TRUNCATIONS OF THE RING OF NUMBER-THEORETIC FUNCTIONS

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Abstract

We study the ring $\Gamma$ of all functions $\mathbb{N}^+ \to K$, endowed with the usual convolution product. $\Gamma$, which we call the ring of number-theoretic functions, is an inverse limit of the “truncations”

$$\Gamma_n = \{ f \in \Gamma | \forall m > n : f(m) = 0 \}.$$

Each $\Gamma_n$ is a zero-dimensional, finitely generated $K$-algebra, which may be expressed as the quotient of a finitely generated polynomial ring with a stable (after reversing the order of the variables) monomial ideal. Using the description of the free minimal resolution of stable ideals given by Eliahou-Kervaire, and some additional arguments by Aramova-Herzog and Peeva, we give the Poincaré-Betti series for $\Gamma_n$.

1. Introduction

Cashwell and Everett [2] studied “the ring of number-theoretic functions”

$$\Gamma = \{ f|\mathbb{N}^+ \to K \}$$ (1)

where $\mathbb{N}^+$ is the set of positive natural numbers (we denote by $\mathbb{N}$ the set of all natural numbers) and $K$ is a field containing the rational numbers. $\Gamma$ is endowed with component-wise addition and multiplication with scalars, and with the convolution (or Cauchy) product

$$fg(n) = \sum_{(a,b) \in (\mathbb{N}^+)^2} f(a)g(b)$$ (2)

With these operations, $\Gamma$ becomes a commutative $K$-algebra. It is immediate that it is a local domain; less obvious is the fact that it is a unique factorisation domain. Cashwell and Everett proved this in [2] using the isomorphism

$$\Phi : \Gamma \to K[[X]]$$

$$f \mapsto \sum f(n)x_1^{a_1}x_2^{a_2}\cdots$$ (3)

where $X = \{ x_1, x_2, x_3, \ldots \}$, $K[[X]]$ is the “large” power series ring of all functions from the free abelian monoid $M = [X]$ (the free abelian monoid generated by $X$) to $K$, and where the summation extends over all $n = p_1^{a_1}p_2^{a_2}\cdots \in \mathbb{N}^+$. Here, and henceforth, we denote by $p_i$ the $i$'th prime number, with $p_1 = 2$, and by $\mathcal{P}$ the set of all prime numbers. That (3) is an isomorphism is immediate from the following isomorphism of commutative monoids, implied by the fundamental theorem of arithmetics:

$$\mathbb{N}^+ \cong \coprod_{p \in \mathcal{P}} (\mathbb{N}, +)$$ (4)

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The following number-theoretic functions are of particular interest (whenever possible, we use the same notation as in [2]):

1. The multiplicative unit \( e \) given by \( e(1) = 1, e(n) = 0 \) for \( n > 1 \),

2. \( \lambda : \mathbb{N}^+ \to \mathbb{N} \) given by \( \lambda(1) = 0, \lambda(q_1 \cdots q_l) = l \) if \( q_1, \ldots, q_l \) are any (not necessarily distinct) prime numbers.

3. \( \lambda : \mathbb{N}^+ \to \mathbb{N} \) given \( \lambda(1) = 0, \lambda(p_1^{a_1} \cdots p_r^{a_r}) = \sum a_r p_r \).

4. The Möbius function \( \mu(1) = 1, \mu(n) = (-1)^v \) if \( n \) is the product of \( v \) distinct prime factors, and 0 otherwise,

5. For any \( i \in \mathbb{N}^+, \chi_i(p_i) = 1, \) and \( \chi_i(m) = 0 \) for \( m \neq p_i \). Note that under the isomorphism (3), \( \Phi(\chi_i) = x_i \).

