim_{t \to 1} \left( \frac{v'(t)}{v(t)} - \frac{a}{2A_1} \right) = 0.
\end{aligned}
\]
There exists a function \( v(t) \) which satisfies the conditions \((H1), (H2)\). For instance \( v(t) = e^{\frac{a}{t^p}(t-1)} \), where \( a, b, A, B, A_1, A_2, c_1, d_1 \) are the constants from the above example.

Let \( N \) be the set
\[
N = \left\{ u(t, r) : u(t, r) \in C^1([0, 1]) \quad \forall r \geq r_0, \right. \\
\left. u(t, \infty) = u_r(t, \infty) = 0 \quad \forall t \in [0, 1], \right. \\
\left. r^\alpha |\partial_r^2 u(t, r) | \leq 1 \quad \forall t \in [0, 1], \forall r \geq r_0, \right. \\
\left. \forall \alpha \in \mathcal{N} \cup \{0\}, \beta = 0, 1, \right. \\
\left. u(t, r) \geq \frac{1}{r} \quad \forall t \in [0, 1], \forall r \in [c_1, d_1], \right. \\
\left. u(t, r) \in L^2([r_0, \infty)) \quad \forall t \in [0, 1] \right\}.
\]

For \( u \in N \), fixed \( n \geq 1 \) and for every fixed \( t \in [0, 1] \) we define the operator
\[
(2.1^{**}) \quad P(u) = \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \left( v^n(t) u - f(t, \tau, u) \right) d\tau ds, \quad r \geq r_0.
\]

We put
\[
u_0(r) = f_r^\infty \frac{1}{l^n} \int_s^\infty \tau^n \left( v^n(0) u_0 - f(0, \tau, u_0) \right) d\tau ds, \quad r \geq r_0.
\]

Bellow we will prove that \( u_0(r) \in L^2([r_0, \infty)) \) exists.

**Theorem 2.1.** Let \( n \geq 2 \) be fixed, \( r_0 \geq 1 \) be fixed, the positive constants \( a, b, c \leq b \), are fixed, \( f(t, |x|, u) \in C^1([0, 1]) \times C^1([0, \infty)) \times C^1(\mathbb{R}^1) \), \( a|u| \leq f_1(t, |x|, u) \leq bx \) for every \( t \in [0, 1] \), for every \( x \geq r_0 \), \( f(t, |x|, 0) = 0 \) for every \( t \in [0, 1] \) and for every \( x \geq r_0 \). Let also the positive constants \( c_1, d_1, A_1, A_2, A, B \) be fixed which satisfy the conditions \((i1)\), the function \( v(t) \) is fixed which satisfies the hypotheses \((H1), (H2)\). Then the Cauchy problem \((2.1), (2.2)\) has one unique solution \( u(t, r) \) in the form \( u(t, r) = v(t)\omega(r) \) for which \( u(t, r) \in N \), \( \lim_{t \to 1} \| u \|_{L^2([r_0, \infty))} = \infty \).

**Proof.** Here and bellow we will suppose that \( t \in [0, 1] \) is fixed.

First we will prove that \( P : N \to N \). Let \( u \in N \) is fixed.

1) Since \( u(t, r) \in C^1([0, 1]) \) for every \( r \geq r_0 \), \( f(t, t, u) \in C^1([0, 1]) \times C^1([0, \infty)) \times C^1(\mathbb{R}^1) \), \( v(t) \in C^1([0, 1]) \), we have \( P(u) \in C^1([0, 1]) \) for every \( r \geq r_0 \). Also
\[
P(u)_{\tau = \infty} = 0, \quad \frac{dP(u)}{d\tau} = -\frac{1}{\tau^n} \int_r^\infty \tau^n \left( v^n(t) u - f(t, \tau, u) \right) d\tau = 0.
\]

We note that from the conditions of the Theorem 2.1 we have \( f_1(t, \tau, u) \leq bu \).

From here and from \( f(t, \tau, 0) = 0 \) we get \( f(t, \tau, u) \leq \frac{b}{\tau^2} u^2 \). From the definition of the set \( N \) we have \( u \leq \frac{b}{\tau^2} \). Therefore \( f(t, \tau, u) \leq \frac{b}{\tau^2} u \) for every \( t \in [0, 1], \tau \geq r_0 \). Also we have \( f(t, r, u) \geq \frac{b}{\tau^2} u^2 \) for every \( t \in [0, 1] \) and for every \( r \geq r_0 \).

Let \( \alpha \in \mathcal{N} \cup \{0\} \) is arbitrary chosen and fixed, \( k \in \mathcal{N} \) is enough large such that
\[
k > \alpha + 3, \quad A_2 + \frac{b}{\tau^2} \leq k - 1.
\]
Then for \( r \geq r_0 \) we have
\[
|r^\alpha P(u)| = \left| r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \left( \frac{v'(t)}{v(t)} u - f(t, \tau, u) \right) d\tau ds \right| \leq
\]
\[
\leq r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \left( \frac{v'(t)}{v(t)} u + f(t, \tau, u) \right) d\tau ds \leq
\]
here we use that \( f(t, \tau, u) \leq \frac{b}{\tau u} \)
\[
\leq r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n A_2 u + \frac{b}{\tau u} \) d\tau ds =
\]
\[
= r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \left( A_2 + \frac{b}{\tau u} \right) u d\tau ds =
\]
\[
= \left( A_2 + \frac{b}{\tau u} \right) r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^{k+n} \) d\tau ds \leq
\]
here we use that from the definition of the set \( N \) we have \( \tau^{k+n} u \leq 1 \)
\[
\leq \left( A_2 + \frac{b}{\tau u} \right) r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \frac{1}{s^n} \) d\tau ds =
\]
\[
= \frac{1}{(k-1)(k+n-2)} \left( A_2 + \frac{b}{\tau u} \right) r^{k+n-1} \leq
\]
\[
\leq \frac{1}{(k-1)(k+n-2)} \left( A_2 + \frac{b}{\tau u} \right) \frac{1}{r_0^{k+n-2}} \leq 1.
\]
In the last inequality we use our choice of the constant \( k \).

