A HOMOTOPY LIE-RINEHART RESOLUTION AND CLASSICAL BRST COHOMOLOGY

LARS KJESETH

(communicated by Larry Lambe)

Abstract

We use an interlaced inductive procedure reminiscent of the integration process from traditional deformation theory to construct a homotopy Lie-Rinehart resolution for the Lie-Rinehart pair which arises as an exercise in Poisson reduction in the context of the BFV construction of classical BRST cohomology. We show that the associated homotopy Rinehart algebra and the BRST algebra are isomorphic as graded commutative algebras. In the irreducible case, the two have the same cohomology.

1. Introduction

In this paper, we utilize a strategy akin to the process of integration found in traditional deformation theory (see, for example, [GS90]) to construct a homotopy Lie-Rinehart resolution \((K_{A/I}, K_{I/I^2})\) for the Lie-Rinehart pair \((A/I, I/I^2)\) that appears in the BFV formulation of classical BRST cohomology. A Lie-Rinehart pair is a couple \((B, s\gamma)\) which admits a structure analogous to that shared by the associative commutative algebra \(C^\infty(M)\) of smooth functions and the Lie algebra \(\Gamma(TM)\) of smooth vector fields on a smooth manifold \(M\). The term Lie-Rinehart pair is not widely used. More often, \(s\gamma\) has been called a \((B, k)\)-Lie algebra, \([Rin63]\) \([Pal61]\) and \([Her72]\). More recently, Lie-Rinehart pairs have appeared as Lie algebroids (see \([dSW]\)).

An associative algebra \(A\) is a Poisson algebra if it admits a Lie bracket \(\{\ , \}\) such that for any \(a \in A\), the map \(\{a, \ \}\) is a graded derivation with respect to the multiplication, i.e., \(\{a, bc\} = \{a, b\}c \pm b\{a, c\}\). A multiplicative ideal \(I\) of a Poisson algebra \(A\) is coisotropic if it is closed under the Poisson bracket on \(A\). Poisson reduction of a Poisson algebra \(A\) by a coisotropic ideal \(I\) is no more than the observation that while the quotient \(A/I\) is not a Poisson algebra (unless \(I\) is a Poisson ideal), the subset of \(I\)-invariant classes in \(A/I\) is again a Poisson algebra. The \(I\)-invariant classes comprise the zeroth cohomology of the Rinehart complex \(R\) for the Lie-Rinehart pair \((A/I, I/I^2)\).
Mathematicians and physicists (see, for example [Sta88], [KS87], [FHST89], [HT92], [Kim92a], [Kim92b], [Sta92], [Kim93] and [Sta96]) recognized the classical analogue of the Batalin, Fradkin and Vilkovisky (BFV) construction of the quantum BRST complex ([BV77], [BF83], [BV83] and [BV85]) as something new and interesting because BRST cohomology performs Poisson reduction without passing first to the quotient $A/I$. The BFV construction of the classical BRST algebra (see, for example, [Kim93]) begins by replacing $A/I$ with the Koszul-Tate resolution [Tat57] and adjoins formal (ghost) variables to the Koszul-Tate resolution. They then exploit a graded Poisson bracket to construct a differential. Under certain conditions, the BRST algebra $(A, D)$ is a cohomological model for the Rinehart complex of $(A/I, I/I^2)$.

In general, constructing cohomological models for the Rinehart complex $(B, sg)$ by the traditional homological means of replacing both $B$ and $sg$ with resolutions fails unless the Lie-Rinehart structure of the pair $(B, sg)$ is preserved. In [Kje01], we defined homotopy Lie-Rinehart pairs and the associated homotopy Rinehart algebra in the context of coalgebras. We defined homotopy Lie-Rinehart resolutions and presented conditions under which the associated homotopy Rinehart algebra is a cohomological model for the Rinehart algebra of the resolved Lie-Rinehart pair.

Summary

The coalgebra setting is used throughout this paper. First, we revisit the definitions of the Lie algebras of subordinate derivation sources, resting coderivations and shared Lie modules, which lie behind the coalgebra realizations of both homotopy and non-homotopy Lie algebras, Lie algebra modules, Lie-Rinehart pairs and Rinehart cohomology. We review homotopy Lie-Rinehart resolutions for Lie-Rinehart pairs and the conditions under which the homotopy Rinehart algebra for a homotopy Lie-Rinehart resolution is a model for the Rinehart cohomology complex for a Lie-Rinehart pair (§2). Next, using an interlaced inductive process, we construct a homotopy Lie-Rinehart resolution for the Lie-Rinehart pair in classical BRST algebra (§3). One surprising result is that the homotopy Rinehart complex associated with the homotopy Lie-Rinehart resolution is isomorphic to the BRST algebra (§4).

We have omitted all sign arguments from the proofs in this paper, as they are not generally instructive. All vector spaces, algebras and coalgebras in this paper are over a field $k$ of characteristic zero. All tensor products are over $k$ and all maps are at least $k$-linear or $k$-multilinear.

Note that $sg$ is the suspension of the Lie algebra $g$, i.e., all elements of $g$ are assigned degree zero and hence, all elements of $sg$ have degree 1. Throughout this paper, we will identify all Lie algebras (and strongly homotopy Lie algebras) with their suspensions. At first, this identification will be explicit; later, when we work with $I/I^2$ and $K_{I/I^2}$, we will hide the suspension.
2. Necessary Background

Review of Chavaliell-Eilenberg and Rinehart cohomologies

For any Lie algebra \( \mathfrak{g} \) and \( \mathfrak{g} \)-module \( B \), an \( n \)-multilinear function \( f_n : (\mathfrak{g})^n \to B \) is alternating if

\[
f_n(sx_1, ..., sx_i, sx_{i+1}, ..., sx_n) = -f_n(sx_1, ..., sx_{i+1}, sx_i, ..., sx_n).
\]

The Chevalley-Eilenberg complex is the set of all alternating multilinear functions \( \text{Alt}_n(\mathfrak{g}, B) \), graded by \( n \), and equipped with a degree +1 differential \( \delta_{CE} : \text{Alt}_n(\mathfrak{g}, B) \to \text{Alt}_{n+1}(\mathfrak{g}, B) \) given by

\[
\delta_{CE} f_n(sx_0, ..., sx_n) = \sum_{i=0}^{n} (-1)^i \omega(sx_i \otimes f_n(sx_0, ..., \widehat{sx}_i, ..., sx_n))
\]

\[
- \sum_{i<j} (-1)^{i+j-1} f_n([sx_i, sx_j], sx_0, ..., \widehat{sx}_i, ..., \widehat{sx}_j, ..., sx_n),
\]

where \( \widehat{sx}_i \) indicates that \( sx_i \) should be omitted. The map \( \omega \) is the \( \mathfrak{g} \)-module action of \( \mathfrak{g} \) on \( B \). Any element \( b \in B \) is considered an element of \( \text{Alt}_0(\mathfrak{g}, B) \). The image of \( b \) under \( \delta_{CE} \) is defined by setting \( \delta_{CE} b(sx) = \omega(sx \otimes b) \). When \( B \) is an algebra, the Chevalley-Eilenberg complex is a differential graded commutative algebra. For \( f_n \) and \( g_m \) in \( \text{Alt}_k(\mathfrak{g}, B) \), the product \( f_n \circ g_m \) is given by

\[
(f_n \circ g_m)(sx_1, ..., sx_{n+m}) = \sum \sum_{\sigma \text{ unshuffles}} f_n(sx_{\sigma(1)}, ..., sx_{\sigma(n)}) g_m(sx_{\sigma(n+1)}, ..., sx_{\sigma(n+m)}).
\]

An \((n, m)\)-unshuffle is any permutation \( \sigma \) in the symmetric group \( \Sigma_{n+m} \) such that

\[
\sigma(1) < \cdots < \sigma(n) \quad \text{and} \quad \sigma(n+1) < \cdots < \sigma(n+m),
\]

where \( \sigma(j) \) is the element of the set \( \{1, ..., n+m\} \) moved to the \( j^{\text{th}} \) position under \( \sigma \). The differential \( \delta_{CE} \) acts as a derivation with respect to this multiplication. The cohomology of this complex with respect to \( \delta_{CE} \) is the Chevalley-Eilenberg cohomology of \( g \) with coefficients in \( B \) [CE48].

**Definition 2.1.** [Rin63] Let \( B \) be an algebra and \( \mathfrak{g} \) be a Lie algebra, both modules over an algebra \( A \) over a field \( k \) of characteristic zero and modules over each other. We denote the left \( B \)-module action \( \mu \) on \( \mathfrak{g} \) by \( \mu(a \otimes sx) := asx \). Let \( \omega : \mathfrak{g} \otimes B \to B \) (or, alternatively, \( \omega : \mathfrak{g} \to \text{Der}(B) \)) denote the \( \mathfrak{g} \)-module action on \( B \). The pair \((B, \mathfrak{g})\) is a Lie-Rinehart pair, provided the Lie-Rinehart relations (LRa) and (LRb) are satisfied for all \( a, b \in B \) and \( sx, sy \in \mathfrak{g} \):

**LRa:** \( \omega(asx \otimes b) = a \cdot \omega(sx \otimes b) \), where \( \cdot \) indicates the multiplication on \( B \).

**LRb:** \( [sx, asy] = a[sx, sy] + \omega(sx \otimes a) sy \).

Suppose \((B, \mathfrak{g})\) is a Lie-Rinehart pair and the alternating function \( f_n \) is a \( B \)-multilinear, i.e., \( f_n(a_1 sx_1, ..., a_n sx_n) = a_1 \cdots a_n f_n(sx_1, ..., sx_n) \). Because the Lie action map \( \omega \) maps \( \mathfrak{g} \) into the derivations of \( B \) and as a result of the Lie-Rinehart relations, the image of \( f_n \) under the Chevalley-Eilenberg differential \( \delta_{CE} \) is again
Coalgebras and Subcoalgebras

A graded coassociative coalgebra is a pair \((C, \Delta)\), where \(C\) is a graded module over \(k\) together with a 0-degree coassociative comultiplication \(\Delta : C \to C \otimes C\). A function \(f\) with degree \(|f| = r\) is a coderivation on \(C\) if \((f \otimes 1 + 1 \otimes f)\Delta = \Delta f\). The set of all coderivations on a coalgebra \(C\), denoted \(\text{Coder}(C)\), is a graded Lie algebra under the graded commutator bracket, that is to say, \([f, g] = fg - (-1)^{|f||g|}gf\) for all \(f\) and \(g\) in \(\text{Coder}(C)\), where \(|f|\) and \(|g|\) are the degrees of \(f\) and \(g\).