The topic of this article is the study of the “truncations” \( \Gamma_n \), where for each \( n \in \mathbb{N}^+ \),

\[
\Gamma_n = \{ f \in \Gamma | m > n \implies f(m) = 0 \}
\]

With the modified multiplication given by

\[
fg(n) = \sum_{(a,b) \in \{1, \ldots, n\} \times \{1, \ldots, n\}} f(a)g(b)
\]

\( \Gamma_n \) becomes a \( K \)-algebra, isomorphic to \( \Gamma/J_n \), where \( J_n \) is the ideal

\[
J_n = \{ f \in \Gamma | \forall m \leq n : f(m) = 0 \}.
\]

If we define

\[
\pi_n : \Gamma \to \Gamma_n
\]

\[
\pi_n(f)(m) = \begin{cases} f(m) & m \leq n \\ 0 & m > n \end{cases}
\]

then \( \pi_n \) is a \( K \)-algebra epimorphism, and \( J_n \) is the kernel of \( \pi_n \). We note furthermore that \( J_n \) is generated by monomials in the elements \( \chi_i \).

To describe the main idea of this paper, we need a few additional definitions. First, for any \( n \in \mathbb{N}^+ \) we denote by \( r(n) \in \mathbb{N} \) the largest integer such that \( p_{r(n)} \leq n \). In other words, \( r(n) \) is the number of prime numbers \( \leq n \) (this number is often denoted \( \pi(n) \)). Secondly, for a monomial \( m = x_1^{\alpha_1} \cdots x_w^{\alpha_w} \), we define the support \( \text{Supp}(m) \) as the set of positive integers \( i \) such that \( \alpha_i > 0 \). We define \( \max(m) \) and \( \min(m) \) as the maximal and minimal elements in the support of \( m \).

**Definition 1.1.** A monomial ideal \( I \subset K[x_1, \ldots, x_r] \) is said to be strongly stable if whenever \( m \) is a monomial such that \( x_j m \in I \), then \( x_j m \in I \) for all \( i \leq j = \max(m) \) then \( I \) is said to be stable.

We can now state our main theorem:

**Theorem 1.2.** Let \( n \in \mathbb{N}^+ \) and \( r = r(n) \). Then the following holds:

1. \( \Gamma_n \cong \frac{K[x_1, \ldots, x_r]}{I_n} \), where \( I_n \) is a strongly stable monomial ideal, with respect to the reverse order of the variables.

2. \( \Gamma_n \) is artinian, with \( \dim_K(\Gamma_n) = n \). Furthermore, if it is given the natural grading with \( |\chi_i| = 1 \), then its Hilbert series is \( \sum_i d_i t^i \) where \( d_i \) is the number of \( w \leq n \) with \( \lambda(w) = i \).

3. There is a 1-1 bijection between the minimal monomial generators of \( I_n \) of minimal support \( v \), and the solutions in non-negative integers to the equation

\[
\log n - \log p_v < \sum_{i=v} b_i \log p_i \leq \log n
\]
Lemma 2.1.

If we denote by $C_{n,v}$ the number of such solutions, then the Poincaré-Betti series of the free minimal resolution of $K$ as a cyclic module over $\Gamma_n$ is the following rational function:

$$P(\text{Tor}^{\Gamma_n}_*(K, K), t) = \frac{(1 + t)^r}{1 - t^2 (\sum_{i=1}^r (1 + t)^{(i-1)C_{n,r-i+1}})}$$ (10)

We will show this result, and also give the graded Poincaré-Betti series. For this, we define the number $C_{n,v,d}$ which counts the number of minimal generators of $I_n$ of minimal support $v$ and total degree $d$. We determine some elementary properties of the numbers $C_{n,v,d}$ and $C_{n,v}$.

