On the quasilinear elliptic problem with a Hardy-Sobolev critical exponent

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Communicated by Shiyi Chen, received April 9, 2010.

Abstract. In this article, we consider a quasilinear elliptic equation involving Hardy-Sobolev critical exponents and superlinear nonlinearity. The right hand side nonlinearity \( f(x, u) \) which is \( (p-1) \)-superlinear nearby 0. However, it does not satisfy the usual Ambrosetti-Rabinowitz condition (AR-condition). Instead we employ a more general condition. Using a variational approach based on the critical point theory and the Ekeland variational principle, we show the existence of two nontrivial positive solutions. Moreover, the obtained results extend some existing ones.

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1. Introduction and main results

We will consider the following problem

\[
\begin{cases}
-\Delta_p u = \mu \frac{|u|^{p^*(s) - 2}}{|x|^s} u + \lambda f(x, u), & x \in \Omega \setminus \{0\}, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]

where \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) denotes the \( p \)-laplacian differential operator, \( \Omega \) is an open bounded domain in \( \mathbb{R}^N (N \geq 3) \) with smooth boundary \( \partial \Omega \) and \( 0 \in \Omega, 0 \leq \lambda < \frac{p^*(s)}{s} \).

Mathematics Subject Classification. 35A15, 35K91.

Key words and phrases. p-Laplacian; Hardy-Sobolev critical exponent; (PS)_c-condition; Mountain pass lemma; Ekeland variational principle.

Research supported by the Specialized Fund for the Doctoral Program of Higher Education and the National Natural Science Foundation of China.

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s < p, 1 < p < N, 0 < µ < ∞, \( p^*(s) = \frac{p(N-s)}{N-p} \) is the Hardy-Sobolev critical exponent and \( p^* = p^*(0) = \frac{Np}{N-p} \) is the Sobolev critical exponent, \( \lambda > 0 \) is a real parameter. \( W_0^{1,p}(\Omega) \) is the Sobolev space with the norm
\[
\|u\| := \left( \int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}},
\]
which is equivalent to the usual norm of \( W_0^{1,p}(\Omega) \) due to the Poincaré inequality and
\[
A_s(\Omega) := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|^p}{\left( \int_{\Omega} |u|^{p^*(s)} dx \right)^{\frac{1}{p^*(s)}}} \tag{1.2}
\]
is the best Hardy-Sobolev constant.

In the case \( s = 0 \) and \( p = 2 \), this problem has been widely studied (see \cite{1, 2, 11} and the references therein). In the case \( s = 0 \), Goncalves and Alves in \cite{8} have studied Problem (1.1) in \( \mathbb{R}^N \) involving \( f(x,u) = h(x)u^r \), \( u \geq 0 \) and \( u \neq 0 \) to obtain existence of positive solutions where \( 2 \leq p < N, 0 < q < p-1 \) or \( p-1 < q < q^*-1 \) and a suitable \( h \). Ghoussoub and Yuan have studied problem (1.1)(see \cite{9}), when \( f(x,u) = |u|^{q-2}u, p \leq r \leq p^* \). For other relevant papers see \cite{6, 10, 7, 12} and the references herein.

A direct extension of these methods to the case \( p = 2 \) is faced with serious difficulties. Such as, the energy functional associated to (1.1) is defined on \( W_0^{1,p}(\Omega) \), which is not a Hilbert space for \( p \neq 2 \). Due to the lack of compactness of the embedding in \( W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \) and \( W_0^{1,p}(\Omega) \hookrightarrow L^{p^*(s)}(\Omega, |x|^{-s} dx) \), we cannot use the standard variational argument directly. The corresponding energy functional fails to satisfy the classical Palais-Smale \((PS)\) for short) condition in \( W_0^{1,p}(\Omega) \). However, a local \((PS)\) condition can be established in a suitable range. Then the existence result is obtained via constructing a minimax level within this range and the Mountain Pass Lemma due to A. Ambrosetti and P.H. Rabinowitz (see also \cite{15}).

\( F(x,t) \) is a primitive function of \( f(x,t) \) defined by \( F(x,t) := \int_0^t f(x,s) ds \) for \( x \in \Omega, t \in \mathbb{R} \). For problem (1.1) we have the following assumptions:

\( (A_1) f \in C(\overline{\Omega} \times \mathbb{R}^+), f(x,0) \equiv 0, \lim_{t \to 0^+} \frac{f(x,t)}{t^{p-1}} = +\infty \) and \( \lim_{t \to +\infty} \frac{f(x,t)}{t^{p^*(s)-1}} = 0 \)
uniformly for \( x \in \overline{\Omega} \).

\( (A_2) f : \Omega \times \mathbb{R}^+ \to \mathbb{R} \) is nondecreasing with respect to the second variable.

\( (A_3) N > p \geq \max \left\{ 2, \frac{3N}{N+3-s}, \frac{s-1+\sqrt{(1-s)^2+4N}}{2} \right\} \).

In what follows, \( \| \cdot \|_p \) denotes the norm in \( L^p(\Omega) \). Now, our main results are as follows:

**Theorem 1.** Suppose that \( N \geq 3, 0 < \mu < \infty, 0 \leq s < p, 1 < p < N \). Assume \((A_1)\) holds, then there exists \( \lambda^* > 0 \) such that problem (1.1) has at least one nontrivial positive solution \( u_\lambda \) for every \( \lambda \in (0, \lambda^*) \).