We will work with the tensor coalgebra \(T^c(sV) = \bigotimes(sV)\), where \(sV\) is the suspension a graded module \(V\) over a \(k\)-algebra \(A\), where \(k\) is a field of characteristic 0. We will let \(sv_{[1 \text{ to } n]}\) denote the element \(sv_1 \otimes \cdots \otimes sv_n \in (sV)^{\otimes n}\). The internal graded action \(\rho_n^\sigma\) of the symmetric group \(\Sigma_n\) on \((sV)^{\otimes n}\) is given by \(\sigma \cdot (sv_{[1 \text{ to } n]}) = \kappa_{\text{id}}(\sigma)sv_{[1 \text{ to } n]}\) for all \(n\), \(\sigma \in \Sigma_n\) and \(sv_{[1 \text{ to } n]} \in (sV)^{\otimes n}\). The symbol \(sv_{\sigma[1 \text{ to } n]}\) is shorthand for \(sv_{\sigma(1)} \otimes \cdots \otimes sv_{\sigma(n)}\), where \(\sigma(i)\) is the element of the ordered set \(\{1, \ldots, k\}\) which moves to the \(i\)th position under \(\sigma\). The factor \(\kappa_{\text{id}}(\sigma)\) is the sign produced by rearranging the \(sv_i\)’s into the \(\sigma\) order, following the Koszul sign convention, which states that exchanging two objects of homogeneous degrees \(p\) and \(q\) (whether elements or maps) introduces a factor of \((-1)^{pq}\). The action \(\rho^\wedge = \{\rho_n^\wedge\}_{n \geq 1}\) of \(\{\Sigma_n\}_{n \geq 1}\) on \(T^c(sV)\) is defined in the obvious way. The \(\rho^\wedge\)-invariant subcoalgebra is well-known as the graded commutative coalgebra \(\Lambda(sV)\) and is generated by elements of the form

\[
\sum_{\sigma \in \Sigma_n} \kappa_{\text{id}}(\sigma)sv_{[1 \text{ to } n]},
\]

which we will denote by \(sv_n^\wedge\). The coassociative comultiplication on \(\Lambda(sV)\) is given by

\[
\Delta(sv_n^\wedge) = \sum_{j=0}^n \sum_{(j, n-j)} \kappa_{\text{id}}(\rho) \left(sv_{[1 \text{ to } j]}^\wedge \otimes (sv_{[j+1 \text{ to } n]}^\wedge)\right),
\]

where it is understood that the second sum is over all \((j, n-j)\)-unshuffles.

Shared Lie modules, Subordinate and resting coderivations

Three algebraic objects are vital for what follows. The graded Lie algebra \(\text{Coder}^W_W(\Lambda(sV))\) is generated by the set of coderivations \(l_n\) on \(\Lambda(sV)\), each of which rests on a specific subordinate coderivation \(m_n\) in the Lie algebra \(\text{Coder}^W_W(\Lambda(sV) \otimes W)\). For a graded commutative algebra \(W\), the Lie algebra \(\text{Coder}^W_W(\Lambda(sV) \otimes W)\) is generated by the set of all coderivations \(m_n\), each of which is both subordinate to a specific resting coderivation \(l_n\) and a \(W\)-derivation source. Finally, the graded commutative algebra \(\text{Hom}^W_W(\Lambda(sV), \Lambda(sV) \otimes W)\) is generated by the set of all \(W\)-linear maps \(f_n\) from \(\Lambda(sV)\) into \(\Lambda(sV) \otimes W\). This algebra admits a shared Lie module structure over both \(\text{Coder}^W_W(\Lambda(sV) \otimes W)\) and \(\text{Coder}^W_W(\Lambda(sV))\). We review what
the terms subordinate coderivation, \( W \)-derivation source, resting coderivation and shared Lie module mean. For a more detailed development, see [Kje01].

We extend any map \( l_n : (sV)^{\wedge n} \to sV \) to a coderivation on the coalgebra \( \Lambda(sV) \) by setting

\[
l_n(sv_{[1]}^\wedge to k)) = \sum_{(n,k-n)} \text{Kid}(\rho)l_n(sv_{[1]}^\wedge to n) \wedge sv_{[n+1]}^\wedge to k). \tag{2}
\]

**Definition 2.2.** A map \( m_n \) on \( \Lambda(sV) \otimes W \) is an \( l_n \)-subordinate coderivation if the degree of \( m_n \) and \( l_n \) agree and \( m_n \), as the extension of a map \( m_n : (sV)^{\wedge n-1} \otimes W \to W \) as a coderivation on \( \Lambda(sV) \otimes W \), is defined on \( (sV)^k \otimes W \) by setting

\[
m_n(sv_{[1]}^\wedge to k) \otimes w = l_n(sv_{[1]}^\wedge to k) \otimes w + \sum_{(k-n+1,n-1)} \text{Kid}(\sigma) \text{Kid}(m_n; sv_{[1]}^\wedge to k-n+1) \cdot sv_{[1]}^\wedge to k-n+1 \otimes m_n(sv_{[1]}^\wedge to k-n+2 \otimes k) \otimes w, \tag{3}
\]

when \( k \geq n-1 \) and setting \( m_n = 0 \) when \( k < n-1 \).

The set of all subordinate coderivations \( \text{Coder}(\Lambda(sV) \otimes W) \) forms a graded Lie algebra under the graded commutator bracket. The bracket respects subordination, that is to say, if \( m_i \) and \( m_j \) are \( l_i \) and \( l_j \)-subordinate coderivations respectively, then \([m_i, m_j] \) is a coderivation subordinate to \([l_i, l_j]\).

**Definition 2.3.** An \( l_n \)-subordinate map \( m_n \) is a \( W \)-derivation source if for every \( sv_{[1]}^\wedge to n-1 \) in \( (sV)^{\wedge n-1} \), the map \( m_n(sv_{[1]}^\wedge to n-1) \otimes (\_)) : W \to W \) is a graded derivation on \( W \).

The subset of all \( W \)-derivation sources in \( \text{Coder}(\Lambda(sV) \otimes W) \) forms a graded Lie subalgebra denoted by \( \text{Coder}^W(\Lambda(sV) \otimes W) \). The subset of all coderivations \( l_n \in \text{Coder}(\Lambda(sV)) \) which admit a subordinate \( W \)-derivation source \( m_n \) forms a graded Lie subalgebra which we will denote by \( \text{Coder}^W(\Lambda(sV)) \).

We can extend any map \( f_n : (sV)^{\wedge n} \to W \) to function from \( \Lambda(sV) \) into \( \Lambda(sV) \otimes W \) by setting \( f_n = 0 \) for \( k < n \) and, for \( k \geq n \), setting \( f_n(sv_{[1]}^\wedge to k) = \sum_{(k-n,n)} \text{Kid}(\sigma) \text{Kid}(f_n; sv_{[1]}^\wedge to k-n) sv_{[1]}^\wedge to k-n \otimes f_n(sv_{[1]}^\wedge to k-n+1 \otimes k)). \tag{4} \)

The set of all such maps is denoted by \( \text{Hom}(\Lambda(sV), \Lambda(sV) \otimes W) \). The algebra structure on \( W \) extends to a graded commutative algebra structure on \( \Lambda(sV) \otimes W \), which provides \( \text{Hom}(\Lambda(sV), \Lambda(sV) \otimes W) \) with a cup product. If \( f_n \) and \( g_s \) are maps in \( \text{Hom}(\Lambda(sV), \Lambda(sV) \otimes W) \). Then \( f_n \sim g_s \) is given by setting \((f_n \sim g_s)(sv_{[1]}^\wedge to n) = \sum_{(n,s)} \text{Kid}(\rho) \text{Kid}(g_s; sv_{[1]}^\wedge to n) f_n(sv_{[1]}^\wedge to n) \cdot g_s(sv_{[n+1]}^\wedge to n) \) on \((sV)^{\wedge n+s}\) \(, where \cdot \) represents the the multiplication on \( W \) (henceforth, the \cdot \) will be suppressed). The map \( f_n \sim g_s \) is then extended to all of \( \Lambda(sV) \) as in equation (4).
If the graded module $sV$ is a module over the graded commutative algebra $W$, then $\wedge(sV)$ is a module over the tensor algebra $TW$. We will follow our general convention and denote $w_1sv_1 \wedge \cdots \wedge w_nsv_n$ by $wsv_{[1 \to n]}$. The unraveling map

$$U : \wedge(sV) \to W \otimes \wedge(sV)$$

$$wsv_{[1 \to n]} \mapsto u([1 \to n])w_{[1 \to n]} \otimes sv_{[1 \to n]},$$

where $u([1 \to n])$ is the sign produced when moving the $w_i$‘s past the $sv_j$‘s. The map $U$ respects the coalgebra structure of $\wedge(sV)$.

**Definition 2.4.** A map $f_n : (sV)^{\wedge n} \to W$ is $W$-linear if

$$f_n(ws^v_{[1 \to n]}) = u([1 \to n])K_{id}(f_n; w_{[1 \to n]}^i)w_{[1 \to n]}^i f_n(sv_{[1 \to n]}).$$

Likewise, a map $m_n : (sV)^{\wedge n-1} \otimes W \to W$ is $W$-linear if

$$m_n(ws^v_{[1 \to n-1]} \otimes w) = u([1 \to n-1])K_{id}(m_n; w_{[1 \to n-1]}^i)w_{[1 \to n-1]}^i m_n(sv_{[1 \to n-1]} \otimes w).$$

For a map $m_n : (sV)^{\wedge n-1} \otimes W \to W$, we define the $V$-extension map $m_n^e : (sV)^{\wedge n} \to V$ by setting

$$m_n^e(ws^v_{[1 \to n]}) = \sum_{(n, p) \in \mathbb{Z}} K_{id}(\rho)m_n(ws^v_{[1 \to n-1]} \otimes w_{p(1 \to n)})sv_{p(n)}.$$

**Definition 2.5.** Given a $W$-linear, $W$-derivation source $m_n \in \text{Coder}^W(\wedge(sV) \otimes W)$ which is subordinate to $l_n \in \text{Coder}^W(\wedge(sV))$, the map $l_n$ is said to rest on $m_n$ if $l_n(ws^v_{[1 \to n]}) = u([1 \to n])K_{id}(l_n; w_{[1 \to n]}^i)w_{[1 \to n]}^i l_n(sv_{[1 \to n]}) + m_n^e(ws^v_{[1 \to n]}).$

The subset of $\text{Coder}^W(\wedge(sV))$ of all coderivations $l_p$ which rest on $W$-linear $W$-derivation sources $m_p$ forms $\text{Coder}^W(\wedge(sV))$. Similarly, the subset of $\text{Coder}^W(\wedge(sV) \otimes W)$ consisting of all $W$-linear $W$-derivation sources $m_p$ subordinate to maps $l_p$, which in turn rest on $m_p$, forms $\text{Coder}^W(\wedge(sV) \otimes W)$.