2. The ring of number-theoretic functions and its truncations

2.1. Norms, degrees, and multiplicativity

For a monomial $M \ni m = x_1^{a_1} \ldots x_n^{a_n}$ we define the weight of $m$ as $w(m) = p_1^{a_1} \ldots p_n^{a_n}$ (we put $w(1) = 1$). Hence $w$ gives a bijection between $M$ and $\mathbb{N}^+$. Furthermore, we can define a term order on $M$ by $m > m'$ iff $w(m) > w(m')$. If we define the initial monomial $\text{in}(f)$ of $f \in K[[X]]$ as the monomial in $\text{Supp}(f)$ minimal with respect to $>$, then $\text{in}(f)$ is easily seen to correspond to the norm $N(\alpha)$ of a number-theoretic function $\alpha$, defined as the smallest $n$ such that $\alpha(n) \neq 0$. Here, we must use $w$ and $\Phi$ to identify $M$ and $\mathbb{N}^+$ and $K[[X]]$ and $\Gamma$. As observed in [2], the norm is multiplicative: $N(\alpha \beta) = N(\alpha)N(\beta)$.

Cashwell and Everett also define the degree $D(\alpha)$ to mean the smallest $d$ such that there exists an $n$ with $\lambda(n) = d$ and $\alpha(n) \neq 0$. This corresponds to the smallest total degree of a monomial in $\text{Supp}(f)$. Furthermore, the norm $M(\alpha)$, defined as the smallest integer $n$ with $\lambda(n) = D(\alpha)$, $\alpha(n) \neq 0$, corresponds to the initial monomial of $f$ under the term order obtained by refining the total degree partial order with the term order $>$.

A multiplicative function is an element $\alpha \in \Gamma$ such that $\alpha(1) = 1$ and $\alpha(ab) = \alpha(a)\alpha(b)$ whenever $a$ and $b$ are relatively prime. Cashwell and Everett observe that a multiplicative function is necessarily a unit in $\Gamma$. One can further observe that if $\alpha$ is multiplicative, then $f = \Phi(\alpha)$ can be written

$$f(x_1, x_2, x_3, \ldots) = f_1(x_1)f_2(x_2)f_3(x_3)\cdots$$

where each $f_i(x_i) \in K[[x_i]]$ is invertible. In particular, the constant function $\Gamma \ni \nu_0$ with $\nu_0(n) = 1$ for all $n$, corresponds to

$$\sum_{m \in M} m = \frac{1}{1 - x_1} \frac{1}{1 - x_2} \frac{1}{1 - x_3} \cdots$$

Since the Möbius function is defined to be the inverse of this function, we get that it corresponds to

$$(1 - x_1)(1 - x_2)(1 - x_3)\cdots = 1 - (\sum_{i=1}^\infty x_i) + (\sum_{i<j} x_ix_j) - (\sum_{i<j<k} x_ix_jx_k) + \cdots$$

2.2. Truncations of the ring of number-theoretic functions

Let $n, n' \in \mathbb{N}^+$, $n' > n$. Then there is a $K$-algebra epimorphism

$$\varphi''_{n'} : \Gamma_{n'} \to \Gamma_n$$

$$\varphi''_{n'}(f)(m) = \begin{cases} f(m) & m \leq n \\ 0 & m > n \end{cases}$$

Hence, the $\Gamma_n$’s form an inverse system.

Lemma 2.1. $\lim \Gamma_n \simeq \Gamma$. 

Proof. Given any \( f \in \Gamma \), the sequence \((\pi_1(f), \pi_2(f), \pi_3(f), \ldots)\) is coherent. Conversely, given any coherent sequence \((g_1, g_2, g_3, \ldots)\), we can define \( g : \mathbb{N} \to K \) by \( g(m) = g_i(m) \) where \( i \geq m \). □

As a side remark, we note that

Lemma 2.2. The decreasing filtration

\[ J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots \tag{11} \]

is separated, that is, \( \cap_n J_n = (0) \).

Definition 2.3. We define

\[ I_n = K[[X]] \{ m \in \mathcal{M} \mid w(m) > n \}, \tag{12} \]

that is, as the monomial ideal in \( K[[X]] \) generated by all monomials of weight strictly higher than \( n \). We put \( A_n = \frac{K[[X]]}{I_n} \).