**Theorem 2.** Suppose that \( N \geq 3, 0 < \mu < \infty, \max \left\{ 0, \frac{p^2-N}{p-1} \right\} < s < p \). If \( (A_1) - (A_3) \) all hold, then there exists \( \lambda^* > 0 \) such that problem (1.1) has at least two nontrivial positive solutions for every \( \lambda \in (0, \lambda^*) \).
Remark 1. Here we give some examples of the nonlinearity satisfying \((A_1)\) and \((A_2)\).

1. \(f(x, t) = t^{q-1}, \ t \geq 0\) with \(1 < q < p\).
2. \(f(x, t) = v(x)t^r + h(x)t^\nu, \ t \geq 0\), where \(v(x), \ h(x) \in L^\infty(\Omega)\), \(v(x), \ h(x) > 0\), \(0 \leq r < p - 1\) and \(p - 1 < \nu < p^*(s) - 1\).

This paper is organized as follows. In Section 2, we manage to give the proof of Theorem 1. The proof of Theorem 2 is given in Section 3. Throughout the article the letters \(C\) or \(C_i\) \((i = 1, 2, 3, \ldots)\) will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem.

2. Proof of Theorem 1

It is obvious that the values of \(f(x, t)\) for \(t < 0\) are irrelevant in Theorem 1-2, and we may define

\[f(x, t) \equiv 0 \text{ for } x \in \Omega, \ t \leq 0.\]

Let \(u := \max\{\pm u, 0\}\). The functional corresponding to (1.1) is

\[I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\mu}{p^*(s)} \int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s} dx - \lambda \int_\Omega F(x, u^+) dx, \ u \in W^{1,p}_0(\Omega).\]

By Hardy-Sobolev inequalities (see \([4, 9]\)) and \((A_1)\), \(I \in C^1(W^{1,p}_0(\Omega), R)\). Now it is well known that there exists a one-to-one correspondence between the weak solutions of problem (1.1) and the critical points of \(I\) on \(W^{1,p}_0(\Omega)\). More precisely we say that \(u \in W^{1,p}_0(\Omega)\) is a weak solution of problem (1.1), if for any \(v \in W^{1,p}_0(\Omega)\), there holds

\[\langle I'(u), v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx - \mu \int_\Omega \frac{(u^+)^{p^*(s)-1}}{|x|^s} v dx - \lambda \int_\Omega f(x, u^+) v dx = 0.\]

Proof of Theorem 1. Let \(X := W^{1,p}_0(\Omega)\). From the Sobolev and Hardy-Sobolev inequalities, we can easily get

\[\|u\|^p \leq C\|u\|^p; \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} dx \leq C\|u\|^{p^*(s)}; \|u\|_{p^*} \leq C\|u\|^{p^*}, \forall u \in X. \quad (2.1)\]

It follows from \((A_1)\) that

\[\exists \delta > 0 \text{ such that } |F(x, t)| < \frac{t^{p^*(s)}}{p^*(s)|x|^s} \text{ for } t > \delta,\]

\[\exists M_1 > 0 \text{ such that } |F(x, t)| \leq M_1 \text{ for all } t \in [0, \delta],\]

uniformly for all \(x \in \Omega \setminus \{0\}\). Therefore, we deduce that

\[|F(x, t)| \leq M_1 + \frac{p^{p^*(s)}}{p^*(s)|x|^s} \quad (2.2)\]

for all \(t \in R\) and for \(x \in \Omega \setminus \{0\}\). By (2.1) and (2.2), we have