Suppose $m_p \in \text{Coder}^W(\wedge(sV) \otimes W)$ is subordinate to $l_p \in \text{Coder}(\wedge(sV))$. Remarkably, the map

$$\langle m_p, \cdot \rangle : \text{Hom}_W(\wedge(sV), \wedge(sV) \otimes W) \to \text{Hom}_W(\wedge(sV), \wedge(sV) \otimes W),$$

given by setting

$$\langle m_p, f_n \rangle = m_pf_n - K_{id}(m_p; f_n)f_nl_p$$

on $(sV)^{\wedge n+p-1}$ and extending $\langle m_p, f_n \rangle$ to all of $\wedge(sV)$ as in equation (4) is well-defined [Kje01]. This map exposes the shared Lie module structure on $\text{Hom}_W(\wedge(sV), \wedge(sV) \otimes W)$ [Kje01]. In a sense, it is simultaneously a Lie-module over both $\text{Coder}^W(\wedge(sV) \otimes W)$ and $\text{Coder}^W(\wedge(sV))$ in that $\langle [m_i, m_j], \cdot \rangle = \langle m_i, [m_j, \cdot] \rangle - K_{id}(m_i; m_j)(m_j, [m_i, \cdot])$. The map $\langle m_p, \cdot \rangle$ acts as a derivation with respect to the cup product.
Homotopy (and non-homotopy) Lie algebras and their modules

Strongly homotopy Lie algebras (shLie algebras) first appeared implicitly in [Su77] and explicitly in [SS] in the context of deformation theory. A concise introduction to shLie algebras is found in [LM95]. The definitions below were lifted directly from [LM95] and [LS93] and then modified in [Kje01] to fit the language of this paper. The machinery used in deformation theory (see, for example, [GS90]) will prove useful in the process of constructing an shLie structure and an shLie module structure on the resolution of the Lie-Rinehart pair in §3, although we will not use the language in the traditional way.

Definition 2.6. An \( \mathcal{L}(m) \)-structure on a graded module \( sL \) consists of a system of coderivations \( \{ l_k : \bigwedge(sL) \to \bigwedge(sL) : 1 \leq k \leq m \leq \infty, k \neq \infty \} \) which are extensions of maps \( l_k : (sL)^{\wedge k} \to sL \). Each \( l_k \) has degree \(-1\). Moreover, the following generalized form of the Jacobi identity is satisfied for \( n \leq m \):

\[
\frac{1}{2} \sum_{i+j=m+1 \atop i,j \geq 1} [l_j, l_i] = 0
\]

on \( \bigwedge(sL) \). If \( L \) admits an \( \mathcal{L}(\infty) \)-structure, then \( L \) is a strongly homotopy Lie algebra (an shLie algebra).

For this paper, we will view a standard Lie algebra \( s\mathfrak{g} \) as a degenerate shLie algebra \( \bigwedge(s\mathfrak{g}) \) where all coderivations \( \hat{l}_i \) are zero except \( \hat{l}_2 \). We decorate the bracket \( \hat{l}_2 \) on \( s\mathfrak{g} \) to distinguish it from \( l_2 \) on the shLie algebra which resolves \( \bigwedge(s\mathfrak{g}) \).

The following maps will be useful in §3, where we will use a deformation theoretic approach to construct an shLie algebra together with an shLie module.

Definition 2.7. For \( n > 1 \), the \( n \)-th Jacobi coderivation \( J_n : \bigwedge(sL) \to \bigwedge(sL) \) is given by

\[
J_n = -\frac{1}{2} \sum_{i+j=m+1 \atop i,j \geq 1} [l_j, l_i].
\]

For an \( \mathcal{L}(m) \)-algebra, \( J_n = [l_1, l_n] \) for every \( 2 < n < m \). Since \( [l_1, l_n] = l_1 l_n + (-1)^{n-1} l_n l_1 \) and \( l_1 \) is a differential on \( L \), it will be useful to have a name for the following map even though it is not a coderivation on \( \bigwedge(sL) \).

Definition 2.8. For \( n > 1 \), the \( n \)-th Jacobi obstruction map \( J_{\text{Obst}}^n : (sL)^{\wedge n} \to sL \) is given by

\[
J_{\text{Obst}}^n = J_n - l_n l_1 - \sum_{i+j=m+1 \atop i > 0, j \geq 1} l_j l_i
\]

In the language of deformation theory (see, among others, [Ger63], [Ger64], [Ger66], [Ger68], [GS90], [GS88], [FGV95] and [FGV]), an \( \mathcal{L}(m) \)-structure on \( L \) may be extended to an \( \mathcal{L}(m+1) \)-structure if there is a linear map \( l_{m+1} : L^{\wedge m+1} \to L \) such that \( J_{\text{ID}}^{m+1} = 0 \) or, equivalently, such that \( \frac{1}{2} [l_1, l_{m+1}] = J_{m+1} \). What we are
We substitute \( l_{i,j} \) for \( l_{ij} \).\(^{2,10}\) Therefore, since deforming here is a bit obscure and less important than the process of extending an \( L(m) \)-structure. The graded Lie algebra \( \text{Coder}(\Lambda L) \), together with the differential given by \([l_1, \_] \), governs the extension theory of \( L(m) \)-structures on \( \Lambda L \). If we were viewing the construction of an \( L(\infty) \)-structure strictly as a deformation theory problem rather than simply borrowing the machinery, the map \( J \) would be called the \( n \)-th obstruction. However, in the context of our construction in §3, the use of the map \( J_{\text{OBST}} \) is more consistent with the common notion of an obstruction. The following lemmas concerning \( J \) and \( J_{\text{OBST}} \) will also be useful in §3.

**Lemma 2.9.** The coderivation \( J \) is a cocycle, i.e., \([l_1, J_n] = 0\).

**Proof of Lemma 2.9.** We will show that \([l_1, -4J_n] = 0\).

\[
[l_1, -4J_n] = \left[ l_1, \sum_{i+j=n+1, i,j \geq 1} 2[l_j, l_i] \right] = \left[ l_1, - \sum_{i+j=n+1, i,j \geq 1} ([l_j, l_i] + [l_i, l_j]) \right]
\]

\[
= \sum_{i+j=n+1, i,j \geq 1} ([l_1, [l_j, l_i]] + [l_1, [l_i, l_j]]).
\]

Since \([l_1, [l_j, l_i]] = ([l_1, l_j], l_i) - ([l_1, l_i], l_j)\), it follows that \([l_1, [l_j, l_i]] + [l_1, [l_i, l_j]]\) equals

\[
= ([l_1, [l_j, l_i]] - ([l_1, l_i], l_j]) + ([l_1, [l_i, l_j]] - ([l_1, l_j], l_i)]
= 0.
\]

Therefore, \([l_1, J_n] = -\frac{1}{4}[l_1, -4J_n] = 0.\)

**Lemma 2.10.** If \( sL \) admits an \( L(n) \)-structure, then \( l_1 J_{\text{OBST}} = 0 \) on \((sL)^\wedge n\).

**Proof of Lemma 2.10.** The map \( J_{\text{OBST}} = - \sum_{i+j=n+1, i>0, j>1} (-1)^{(i+1)}l_j l_i \), so

\[
l_1 J_{\text{OBST}} = l_1 \left( - \sum_{i+j=n+1, i>0, j>1} l_j l_i \right) = - \sum_{i+j=n+1, i>0, j>1} l_1 (l_j l_i)
= - \sum_{i+j=n+1, i>0, j>1} (l_1 l_j) l_i.
\]

We substitute \( l_1 l_j \) with \( \sum_{i+j=n+1, i>0, j>1} l_r l_s \), and reassociate so that

\[
\sum_{i+j=n+1, i>0, j>1} \left( \sum_{i+j=n+1, i>0, j>1} l_r l_s \right) l_i = \sum_{i+j=n+2, i,s>0, r>1} l_r (l_s l_i).
\]
If we let \( i + s = t + 1 \), the sum above becomes

\[
\sum_{t>0,r>1}^{t+r=m+1} l_r \left( \sum_{t=s+1}^{t+r=m+1} l_sl_i \right) = \sum_{t>0,r>1}^{t+r=m+1} l_r \mathcal{J}_{ID_t}.
\]

Since \( \mathcal{J}_{ID_t} = 0 \) on \((sL)^{\wedge t}\) for all \( 1 \leq t \leq n - 1 \), the final sum above is zero. \( \square \)

**Definition 2.11.** [LM95] Let \((sL, l_i)\) be an \( \mathcal{L}(p) \)-algebra \((0 < p < \infty)\) and let \( M \) be a differential graded module with differential \( m_1 \). Then a left \( \mathcal{L}(k) \)-module structure over \( sL \) on \( M \) \((\text{for } k \leq p)\) is a collection of coderivations \( \{ m_n : \wedge(sL) \otimes M \to \wedge(sL) \otimes M : 1 \leq n \leq k, n \neq \infty \} \). Each \( m_n \) is \( l_n \)-subordinate and the extension of a map \( m_n : \wedge^{n-1} \otimes M \to M \) such that the \( n^{th} \) action identity map

\[
\text{ACT}_{ID_n} = \frac{1}{2} \sum_{i+j=m+1, i,j \geq 1} [m_j, m_i] = 0
\]
on \( \wedge(sL) \otimes M \). The differential graded module \( M \) is a strongly homotopy Lie module over \( sL \) \((\text{or an } sL\text{-shLie module})\) if \( sL \) admits an \( \mathcal{L}(\infty) \)-structure and \( M \) is a module with respect to that \( \mathcal{L}(\infty) \)-structure.

Definition 2.11 implies that the differential \( m_1 \) on \( M \) must be \( l_1 \)-subordinate. It is simple to verify that \( m_1 \) is a differential on \( \wedge L \otimes M \).

When \( M \) is a differential graded commutative algebra, we will insist the maps \( m_i \) be \( M \)-derivation sources, that is to say, the map \( m_i(v_{[1]}^{t_1} \text{ to } i-1 \otimes (\cdot)) : M \to M \) is in \( \text{Der}(M) \) for every \( v_{[1]}^{t_1} \text{ to } i-1 \in L^{\wedge^{t_1}-1} \).

In this paper a module \( B \) over a Lie algebra \( g \) is a degenerate shLie module \( \wedge(sg) \otimes B \) over the shLie algebra \( \wedge(sg) \), where all the coderivations \( m_i \) are zero except for \( m_2 \) (again, we decorate this particular map).

Below, we define the maps \( \text{ACT}_n \) and \( \text{ACT}_{\text{OBST}}_n \), the analogs of \( \mathcal{J}_n \) and \( \mathcal{J}_{\text{OBST}}_n \).

**Definition 2.12.** For \( n > 1 \), the \( n^{th} \) action map \( \text{ACT}_n : \wedge(sL) \otimes M \to \wedge(sL) \otimes M \) is given by

\[
\text{ACT}_n = -\frac{1}{2} \sum_{i+j=m+1, i,j \geq 1} [m_j, m_i].
\]

**Definition 2.13.** For \( n > 1 \), the \( n^{th} \) action obstruction map \( \text{ACT}_{\text{OBST}}_n : (sL)^{\wedge n-1} \otimes M \to M \) is given by

\[
\text{ACT}_{\text{OBST}}_n = \text{ACT}_n - m_nm_1 = -\sum_{i+j=m+1, i>0, j>1} m_jm_i.
\]

If \( sL \) is an \( \mathcal{L}(p) \)-algebra and \( M \) is an \( \mathcal{L}(n) \)-module over \( sL \) with \( n < p - 1 \) \((\text{or } n < \infty \text{ if } p = \infty)\), we can extend the \( \mathcal{L}(n) \)-module structure to an \( \mathcal{L}(n+1) \)-module structure if an \( l_{n+1} \)-subordinate \( M \)-derivation source \( m_{n+1} : (sL)^{\wedge n} \otimes M \to M \) can be found such that \( \text{ACT}_{ID_{n+1}} = 0 \). Again, what exactly we are deforming here is less important than the language and machinery of deformation theory.
The following lemmas and their proofs mimic lemmas 2.9 and 2.10 and their proofs.