Let \( k \) is the same as above. Then for \( r \geq r_0 \) we have
\[
\left| r^\alpha \frac{\partial P(u)}{\partial r} \right| = \left| r^\alpha \frac{1}{s^n} \int_s^\infty \tau^n \left( \frac{v'(t)}{v(t)} u - f(t, \tau, u) \right) d\tau \right| \leq
\]
\[
\leq r^\alpha \frac{1}{s^n} \int_s^\infty \tau^n \left( \frac{v'(t)}{v(t)} u + f(t, \tau, u) \right) d\tau \leq
\]
here we use that \( f(t, \tau, u) \leq \frac{b}{\tau u} \)
\[
\leq r^\alpha \frac{1}{s^n} \int_s^\infty \tau^n A_2 u + \frac{b}{\tau u} \) d\tau ds =
\]
\[
= r^\alpha \frac{1}{s^n} \int_s^\infty \tau^n \left( A_2 + \frac{b}{\tau u} \right) u d\tau ds =
\]
\[
= \left( A_2 + \frac{b}{\tau u} \right) r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^{k+n} \) d\tau ds \leq
\]
here we use that from the definition of the set \( N \) we have \( \tau^{k+n} u \leq 1 \)
\[
\leq \left( A_2 + \frac{b}{\tau u} \right) r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \frac{1}{s^n} \) d\tau =
\]
\[
= \left( A_2 + \frac{b}{\tau u} \right) \frac{1}{(k-1)(k+n-2)} \left( A_2 + \frac{b}{\tau u} \right) \frac{1}{r_0^{k+n-2}} \leq 1.
\]
In the last inequality we use our choice of the constant \( k \).

2) Now we will prove that for every fixed \( t \in [0, 1] \) and for every \( r \geq r_0 \) we have \( P(u) \geq 0 \).

Really, for \( k \in \mathbb{N} \) for which \( \frac{1}{r_0^{k+n}} < 1 \) we have
\[
P(u) = \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \left( \frac{v'(t)}{v(t)} u - f(t, \tau, u) \right) d\tau ds =
\]
now we apply the middle point theorem
\[
= \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \left( \frac{v'(t)}{v(t)} u - f'_s(t, \tau, \xi) u \right) d\tau ds \geq
\]
Here we use that $f'_u(t, \tau, \xi) \leq b \xi \leq bu$

$$\geq \int_r^{\infty} \int_s^{\infty} \tau^n \left( \frac{d(t)}{t(t)} - bu \right)ud\tau ds =$$

$$= \int_r^{\infty} \int_s^{\infty} \tau^n \left( \frac{d(t)}{t(t)} - b \frac{u}{t(t)} \right)ud\tau ds \geq$$

Here we use $\tau^k u \leq 1$

$$\geq \int_r^{\infty} \int_s^{\infty} \tau^n \left(A_1 - \frac{b}{r^{nk}_0}\right)ud\tau ds,$$

i.e.

(2.2')

$$P(u) \geq \int_r^{\infty} \frac{1}{s^n} \int_s^{\infty} \tau^n \left(A_1 - \frac{b}{r^{nk}_0}\right)ud\tau ds.$$  

Since $u(t, r) \geq 0$ for every fixed $t \in [0, 1]$ and for every $r \geq r_0$ and from our choice of the constant $k$ we have that $P(u) \geq 0$ for every $t \in [0, 1]$ and for every $r \geq r_0$.

3 ) Now we will see that for every fixed $t \in [0, 1]$ and for every $r \in [c_1, d_1]$ we have $P(u) \geq \frac{1}{k}$. We suppose that $k$ is same as in 2). Let

$$g(u) = \int_r^{\infty} \frac{1}{s^n} \int_s^{\infty} \tau^n \left(A_1 - \frac{b}{r^{nk}_0}\right)ud\tau ds.$$

Then

$$g'(u) = \int_r^{\infty} \frac{1}{s^n} \int_s^{\infty} \tau^n \left(A_1 - \frac{b}{r^{nk}_0}\right)d\tau ds \geq 0.$$

In the last inequality we use our choice of the constant $k$. Consequently $g(u)$ is increase function of $u$. Since for every fixed $t \in [0, 1]$ and for every $r \in [c_1, d_1]$ we have $u \geq \frac{1}{k}$ we get

$$g(u) \geq g\left(\frac{1}{k}\right) =$$

$$= \int_r^{\infty} \frac{1}{s^n} \int_s^{\infty} \tau^n \left(A_1 - \frac{b}{r^{nk}_0}\right)d\tau ds \geq$$

$$\geq \int_{d_1}^{d_1+1} \int_{d_1+1}^{d_1+1} \tau^n \left(A_1 - 1\right)d\tau ds \geq$$

$$\geq \left(A_1 - 1\right) \frac{1}{(d_1+1)^n} \frac{1}{A} \geq \frac{1}{A}.$$

In the last inequality we use the conditions (i1). From here and from (2.2') we get that $P(u) \geq \frac{1}{k}$ for every fixed $t \in [0, 1]$ and for every $r \in [c_1, d_1]$.