\[I(u) \geq \frac{1}{p} \|u\|^p - C_1\|u\|^{p^*(s)} - \lambda M_1|\Omega|\]
for all \( \lambda \in (0,1] \) and some \( C_1 = \frac{C_\mu}{p^*(s)} \), so there exist \( \rho > 0 \) and \( \lambda^* \in (0,1] \) such that
\[
I(u) > 0 \quad \text{if} \quad \|u\| = \rho, \quad \text{and} \quad I(u) \geq -C_2 \quad \text{if} \quad \|u\| \leq \rho
\]
for every \( 0 < \lambda < \lambda^* \), where \( C_2 = C_1 \rho p^*(s) + \lambda^* M_1(\Omega) \). Choose \( u_0 \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) such that \( u_0^+ \neq 0 \). Let \( M_2 := \|u_0\|^p/(\lambda^* \|u_0^+\|^p) \). From (A1), there exists \( \delta_1 \) such that
\[
|F(x,t)| \geq \frac{2M_2}{p} |t|^p, \quad 0 < t < \delta_1.
\]
Hence we have
\[
I(ru_0) = \frac{r^p}{p} \|u_0\|^p - \frac{\rho p^*(s)}{p^*(s)} \int_\Omega \frac{(\rho u_0^*)^{p^*(s)}}{|x|^s} dx - \lambda \int_\Omega F(x,ru_0^+) dx
\]
\[
\leq \frac{r^p}{p} \|u_0\|^p - 2\frac{\rho}{p} \lambda M_2 \|u_0^+\|^p
\]
\[
= -\frac{r^p}{p} \|u_0\|^p < 0
\]
for every \( 0 < \lambda < \lambda^* \) and \( 0 < r < \min\{\rho, \delta_1/\|u_0^+\|_\infty\} \). Thus there exists \( u \) small enough such that \( I(u) < 0 \). Then we deduce that
\[
\inf_{u \in \overline{B}_\rho(0)} I(u) < 0 < \inf_{u \in \partial \overline{B}_\rho(0)} I(u).
\]
By applying Ekeland’s variational principle (see [13], Theorem 4.1) in \( \overline{B}_\rho(0) \), there is a minimizing sequence \( \{u_n\} \subset \overline{B}_\rho(0) \) such that
\[
I(u_n) \leq \inf_{u \in \overline{B}_\rho(0)} I(u) + \frac{1}{n}, \quad I(\omega) \geq I(u_n) - \frac{1}{n} \|\omega - u_n\|, \quad \omega \in \overline{B}_\rho(0).
\]
Therefore, we have
\[
\|I'(u_n)\| \to 0 \quad \text{and} \quad I(u_n) \to c_\lambda \quad \text{as} \quad n \to \infty,
\]
where \( c_\lambda \) stands for the infimum of \( I(u) \) on \( \overline{B}_\rho(0) \). Since \( \{u_n\} \) is bounded and \( \overline{B}_\rho(0) \) is a closed convex set, there exist \( u_\lambda \in \overline{B}_\rho(0) \subset W^{1,p}_0(\Omega) \). Going if necessary to a subsequence, one can get that (see [9])
\[
\begin{align*}
\{u_n\} &\to u_\lambda \quad \text{weakly in} \quad W^{1,p}_0(\Omega), \\
\{u_n\} &\to u_\lambda \quad \text{strongly in} \quad L^\gamma(\Omega), \quad 1 < \gamma < p^*, \\
\nabla u_n &\to \nabla u_\lambda \quad \text{a.e. in} \quad \Omega, \\
\frac{u_n}{\rho} &\to \frac{u_\lambda}{\rho} \quad \text{weakly in} \quad L^p(\Omega), \\
\int_\Omega |u_n|^{p^*(s)-2} u_n v dx &\to \int_\Omega |u_\lambda|^{p^*(s)-2} u_\lambda v dx, \quad \forall \ v \in W^{1,p}_0(\Omega).
\end{align*}
\]
Consequently, passing to the limit in \( \langle I'(u_n), v \rangle \), as \( n \to \infty \), we have
\[
\int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \nabla v dx - \mu \int_\Omega \frac{(u_\lambda^+)^{p^*(s)-1} u_\lambda}{|x|^{s}} dx - \lambda \int_\Omega f(x,u_\lambda^+) v dx = 0
\]
for all \( v \in W^{1,p}_0(\Omega) \). That is, \( \langle I'(u_\lambda), v \rangle = 0 \). Thus \( u_\lambda \) is a critical point of the functional \( I \). Since \( \|u_\lambda\|^p = -\langle I'(u_\lambda), u_\lambda^- \rangle = 0 \), thus \( u_\lambda = u_\lambda^+ \geq 0 \). Moreover, we deduce from (A1) and the boundedness of \( \Omega \) that
\[
\exists \ M_3 > 0 \quad \text{such that} \quad |f(x,t)| < \frac{\mu}{\lambda} \frac{p^*(s)-1}{|x|^{s}} \quad \text{for} \quad t > M_3,
\]
\[
\exists \ \delta_2 \in (0, M_3) \quad \text{such that} \quad |f(x,t)| < 0 \quad \text{for} \quad 0 < t < \delta_2.
\]
Moreover, by the Maximum Principle, 
\[ v > u \] 
and solutions of problem (3.1). That is, if solution of (1.1) and 
\[ J \] 
for all \( t \in R^+ \) and for \( x \in \overline{\Omega} \setminus \{0\} \). From (1.1) and (2.3), we have
\[ -\Delta_p u_\lambda + \lambda M_\delta \delta_2^{-1} u_\lambda \geq 0. \]
From the strong maximum principle, we deduce that \( u_\lambda > 0 \). So Theorem 1 is proved. \( \Box \)

3. Proof of Theorem 2

The first positive solution \( u_\lambda \) of problem (1.1) have been obtained in previous section, we can look for the second positive solution by a translated functional as in [1]. For fixed \( \lambda \in (0, \lambda^*) \), we look for the second solution of problem (1.1) of the form \( u = u_\lambda + v \), where \( u_\lambda \) is the first positive solution obtained in previous section. The corresponding equation for \( v \) is 
\[ \begin{cases} 
-\Delta_p v = \mu \frac{(u_\lambda + v)^{\rho^*(s)-1}}{|x|^s} - \mu \frac{(u_\lambda)^{\rho^*(s)-1}}{|x|^s} + \lambda f(x, u_\lambda + v) - \lambda f(x, u_\lambda), & x \in \Omega \setminus \{0\}, \\
v = 0, & x \in \partial \Omega. 
\end{cases} \]
(3.1)

Let us define 
\[ g(x, t) = \begin{cases} 
\mu \frac{(u_\lambda + t)^{\rho^*(s)-1}}{|x|^s} - \mu \frac{(u_\lambda)^{\rho^*(s)-1}}{|x|^s} + \lambda f(x, u_\lambda + t) - \lambda f(x, u_\lambda), & t \geq 0, \\
0, & t < 0, 
\end{cases} \]
and 
\[ G(x, t) = \int_0^t g(x, s)ds \]
(3.2)

and
\[ J(v) = \frac{1}{p} \int_\Omega |\nabla v|^p dx - \int_\Omega G(x, v^+)dx \]
\[ = \frac{1}{p} \|v\|^p - \mu \frac{(u_\lambda + v^+)^{\rho^*(s)}}{|x|^s} - \frac{(u_\lambda)^{\rho^*(s)}}{|x|^s} - p^* \left( \frac{(u_\lambda v^+)^{\rho^*(s)-1}}{|x|^s} - \frac{(u_\lambda)^{\rho^*(s)-1}}{|x|^s} \right) \] 
\[ - \lambda \int_\Omega [F(x, u_\lambda + v^+) - F(x, u_\lambda) - f(x, u_\lambda) v^+] dx. \]