**Lemma 2.14.** The coderivation $ACT_n$ is a cocycle, i.e., $[m_1, ACT_n] = 0$.

**Lemma 2.15.** If $L$ admits an $L(p)$-structure and $M$ is an $L$-$L(n)$-module for $n < p - 1$ (or $n < \infty$ if $p = \infty$), then $m_1 ACT_{OBST} = 0$ on $L^{n-1} \otimes M$.

**Proposition 2.16.** [LS93] If $(sL, l_i)$ is an shLie algebra, the map $D_{sL} = \sum_{i=1}^\infty l_i$ is a differential on $\wedge(sL)$, i.e. it is a map of degree -1 such that $D_{sL} \circ D_{sL} = 0$.

**Proof of Proposition 2.16.** A proof can be found in [LS93]. Here the proof is essentially the same but takes advantage of the bracket of coderivations:

$$D_{sL} \circ D_{sL} = \left( \sum_{j=1}^\infty l_j \right) \circ \left( \sum_{i=1}^\infty l_i \right) = \sum_{n=1}^\infty \sum_{i,j=1}^{n+1} l_j l_i.$$

Using equation in definition 2.6, we see that

$$\sum_{n=1}^\infty \sum_{i,j=1}^{n+1} l_j l_i = \frac{1}{2} \sum_{n=1}^\infty \sum_{i,j=1}^{n+1} [l_j, l_i] = \frac{1}{2} \sum_{n=1}^\infty \mathcal{J}_{ID} n = 0.$$

Note also that $D_{sL}$ is a degree -1 coderivation, so $D_{sL} \circ D_{sL} = \frac{1}{2} [D_{sL}, D_{sL}]$ and the graded skew-commutativity of the bracket ensures that $[D_{sL}, D_{sL}] = 0$. □

For the Lie algebra $\wedge(sg)$, the differential $D_{sg} = \hat{t}_2$.

**Proposition 2.17.** The map $D_M = \sum_{i=1}^\infty m_i$ is a differential on $\wedge(sL) \otimes M$, i.e., $D_M \circ D_M = 0$.

**Proof of Proposition 2.17.** Isomorphic to the proof of proposition 2.16. Notice that $D_M$ is $D_{sL}$-subordinate and that $D_M \circ D_M = \frac{1}{2} [D_M, D_M] = 0$. □

For the Lie module $\wedge(sg) \otimes B$, the differential $D_B = \hat{m}_2$.

**Homotopy Chevalley-Eilenberg cohomology**

**Definition 2.18.** A homotopy Chevalley-Eilenberg pair $(M, L)$ consists of an shLie algebra $L$ and an $L$-shLie module $M$.

A multi-linear graded alternating function $F_n : L^{\times n} \to M$ can be seen as a linear function $F_n : (sL)^{\times n} \to M$. So the homotopy Chevalley-Eilenberg algebra $Alt_h(L, M)$ is isomorphic to $\text{Hom}_h(\wedge(sL), \wedge(sL) \otimes M)$ with the differential $\delta_{hCE} = \langle D_M, \rangle$.

**Proposition 2.19.** $\delta_{hCE} \circ \delta_{hCE} = 0$.

**Proof of Proposition 2.19.** $\delta_{hCE} \circ \delta_{hCE} = \langle D_M, \langle D_M, \rangle \rangle = \langle \frac{1}{2} [D_M, D_M], \rangle = 0$. □
The cohomology with respect to \( \delta_{hCE} \) is the homotopy Chevalley-Eilenberg cohomology of \( sL \) with coefficients in \( M \). A quick check shows that the Chevalley-Eilenberg cohomology complex for the pair \( (B, sg) \) is simply a degenerate form of the homotopy version where \( \delta_{CE} = \delta_{hCE} = \langle \hat{m}_2, \hat{l}_i \rangle \) because \( \hat{m}_i \) and \( \hat{l}_i \) are zero for all \( i \neq 2 \).

When \( M \) is a differential graded commutative algebra, the homotopy Chevalley-Eilenberg complex \( \text{Hom}_k(\Lambda(sL), \Lambda(sL) \otimes M) \) is a differential graded commutative algebra. The differential \( \delta_{hCE} \) acts as a derivation with respect to this multiplication.

**Homotopy Lie-Rinehart pairs and homotopy Rinehart cohomology**

The homotopy Rinehart complex is a straightforward generalization of the Rinehart complex in the ungraded setting. Since the Rinehart complex is defined only for Lie-Rinehart pairs \((B, sL)\), we must define what constitutes a homotopy Lie-Rinehart pair \((M, sL)\). The subset \( \text{Alt}_M(sL, M) \) of \( \text{Alt}_k(sL, M) \) consisting of all \( M \)-linear graded alternating functions is isomorphic to \( \text{Hom}_M(\Lambda(sL), \Lambda(sL) \otimes M) \).

**Definition 2.20.** A homotopy Lie-Rinehart pair \((M, sL)\) consists of a differential graded commutative algebra \((M, m_i)\) which is a shLie module over the shLie algebra \((sL, l_i)\), which in turn is an \( M \)-module. Moreover, the following two homotopy Lie-Rinehart relations must be satisfied for all \( i \geq 1 \):

- \((hLRa_i)\): The shLie module structure map \( m_i \) is \( M \)-linear.
- \((hLRb_i)\): The shLie structure map \( l_i \) rests on \( m_i \) (definition 2.5).

An \( \mathcal{L}(p) \)-Lie-Rinehart pair \(((M, m_i), (sL, l_i))\) has maps \( m_i \) and \( l_i \) which satisfy \((hLRa_i)\) and \((hLRb_i)\) for \( 1 \leq i \leq p \).

The proposition below follows from the fact that, for every \( i \), the image of \( F \) under the map \( \langle \hat{m}_i, \hat{l}_i \rangle \) is again a map in \( \text{Hom}_M(\Lambda(sL), \Lambda(sL) \otimes M) \) for all \( i \) [Kje01].

**Proposition 2.21.** If \( F \) is \( M \)-linear, then so is \( \delta_{hCE} F \).

We conclude that the subset of all \( M \)-linear functions in \( \text{Hom}_k(\Lambda(sL), \Lambda(sL) \otimes M) \) forms a subcomplex \( \mathcal{R} = \text{Hom}_M(\Lambda(sL), \Lambda(sL) \otimes M) \) with differential \( \delta_{\mathcal{R}} = \langle D_M, \hat{m} \rangle \). The differential, together with the cup product, provides the homotopy Rinehart complex \( \mathcal{R} \) with the structure of a differential graded commutative algebra. The cohomology of \( \mathcal{R} \) with respect to \( \delta_{\mathcal{R}} \) is the homotopy Rinehart cohomology of \( sL \) with coefficients in \( M \).

The following two propositions for general homotopy Lie-Rinehart pairs \((M, sL)\) are observations based on the fact that both \( \text{Coder}^M_M(\Lambda(sL) \otimes M) \) and \( \text{Coder}^M_M(\Lambda(sL)) \) are graded Lie algebras.

**Proposition 2.22.** The maps \( \text{ACT}_{\mathcal{ID}_n} \) and \( \text{ACT}_{\mathcal{J}_n} \) are \( \mathcal{J}_{\mathcal{ID}_n} \) and \( \mathcal{J}_{\mathcal{J}_n} \)-subordinate, respectively; both are \( M \)-linear and \( M \)-derivation sources.

The significance of this proposition lies in recognizing that, as a result, each of the maps \( \text{ACT}_{\mathcal{ID}_n} \) and \( \text{ACT}_{\mathcal{J}_n} \) are completely determined by their image on a generating set for \( \Lambda(sL) \) as an \( M \)-module and a generating set for \( M \), so long as the maps \( \mathcal{J}_{\mathcal{ID}_n} \) and \( \mathcal{J}_{\mathcal{J}_n} \) are available.
Proposition 2.23. The maps $J_{ID_n}$ and $J_n$ rest on $ACT_{ID_n}$ and $ACT_n$, respectively.

Here again, the implication is that the maps $J_{ID_n}$ and $J_n$ are completely determined by their image on a generating set of $\Lambda(sL)$ as an $M$-module.

Homotopy Lie-Rinehart resolutions of Lie-Rinehart pairs

Let $(B, s\g)$ be a Lie-Rinehart pair over a $k$-algebra $A$ with module structure maps $\hat{\mu} : B \otimes s\g \to s\g$ and $\hat{\mu}_2 : s\g \otimes B \to B$. Let $\hat{l}_2$ denote the bracket on $s\g$ and $\pi_B$ denote the multiplication on $B$. The basic ingredients of a homotopy Lie-Rinehart resolution for a Lie-Rinehart pair $(B, s\g)$ are $(sL, l_i)$ and $(M, m_i, \pi_M)$ where $H_l^1(sL) = s\g$ and $H_m^1(M) = B$.

Definition 2.24. A homotopy Lie-Rinehart resolution of a Lie-Rinehart pair $(B, s\g)$ over an algebra $A$ is a homotopy Lie-Rinehart pair $(M, sL)$ over $A$, such that

(a) the shLie algebra $(sL, l_i)$, seen as the coalgebra $\Lambda(sL)$, resolves $\Lambda(s\g)$, i.e.,

\[ H_l^1(\Lambda(sL)) = \Lambda(s\g), \]

where (following the physicists’ notation) $H_l^1$ denotes the homology with respect to the differential $l_1$,

(b) the differential graded commutative algebra $(M, m_i, \pi_M)$ satisfies

\[ H_m^1(\Lambda(sL) \otimes M) = \Lambda(s\g) \otimes B, \]

(c) the dgca $(M, m_i, \pi_M)$ also satisfies

\[ H_m^1(M \otimes M) = B \otimes B. \]

Furthermore, the following conditions hold:

i. $H_l^1(l_2) = \hat{l}_2$ on $\Lambda(s\g)$.

ii. $H_m^1(m_2) = \hat{\mu}_2$ on $\Lambda(s\g) \otimes B$.

iii. $H_m^1(\pi_M) = \pi_B$ on $B \otimes B$.

In the proposition below, we state conditions on the homotopy Lie-Rinehart pair that guarantee it satisfies definition 2.24. The proof is found in [Kje01].

Proposition 2.25. Let $(M, sL)$ be a homotopy Lie-Rinehart pair. Suppose the differential graded algebra $(M, m_i, \pi_M)$ is a projective resolution of $B$ over $A$ which respects the algebra structure on $B$, i.e., condition iii of definition 2.24 is satisfied. Suppose $(sL, l_i)$ is a projective resolution $s\g$. Furthermore,

(a) $H_l^1(l_2) = \hat{l}_2$ on $s\g \wedge s\g$ and

(b) $H_m^1(m_2) = \hat{\mu}_2$ on $s\g \otimes B$.