4 ) Now we will prove that for every fixed $t \in [0, 1]$ and for every $r \geq r_0$ we have $P(u) \leq \frac{1}{B}$. Let $k \in N$ is chosen such that

$$\left(A_2 + \frac{b}{2B}\right) \frac{1}{(k-1)(n+k-2)r^{n+k-2}_0} \leq \frac{1}{B}, \quad k \geq 2.$$

Then

$$P(u) = \int_r^{\infty} \frac{1}{s^n} \int_s^{\infty} \tau^n (\frac{d(t)}{t(t)} u - f(t, \tau, u))d\tau ds \leq$$

$$\leq \int_r^{\infty} \frac{1}{s^n} \int_s^{\infty} \tau^n \left(A_2u + f(t, \tau, u)\right)d\tau ds =$$

$$\leq \int_r^{\infty} \frac{1}{s^n} \int_s^{\infty} \tau^n \left(A_2u + \frac{b}{2B}u\right)d\tau ds =$$

$$= \int_r^{\infty} \frac{1}{s^n} \int_s^{\infty} \left(A_2 + \frac{b}{2B}\right) \tau^{n+k} d\tau ds \leq$$

Here we use $\tau^{n+k} u \leq 1$

$$\leq \left(A_2 + \frac{b}{2B}\right) \frac{1}{s^n} \int_s^{\infty} \left(A_2 + \frac{b}{2B}\right) \tau^{n+k} d\tau ds \leq$$

$$\leq \left(A_2 + \frac{b}{2B}\right) \frac{1}{(k-1)(n+k-2)r^{n+k-2}_0} \leq \frac{1}{B}.$$  

In the last inequality we use our choice of the constant $k$. 

BLOW UP OF THE NONLINEAR PARABOLIC EQUATION 5
5) Now we will prove that \( P(u) \in L^2([r_0, \infty)) \) for every fixed \( t \in [0, 1] \). We choose \( k \in \mathcal{N} \) such that \( n + k - 4 > 0 \).

\[
\|P(u)\|_{L^2([r_0, \infty))}^2 = \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( \frac{f(t, \tau, u)}{\tau^n} \right) dr ds \right)^2 dr \\
\leq \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( A_2 + \frac{b}{2r^n} \right) dr ds \right)^2 dr = \\
\left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \sqrt{\tau^{2k+2n} u^2} dr ds \right)^2 dr \\
\leq \left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \sqrt{\tau^{2k+2n} u^2} dr ds \right)^2 dr \\
\leq \left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s u^2 dr \right)^{\frac{1}{2}} \right)^2 dr \\
\leq \left( A_2 + \frac{b}{2r^n} \right)^2 \frac{1}{(\frac{2}{k-1})^2 (2n+2k-n-\frac{2}{2})} \frac{1}{r_0^{2n+2k-\frac{2}{2}}} < \infty.
\]

From 1), 2), 3), 4), 5) we conclude that \( P : \mathcal{N} \rightarrow \mathcal{N} \) for every fixed \( t \in [0, 1] \). Now we will prove that the operator \( P \) has one unique nontrivial fixed point in the set \( \mathcal{N} \).

Let \( t \in [0, 1] \) is fixed. Let also \( u_1 \in \mathcal{N} \), \( u_2 \in \mathcal{N} \) are fixed and \( \alpha = \|u_1 - u_2\|_{L^2([r_0, \infty))} \neq 0 \).

We choose \( k \in \mathcal{N} \), \( k > 3 \) enough large so that

\[
\frac{1}{\alpha} Q_1 = \frac{2 \left( A_2 + \frac{b}{2r^n} \right)^2}{\left( \frac{2}{k-1} \right)^2 (2n+2k-n-\frac{2}{2})} < 1, \quad Q_1 < 1.
\]

Then

\[
\|P(u_1) - P(u_2)\|_{L^2([r_0, \infty))}^2 = \\
\int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( \frac{f(t, \tau, u_1) - f(t, \tau, u_2)}{\tau^n} \right) dr ds \right)^2 dr \\
\leq \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| + |f(t, \tau, u_1) - f(t, \tau, u_2)| \right) dr ds \right)^2 dr \\
\leq \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| + f(t, \tau, u_1) - f(t, \tau, u_2) \right) dr ds \right)^2 dr \\
\leq \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| + f(t, \tau, u_1) - f(t, \tau, u_2) \right) dr ds \right)^2 dr \\
\leq 2 \left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| \right) dr ds \right)^2 dr \\
\leq 2 \left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| \right) dr ds \right)^2 dr \\
\leq 2 \left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| \right) dr ds \right)^2 dr \\
\leq 2 \left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| \right) dr ds \right)^2 dr \\
\leq 2 \left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| \right) dr ds \right)^2 dr \\
\leq 2 \left( A_2 + \frac{b}{2r^n} \right)^2 \int_{r_0}^\infty \left( \int_r^\infty \int_s^\infty \tau^n \left( f_s |u_1 - u_2| \right) dr ds \right)^2 dr \\
\leq \|u_1 - u_2\|_{L^2([0, \infty))}.
\]

i.e.

\[
\|P(u_1) - P(u_2)\|_{L^2([r_0, \infty))}^2 \leq Q_1 \|u_1 - u_2\|_{L^2([r_0, \infty))}.
\]
Since we choose the constant $k$ so that $\frac{1}{\alpha}Q_1 < 1$ we have

$$||P(u_1) - P(u_2)||^2_{L^2(\mathbb{R}_0, \infty)} \leq \frac{Q_1}{\alpha}||u_1 - u_2||^2_{L^2(\mathbb{R}_0, \infty)} \leq \frac{Q_1}{\alpha}||u_1 - u_2||^2_{L^2(\mathbb{R}_0, \infty)}.$$ 

From here and from the following theorem

**Theorem** [5, p. 294] Let $B$ be the completely metric space for which $AB \subset B$ and for the operator $A$ is hold the following condition

$$\rho(Ax, Ay) \leq L(\alpha, \beta) \rho(x, y), \quad x, y \in B, \alpha \leq \rho(x, y) \leq \beta,$$

where $L(\alpha, \beta) < 1$ for $0 < \alpha \leq \beta < \infty$. Then the operator $A$ has exactly one fixed point in the space $B$.