Proposition 2.4. A \( K \)-basis of \( A_n \) is given by all monomials of weight \( \leq n \). Hence \( A_n \) is an artinian algebra, with \( \dim_K(A_n) = n \). Putting \( r = r(n) \), we have that

\[ A_n = \frac{K[[X]]}{I_n} \simeq \frac{K[x_1, \ldots, x_r]}{I_n \cap K[x_1, \ldots, x_r]} \tag{13} \]

Proof. As a vector space, \( K[[X]] \simeq U \oplus I_n \), where \( U \) consists of all functions supported on monomials of weight \( \leq n \). It follows that \( A_n \simeq U \) as \( K \)-vector spaces. Of course, there are exactly \( n \) monomials of weight \( \leq n \). Finally, if \( s > r \) then \( w(x_s) = p_s > n \), hence \( x_s \in I_n \).

We will abuse notations and identify \( I_n \) and its contraction \( I_n \cap K[x_1, \ldots, x_r] \).

Lemma 2.5. \( \Gamma_n \simeq A_n \).

Proof. Since \( A_n \) has a \( K \)-basis is given by all monomials of weight \( \leq n \), the two \( K \)-algebras are isomorphic as \( K \)-vector spaces. The multiplication in \( A_n \) is induced from the multiplication in \( K[[X]] \), with the extra condition that monomials of weight \( > n \) are truncated. This is the same multiplication as in \( \Gamma_n \). □

Proposition 2.6. \( I_n \) is a strongly stable ideal, with respect to the reverse order of the variables.

Proof. We must show that if \( m \in I_n \), and \( x_i \mid m \), then \( mx_j / x_i \in I \) for \( i \leq j \leq r \). We have that \( w(mx_j / x_i) = w(m) p_j / p_i > w(m) > n \).

Part I of the main theorem is now proved.

We give \( K[x_1, \ldots, x_r] \) an \( \mathbb{N}^2 \)-grading by giving the variable \( x_i \) bi-degree \((1, p_i)\). Since each \( I_n \) is bihomogeneous, this grading is inherited by \( A_n \).

Theorem 2.7. The bi-graded Hilbert series of \( A_n \) is given by

\[ A_n(t,u) = \sum_{i,j} c_{i,j} t^i u^j, \]

where \( c_{i,j} \) is the number of \( p_{i,1} \cdots p_{i,r} \leq n \) with \( \sum a_r = i \) and \( \sum a_r p_r = j \). Furthermore,

\[ A_n(t,1) = \sum_i d_i t^i \]
\[ A_n(1,u) = \sum_j e_j u^j \]

where \( d_i \) is the number of \( w \leq n \) with \( \lambda(w) = i \), and \( e_i \) is the number of \( w \leq n \) with \( \tilde{\lambda}(w) = i \). In particular, the \( t^i \)-coefficient of \( A_n(t,1) \) is the number of prime numbers \( \leq n \).
Proof. The monomial $x_1^{a_1} \cdots x_n^{a_n}$ has bi-degree $(\sum_{i=1}^n a_i, \sum a_ip_i)$.

This establishes part II of the main theorem.

3. Minimal generators for $I_n$

Let $n \in \mathbb{N}^+$, and let $r = r(n)$. We have that

$$x_1^{a_1} \cdots x_r^{a_r} = m \in I_n \iff w(m) > n \iff \prod_{i=1}^r p_i^{a_i} > n. \quad (14)$$

We denote by $G(I_n)$ the set of minimal monomial generators of $I_n$. For $m = x_1^{a_1} \cdots x_r^{a_r}$ to be an element of $G(I_n)$ it is necessary and sufficient that $m \in I_n$ and that for $1 \leq v \leq r$,

$$x_v | m \implies m / x_v \notin I_n.$$

In other words,

$$1 \leq j \leq n, a_j > 0 \implies n < \prod_{i=1}^r p_i^{a_i} \leq p_j n. \quad (15)$$