Then $(M, sL)$ is a homotopy Lie-Rinehart resolution of $(B, s\g)$.

We used a spectral sequence argument in [Kje01] to prove the following theorem.

Theorem 2.26. If the pair $(M, sL)$ is a projective homotopy Lie-Rinehart resolution of the Lie-Rinehart pair $(B, s\g)$, then $(R, (D_M, \langle D_M, \rangle))$ is a cohomological model for $(R, (\hat{\mu}_2, \langle \hat{\mu}_2, \rangle))$. 
3. Constructing the homotopy Lie-Rinehart pair 
\((K_{A/I}, K_{I/I^2})\) and classical BRST cohomology

Constructing homotopy Lie-Rinehart resolutions for a Lie-Rinehart pair is not always easy, as we will see as we construct a homotopy Lie-Rinehart resolution 
\((K_{A/I}, K_{I/I^2})\) for the Lie-Rinehart pair \((A/I, I/I^2)\) from classical BRST cohomology. Details which are not instructive (especially sign arguments) are omitted, but can be found in [Kje96].

The Lie algebra \(I/I^2\) has already been suspended, i.e., every element of \(I/I^2\) has degree +1 and the bracket \(\hat{I}_2\) is a coderivation on \(\bigwedge I/I^2\). However, we will suppress the suspension indicator “\(s\)” both here and in the shLie algebra \(K_{I/I^2}\).

Let the set \(y_{\alpha} = \{y_1, \ldots, y_n\}\) generate the coisotropic ideal \(I\) as an finitely presented \(A\)-module. If \(y_{\alpha}\) forms a basis of \(I\) over the field \(k\), there is a unique set of structure constants \(C_{\alpha\beta}^{\gamma} \in k\), each of which is antisymmetric in its lower indices, such that \(\{y_{\alpha}, y_{\beta}\} = C_{\alpha\beta}^{\gamma} y_{\gamma}\). (Note: We will use Einstein’s summation convention throughout §3 and §4.) With structure constants, the Jacobi identity is a statement about coefficients, namely \(C_{\alpha\beta}^{\gamma} C_{\gamma\delta}^{\varepsilon} + C_{\alpha\delta}^{\gamma} C_{\gamma\beta}^{\varepsilon} + C_{\delta\beta}^{\gamma} C_{\alpha\gamma}^{\varepsilon} = 0\) for every \(\epsilon\).

When the generators do not form a basis for \(I\) over \(k\), we can still find (not necessarily unique) structure functions \(C_{\alpha\beta}^{\gamma}\) in \(A\) such that \(\{y_{\alpha}, y_{\beta}\} = C_{\alpha\beta}^{\gamma} y_{\gamma}\). (In the BRST context, the elements of the Poisson algebra \(A\) are functions on a symplectic manifold, hence the term structure functions.) The Jacobi identity becomes \((C_{\alpha\beta}^{\gamma} C_{\delta\gamma}^{\varepsilon} + \{C_{\alpha\delta}^{\gamma}, y_{\gamma}\} + \text{c.p.})y_{\epsilon} = 0\), where c.p. stands for cyclic permutations of \(\alpha\), \(\beta\), and \(\gamma\). For convenience we will set \(J'\) equal to \((C_{\alpha\beta}^{\gamma} C_{\delta\gamma}^{\varepsilon} + \{C_{\alpha\delta}^{\gamma}, y_{\gamma}\} + \text{c.p.})\), so that the Jacobi identity can be written as \(J'y_{\epsilon} = 0\). The Jacobi identity is now a relation among the generators of \(I\).

**Definition 3.1.** [Kim93], [FHST89] Let \(I\) be finitely generated by the set \(y_{\alpha} = \{y_1, \ldots, y_n\}\). The set is irreducible if \(f^\alpha y_{\alpha} = 0\) implies \(f^\alpha = g^{\alpha\beta} y_{\beta}\), where each coefficient \(g^{\alpha\beta}\) is antisymmetric with respect to its indices. An ideal which has a set of irreducible generators is called irreducible.

**Definition 3.2.** [Kim93], [FHST89] An ideal \(I\) which is finitely generated by the set \(y_{\alpha} = \{y_1, y_2, \ldots, y_n\}\) admits a complete set of reducibility functions if there exists \(Z^{\alpha_{n-1}}\) in the complement of \(I\), such that for \(n = 2\), \(Z^{\alpha_1}_{\alpha_2} y_{\alpha_1} = 0\) and \(f^\alpha y_{\alpha_1} = 0\) implies \(f^\alpha = g^{\alpha\beta} y_{\beta}\), where \(\forall \alpha, \beta \in \{1, 2\}\). For \(n > 2\), \(Z^{\alpha_{n-2}}_{\alpha_{n-1}} Z^{\alpha_{n-1}}_{\alpha_{n-2}} \equiv 0 \mod I\) and \(f^\alpha_{\alpha_{n-1}} Z^{\alpha_{n-2}}_{\alpha_{n-1}} \equiv 0 \mod I\) implies \(f^\alpha_{\alpha_{n-1}} \equiv g^{\alpha\beta} Z^{\alpha_{n-1}}_{\alpha_{n-1}} \mod I\). An ideal which admits a complete set of reducibility functions is reducible.

We consider only ideals which are either irreducible or reducible. Since the Jacobi identity is a relation among the generators of \(I\), when the ideal is irreducible, we have \(J'y_{\epsilon} = 0\) implies \(J' = g^{\alpha\beta} y_{\beta}\). When the ideal is completely reducible, we have \(J'y_{\epsilon} = 0\) implies \(J' = g^{\alpha\beta} Z^{\alpha}_{\alpha} + \text{c.p.}\).

The quotient \(I/I^2\) inherits a Lie structure from \(I\) and is both an \(A\) and an \(A/I\)-module. There is a Lie action of \(I\) on \(A\) given by \(z \cdot f = \{z, f\}\), which passes to the quotient so that \(A/I\) is a Lie-module over both \(I\) and \(I/I^2\). Although \((A/I, I)\) is
not a Lie-Rinehart pair, the pair \((A/I, I/I^2)\) is. The actions \(\mu\) and \(\omega\) are given by

\[
\mu : A/I \otimes I/I^2 \to I/I^2 \quad \text{and} \quad \omega : I/I^2 \otimes A/I \to A/I,
\]

where \(\sim\) and \(\bar{\sim}\) denote equivalence classes. It is a straightforward exercise to show that the Lie-Rinehart relations are satisfied.

**The construction of \((K_{A/I}, K_{I/I^2})\)**

Before we describe how we construct the homotopy Lie-Rinehart resolution \((K_{A/I}, K_{I/I^2})\) for the pair \((A/I, I/I^2)\), we skip ahead to STEP 1, where we review the features of the Koszul-Tate resolution \(K_{A/I}\) of \(A/I\) (see [Tat57] and [Joz72]), a free differential graded commutative algebra over \(A\) which is a projective resolution of \(A/I\).

**STEP 1: The Koszul-Tate resolution \(K_{A/I}\)**

The Koszul complex \(K_{(0)}\) is a dgca over \(A\) and is isomorphic to \(A \otimes \Lambda \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_s\}\), where the \(\mathcal{P}_i\)'s are assigned degree 1 and are in one-to-one correspondence with the generators \(y_i\). (Following the physics literature, we call the \(\mathcal{P}_i\)'s antighosts [HT92].) The differential \(m_1\) maps each \(\mathcal{P}_i\) to \(y_i\) and \(m_1\) is extended as an \(A\)-linear graded derivation. The zeroth homology of the Koszul complex is \(A/I\) even if higher homologies do not vanish. Using the inductive method Tate introduced in [Tat57], we kill an unwanted nontrivial homology class \([z_0]\) of degree \(q\) by adjoining a formal variable \(\mathcal{P}_{z_0}\) of degree \(q+1\) to the existing Koszul complex. The variable \(\mathcal{P}_{z_0}\) maps to \(z_0\) under the differential, killing the unwanted homology. But these new variables may introduce new nontrivial homology on higher levels, which must in turn be killed. More formally, if \(I\) is reducible and generated by the set \(y_{a_1} = \{y_1, \ldots, y_s\}\), the zeroth homology of the Koszul complex \(K_{(0)} = A \otimes \Lambda \{\mathcal{P}_{a_1}\}\) is \(A/I\), but \(H_1(K_{(0)})\) may not be zero. The unwanted homology classes form a module generated by the cycles \(z_{a_2} = \mathcal{P}_{z_0}\mathcal{P}_{a_1}\), so we adjoin degree 2 antighosts \(\mathcal{P}_{a_2}\) and setting \(m_1(\mathcal{P}_{a_2}) = z_{a_2}\).

The resulting differential graded commutative algebra \(K_{(1)} = A \otimes \Lambda \{\mathcal{P}_{a_1}, \mathcal{P}_{a_2}\}\) has homology \(H_0(K_{(1)}) = A/I\) and \(H_1(K_{(1)}) = 0\). Again, \(H_2(K_{(1)})\) may contain nontrivial classes, either as a result of introducing the level two antighosts or because they were already present in \(K_{(0)}\). Regardless of origin, we choose a generating set \(z_{a_3}\) for the nontrivial 2-cycles and adjoin degree 3 antighosts \(\mathcal{P}_{a_3}\) to kill them. The third homology of \(K_{(2)}\) may be nonzero, so we continue the process. The complete reducibility of \(I\) guarantees that a representative for each nontrivial \(n\)-class can be chosen so that each \(z_{a_{n+1}} = f_{\alpha_{n+1}}^a \mathcal{P}_{a_{n+1}} + \text{“more”}\). The limit \(K_{(\infty)}\) is the Koszul-Tate projective resolution \(K_{A/I}\) of \(A/I\) and is isomorphic to \(A \otimes \Lambda P\), where \(P\) is the graded vector space with basis \(\{\mathcal{P}_{a_1}, \mathcal{P}_{a_2}, \mathcal{P}_{a_3}, \ldots\}\). The product \(\pi_K\) on \(K_{A/I}\) is a chain map. It is straightforward to check that \(H_{m_1}(\pi_K) = \pi_{A/I}\) that is to say, condition (iii) in definition 2.24 is satisfied.

A chain \(f^I\mathcal{P}_I\) in \(K_{A/I}\), where \(\mathcal{P}_I = \mathcal{P}_{a_{n_1}} \cdots \mathcal{P}_{a_{n_I}}\), has homogeneous degree, which we again denote by \(I\). In context, we will not confuse the index \(I\) with the degree \(I\). The boundary of an antighost \(\mathcal{P}_{a_n}\) will be denoted as \(Z_{\alpha_{n-1}, n}^I \mathcal{P}_{a_{n-1}} + Z_{\alpha_n}^I \mathcal{P}_I\) whenever the “linear” term plays an important role. Otherwise, we will set
\[ m_1(P_{\alpha n}) = Z^f_{\alpha n} P_J. \]
Whenever possible, an element of \( K_{A/I} \) will be denoted simply by \( \lambda \).