Now we have one-to-one correspondence between critical points of \( J \) in \( W_0^{1,p}(\Omega) \) and solutions of problem (3.1). That is, if \( v \in W_0^{1,p}(\Omega) \), \( v \neq 0 \) is a critical point of \( J \), then \( v \) is a solution of (3.1). Since \( \|v^-\|^p = -\langle J'(v), v^- \rangle = 0 \), thus \( v = v^+ \geq 0 \). Moreover, by the Maximum Principle, \( v > 0 \) in \( \Omega \). Here \( u = u_\lambda + v \) is a positive solution of (1.1) and \( u \neq u_\lambda \). We will prove the existence of a second positive solution of (1.1) by contradiction. Assume that \( v = 0 \) is the only critical point of \( J \) in \( W_0^{1,p}(\Omega) \).

**Lemma 1.** \( v = 0 \) is a local minimum of \( J \) in \( W_0^{1,p}(\Omega) \).

**Proof.** For any \( v \in W_0^{1,p}(\Omega) \), write \( v = v^+ - v^- \). From the expression of \( J \) and direct computation, we obtain that
\[ J(v) = \frac{1}{p} \|v^-\|^p + I(u_\lambda + v^+) - I(u_\lambda). \]
(3.3)
Therefore, one gets

\[ J(v) \geq \frac{1}{p} \|v\|^{p} \]

as long as \( \|v\| \leq \varepsilon \) for \( \varepsilon \) small enough. \( \Box \)

**Lemma 2.** (9) Suppose \( 1 < p < N, \ 0 \leq s < p \). Then we have the following:
(i) \( A_{s}(\Omega) \) is independent of \( \Omega \) (and will henceforth be denoted by \( A_{s} \)).
(ii) \( A_{s} \) is attained when \( \Omega = \mathbb{R}^{N} \) by the functions

\[ l_{\varepsilon}(x) = \left[ \varepsilon(N-s) \left( \frac{N-p}{p-1} \right)^{p-1} \right]^{\frac{N-s}{2N-s}} \left( \varepsilon + |x|^{\frac{-s}{N-s}} \right)^{\frac{p-s}{2N-s}} \]

for some \( \varepsilon > 0 \). Moreover the functions \( l_{\varepsilon}(x) \) are the only positive radial solutions of

\[-\Delta_{p}u = \frac{u^{p^{*}(s)-1}}{|x|^{s}} \]

in \( \mathbb{R}^{N} \), and satisfy

\[ \int_{\mathbb{R}^{N}} |\nabla l_{\varepsilon}|^{p} dx = \int_{\mathbb{R}^{N}} \frac{|l_{\varepsilon}|^{p^{*}(s)}}{|x|^{s}} dx = A_{s}^{\frac{N-s}{2N-s}}. \]

**Lemma 3.** Suppose \( 0 < \mu < \infty \), \( f \) satisfies \((A_{1})-(A_{3})\). Assume that \( v=0 \) is the only critical point of \( J \). Let \( \{v_{n}\} \) be a (PS)_c sequence with \( 0 < c < \frac{\mu-N}{p(N-s)}A_{s}^{\frac{N-s}{2N-s}} \). Then we have

\[ v_{n} \to 0 \mbox{ in } W_{0}^{1,p}(\Omega) \mbox{ as } n \to \infty. \]

**Proof.** Let \( v_{n} \) be a sequence in \( W_{0}^{1,p}(\Omega) \) such that

\[ J(v_{n}) \to c < \frac{p-s}{p(N-s)}A_{s}^{\frac{N-s}{2N-s}} \mu^{\frac{N-s}{2N-s}} \mbox{ and } J'(v_{n}) \to 0 \mbox{ in } \left( W_{0}^{1,p}(\Omega) \right)^{\ast}. \]

Then from (3.3) and (3.4), we have

\[ J(v_{n}) = \frac{1}{p} \|v_{n}\|^{p} + I(u_{\lambda} + v_{n}^{+}) - I(u_{\lambda}) = c + o(1), \]

(3.5)

\[ \langle J'(v_{n}), u_{\lambda} + v_{n}^{+} \rangle = \int_{\Omega} |\nabla v_{n}^{+}|^{p-2} \nabla v_{n}^{+} \nabla u_{\lambda} dx + \langle I'(u_{\lambda} + v_{n}^{+}), u_{\lambda} + v_{n}^{+} \rangle = o(1) \|u_{\lambda} + v_{n}^{+}\|. \]

It yields that

\[ J(v_{n}) - \frac{1}{p} \langle J'(v_{n}), u_{\lambda} + v_{n}^{+} \rangle \]

\[ = \frac{1}{p} \left( \|v_{n}^{+}\|^{p} - \int_{\Omega} |\nabla v_{n}^{+}|^{p-2} \nabla v_{n}^{+} \nabla u_{\lambda} dx - \langle I'(u_{\lambda} + v_{n}^{+}), u_{\lambda} + v_{n}^{+} \rangle \right) \]

\[ + \lambda \int_{\Omega} \left[ \frac{1}{p} f(x, u_{\lambda} + v_{n}^{+})(u_{\lambda} + v_{n}^{+}) - F(x, u_{\lambda} + v_{n}^{+}) \right] dx \]

\[ \leq I(u_{\lambda}) + c + 1 + o(1) \|u_{\lambda} + v_{n}^{+}\|. \]