Definition 3.1. For $n, v, d$ positive integers, we define:

$$C_n = \# G(I_n) \quad (16)$$

$$C_{n,v} = \# \{ m \in G(I_n) \mid \min(m) = v \} \quad (17)$$

$$C_{n,v,d} = \# \{ m \in G(I_n) \mid \min(m) = v, |m| = d \} \quad (18)$$

Theorem 3.2. $C_{n,v}$ is the number of solutions $(b_1, \ldots, b_r) \in \mathbb{N}^r$ to the equation

$$\log n - \log p_v < \sum_{i=v}^r b_i \log p_i \leq \log n. \quad (19)$$

Equivalently, $C_{n,v}$ is the number of integers $x$ such that $n/p_v < x \leq n$ and such that no prime factors of $x$ are smaller than $p_v$.

Similarly, $C_{n,v,d}$ is the number of solutions $(b_1, \ldots, b_r) \in \mathbb{N}^r$ to the system of equations

$$\log n - \log p_v < \sum_{i=v}^r b_i \log p_i \leq \log n$$

$$\sum_{i=1}^r b_i = d - 1. \quad (20)$$

or equivalently, $C_{n,v,d}$ is the number of integers $x$ such that $n/p_v < x \leq n$ and such that no prime factors of $x$ are smaller than $p_v$, and with the additional constraint that $\lambda(x) = d$.

Proof. We have that $a_v > 0, a_w = 0$ for $w < v$. Hence equation (15) implies that

$$n < \prod_{j=v}^r p_j^{a_j} \leq p_v n.$$

Putting $b_v = a_v - 1, b_j = a_j$ for $j > v$ we can write this as

$$n < p_v \prod_{j=v}^r p_j^{b_i} \leq p_v n \iff n/p_v < \prod_{j=v}^r p_j^{b_i} \leq n$$

from which (19) follows by taking logarithms. This implies (20) as well.

We have now proved part III of the main theorem.
Figure 1: The numbers $C_n$ and $C_{n,i}$.

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Figure 2: The numbers $C_{n,i,g}$.

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Example 3.3. The first few \( I_n \)'s are as follows: \( I_2 = (x_1^2) \), \( I_3 = (x_1^2, x_2^2, x_1x_2) \), \( I_4 = (x_1^3, x_2^2, x_1x_2) \), \( I_5 = (x_1^3, x_2^2, x_1x_2, x_3^2, x_1x_3, x_2x_3) \).

We tabulate \( C_{n,i} \) and \( C_{n,i,d} \), the latter in form of the polynomial \( u^{-2} \sum_j C_{n,i,j}u^j \) in the tables 1 and 2.

Theorem 3.4. (1) \( C_{n,v} = 0 \) for \( v > r(n) \)
(2) \( \forall n \in \mathbb{N} \) : \( \forall v \leq r(n) : C_{n,1+r(n)−v} \geq v \),
(3) \( \forall n \in \mathbb{N} : C_n \geq \left(\frac{r(n)+1}{2}\right) \),
(4) \( \forall v \in \mathbb{N} : \exists N : \forall n \geq N : C_{n,1+r(n)−v} = v \).
(5) If \( n \) is even, then \( C_{n,v} = C_{n-1,v} \) for all \( v \),
(6) \( C_{n,1} = \lfloor n/2 \rfloor \).

Proof. (1) Obvious.
(2) and (3) It suffices to show that for any subset \( S \subset \{1, \ldots, r\} \) of cardinality 1 or 2, there is an \( m \in G(I_n) \) with \( \text{Supp}(m) = S \). If \( S = \{i\} \) then there is an unique positive integer \( a \) such that \( p_i^{b_1} \leq n < p_i^{b_1} \), and \( m = x_i^a \) is the desired generator. If \( S = \{i, j\} \) with \( i < j \) then we claim that there is a positive integer \( a \) such that \( x_i^ax_j \in G(I_n) \). Namely, choose \( b \) such that \( p_i^{b_1} \leq n < p_i^{b_1} \), then since \( p_i < p_j \), one has \( n < p_i^{b_1}p_j \). Hence \( x_i^{b_1}x_j \in I_n \), so it is a multiple of some minimal generator. By the definition of \( b \), this minimal generator must be of the form \( x_i^ax_j \) for some \( a \), which establishes the claim.