When \( A \) is a Poisson algebra and the ideal \( I \) is a multiplicative ideal, the Koszul-Tate resolution \( K_{A/I} \) admits a graded Poisson bracket given by \( \{ f^I P_J, g^J P_J \} = \{ f^I, g^J \} P_I P_J \).

**The Procedure**

The construction of \((K_{A/I}, K_{I/I^2})\) relies on four features: the Poisson structure on the Koszul-Tate resolution \( K_{A/I} \), the relations among the generators of \( I \), the fact that both \( K_{A/I} \) and \( K_{I/I^2} \) are projective resolutions of \( A/I \) and \( I/I^2 \), respectively, and an \( A \)-module coderivation \( \Psi \) from \( K_{A/I} \) to \( K_{I/I^2} \). A general outline of the process follows:

**STEP 1:** Choose a Koszul-Tate resolution \( K_{A/I} \) for \( A/I \). The differential on \( K_{A/I} \) is \( m_1 \), which must satisfy condition (iii) in definition 2.24. (Already done!)

**STEP 2:** Construct a differential graded \( K_{A/I} \)-module \( K_{I/I^2} \) which is a projective resolution of \( I/I^2 \) (See propositions 3.3 and 3.4). The action \( \mu : K_{A/I} \otimes K_{I/I^2} \to K_{I/I^2} \) is free, i.e., \( K_{I/I^2} \) is isomorphic to \( K_{A/I} \otimes \Phi \), where \( \Phi \) is a graded vector space over \( k \) which is isomorphic to the graded vector space \( P \). (We have already suspended the graded vector space \( \Phi \). Therefore, \( K_{I/I^2} \) is already suspended.) We construct the differential on \( K_{I/I^2} \) using the coderivation \( \Psi \). The differential \( l_1 \) rests on \( m_1 \), so the pair \((K_{A/I}, K_{I/I^2})\) has an \( \mathcal{L}(1) \)-Lie-Rinehart structure.

**STEP 3:** The loop—defining the homotopy Lie-Rinehart structure maps \( m_n \) and \( l_n \). For \( n \geq 2 \), the loop extends the \( \mathcal{L}(n-1) \)-Lie-Rinehart structure to an \( \mathcal{L}(n) \)-Lie-Rinehart structure.

\((a_n)\). Constructing \( m_n \). Since the map \( m_n \) must satisfy the homotopy Lie-Rinehart relation \((hLRa_n)\) and be a \( K_{A/I} \)-derivation source, the map \( m_n \) is completely determined once we define it on \( \Phi^{>n-1} \otimes A \) and the basis elements \( \varphi^{[1 \to n-1]} \otimes P_n \) of \( \Phi^{>n-1} \otimes P \), such that \( \varphi_1 \leq \cdots \leq \varphi_n \) with respect to a degree-preserving total ordering of the prehosts:

i. Define \( m_n \) on \( \Phi^{>n-1} \otimes A \) by setting \( m_n(\varphi^{[1 \to n-1]} \otimes f) = (-1)^n \{ m_{n-1}(\varphi^{[1 \to n-2]} \otimes P_{n-1}), f \}. \)

Here, the inductive assumption (step \( \alpha_{n-1} \)iii “above”) guarantees that \( m_n \) is well-defined on \( \Phi^{>n-1} \otimes A \). The extended Poisson bracket on \( K_{A/I} \) in the definition guarantees that \( m_n \) will be an \( A \)-derivation source. (Does not require proof.)

ii. Verify that \( ACT_{IDn} = 0 \) on \( \Phi^{>n-1} \otimes A \) by showing that \( m_1 m_n = ACT_{OBST} \) on that subspace. (Requires proof. See proposition 3.5.)

iii. Ensure that \( m_n \) is well-defined and that \( ACT_{IDn} = 0 \) on \( \Phi^{>n-1} \otimes P \) by exploiting the acyclicity of \( K_{A/I} \) to define \( m_n \) on the basis \( \varphi^{[1 \to n-1]} \otimes P_n \) of \( \Phi^{>n-1} \otimes P \) with \( \varphi_1 \leq \cdots \leq \varphi_n \). (Requires proof. See proposition 3.6.)
iv. Since \(ACT_{ID_n}\) is completely determined by its image on the generating set \(\Phi^{\wedge n-1} \otimes A\) and \(\Phi^{\wedge n-1} \otimes P\) (recall Proposition 2.22), it follows that \(ACT_{ID_n} = 0\) on all of \(K_{I/I^2}^{\wedge n-1} \otimes K_{A/I}\) and therefore \(ACT_{ID_n} = 0\) on \(\bigwedge K_{I/I^2} \otimes K_{A/I}\). (Does not need proof.)

(\(b_n\)). Constructing \(l_n\). The map \(l_n\) must rest on \(m_n\) and is therefore completely determined once we define it on \(\Phi^{\wedge n}\).

i. Set \(l_n(\varphi_{[1 \to n]}^i) = \Psi m_n(\varphi_{[1 \to n-1]}^i \otimes P_n)\). Since \(m_n\) is well-defined on \(\Phi^{\wedge n-1} \otimes P\), it follows that the map \(l_n\) is well-defined on \(\Phi^{\wedge n}\). (Does not need proof.)

ii. Verify that \(J_{ID_n} = 0\) on \(\Phi^{\wedge n}\). We do so by showing that on the space \(\Phi^{\wedge n-1} \otimes P\), the sequence of equalities

\[ l_1 l_n(1^{\wedge n-1} \otimes \Psi) = \Psi m_1 m_n = \Psi ACT_{OBST} = J_{OBST}(1^{\wedge n-1} \otimes \Psi) \]

holds. (Requires proof. See proposition 3.7.)

iii. Since \(J_{ID_n}\) rests on \(ACT_{ID_n}\) (recall Proposition 2.23) and \(J_{ID_n} = 0\) on \(\Phi^{\wedge n}\), it follows that \(J_{ID_n} = 0\) on all of \(K_{I/I^2}^{\wedge n}\) and therefore \(J_{ID_n} = 0\) on \(\bigwedge K_{I/I^2}\). (Does not need proof.)

STEP 4: Verify that the conditions (i) and (ii) in definition 2.24 are met.

STEP 2: Constructing \(K_{I/I^2}\)

The shLie algebra \(K_{I/I^2}\) is the free \(K_{A/I}\)-module \(K_{A/I} \otimes \Phi\). The graded vector space \(\Phi\) is spanned by the graded basis \(\{\varphi_{o_1}, \varphi_{o_2}, \varphi_{o_3}, \ldots\}\), where there is a one-to-one correspondence between the preghosts \(\varphi_{o_1}\), and the antighosts \(P_{o_1}\). Each \(\varphi_{o_1}\) is assigned degree \(n\). (The vector space \(\Phi\) is the suspension of the vector space \(\Phi\) found in [Kje96].) A typical element has the form \(X^i P_i^{\alpha_n} \varphi_{o_1}\), where \(X^i P_i^{\alpha_n}\) is an element of \(K_{A/I}\). As before, the degree of \(X^i P_i^{\alpha_n}\) in \(K_{A/I}\) will be denoted by \(I\).

When \(I\) is irreducible, the map \(l_1 = m_1 \otimes 1\) is a differential on \(K_{I/I^2} \approx K_{A/I} \otimes \Phi\), whose homology is \(A/I \otimes \Phi\), which is isomorphic to \(I/I^2\) as \(A/I\)-modules. The isomorphism is given by sending \(f^{\alpha_n} \varphi_{o_1} \mapsto f^{\alpha_n} y_{o_1}\). The details are left to the reader. It follows that \(K_{I/I^2}\) is a resolution of \(I/I^2\).

We define a degree 0 \(A\)-linear map \(\Psi : K_{A/I} \rightarrow K_{I/I^2}\) by setting \(\Psi(P_{\alpha_n}) = \varphi_{\alpha_n}\) and extending \(\Psi\) as a graded derivation, i.e., \(\Psi(P_{\alpha}, P_{\alpha_j}) = (-1)^{i+j} P_{\alpha_j} \varphi_{\alpha_i} + P_{\alpha_i} \varphi_{\alpha_j}\).

In the reducible case, the map \(\Psi\) is a chain map.

In the reducible case, the graded module \(K_{I/I^2}\) does not come equipped with a differential; we must build one. The tool we need is the map \(\Psi\). We create \(l_1\) on \(K_{I/I^2}\) so that \(\Psi\) is a chain map. We set \(l_1(\varphi_{\alpha_n}) = \Psi m_1(P_{\alpha_n})\) and extend \(l_1\) to all of \(K_{I/I^2}\) so that it rests on \(m_1\). The map \(\Psi\) is a chain map provided \(l_1\) is a differential on \(K_{I/I^2}\).
Proposition 3.3. \( l_1 \circ l_1 = 0 \) on \( K_{I/I^2} \).

Proof of Proposition 3.3. We begin by showing that \( l_1(l_1(\varphi_{\alpha_n})) = 0 \). Recall that \( \Psi \) is a chain map. It follows that \( l_1(l_1(\varphi_{\alpha_n})) = l_1(\Psi(m_1(P_{\alpha_n}))) = \Psi(m_1(m_1(P_{\alpha_n}))) = 0 \). Since \( l_1 \) rests on \( m_1 \), it follows that

\[
\begin{align*}
l_1(l_1(\alpha^n f_{\alpha_n})) &= l_1(m_1(\alpha^n f_{\alpha_n})) + (-1)^n \alpha^n l_1(\varphi_{\alpha_n}) \\
&= m_1(m_1(\alpha^n f_{\alpha_n})) + (-1)^{n-1} m_1(\alpha^n f_{\alpha_n}) l_1(\varphi_{\alpha_n}) \\
&
\end{align*}
\]

Since the \( \{\alpha^n f_{\alpha_n}\} \) term collapses to one column, the homology of \( K_{I/I^2} \) resolves \( I/I^2 \) as modules.

Proposition 3.4. \( H_l(K_{I/I^2}) = I/I^2 \).

Proof of Proposition 3.4. The module \( K_{I/I^2} \) is naturally bigraded. The bidegree of the element \( \alpha^n f_{\alpha_n} \) is \((\alpha_n, n)\). The differential decomposes as \( l_1 = m_1(1 + d^{(1)}) + d^{(2)} + \cdots \) with respect to the filtration of \( K_{I/I^2} \) by preghost degree, where \( m_1 \) has bidegree \((-1, 0)\) and \( d^{(i)} \) has bidegree \((i-1, i)\). The differential on the \( E_0 \) term is \( m_1 \). The \( k^{th} \) row of the \( E_0 \) term is \( K_{A/I} \otimes \Phi^{(k)} \) where \( \Phi^{(k)} \) is the vector space spanned by the \( \varphi_{\alpha_k} \)'s. Each row is exact except in the first slot. Therefore the \( E_1 \) term of the spectral sequence is concentrated in the first column:

\[
\begin{array}{c}
E_1:
\end{array}
\]

\[
\begin{array}{c}
A/I \otimes \Phi^{(3)} \\
A/I \otimes \Phi^{(2)} \\
A/I \otimes \Phi^{(1)} \\
0
\end{array}
\]

Since the \( E_1 \) term collapses to one column, the homology of \( K_{I/I^2} \) with respect to \( l_1 \) will be the homology of the \( E_1 \) term.