We conclude that the operator $P$ has one unique fixed point $u$ in the set $N$. We note that the set $N$ is a closed subset of the space $L^2([\mathbb{R}_0, \infty])$ for every fixed $t \in [0, 1]$ (see lemma 3.1 in the appendix of this paper) As in the proof of the Proposition 2.1, 2.2 [4] we have that the fixed point $u$ satisfies the equation (2.1) with initial data

$$u_0 = \int_0^\infty \frac{1}{\sqrt{\pi}} \int_0^\infty \tau^n \left(\frac{v(t)}{v(t)} u_0 - f(0, \tau, u_0)\right) d\tau dr, r \geq r_0.$$ 

We have that $u_0 \in L^2([\mathbb{R}_0, \infty])$.

Now we will prove that

$$\lim_{t \to 1} ||u||_{L^2(\mathbb{R}_0, \infty)} = \infty.$$ 

For $k \in N$ we put

$$Q_2 = \left(A_2 + \frac{1}{2\pi}\right)^2 \frac{1}{(k - 1)\frac{\pi}{2} (n + k - \frac{1}{2})(2n + 2k - \frac{5}{2})},$$

$$Q_3 = 2A^2 \left(\frac{\alpha}{\alpha} - \frac{\beta}{\beta}\right)^2 d_2^n (d_1 - c_1)^3 \frac{1}{c_1^2},$$

$$Q_4 = \left(A_1 - \frac{1}{2\pi}\right) \frac{\alpha}{\alpha} (d_1 - c_1)^2.$$ 

We choose the constant $k \in N$ such that

$$1 - 10\frac{Q_2}{Q_4} > 0.$$ 

Then

$$||u||^2_{L^2(\mathbb{R}_0, \infty)} = \int_{r_0}^\infty \left(\int_r^\infty \frac{1}{\sqrt{\pi}} \int_s^\infty \tau^n \left(\frac{v(t)}{v(t)} u - f(t, \tau, u)\right) d\tau dr\right)^2 dr =$$

$$= \int_{r_0}^\infty \left(\int_r^\infty \frac{1}{\sqrt{\pi}} \int_s^\infty \tau^n \left(\frac{v(t)}{v(t)} u - f(t, \tau, u)\right) d\tau dr\right)^2 dr +$$

$$+ \int_{r_0}^\infty \left(\int_r^\infty \frac{1}{\sqrt{\pi}} \int_s^\infty \tau^n \left(\frac{v(t)}{v(t)} u - f(t, \tau, u)\right) d\tau dr\right)^2 dr.$$ 

Let

$$J_1 := \int_{r_0}^\infty \left(\int_r^\infty \frac{1}{\sqrt{\pi}} \int_s^\infty \tau^n \left(\frac{v(t)}{v(t)} u - f(t, \tau, u)\right) d\tau dr\right)^2 dr,$$

$$J_2 := \int_{r_0}^\infty \left(\int_r^\infty \frac{1}{\sqrt{\pi}} \int_s^\infty \tau^n \left(\frac{v(t)}{v(t)} u - f(t, \tau, u)\right) d\tau dr\right)^2 dr.$$ 

Then

$$\int_{r_0}^\infty \left(\int_r^\infty \frac{1}{\sqrt{\pi}} \int_s^\infty \tau^n \left(\frac{v(t)}{v(t)} u - f(t, \tau, u)\right) d\tau dr\right)^2 dr = J_1 + J_2.$$
For $J_1$ we have the following estimate
\[ J_1 \leq J_{r_0}^\infty \left( \int_r^\infty \frac{1}{s^2} s^\tau \left( \frac{\omega(t)}{t(t)} u + f(t, \tau, u) \right) d\tau ds \right)^2 d\tau \leq \int_r^\infty \left( \int_r^\infty \frac{1}{s^2} s^\tau (A_2 u + \frac{1}{2} u) d\tau ds \right)^2 d\tau = \left( A_2 + \frac{1}{2} \right)^2 \int_r^\infty \left( \int_r^\infty \frac{1}{s^2} s^\tau \frac{1}{\sqrt{\tau^2 + 2u}} d\tau ds \right)^2 d\tau \leq \left( A_2 + \frac{1}{2} \right)^2 \int_r^\infty \left( \int_r^\infty \frac{1}{s^2} \left( \int_s^\infty t^{-\frac{1}{2}} u^2 d\tau \right)^\frac{1}{2} ds \right)^2 d\tau \leq Q_2 \| u \|_{L^2((r_0, \infty))}.
\]
(2.4)

Now we consider $J_2$. For it we have
\[ J_2 = J_{c_1} \left( \int_{c_1}^\infty \frac{1}{s^2} s^\tau \left( \frac{\omega(t)}{t(t)} u + f(t, \tau, u) \right) d\tau ds \right)^2 d\tau = \int_{c_1}^\infty \left( \int_{c_1}^\infty \frac{1}{s^2} s^\tau \left( \frac{\omega(t)}{t(t)} u + f(t, \tau, u) \right) d\tau ds \right)^2 d\tau + \int_{c_1}^\infty \frac{1}{s^2} s^\tau \left( \frac{\omega(t)}{t(t)} u + f(t, \tau, u) \right) d\tau ds \leq \left( A_2 + \frac{1}{2} \right)^2 \int_{c_1}^\infty \left( \int_{c_1}^\infty \frac{1}{s^2} s^\tau \frac{1}{\sqrt{\tau^2 + 2u}} d\tau ds \right)^2 d\tau \leq Q_2 \| u \|_{L^2((r_0, \infty))}.
\]