(6) We must show that the number of solutions in \( \mathbb{N}^r \) to

\[
\frac{n}{2} < \prod_{i=1}^{r} p_i^{b_i} \leq n
\]

is precisely \( \lceil \frac{n}{2} \rceil \). Obviously, any integer \( \in (\frac{n}{2}, n] \) fits the bill; there are \( \lceil \frac{n}{2} \rceil \) of those.

(5) The case \( v = 1 \) follows from (6). Hence, it suffices to show that if \( v > 1 \), \( x \in (\frac{n}{p_v}, n] \cap \mathbb{N} \), and if \( x \) has no prime factor \( < p_v \), then \( x \in (\frac{n}{p_v}, n-1] \cap \mathbb{N} \). The only way this can fail to happen is if \( x = n \), but then \( x \) is even, and has the prime factor \( 2 = p_1 < p_v \), a contradiction.

(4) For large enough \( n \), the only integers \( x < n \) with all prime factors \( \geq 1 + r(n) - v \) are \( p_1, p_2, \ldots, p_{r(n)} \). There is \( v \) of these, and they are all \( > \frac{n}{p_v} \).

\[
\text{Theorem 3.5.} \quad 1. \quad C_{n,v,d} = 0 \quad \text{for} \quad v > r(n), \quad \text{and} \quad d < 2,
2. \quad \forall v \in \mathbb{N} : \exists N : \forall n \geq N : C_{n,1+r(n)−v,2} = v, \quad C_{n,1+r(n)−v,d} = 0 \quad \text{for} \quad d \neq 2,
3. \quad \binom{r(n)}{v} = \# \left\{ m \in \mathbb{N}^+ \mid n \leq m, \lambda(m) = 2 \right\}.
\]

Proof. The first and the last assertions are obvious. The second one follows from the proof of (4) in the previous lemma.

4. Poincaré series

In [3], a minimal free multi-graded resolution of a \( I \) over \( S \) is given, where \( S = K[x_1, \ldots, x_r] \) is a polynomial ring, and \( I \subset (x_1, \ldots, x_r)^2 \) is a stable ideal. As a consequence, the following formula for the Poincaré-Betti series is derived:

\[
P(\text{Tor}^S_r(I, K), t) = \sum_{a \in G(I)} (1 + t)^{\max(a)-1}
\]

(21)

where \( G(I) \) is the minimal generating set of \( I \). Since the resolution is multi-graded, (21) can be modified to yield a formula for the graded Poincaré-Betti series (we here consider \( S \) as \( \mathbb{N} \)-graded, with each variable given weight 1):

\[
P(\text{Tor}^S_r(I, K), t, u) = \sum_{a \in G(I)} u^{\left| a \right|} (1 + t)^{\max(a)-1}
\]

(22)
We will use the following variant of this result:

**Theorem 4.1 (Eliahou-Kervaire).** Let $I \subset (x_1, \ldots, x_r)^2 \subset K[x_1, \ldots, x_r] = S$ be a stable monomial ideal. Put

\[
b_{i,d} = \# \{ m \in G(I) | \max(m) = i, |m| = d \}
\]

(23)

\[
b_i = \# \{ m \in G(I) | \max(m) = i \}
\]

(24)

Then

\[
P(\text{Tor}_*^S(I, K), t) = \sum_{i=1}^r b_i (1 + t)^{i-1}
\]

(25)

\[
P(\text{Tor}_*^S(I, K), t, u) = \sum_{i=1}^r \left( (1 + tu)^{(i-1)} \sum_j b_iu^j \right).
\]

(26)

For the Betti-numbers we have that

\[
\beta_q = \dim_K (\text{Tor}_q^S(I, K)) = \sum_{i=1}^r b_i \binom{i-1}{q}.
\]

(27)