Let \( K \) be the kernel of the map \( \pi: A/I \otimes \Phi^{(1)} \rightarrow I/I^2 \) which sends \( f_{\alpha_1} \otimes \varphi_{\alpha_1} \) to \( f_{\alpha_1} \). We need to show that \( d^{(1)}(A/I \otimes \Phi^{(1)}) = K \). \((\subseteq)\): Since \( l_1 \varphi_{\alpha_n} = Z_{\alpha_n} \varphi_{\alpha_{n-1}} + \ldots \) it follows that \( d^{(1)} f_{\alpha_1} \varphi_{\alpha_n} = f_{\alpha_1} Z_{\alpha_n} \varphi_{\alpha_{n-1}} \). Consider any element \( f_{\alpha_2} \varphi_{\alpha_2} \) of \( A/I \otimes \Phi^{(2)} \). The image under \( \pi d^{(1)} \) is zero because for all \( \alpha_2 \) the sum \( Z_{\alpha_2} / y_{\alpha_1} = 0 \). \((\supseteq)\): Suppose \( \pi(f_{\alpha_1}) = f_{\alpha_1} \). Recalling that the ideal
\( \mathcal{I} \) admits a complete set of reducibility functions, we see that
\[
\begin{align*}
  f^{\alpha_1} y_{\alpha_1} = h^{\alpha_1; \beta_1} y_{\beta_1} y_{\alpha_1} & \Rightarrow (f^{\alpha_1} - h^{\alpha_1; \beta_1} y_{\beta_1}) y_{\alpha_1} = 0 \\
  & \Rightarrow f^{\alpha_1} \equiv g^{\alpha_2} Z^{\alpha_2} \text{ mod } \mathcal{I}.
\end{align*}
\]

It then follows that \( \tilde{d}^{(1)}(g^{\alpha_2} \varphi_{\alpha_2}) = g^{\alpha_2} Z^{\alpha_2} \varphi_{\alpha_1} = f^{\alpha_1} \varphi_{\alpha_1} \). We conclude that the zeroth cohomology of \( K_{\mathcal{I}/T^2} \) with respect to \( l_1 \) is \( (A/\mathcal{I} \otimes \Phi^{(1)})/K \), which is isomorphic to \( I/I^2 \) as modules. For higher cohomologies, suppose \( f^{\alpha_n} \varphi_{\alpha_n} \) is a cocycle. Then
\[
\begin{align*}
  f^{\alpha_n} Z^{\alpha_{n-1}} \varphi_{\alpha_{n-1}} = 0 & \Rightarrow f^{\alpha_n} Z^{\alpha_{n-1}} \in \mathcal{I} \\
  & \Rightarrow f^{\alpha_n} \equiv g^{\alpha_{n+1}} Z^{\alpha_{n+1}} \text{ mod } \mathcal{I} \\
  & \Rightarrow f^{\alpha_n} \varphi_{\alpha_n} = g^{\alpha_{n+1}} Z^{\alpha_{n+1}} \varphi_{\alpha_n} = \tilde{d}^{(1)} g^{\alpha_{n+1}} \varphi_{\alpha_{n+1}}.
\end{align*}
\]

Hence, the higher cohomologies are zero. \( \square \)

The map \( \Psi \) is quite useful. We constructed \( l_1 \) so that \( \Psi \) was a chain map. In the notation of shLie algebras and shLie modules, we were able to show that \( \tilde{J}_{ID1} = 0 \) on \( \Phi \) because \( \Psi \cdot ACT_{ID1} = \tilde{J}_{ID1} \Psi \) and \( ACT_{ID1} = 0 \) on \( \Psi^{-1} \Phi \). We shall see this pattern of proof again in proposition 3.7.

Let \( \mathbf{P}_I = \mathbf{P}_{I_1} \cdots \mathbf{P}_{I_n} \in K_{A/T} \). Then \( \Psi(\mathbf{P}_I) = \kappa(I_i) \mathbf{P}_I^{i+1} \varphi_{I_i} \), where \( \mathbf{P}_I^{i+1} = \mathbf{P}_{I_i} \cdots \mathbf{P}_{I_{i-1}} \mathbf{P}_{I_{i+1}} \cdots \mathbf{P}_{I_{n-1}} \) and \( \mathbf{P}_{I_i} \) indicates that this factor is omitted. The sign \( \kappa(I_i) \) is the sign produced by moving \( \mathbf{P}_{I_i} \) past \( \mathbf{P}_{I_{i-1}} \cdots \mathbf{P}_{I_{i+1}} \cdots \mathbf{P}_{I_{n-1}} \). Recalling that \( m_1 \mathbf{P}_{I_1} = Z_{I_1} \mathbf{P}_{I_1} \), we can write \( l_1(\varphi_{I_1}) = \kappa(I_1) Z_{I_1} \mathbf{P}_I^I \varphi_{I_i} \).

**STEP 3: The loop**

We prove the three propositions needed to complete the inductive step of the loop. Then we will return to \( n = 2 \) and define \( m_2 \) at the one location where both \( K_{A/T} \) and \( K_{I/T^2} \) are not exact.

**Proposition 3.5. (3a:nii)** If we set \( m_n(\varphi_{[1 \to n-1]} \otimes f) = (-1)^n \{ m_{n-1}(\varphi_{[1 \to n-2]} \otimes P_{n-1}), f \} \), for \( \varphi_1 \leq \cdots \leq \varphi_{n-1} \), then
\[
m_1(m_n(\varphi_{[1 \to n-1]} \otimes f)) = ACT_{\text{obst}} n(\varphi_{[1 \to n-1]} \otimes f).
\]

**Proof of Proposition 3.5.** We omit sign arguments and specific signs, opting instead for ±. Using the definition, \( m_1 m_n(\varphi_{[1 \to n-1]} \otimes f) \)
\[
\begin{align*}
  & = \pm m_1 \{ m_{n-1}(\varphi_{[1 \to n-2]} \otimes P_{n-1}), f \} \\
  & = \pm \{ m_1(m_{n-1}(\varphi_{[1 \to n-2]} \otimes P_{n-1})), f \} \pm Z_{[1 \to n-1]}^A \{ m_1(P_A), f \},
\end{align*}
\]
where \( m_n-1(\varphi_{[1 \to n-2]} \otimes P_{n-1}) = Z_{[1 \to n-1]}^A \mathbf{P}_A \). Now, \( ACT_{\text{obst}} n(\varphi_{[1 \to n-1]} \otimes f) \) contains one term \( \pm m_2(l_{n-1}(\varphi_{[1 \to n-1]} \otimes f)) \) which uses the \((n-1,0)\)-shuffle. But since
\[
l_{n-1}(\varphi_{[1 \to n-1]}) = \Psi m_{n-1}(\varphi_{[1 \to n-1]} \otimes \mathbf{P}_{n-1}) = Z_{[1 \to n-1]}^A \mathbf{P}_A \varphi_{\beta}.
\]
and $m_2$ is $K_{A/I}$-linear,

$$m_2(Z_{[1 \text{ to } n-1]}^A P_A^\beta \otimes \varphi) = \pm Z_{[1 \text{ to } n-1]}^A P_A^\beta \{m_1 P_A, \varphi\} = \pm Z_{[1 \text{ to } n-1]}^A \{m_1 (P_A), \varphi\}. $$

Since $m_1 m_{n-1}$ is equal to $ACT_{OBST_{n-1}}$, we may rewrite $\pm \{m_1 m_{n-1} (\varphi_{[1 \text{ to } n-2]} \otimes P_{n-1}), f\}$ as $\pm \{ACT_{OBST_{n-1}} (\varphi_{[1 \text{ to } n-2]} \otimes P_{n-1}), f\}$, which must equal the remaining terms in the map $ACT_{OBST_{n}} (\varphi_{[1 \text{ to } n-1]} \otimes f)$.

We organize the remaining terms by unshuffle. Let $\varphi$ be an $(i, j-2)$-unshuffle. Applying $\sigma$ to $\varphi_{[1 \text{ to } n-1]}$ produces two hands $\varphi_{[1 \text{ to } i]}$ and $\varphi_{[i+1 \text{ to } n-1]}$. In $ACT_{OBST_{n}} (\varphi_{[1 \text{ to } n-1]} \otimes f)$, there are two terms in which the $\varphi_i$’s appear in the $\sigma$ order. They are

$$(\tilde{M}_i \tilde{L}_i \sigma) = \pm m_j (l_i (\varphi_{[1 \text{ to } i]} \otimes \varphi_{[i+1 \text{ to } n-1]} \otimes f)$$

and

$$(\tilde{M}_{i+1} \tilde{M}_{j-1} \sigma) = \pm m_{i+1} (\varphi_{[1 \text{ to } i]} \otimes \varphi_{[i+1 \text{ to } n-1]} \otimes f).$$

There is a unique $(j-2, i)$-unshuffle $\rho$ which switches the $\sigma$ hands, i.e., $\varphi_{\rho(j-1) \text{ to } n-1]} = \varphi_{[1 \text{ to } i]}$ and $\varphi_{[j-2] \text{ to } n-1]} = \varphi_{[i+1 \text{ to } n-1]}$. The two terms in which the $\varphi_i$’s appear in the $\rho$ order, written in terms of $\sigma$ are

$$(\tilde{M}_{i+2} \tilde{L}_{j-2} \sigma) = \pm m_{i+2} (l_j (\varphi_{[i+1 \text{ to } n-1]} \otimes \varphi_{[1 \text{ to } i]} \otimes f)$$

and

$$(\tilde{M}_{j-1} \tilde{M}_{i+1} \sigma) = \pm m_{j-1} (\varphi_{[1 \text{ to } i]} \otimes m_{i+1} (\varphi_{[1 \text{ to } i]} \otimes f)).$$

Organized in this way, the terms remaining in $ACT_{OBST_{n}} (\varphi_{[1 \text{ to } n-1]} \otimes f)$ can be rewritten as

$$\frac{1}{2} \sum_{\substack{i+j=n-1 \\text{ and } \rho \text{ is } (j-2, i) \text{-unshuffle} \\text{ order} \\text{ and } \rho \text{ is } (j-2, i) \text{-unshuffle}} \text{ order}} \sum_{2 \leq j \leq n} \tilde{M}_i \tilde{L}_i \sigma + \tilde{M}_{i+1} \tilde{M}_{j-1} \sigma + \tilde{M}_{i+2} \tilde{L}_{j-2} \sigma + \tilde{M}_{j-1} \tilde{M}_{i+1} \sigma.$$