Then
\[ J_2 \leq 10Q_2 \| u \|_{L^2((r_0, \infty))} + Q_3 \| u \|_{L^2((r_0, \infty))}.
\]
From here and from (2.3), (2.4) we get
\[ \| u \|_{L^2((r_0, \infty))} \leq 10Q_2 \| u \|_{L^2((r_0, \infty))} + Q_3 \| u \|_{L^2((r_0, \infty))}.
\]
(2.5)
Also we note
\[ ||u||_{L^2([r_0, \infty))}^2 = \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( \frac{\nu(t)}{\tau(t)} u - f(t, \tau, u) \right)^2 \tau \, dr \right) ds \right)^{\frac{1}{2}} \geq \]
\[ \geq \left( \int_{r_1}^{\infty} \left( \int_{r_1}^{\infty} \tau^n \left( \frac{\nu(t)}{\tau(t)} u - f(t, \tau, u) \right)^2 \tau \, dr \right) ds \right)^{\frac{1}{2}} \geq \]
\[ \geq \left( \int_{r_1}^{\infty} \left( \int_{r_1}^{\infty} \tau^n \left( A \frac{1}{\tau(t)} u - \frac{b}{\tau(t)} \right)^2 \tau \, dr \right) ds \right)^{\frac{1}{2}} \geq \]
\[ \geq \frac{\tau}{\tau(t)} \left( A \frac{1}{\tau(t)} u - \frac{b}{\tau(t)} \right)^2 \frac{d\tau}{d\tau}. \]
From here and from (2.5) we get
\[ Q_3 \geq 10Q_3 \left( ||u||_{L^2([r_0, \infty))}^2 \right) \leq 10Q_3 \left( ||u||_{L^2([r_0, \infty))}^2 \right) + Q_3 \left( ||u||_{L^2([r_0, \infty))}^2 \right). \]
Then
\[ \frac{(Q_4 - 10Q_3 ||u||_{L^2([r_0, \infty))})}{Q_3} \leq \left( ||u||_{L^2([r_0, \infty))}^2 \right). \]
from where
\[ \lim_{t \to 1} ||u||_{L^2([r_0, \infty))} = \infty, \]
because \( \lim_{t \to 1} Q_3 = 0 \) (see (H2)).

**Theorem 2.2.** Let \( n \geq 2 \) be fixed, \( r_0 \geq 1 \) be fixed, the positive constants \( a, b, \) \( a \leq b, \) are fixed, \( f(t, |x|, u) \in C^1([0, \infty]) \times C^1([r_0, \infty)) \times C^1(\mathbb{R}^1), \) \( a|u| \leq f(t, |x|, u) \leq b|u| \) for every \( t \in [0, 1] \) for every \( |x| \geq r_0, \) \( f(t, |x|, 0) = 0. \) Let also the positive constants \( c_1, b, A, B \) are fixed which satisfy the conditions (i), the function \( v(t) \) is the same function as in the Theorem 2.1. Then the Cauchy problem (2.1), (2.2) has one unique solution \( u(t, r) \) in the form \( u(t, r) = v(t)w(r) \) for which \( u(t, r) \in N, \) \( u(t, r) \in H^1([r_0, \infty)) \) for \( \forall t \in [0, 1], \) and the solution map is not uniformly continuous.

**Proof.** In the Theorem 2.1 was proved that the equation (2.1) has one unique nontrivial solution \( \bar{u}(t, r) = v(t)w(r) \) for which \( \bar{u}(t, r) \in N. \) Also, for every \( k \in \mathbb{N} \) and for every fixed \( t \in [0, 1] \) we have
\[ \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( \frac{\nu(t)}{\tau(t)} \bar{u} - f(t, \tau, \bar{u}) \right)^2 \tau \, dr \right) ds \right)^{\frac{1}{2}} \leq \]
\[ \leq \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( \frac{\nu(t)}{\tau(t)} \bar{u} + f(t, \tau, \bar{u}) \right)^2 \tau \, dr \right) ds \right)^{\frac{1}{2}} \leq \]
\[ \leq \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( A_2 + \frac{b}{\tau(t)} \right) \bar{u} \right) ds \right)^{\frac{1}{2}} dr = \]
\[ \left( A_2 + \frac{b}{\tau(t)} \right)^2 \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( \frac{\nu(t)}{\tau(t)} \bar{u} \right)^2 \tau \, dr \right) ds \right)^{\frac{1}{2}} \leq \]
\[ \leq \left( A_2 + \frac{b}{\tau(t)} \right)^2 \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( A_2 + \frac{b}{\tau(t)} \right) \bar{u} \right) ds \right)^{\frac{1}{2}} dr \]
\[ \leq \left( A_2 + \frac{b}{\tau(t)} \right)^2 \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( \frac{\nu(t)}{\tau(t)} \bar{u} \right)^2 \tau \, dr \right) ds \right)^{\frac{1}{2}} \leq \]
\[ \leq \left( A_2 + \frac{b}{\tau(t)} \right)^2 \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( A_2 + \frac{b}{\tau(t)} \right) \bar{u} \right) ds \right)^{\frac{1}{2}} dr \]
\[ \leq \left( A_2 + \frac{b}{\tau(t)} \right)^2 \left( \int_{r_0}^{\infty} \left( \int_{r_0}^{\infty} \tau^n \left( \frac{\nu(t)}{\tau(t)} \bar{u} \right)^2 \tau \, dr \right) ds \right)^{\frac{1}{2}} \leq \]
because \( \bar{u} \in L^2([r_0, \infty)). \) Consequently \( \bar{u} \in H^1([r_0, \infty)) \) for every fixed \( t \in [0, 1]. \)
Now we suppose that the solution map \((u_0, u_1) \to u(t, r)\) is uniformly continuous.