Therefore, one gets

\[ \frac{1}{p} \left( \|v_{n}^{+}\|^{p} - \int_{\Omega} |\nabla v_{n}^{+}|^{p-2} \nabla v_{n}^{+} \nabla u_{\lambda} dx \right) + \mu \left( \frac{1}{p} - \frac{1}{p^{*}(s)} \right) \int_{\Omega} \frac{u_{\lambda} + v_{n}^{+}}{|x|^{s}}^{p^{*}(s)} dx \]

\[ + \lambda \int_{\Omega} \left[ \frac{1}{p} f(x, u_{\lambda} + v_{n}^{+})(u_{\lambda} + v_{n}^{+}) - F(x, u_{\lambda} + v_{n}^{+}) \right] dx \]

\[ \leq I(u_{\lambda}) + c + 1 + o(1) \|u_{\lambda} + v_{n}^{+}\|. \]
By (A1) and the boundedness of $\Omega$, for any $\varepsilon > 0$, there exists $M_5 = M_5(\varepsilon) > 0$ such that

$$
|f(x, t)| \leq \frac{\varepsilon |t|^{p^*(s)}}{|x|^s}, \quad x \in \Omega \setminus \{0\}, \quad |t| > M_5,
$$

$$
|f(x, t)| \leq C_3(\varepsilon), \quad x \in \Omega, \quad |t| \in [0, M_5],
$$

$$
|F(x, t)| \leq \frac{\varepsilon |t|^{p^*(s)}}{|x|^s}, \quad x \in \Omega \setminus \{0\}, \quad |t| > M_5,
$$

$$
|F(x, t)| \leq C_4(\varepsilon), \quad x \in \Omega, \quad |t| \in [0, M_5],
$$

where $C_3(\varepsilon), \ C_4(\varepsilon) > 0$. Therefore, we have

$$
|f(x, t)| \leq C_3(\varepsilon) + \frac{\varepsilon |t|^{p^*(s)}}{|x|^s}, \quad (x, t) \in (\Omega \setminus \{0\}) \times R,
$$

(3.7)

$$
|F(x, t)| \leq C_4(\varepsilon) + \frac{\varepsilon |t|^{p^*(s)}}{|x|^s}, \quad (x, t) \in (\Omega \setminus \{0\}) \times R.
$$

(3.8)

Let $C(\varepsilon) := \frac{1}{p} C_3(\varepsilon) + C_4(\varepsilon)$, combining (3.7) and (3.8), one gets

$$
F(x, t) = \frac{1}{p} f(x, t) \leq C(\varepsilon) + \frac{2\varepsilon |t|^{p^*(s)}}{|x|^s}, \quad (x, t) \in (\Omega \setminus \{0\}) \times R.
$$

(3.9)

From (3.6) and (3.9), we deduce that

$$
\left( \frac{\mu(p-s)}{p(N-s)} - \frac{2\lambda C(\varepsilon)}{p} \right) \int_{\Omega} \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s} dx \leq \frac{\lambda C_5(\varepsilon)}{\lambda} - \frac{1}{p} \|v_n^+\|^p + C_6 \|v_n\|^p - C_9 + o(1)\|u_\lambda + v_n^+\|
$$

where $C_5 = \frac{1}{p} \|u_\lambda\|$, $C_6 = I(u_\lambda) + c + 1$. Let $\varepsilon = \frac{\mu(p-s)}{p(N-s)}$, we have

$$
\int_{\Omega} \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s} dx \leq C_7 \|v_n^+\|^p - C_9 + o(1)\|u_\lambda + v_n^+\|
$$

where $C_7 = \frac{2\mu(p-s)}{p(N-s)} C_5$, $C_8 = \frac{2\mu(p-s)}{p(N-s)} (\lambda C(\varepsilon)\|\Omega\| + C_6)$, which together with (3.3), (3.5) and (3.8) imply that

$$
\|v_n^+\|^p + \frac{1}{p} \left[ (1 - \varepsilon) \|v_n^+\|^p - C_7 \|u_\lambda\|^p - (1 - \varepsilon) \|v_n^+\|^p - C_9 \right]
$$

$$
\leq \frac{1}{p} \|v_n\|^p + \frac{1}{p} \left[ (1 - \varepsilon) \|v_n^+\|^p - C_7 \|u_\lambda\|^p \right]
$$

$$
\leq \frac{1}{p} \|v_n\|^p + \frac{1}{p} \left[ (1 - \varepsilon) \|v_n^+\|^p - C_7 \|u_\lambda\|^p \right]
$$

$$
\leq \frac{1}{p} \|v_n\|^p + \frac{1}{p} \|u_\lambda + v_n^+\|^p
$$

$$
= \frac{\mu}{p^p(s)} \int_{\Omega} \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s} dx + \lambda \int_{\Omega} F(x, u_\lambda + v_n^+) dx + J(v_n) + I(u_\lambda) + o(1)
$$