From Proposition 2.6 we have that the ideals $I_n$ are stable after reversing the order of the variables. Hence, replacing max by min, and hence $b_i$ with $C_{n,1+r-i}$, we get:

**Corollary 4.2.** Let $n \in \mathbb{N}^+$, $r = r(n)$, $S = K[x_1, \ldots, x_r]$. Then

\[
P(\text{Tor}_*^S(I_n, K), t) = \sum_{i=1}^r C_{n,1+r-i}(1 + t)^{i-1}
\]

(28)

\[
P(\text{Tor}_*^S(I_n, K), t, u) = \sum_{i=1}^r (1 + tu)^{(i-1)} \sum_j C_{n,1+r-i,j}u^j.
\]

(29)

For the Betti-numbers we have that

\[
\beta_q = \sum_{i=1}^r C_{n,1+r-i} \binom{i-1}{q}.
\]

(30)

In [6, 1] it is shown that if $S = K[x_1, \ldots, x_r]$ and $I$ is a stable monomial ideal in $S$, then $S/I$ is a Golod ring. Hence, from a result of Golod [4] (see also [5]), it follows that

\[
P(\text{Tor}_*^{S/I}(K, K), t) = \frac{(1 + t)^r}{1 - t^2 P(\text{Tor}_*^S(I, K), t)}
\]

(31)

Regarding $S$ as an $\mathbb{N}$-graded ring, one can show that in fact

\[
P(\text{Tor}_*^{S/I}(K, K), t, u) = \frac{(1 + ut)^r}{1 - t^2 P(\text{Tor}_*^S(I, K), t, u)}
\]

(32)

The following theorem is an immediate consequence:

**Theorem 4.3 (Herzog-Aramova, Peeva).** Let $S = K[x_1, \ldots, x_r]$, and suppose that $I$ is a stable monomial ideal in $S$. Put

\[
b_{i,d} = \# \{ x \in G(I) | \max(x) = i, |x| = d \}
\]

\[
b_i = \# \{ x \in G(I) | \max(x) = i \}
\]
Then, for $R = S/I$, we have that

$$P(\text{Tor}_i^R(K, K), t) = \frac{(1 + t)^r}{1 - t^2 \sum_{i=1}^r (1 + t)^{(i-1)} \sum_j b_i}$$

(33)

$$P(\text{Tor}_i^R(K, K), t, u) = \frac{(1 + t)^r}{1 - t^2 \sum_{i=1}^r (1 + tu)^{(i-1)} \sum_j b_{i,j}u^j}$$

(34)

Specialising to the case of $A_n$, we obtain:

**Corollary 4.4.** Let $n \in \mathbb{N}^+$, and let $r = r(n)$. Regard $A_n$ as a naturally graded $K$-algebra, with each $x_i$ given weight 1, and regard $K$ as a cyclic $A$-module. Then

$$P(\text{Tor}_i^{A_n}(K, K), t) = \frac{(1 + t)^r}{1 - t^2 \sum_{i=1}^r (1 + t)^{(i-1)} C_{n,r-i+1}}$$

(35)

$$P(\text{Tor}_i^{A_n}(K, K), t, u) = \frac{(1 + ut)^r}{1 - t^2 \left( \sum_{i=1}^r \left( (1 + tu)^{(i-1)} \sum_j C_{n,r-i+1,j}u^j \right) \right)}$$

(36)

Part IV of the main theorem is now proved.

**Example 4.5.** We consider the case $n = 5$, then $r = r(n) = 3$, so $S = K[x_1, x_2, x_3]$ and $I = I_5 = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)$. We get that $C_{5,1} = 3$, $C_{5,2} = 2$, $C_{5,3} = 1$. According to our formulas\(^1\) we have

$$P_I^S(t) = 1 + 2(1 + t) + 3(1 + t)^2 = 6 + 8t + 3t^2$$

$$P_K^{S/I} = \frac{(1 + t)^r}{1 - t^2 P_I^S(t)} = \frac{1}{1 - 3t}$$

When we consider the grading by total degree, we have that $C_{5,1,2} = 2$, $C_{5,1,3} = 1$, $C_{5,2,2} = 2$, $C_{5,3,2} = 1$. Hence, our formulas yield