Let
\[
0 < \epsilon < \left( A_1 - \frac{b}{2B} \right)^2 \frac{1}{A^2} (d_1 - c_1) \frac{c_{\infty}^n}{d_1^n},
\]
Let also
\[
u_1 = \tilde{u}, \quad \nu_2 = 0.
\]
Then there exists positive constant \(\delta\) such that
\[
E(0, \nu_1 - \nu_2) \leq \delta
\]
and
\[
E(1, \nu_1 - \nu_2) \leq \epsilon.
\]
From here
\[
\epsilon \geq E(1, \nu_1 - \nu_2) = E(1, \tilde{u}) \geq \int_{r_0}^{\infty} \left( \frac{1}{r} \int_{r}^{\infty} s^n \left( \frac{\nu'(t)}{\nu(t)} \tilde{u} - f(1, s, \tilde{u}) \right) ds \right)^2 dr \geq
\]
\[
\int_{r_0}^{\infty} \left( \frac{1}{r} \int_{r}^{\infty} s^n \left( \frac{\nu'(t)}{\nu(t)} \tilde{u} - \frac{b}{2B} \tilde{u} \right) ds \right)^2 dr \geq
\]
\[
\left( A_1 - \frac{b}{2B} \right)^2 \frac{1}{A^2} (d_1 - c_1) \frac{c_{\infty}^n}{d_1^n}
\]
which is a contradiction with (2.6).

3. Appendix

**Lemma 3.1.** The set \(N\) is a closed subset of \(C([0, 1]L^2([r_0, \infty)))\).

**Proof.** Let \(t \in [0, 1]\) is fixed.

Let also \(\{u_n\}\) is a sequence of elements of the set \(N\) for which
\[
\lim_{n \to \infty} ||u_n - \tilde{u}||_{L^2([r_0, \infty))} = 0,
\]
where \(\tilde{u} \in L^2([r_0, \infty))\). Since \(P(u)\) is a continuous- differentiable function of \(u\), for \(r \in [r_0, r_0 + 1]\) and \(u \in N\) we have
\[
P'(u) = \int_{r_0}^{\infty} \frac{1}{s} \int_{r}^{\infty} s^n \left( \frac{\nu'(t)}{\nu(t)} - f'_u \right) ds dr \geq
\]
\[
\int_{r_0}^{\infty} \frac{1}{s} \int_{r}^{\infty} s^n \left( A_1 - \frac{b}{2B} \right) ds dr \geq
\]
\[
\left( A_1 - \frac{b}{2B} \right) \frac{(n+1)^n}{(n+2)^n}
\]
From here follows that for every \(u \in N\) there exists
\[
L = \min_{r \in [r_0, r_0 + 1]} |P'(u)(r)| > 0.
\]
Let
\[
M_1 = \max_{r \in [r_0, r_0 + 1]} |\frac{\partial}{\partial r} P'(u)(r)|.
\]
Now we will prove that for every \(\epsilon > 0\) there exists \(\delta(\epsilon) > 0\) such that from
\[
|x - y| < \delta \quad \text{we have}
\]
\[
|u_m(x) - u_m(y)| < \epsilon \quad \text{for all } m \in N.
\]
We suppose that there exists $\tilde{\epsilon} > 0$ such that for every $\delta > 0$ there exist natural $m$ and $x, y \in [r_0, \infty)$, $|x - y| < \delta$ for which $|u_m(x) - u_m(y)| \geq \tilde{\epsilon}$. We choose $\tilde{\epsilon}$ such that $0 < \tilde{\epsilon} < L\tilde{\epsilon}$. We note that $P(u_m)(x)$ is uniformly continuous for $x \in [r_0, \infty)$ (for $u \in N$ $P(u)(r)$ is uniformly continuous for $r \in [r_0, \infty)$ because $P(u)(r) \in C([r_0, \infty))$ and as in the proof of the Theorem 2.1 we have that there exists positive constant $C$ such that $\left| \frac{\partial}{\partial r} P(u)(r) \right| \leq C$. Then there exists $\delta_1 = \delta_1(\tilde{\epsilon}) > 0$ such that for every natural $m$ we have

$$|P(u_m)(x) - P(u_m)(y)| < \tilde{\epsilon}, \quad \forall x, y \in [r_0, \infty) : |x - y| < \delta_1.$$  

Consequently we can choose

$$0 < \delta < \min\left\{1, \delta_1, \frac{(L\tilde{\epsilon} - \tilde{\epsilon})B}{M_1}\right\}$$

such that there exist natural $m$ and $x_1, x_2 \in [r_0, \infty)$ for which

$$|x_1 - x_2| < \delta, \quad |u_m(x_1 - x_2 + r_0) - u_m(r_0)| \geq \tilde{\epsilon}.$$  