\leq C_9 \|v_n\|^p - C_{11} \|v_n^+\|^p - C_{12} \|v_n\|^p - C_{13} \|v_n^+\|^p - C_{14} \|u_\lambda\|,

where in the second step we used the fact that, the elementary inequality $|a - b|^t \geq (1 - \varepsilon) a^t - C_7 b^t \ (t \geq 1, \ a, b > 0)$ holds. $C_9 = \left( \frac{\mu}{p^p(s)} + \frac{C_7}{p} \right) C_7$, $C_{10} = \lambda C_3(\varepsilon)\|\Omega\| + \left( \frac{\mu}{p^p(s)} + \frac{C_7}{p} \right) C_8 + I(u_\lambda) + c + o(1)$. Since $\|v_n\|^p + \|v_n^+\|^p - \|v_n\|^p - \|v_n\|^p - \|v_n^+\|^p = \|v_n\|^p - \|v_n\|^p$, then we deduce

$$
\|v_n\|^p - C_{11} \|v_n^+\|^p - C_{12} \|v_n\|^p \leq C_{13} \|v_n\|^p - C_{14} \|u_\lambda\|,
$$
where $C_{11} = 1 + o(1)\frac{1}{1-\varepsilon}$, $C_{11}' = \frac{C_{11}0}{1-\varepsilon}$, $C_{12} = \frac{\|u\|p + pC_{10}}{1-\varepsilon}$. So we get

$$\|v_n\|^p - C_{13}\|v_n\|^{p-1} \leq C_{12} + o(1)\|u\|.$$  

where $C_{13} = C_{11} + C_{11}'$. It shows that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$, going if necessary to a subsequence, one gets that

$$\begin{cases}
    v_n \rightharpoonup v_0 \text{ weakly in } W_0^{1,p}(\Omega), \\
    v_n \rightarrow v_0 \text{ strongly in } L^\gamma(\Omega), \quad 1 < \gamma < p^*, \\
    v_n \rightarrow v_0 \text{ a.e. in } \Omega,
\end{cases} \quad (3.10)$$

as $n \rightarrow \infty$.

In addition, by the Sobolev embedding theorem, there exists $M' > 0$ such that $\|u_\lambda + v_n^+\|_{p^*(s)}^s \leq M'$, denote by $\text{meas}E$ the measure of $E$. By (A1), for any $\varepsilon > 0$, there exists $C_{14}(\varepsilon) > 0$ such that

$$|f(x, t)|t \leq C_{14}(\varepsilon) + \frac{\varepsilon}{2M'}|t|^{p^*(s)}, \quad (x, t) \in \Omega \times R.$$  

Set $\delta := \frac{\varepsilon}{2C_{14}(\varepsilon)} > 0$, when $E \subset \Omega$, $\text{meas}E < \delta$, we have

$$\int_E |f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)| \ dx \leq \int_E |f(x, u_\lambda + v_0^+)(u_\lambda + v_0^+)| \ dx$$

$$\leq \int_E C_{14}(\varepsilon) \ dx + \frac{\varepsilon}{2M'} \int_E |u_\lambda + v_n^+|^{p^*(s)} \ dx$$

$$\leq C_{14}(\varepsilon) \text{meas}E + \frac{\varepsilon}{2} < \varepsilon.$$  

By Vitali’s theorem, we prove that

$$\int_{\Omega} f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+) \ dx \rightarrow \int_{\Omega} f(x, u_\lambda + v_0^+)(u_\lambda + v_0^+) \ dx \text{ as } n \rightarrow \infty.$$  

Hence one has

$$\begin{align}
\int_{\Omega} f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+) \ dx \\
\rightarrow \int_{\Omega} f(x, u_\lambda + v_0^+)(u_\lambda + v_0^+) \ dx \text{ as } n \rightarrow \infty.
\end{align} \quad (3.11)$$

Using the same method, we deduce that

$$\int_{\Omega} F(x, u_\lambda + v_n^+) \ dx \rightarrow \int_{\Omega} F(x, u_\lambda + v_0^+) \ dx, \quad (3.12)$$

$$\int_{\Omega} f(x, u_\lambda + v_n^+) \omega \ dx \rightarrow \int_{\Omega} f(x, u_\lambda + v_0^+) \omega \ dx,$$

as $n \rightarrow \infty$ for $\omega \in W_0^{1,p}(\Omega)$. Hence, similar to the proof of Theorem 1, we have

$$0 = \lim_{n \rightarrow \infty} \langle J'(v_n), \omega \rangle = \langle J'(v_0), \omega \rangle$$

for $\omega \in W_0^{1,p}(\Omega)$, which implies that $J'(v_0) = 0$. Therefore, $v_0$ is a critical point of $J$ in $W_0^{1,p}(\Omega)$. From the assumption that $v = 0$ is the only critical point of $J$, we know that $v_0 = 0$. Now we want to prove $v_0 \rightarrow 0$ strongly in $W_0^{1,p}(\Omega)$. From (3.10), (3.12) and the Brezis-Leib Lemma (see [3]), we have

$$J(v_n) \leq \frac{1}{p}\|v_n\|^p + I(u_\lambda + v_n^+) - I(u_\lambda) = \frac{1}{p}\|v_n\|^p - \|v_n\|_{p^*(s)}^{p} \int_{\Omega} \frac{|v_n^+|^{p^*(s)}}{|x|^s} \ dx + o(1).$$
Therefore, we get
\[ \langle J'(v_n), v_n \rangle = \| v_n \|^p - \mu \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} \, dx + o(1) \to 0, \]
then \( \| v_n \|^p \to 0 \) as \( n \to \infty \). Otherwise, there exists a subsequence (still denoted by \( v_n \)) such that
\[ \lim_{n \to \infty} \| v_n \|^p = k, \quad \lim_{n \to \infty} \mu \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} \, dx = k, \quad k > 0. \]
By (1.2), we deduce that
\[ \| v_n \|^p \geq A_s \left( \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} \, dx \right)^{\frac{p}{p-N}}, \quad \text{for all } n \in N. \]
Then, \( k \geq A_s \left( \frac{c}{\mu} \right)^{\frac{p}{p-N}}, \) that is, \( k \geq A_s^{\frac{p}{p-N}} \mu^{\frac{N-p}{p-N}} \). Thus we get that
\[ c = o(1) + J(v_n) = \frac{1}{p} \| v_n \|^p - \mu \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} \, dx + o(1) = \frac{\mu}{p(N-s)} k + o(1) \geq \frac{\mu}{p(N-s)} A_s^{\frac{N-p}{p-N}} \mu^{\frac{N-p}{p-N}}. \]
This is a contradiction. So \( v_n \to 0 \) strongly in \( W^{1,p}_0(\Omega) \) as \( n \to \infty \). \( \square \)