$$P_I^S(t, u) = u^2 + 2u^2(1 + t) + (2u^2 + u^3)(1 + t)^2$$

$$= 5u^2 + u^3 + (6u^2 + 2u^3)t + (2u^2 + u^3)t^2$$

$$P_K^{S/I}(t, u) = -\frac{1 + tu}{u^3t^2 + 2t^2u^2 + 2tu - 1}$$

We list the first few Poincaré-Betti series $P(\text{Tor}_i^{A_n}(K, K), t, u)$ in table 3.

**Conjecture 4.6.** $P(\text{Tor}_i^{A_n}(K, K), t) = -\frac{(1 + t)^{\ell_1(n)}}{q_n(t)} \cdot q_n(t) = \sum_{i=0}^{\ell_2(n)} h_i(n)t^i$, with

1. $q_n(-1) \neq 0$,
2. $\ell_1(n)$ is the number of odd primes $p$ such that $p^2 \leq n$,
3. $\ell_2(n) = \ell_1(n) + 1$,
4. $h_0(n) = -1$,
5. $h_1(n) = r(n) - \ell_1(n)$,
6. $h_{\ell_2(n)}(n) = C_{n,1} = [n/2]$.

5. Acknowledgements

I am indebted to Johan Andersson for suggesting the idea of studying the homological properties of the truncations $\Gamma_n$. I thank the referee for suggesting a simplified proof of parts of Theorem 3.4.

\(^1\)Here, we have used the abbreviation $P_I^S(t) = P(\text{Tor}_i^S(I, K), t)$, we will also write $P_K^{S/I}(t) = P(\text{Tor}_i^{S/I}(K, K), t)$ et cetera.
\[ \begin{array}{c|c|c}
 n & \text{Graded} & \text{Non-graded} \\
 \hline
 2 & -(tu - 1)^{-1} & -(t - 1)^{-1} \\
 3 & -(2tu - 1)^{-1} & -(2t - 1)^{-1} \\
 4 & -(u^3 + u)^{t+1}u^{-1} & -(2t - 1)^{-1} \\
 5 & -(u^3 + u)^{t+2}u^{-1} & -(3t - 1)^{-1} \\
 6 & -(2u^3 + u)^{t+2}u^{-1} & -(3t - 1)^{-1} \\
 7 & -(u^3 + 2u)^{t+3}u^{-1} & -(4t - 1)^{-1} \\
 8 & -(u^3 + 2u)^{t+3}u^{-1} & -(4t - 1)^{-1} \\
 9 & -(u^3 + u)^{t+4}u^{-1} & -(4t - 1)^{-1} \\
 10 & -(u^3 + u)^{t+4}u^{-1} & -(4t - 1)^{-1} \\
 11 & -(u^3 + u)^{t+5}u^{-1} & -(4t - 1)^{-1} \\
 12 & -(u^3 + u)^{t+5}u^{-1} & -(4t - 1)^{-1} \\
 13 & -(u^3 + u)^{t+5}u^{-1} & -(4t - 1)^{-1} \\
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 23 & -(u^3 + u)^{t+5}u^{-1} & -(4t - 1)^{-1} \\
 24 & -(u^3 + u)^{t+5}u^{-1} & -(4t - 1)^{-1} \\
 25 & -(u^3 + u)^{t+5}u^{-1} & -(4t - 1)^{-1} \\
 \end{array} \]

Figure 3: Graded and non-graded Poincaré-Betti series of the minimal free resolution of \( K \) over \( A_n \).
References


This article may be accessed via WWW at http://www.rmi.acnet.ge/hha/ or by anonymous ftp at ftp://ftp.rmi.acnet.ge/pub/hha/volumes/2000/n2/n2.(dvi,ps,dvi.gz,ps.gz)

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