In particular

$$|P(u_m)(x_1 - x_2 + r_0) - P(u_m)(r_0)| < \tilde{\epsilon}.$$  

Let us suppose for convenience that $x_1 - x_2 > 0$. Then $x_1 - x_2 < 1$ and for every $u \in N$ we have $P(u)(x_1 - x_2 + r_0) \geq L$. Then from the middle point theorem we have

$$P(0) = 0, \quad P(u_m)(x_1 - x_2 + r_0) = P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0),$$

$$\left|\frac{\partial P(u_m)}{\partial r}(r_0)\right| = P'(\xi)(r_0)u_m(r_0),$$

$$\left|\frac{\partial P(u_m)}{\partial r}(x_1 - x_2 + r_0) - P(u_m)(r_0)\right| =$$

$$= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(r_0)u_m(r_0)| =$$

$$= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0) +$$

$$+ P'(\xi)(x_1 - x_2 + r_0)u_m(r_0) - P'(\xi)(r_0)u_m(r_0)| \geq$$

$$\geq \left|P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0)| -$$

$$- |P'(\xi)(x_1 - x_2 + r_0)u_m(r_0) - P'(\xi)(r_0)u_m(r_0)| =$$

$$= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0)| -$$

$$- \frac{\partial}{\partial r} P'(\xi)|x_1 - x_2||u_m(r_0)| \geq$$

$$\geq L\tilde{\epsilon} - M_1\delta_1 \frac{1}{B} \geq \tilde{\epsilon},$$

which is a contradiction with (3.1). Therefore, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that from $|x - y| < \delta$ follows

$$|u_m(x) - u_m(y)| < \epsilon, \quad \forall m \in N.$$  

On the other hand from the definition of the set $N$ we have that for every natural $m$

$$u_m(r) \leq \frac{1}{B}, \quad \forall r \geq r_0.$$  

From (3.2) and (3.3) follows that the set $\{u_m\}$ is a compact subset of the space $C([r_0, \infty))$. Therefore there is a subsequence $\{u_{m_k}\}$ and function $u \in C([r_0, \infty))$ for which

$$|u_{m_k}(x) - u(x)| < \epsilon, \quad \forall x \in [r_0, \infty).$$
Now we suppose that there is no true that \( u = \tilde{u} \) a.e. in \([r_0, \infty)\). Then there exist \( \epsilon_1 > 0 \) and subinterval \( \Delta \subset [r_0, \infty) \) such that \( \mu(\Delta) > 0 \) and
\[
|u - \tilde{u}| > \epsilon_1 \quad \text{for} \quad r \in \Delta.
\]
Let \( \epsilon > 0 \) is chosen such that
\[
(3.4) \quad \epsilon < \frac{\epsilon_1(\mu(\Delta))^{\frac{1}{2}}}{\mu(\Delta)^{\frac{1}{2}} + 1}
\]
Then, for every enough large \( n_k \in N \), we have
\[
\begin{align*}
|u_{n_k} - \tilde{u}| &< \epsilon, \\
\epsilon \mu(\Delta) &< \epsilon \int_\Delta |u_{n_k} - u|\,dx = \int_\Delta |u_{n_k} - \tilde{u} + \tilde{u} - u|\,dx \\
&\geq \int_\Delta |\tilde{u} - u|\,dx - \int_\Delta |u_{n_k} - \tilde{u}|\,dx \\
&\geq \epsilon \mu(\Delta) - \left(\int_\Delta |u_{n_k} - \tilde{u}|^2\,dx\right)^{\frac{1}{2}} \left(\mu(\Delta)\right)^{\frac{1}{2}} \\
&\geq \epsilon \mu(\Delta) - \epsilon \left(\mu(\Delta)\right)^{\frac{1}{2}}
\end{align*}
\]
which is a contradiction with (3.4). From here \( u = \tilde{u} \) a.e. in \([r_0, \infty)\), \(|u_n - u|^2 = |\tilde{u} - u_n|^2\) a.e. in \([r_0, \infty)\), \(|u_n - u|_{L^2([r_0, \infty))} = |u_n - \tilde{u}|_{L^2([r_0, \infty))}\).

Consequently, for every sequence \( \{u_n\} \) from elements of the set \( N \), which is convergent in \( L^2([r_0, \infty)) \), there exists a function \( u \in \mathcal{C}([r_0, \infty)), u \in L^2([r_0, \infty)) \) for which
\[
\lim_{n \to \infty} ||u_n - u||_{L^2([r_0, \infty))} = 0.
\]

Bellow we will suppose that \( \{u_n\} \) is a sequence from elements of the set \( N \), which is convergent in \( L^2([r_0, \infty)) \). Then there exists a function \( u \in \mathcal{C}([r_0, \infty)), u \in L^2([r_0, \infty)) \) for which
\[
\lim_{n \to \infty} ||u_n - u||_{L^2([r_0, \infty))} = 0.
\]

Now we suppose that \( u(t, \infty) \neq 0 \). Then there exist enough large \( Q > 0 \), enough large natural \( m \) and \( \epsilon_2 > 0 \) for which
\[
u_m(t, r) = 0, \quad u(t, r) > \epsilon_2, \quad \forall r \geq Q.
\]
We choose
\[
(3.5) \quad 0 < \epsilon_3 < \epsilon_2.
\]
Then, for every enough large \( n \in N \) we have \(|u_n(t, r) - u(t, r)| < \epsilon_3 \) and
\[
\begin{align*}
\epsilon_3 &> \int_Q^{Q+1} |u_n(t, r) - u(t, r)|\,dr \\
&\geq \int_Q^{Q+1} (|u(t, r)| - |u_n(t, r)|)\,dr = \int_Q^{Q+1} |u(t, r)|\,dr > \epsilon_2,
\end{align*}
\]
which is a contradiction with (3.5). Therefore \( u(t, \infty) = 0 \).