Since \( u_\lambda > 0 \) is a solution of problem (1.1), in a way similar to the proof of Theorem 1.1 in [5], we obtain positive constants \( R \) and \( r_0 \) such that \( B_{2R}(0) \subset \Omega \) and
\[ 0 < r_0 \leq u_\lambda(x), \quad \forall \ x \in B_{2R}(0) \setminus \{0\}. \quad (3.13) \]
In the following, we shall give some estimates for the extremal functions. Let
\[ C_\varepsilon := \left[ \varepsilon (N-s) \left( \frac{N-p}{p-1} \right)^{p-1} \right]^{\frac{N-p}{p-N-s}}, \quad U_\varepsilon(x) := \frac{I_\varepsilon(x)}{C_\varepsilon}. \]
Define a function \( \varphi \in C_0^\infty(\Omega), \ 0 \leq \varphi(x) \leq 1 \) such that
\[ \varphi(x) = \begin{cases} 1, & \text{if } |x| \leq R, \\ 0, & \text{if } |x| \geq 2R, \end{cases} \]
where \( B_{2R}(0) \subset \Omega \). Set
\[ u_\varepsilon(x) := \varphi(x) U_\varepsilon(x), \quad v_\varepsilon(x) := \frac{u_\varepsilon(x)}{\left( \int_{\Omega} |u_\varepsilon(x)|^{p(x)} \, dx \right)^{\frac{1}{p(x)}}}, \]
so that \( \int_{\Omega} |u_\varepsilon(x)|^{p(x)} \, dx = 1 \). Then, by using the argument as [9], we can get the following results:
\[ A_s + C_{15} \varepsilon^{\frac{N-p}{p-N-s}} \leq \| v_\varepsilon \|^p \leq A_s + C_{16} \varepsilon^{\frac{N-p}{p-N-s}} \quad (3.14) \]
and
\[ C_{17} \left( \frac{s-1}{p} \right) \leq \int_{\Omega} |v_\varepsilon|^p \, dx \leq C_{18} \left( \frac{(p-1)^2}{p} \right), \quad p > \frac{N-s}{N-p}(p-1). \quad (3.15) \]
Lemma 4. Suppose that $N \geq 3$, $0 < \mu < \infty$ and $\max \left\{ 0, \frac{2-N}{p-1} \right\} < s < p$. Assume $(A_1)-(A_3)$ and $f(x, 0) \equiv 0$ hold. Then there exists $v_* \in W_0^1(\Omega)$, $v_* \neq 0$, such that

$$\sup_{t \geq 0} J(tv_*) < \frac{p-s}{p(N-s)} A_{\frac{N-s}{p-1}}^{\frac{N-s}{p-1}} \mu^{\frac{N-N}{p-1}}.$$ 

Proof. By (3.2), (3.13)-(3.15), we deduce that it is clear that the equation

$$(a+b)^\gamma \geq a^\gamma + b^\gamma + Ca^{\gamma-t}b^t, \quad \gamma \geq 2, \quad 1 \leq t \leq \gamma - 1, \quad a, b > 0,$$

where $C$ is a positive constant, we have

$$g(x,t) \geq \mu \frac{|p^*(s)-1|}{|x|^s} + C \mu \frac{|p^*-1|}{|x|^s},$$

where $(A_3)$ implies $p^*(s) - 1 \geq 2, \quad 1 \leq p-1 \leq (p^*(s) - 1) - 1$. Therefore, we have

$$G(x,tv_*) \geq \frac{\mu |p^*(s)-1|}{p^*(s) |x|^s} + C \mu_p |v^*_p| |p^*(s)-1|.$$ 

Note that $s > \frac{2-N}{p-1}$ implies $p > \frac{N-s}{N-p}(p-1)$, then (3.15) holds. Therefore, from (3.13)-(3.15), we deduce that

$$J(tv_*) = \frac{\mu}{p} \|v_*\|^p - \int_\Omega G(x,tv_*)dx$$

$$\leq \frac{\mu}{p} \|v_*\|^p - \mu \frac{|p^*(s)-1|}{|x|^s} - C_{19} t^p \int_\Omega |v|^p \frac{|x|}{|x|^s} dx$$

$$\leq \frac{\mu}{p} \|v_*\|^p - \mu \frac{|p^*(s)-1|}{p^*(s)} - C_{20} t^p \frac{(p-1)(p-1)}{p}$$

$$\leq \frac{A}{p} t^p + C_{21} t^p \frac{N-p}{N-p} - \mu \frac{|p^*(s)-1|}{p^*(s)} - C_{20} t^p \frac{(p-1)(p-1)}{p}.$$ 

where $C_{19} = \frac{C_{19} |p^*(s)-1|}{p}$, $C_{20} = C_{17} C_{19}$ and $C_{21} = \frac{C_{21a}}{p}$. Let

$$Q(t) := \frac{A}{p} t^p + C_{21} t^p \frac{N-p}{N-p} - \mu \frac{|p^*(s)-1|}{p^*(s)} - C_{20} t^p \frac{(p-1)(p-1)}{p}.$$ 

It is clear that the equation

$$0 = Q'(t) = A t^{p-1} + p C_{21} t^{p-1} \frac{N-p}{N-p} - \mu |p^*(s)-1| - p C_{20} t^{p-1} \frac{(p-1)(p-1)}{p}$$

has only one positive root

$$t_\varepsilon = \left( A + p C_{21} \frac{N-p}{p} - p C_{20} \frac{(p-1)(p-1)}{p} \right) \frac{1}{p^*(s)-1}.$$
We have
\[
Q(t_\varepsilon) = \frac{1}{p} \left( A_\varepsilon + pC_{21}v^{\frac{N-s}{p}} - pC_{20}v^{\frac{(p-1)(p-s)}{p}} \right) t_\varepsilon^p - \mu \left( \frac{1}{p} - \frac{1}{p^{*}(s)} \right) t_\varepsilon^{p^{*}(s)} - \mu A_1^{\frac{N-s}{p}} t_\varepsilon^{\frac{N-s}{p}} - \mu A_1^{\frac{N-s}{p}} t_\varepsilon^{\frac{N-s}{p}}.
\]

for \( \varepsilon > 0 \) sufficiently small due to the fact that
\[
\frac{N - p}{p - s} \geq (p - s)(p - 1), \text{ for } s > \frac{p^2 - N}{p - 1}.
\]

Noting that \( Q(0) = 0 \) and \( \lim_{t \to +\infty} Q(t) = -\infty \), we have
\[
\sup_{t \geq 0} Q(t) = Q(t_\varepsilon) < \frac{p - s}{p(N - s)} \mu \frac{N - s}{p} A_1^{\frac{N-s}{p}},
\]

for \( \varepsilon > 0 \) sufficiently small. Hence we obtain
\[
\sup_{t \geq 0} J(tv_\varepsilon) \leq \sup_{t \geq 0} Q(t) < \frac{p - s}{p(N - s)} \mu \frac{N - s}{p} A_1^{\frac{N-s}{p}},
\]

for \( \varepsilon > 0 \) sufficiently small, which complete the proof by letting \( v_* = v_\varepsilon \) for \( \varepsilon > 0 \) sufficiently small.

**Proof of Theorem 2.** By contradiction. Assume that \( v = 0 \) is the only critical point of \( J \) in \( W_0^{1,p}(\Omega) \). From Lemma 1, there exists \( \alpha > 0 \) such that \( J(v) > \alpha \) for all \( v \in \partial B_\rho = \{ v \in W_0^{1,p}(\Omega), \| v \| = \rho \}, \) where \( \rho > 0 \) small enough. By Lemma 4 there exists \( v_* \in W_0^{1,p}(\Omega) \), \( v_* \neq 0 \), such that
\[
\sup_{t \geq 0} J(tv_*) < \frac{p - s}{p(N - s)} A_1^{\frac{N-s}{p}} \mu \frac{N-s}{p}.
\]

From (3.8), we easily note that \( \lim_{t \to \infty} J(tv_*) \to -\infty \). Hence we can choose \( t_0 > 0 \) such that \( \|tv_*\| > \rho \) and \( J(t_0v_*) < 0 \). Applying the Mountain Pass Lemma (see [15] or [14]), there is a sequence \( \{v_n\} \subset W_0^{1,p}(\Omega) \) satisfying
\[
J(v_n) \to c \geq \alpha \text{ and } J'(v_n) \to 0,
\]

where
\[
c = \inf_{h \in \Gamma} \max_{t \in [0,1]} J(h(t))
\]
and
\[
\Gamma = \{ h \in C([0,1], X) \mid h(0) = 0, h(1) = t_0v_* \}.
\]

Note that
\[
0 < \alpha \leq c = \inf_{h \in \Gamma} \max_{t \in [0,1]} J(h(t)) \leq \max_{t \in [0,1]} J(tt_0v_*) \leq \sup_{t \geq 0} J(tv_*) < \frac{p - s}{p(N - s)} A_1^{\frac{N-s}{p}} \mu \frac{N-s}{p}.
\]
Together with Lemma 3, we know that $v_n \to 0$ strongly in $W^{1,p}_0(\Omega)$ as $n \to \infty$. Hence one has $0 = J(0) = \lim_{n \to \infty} J(v_n) = c \geq \alpha > 0$, this is a contradiction. So Theorem 2 holds.

□

Appendix A

Here we give the proof of the elementary inequality in Lemma 4, that is,

$$(a + b)^\gamma \geq a^\gamma + b^\gamma + Ca^\gamma - t^\gamma, \gamma \geq 2, 1 \leq t \leq \gamma - 1, a, b > 0,$$

where $C$ is a positive constant.

Proof. Indeed, by scaling it suffices to show that

$$(1 + x)^\gamma \geq 1 + x^\gamma + Cx^t, 0 < x < \infty.$$

Let $\gamma = k + \theta$, $t = m + \eta$, where $k \geq 2, 1 \leq m \leq k - 1$ are integral numbers and $0 \leq \eta \leq \theta < 1$ are real numbers. It is obvious that

$$(1 + x)^\gamma = (1 + x)^{k + \theta} = (1 + x)^k (1 + x)^\theta \geq (1 + x^k + Cx^m)(1 + x)^\theta \geq 1 + x^{k + \theta} + Cx^m (1 + x)^\theta \geq 1 + x^{k + \theta} + Cx^m x^\eta = 1 + x^\gamma + Cx^t.$$

Therefore, this inequality holds. □

References


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