Now we will prove that \( \frac{d}{dr}u(t, r) \) exists for every \( t \in [0, 1] \). Let us suppose that \( r \in [r_0, \infty) \) is fixed and there exists \( t_1 \in [0, 1] \) such that \( \frac{d}{dr}u(t_1, r) \) no exists. Then for every \( h > 0 \), which is enough small, exists \( \epsilon_4 > 0 \) such that
\[
\left|\frac{u(t_1 + h, r) - u(t_1, r)}{h}\right| > \epsilon_4,
\]
and

\[ 0 < \varepsilon_5 < \frac{h}{2} \varepsilon_4, \]

such that

\[ |u_n(t_1 + h, r) - u(t_1, r)| < \varepsilon_5. \]

From here

\[ \varepsilon_5 > |u_n(t_1 + h, r) - u(t_1, r)| = \]

\[ = |u_n(t_1 + h, r) - u(t_1, r) + u(t_1, r) - u(t_1 + h, r)| \geq \]

\[ \geq |u(t_1, r) - u(t_1 + h, r)| \frac{1}{h} |u_n(t_1 + h, r) - u(t_1, r)| \geq \varepsilon_5 h - \varepsilon_5, \]

which is a contradiction of our choice of \( \varepsilon_5 \). Therefore \( \frac{\partial}{\partial r} u(t, r) \) exists for every \( t \in [0, 1] \). As in above we can see that \( u(t, r) \in C^1([0, 1]) \) for every \( r \geq r_0 \), \( u(t, r) \in C^2([r_0, \infty)) \) for every \( t \in [0, 1] \), \( u_r(t, \infty) = 0 \) for every \( t \in [0, 1] \).

Now we suppose that there exists interval \( \Delta_2 \subset [r_0, \infty) \) such that

\[ u(t, r) \geq \frac{1}{B} + \varepsilon_7 \quad \text{for} \quad r \in \Delta_2. \]

Let \( n \in N \) is enough large and \( \varepsilon_8 > 0 \) are chosen such that

\[ |u_n(t, r) - u(t, r)| < \varepsilon_8 \quad \text{for} \quad r \in \Delta_2, 0 < \varepsilon_8 < \varepsilon_7. \]

From here, for \( r \in \Delta_2 \) we have

\[ \varepsilon_8 > |u_n(t, r) - u(t, r)| \geq |u(t, r)| - |u_n(t, r)| \geq \frac{1}{B} + \varepsilon_7 - \frac{1}{B} = \varepsilon_7, \]

which is one contradiction with (3.7). Therefore we have \( u(t, r) \leq \frac{1}{B} \) for every \( r \geq r_0 \).

Now we suppose that there exists interval \( \Delta_3 \subset [c_1, d_1] \) for which \( u(t, r) < \frac{1}{A} \) for every \( r \in \Delta_3 \). From here there exists \( \varepsilon_9 > 0 \) such that \( u(t, r) \leq \frac{1}{A} - \varepsilon_9 \) for \( r \in \Delta_3 \). Also, let

\[ 0 < \varepsilon_10 < \varepsilon_9 \]

and \( n \in N \) is enough large such that \( \varepsilon_10 > |u_n(t, r) - u(t, r)| \) for \( r \in \Delta_3 \). Then for \( r \in \Delta_3 \) we have

\[ \varepsilon_10 > |u_n(t, r) - u(t, r)| \geq |u_n(t, r)| - |u(t, r)| \geq \frac{1}{A} - \frac{1}{A} + \varepsilon_9, \]

which is one contradiction with (3.8). Consequently, for every \( r \in [c_1, d_1] \) we have \( u(t, r) \geq \frac{1}{A} \).

Now we suppose that there exist \( \alpha \in N \cup \{0\} \), interval \( \Delta_4 \subset [r_0, \infty) \) and \( \varepsilon_{11} > 0 \) such that

\[ |r^\alpha u(t, r)| > 1 + \varepsilon_{11} \quad \text{for} \quad r \in \Delta_4. \]

Let \( \varepsilon_{12} > 0 \) and \( n \in N \) are chosen such that

\[ |r^\alpha (u_n(t, r) - u(t, r))| < \varepsilon_{12} \quad \text{for} \quad r \in \Delta_4, \quad 0 < \varepsilon_{12} < \varepsilon_{11}. \]

From here

\[ \varepsilon_{12} > |r^\alpha (u_n(t, r) - u(t, r))| \geq |r^\alpha u(t, r)| - r^\alpha |u_n(t, r)| \geq \varepsilon_{11}, \]

which is a contradiction with (3.9). Therefore for every \( \alpha \in N \cup \{0\} \) and for every \( r \in [r_0, \infty) \) we have \( r^\alpha u(t, r) \leq 1 \). After we use the same arguments we can see that for every \( \alpha \in N \cup \{0\} \) and for every \( r \in [r_0, \infty) \) we have \( r^\alpha |u_r(t, r)| \leq 1 \).
Now we suppose that there exist interval \( \Delta_5 \subset [r_0, \infty) \) and \( \epsilon_{13} > 0 \) such that for \( r \in \Delta_5 \) we have
\[
 u(t, r) < -\epsilon_{13}.
\]
Let \( n \in \mathbb{N} \) is enough large and \( \epsilon_{14} > 0 \) are fixed for which
\[
(3.10) \quad |u_n(t, r) - u(t, r)| < \epsilon_{14} \quad \text{for} \quad r \in \Delta_5, \quad 0 < \epsilon_{14} < \epsilon_{13}.
\]
Then for \( r \in \Delta_5 \) we have
\[
\epsilon_{14} > u_n(t, r) - u(t, r) > \epsilon_{13}
\]
which is one contradiction with (3.10).

References


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