Without loss of generality, we now compute the sum

$$\tilde{M}_i \tilde{L}_i e + \tilde{M}_{i+1} \tilde{M}_{j-1} e + \tilde{M}_{i+2} \tilde{L}_{j-2} e + \tilde{M}_{j-1} \tilde{M}_{i+1} e$$

for the identity $(i, j-2)$-unshuffle $e$ and show that the sum equals the two terms in

$$\pm \{ACT_{OBST_{n-1}} (\varphi_{[1 \text{ to } n-2]} \otimes P_{n-1}), f\}$$

for which $\varphi_{[1 \text{ to } n-2]} \otimes P_{n-1}$ is split into the hands $\varphi_{[1 \text{ to } i]}$ and $\varphi_{[i+1 \text{ to } n-2]} \otimes P_{n-1}$. The terms of the sum $\tilde{M}_i \tilde{L}_i e + \tilde{M}_{i+1} \tilde{M}_{j-1} e + \tilde{M}_{i+2} \tilde{L}_{j-2} e + \tilde{M}_{j-1} \tilde{M}_{i+1} e$ expand to become
\[
\begin{align*}
(\mathcal{M}_{j} \mathcal{L}_{n} e) & = \pm \{m_{j-1}(l_{i}(\varphi_{[1 \to i]} \wedge \varphi_{[i+1 \to n-2]} \otimes P_{n-1}, f) + \{Z_{[1 \to i]}^{A}, f\}m_{j-1}(\varphi_{[i+1 \to n-1]} \otimes P_{A}), \quad (A) \\
(\tilde{\mathcal{M}}_{j-1} \mathcal{M}_{i+1} e) & = \pm \{Z_{[i+1 \to n-1]}^{B}, \{Z_{[1 \to i]}^{A}, f\}P_{B}P_{A} \pm \{Z_{[1 \to i]}^{A}, f\}m_{j-1}(\varphi_{[i+1 \to n-1]} \otimes P_{A}), \quad (B) \\
(\mathcal{M}_{i+1} \mathcal{M}_{j-1} e) & = \pm \{Z_{[1 \to i]}^{A}, \{Z_{[i+1 \to n-1]}^{B}, f\}P_{A}P_{B} \pm \{Z_{[i+1 \to n-1]}^{B}, f\}m_{i+1}(\varphi_{[1 \to i]} \otimes P_{B}), \quad (C) \\
(\tilde{\mathcal{M}}_{i+2} \mathcal{L}_{j-2} e) & = \pm \{Z_{[i+1 \to n-1]}^{B}, m_{i+1}(\varphi_{[1 \to i]} \otimes P_{B}), f\} \pm \{Z_{[i+1 \to n-1]}^{B}, f\}m_{i+1}(\varphi_{[1 \to i]} \otimes P_{B}), \quad (D)
\end{align*}
\]

Terms which are identical up to sign cancel, i.e., \((B)\) cancels with \((D)\), \((F)\) cancels with \((H)\). After exchanging \(P_{B}\) and \(P_{A}\) in \((C)\), we can combine \((C)\) with \((E)\) and use the Jacobi identity to produce \(\pm \{Z_{[1 \to i]}^{A}, Z_{[i+1 \to n-1]}^{B}\}P_{A}P_{B}, f\), where we have brought \(P_{A}P_{B}\) inside the bracket. Since \(m_{i+1}(\varphi_{[1 \to i]} \otimes Z_{[i+1 \to n-1]}^{B}) = \pm \{Z_{[1 \to i]}^{A}, Z_{[i+1 \to n-1]}^{B}\}P_{A}, \) the term above becomes \(\pm \{m_{i+1}(\varphi_{[1 \to i]} \otimes Z_{[i+1 \to n-1]}^{B})P_{B}, f\}. \) To this we add \((G)\), yielding one of the desired terms: \(\pm \{m_{i+1}(\varphi_{[1 \to i]} \otimes m_{j-1}(\varphi_{[i+1 \to n-2]} \otimes P_{n-1}), f\}). \) This term and \((A)\) are the two terms in \(\pm \{ACT_{\text{OBST}_{n-1}}(\varphi_{[1 \to n-2]} \otimes P_{n-1}), f\}\) for which \(\varphi_{[1 \to n-2]} \otimes P_{n-1}\) is split into the hands \(\varphi_{[1 \to i]}\) and \(\varphi_{[i+1 \to n-2]} \otimes P_{n-1}. \) Therefore, for each \((i, j - 2)\)-unshuffle \(\sigma\), we produce a term of the form \(\pm \{m_{j-1}(l_{i}(\text{first} \ \sigma \ \text{hand}) \otimes (\text{second} \ \sigma \ \text{hand}), f\}\) and a term of the form \(\pm \{m_{i+1}(\text{first} \ \sigma \ \text{hand}) \otimes m_{j-1}(\text{second} \ \sigma \ \text{hand}), f\}\) with the appropriate signs. As \(j\) runs from 2 to \(n, j - 1\) runs from 1 to \(n - 1, \) so as we run through all \((i, j - 2)\)-unshuffles and divide by two, we produce \(\pm \{ACT_{\text{OBST}_{n-1}}(\varphi_{[1 \to n-2]} \otimes P_{n-1}), f\}\). \(\square\)

**Proposition 3.6.** (3a_{iii}) For \(n \geq 3, \) one can define \(m_{n}(\varphi_{[1 \to n-1]} \otimes P_{n})\) for each element of the totally ordered basis for \(\Phi^{n-1} \otimes P\) so that \(m_{n}\) is well-defined and so that

\[
m_{1}(m_{n}(\varphi_{[1 \to n-1]} \otimes P_{n})) = ACT_{\text{OBST}_{n}}(\varphi_{[1 \to n-1]} \otimes P_{n}).
\]

**Proof of Proposition 3.6.** We ensure that \(m_{n}\) is well-defined on \(\Phi^{n-1} \otimes P\) by defining \(m_{n}\) on the basis elements \(\varphi_{[1 \to n-1]} \otimes P_{n}\) with \(\varphi_{1} \leq \cdots \leq \varphi_{n}\) with respect to the total ordering of the prehosts. The image of \(\varphi_{[1 \to n-1]} \otimes P_{n}\) under \(ACT_{\text{OBST}_{n}}\) is in \(K_{A/I}\). The map \(ACT_{\text{OBST}_{n}}\) has degree \(-2, \) so for \(n \geq 3, \) the degree of \(ACT_{\text{OBST}_{n}}(\varphi_{[1 \to n-1]} \otimes P_{n})\) is at least one, at which level \(K_{A/I}\) is exact. So if we can show that

\[
m_{1}ACT_{\text{OBST}_{n}}(\varphi_{[1 \to n-1]} \otimes P_{n}) = 0,
\]

then a pre-image exists which we can set equal to \(m_{n}(\varphi_{[1 \to n-1]} \otimes P_{n})\). Since \(ACT_{\text{OBST}_{n}} = ACT_{n} - m_{n}m_{1}\) and that \(ACT_{n}\) is a symmetric chain map, we find
that
\[ m_1\text{ACT}_{OBST,n} = m_1(\text{ACT}_n - m_n m_1) = m_1\text{ACT}_n - m_1 m_n m_1 = \text{ACT}_n m_1 - m_1 m_n m_1 = (\text{ACT}_n - m_1 m_n)m_1. \]

Let us begin by examining \( m_n \) on an ordered basis element \( \varphi[1 \to n-1] \otimes P_n \) with \( |\varphi_i| = 1 \) for all \( i \). For such a basis element we have
\[
m_1\text{ACT}_{OBST,n}(\varphi[1 \to n-1] \otimes P_n) = (\text{ACT}_n - m_1 m_n) m_1(\varphi[1 \to n-1] \otimes P_n) = (\text{ACT}_n - m_1 m_n)(\varphi[1 \to n-1] \otimes y_n).
\]

Since we already know that \( \text{ACT}_{ID,n}(\varphi[1 \to n-1] \otimes y_n) = 0 \), we can replace \( \text{ACT}_n - m_1 m_n \) with \((-1)^{n-1} m_n m_1 \), yielding \( m_n m_1(\varphi[1 \to n-1] \otimes y_n) \), which equals 0.

The proof is completed by strong induction. Let \( \varphi[1 \to n-1] \otimes P_{n_k} \) be the \( k \)th basis element in the ordered list and suppose \( \text{ACT}_{ID,n}(\varphi[1 \to n-1] \otimes P_{n_i}) = 0 \) for all basis elements before the \( k \)th one, where once again,
\[
m_1\text{ACT}_{OBST,n}(\varphi[1 \to n-1] \otimes P_{n_k}) = (\text{ACT}_n - m_1 m_n) m_1(\varphi[1 \to n-1] \otimes P_{n_k}).
\]

The element \( m_1(\varphi[1 \to n-1] \otimes P_{n_k}) \) can only have pieces in \( K^{p\alpha}/I^2 K_{A/I} \) for \( 0 \leq p \leq n-1 \) and, moreover, the pieces in \( K^{\alpha n-1}/I^2 K_{A/I} \) contain only basis elements of order less than that of \( \varphi[1 \to n-1] \otimes P_{n_k} \). So for \( m_1(\varphi[1 \to n-1] \otimes P_{n_k}) \), we can replace \( \text{ACT}_n - m_1 m_n \) with \( m_n m_1 \). It follows that
\[
(\text{ACT}_n - m_1 m_n) m_1(\varphi[1 \to n-1] \otimes P_{n_k}) = m_n(m_1(\varphi[1 \to n-1] \otimes P_{n_k}))) = 0.
\]

**Proposition 3.7.** (3bii) If we set \( l_n(\varphi[1 \to n]) = \Psi m_n(\varphi[1 \to n-1] \otimes P_n) \), then \( \mathcal{J}_{ID,n} = 0 \) on \( \Phi^{\wedge n} \).

**Proof of Proposition 3.7.** As outlined above, we will prove that \( \mathcal{J}_{ID,n} = 0 \) on \( \Phi^{\wedge n} \) by showing that on \( \Phi^{\wedge n-1} \otimes P \), the following equalities hold:
\[
l_1 l_n(1^{\wedge n-1} \otimes \Psi) = \Psi m_1 m_n = \Psi \text{ACT}_{OBST,n} = \mathcal{J}_{OBST,n}(1^{\wedge n-1} \otimes \Psi).
\]

The first equality holds because the map \( \Psi \) is a chain map. Since \( \text{ACT}_{ID,n} = 0 \), it follows that \( m_1 m_n = \text{ACT}_{OBST,n} \). The second equality holds because \( \Psi m_1 m_n = \Psi \text{ACT}_{OBST,n} \). Showing that \( \Psi \text{ACT}_{OBST,n} = \mathcal{J}_{OBST,n}(1^{\wedge n-1} \otimes \Psi) \) is more difficult because \( \Psi \) does not commute with \( m_i \) for \( i > 1 \). We will need to organize \( \text{ACT}_{OBST,n} \) and \( \mathcal{J}_{OBST,n} \) by unshuffle. Let \( \sigma \) be any \((i, i-2)-unshuffle\). The two terms in the sum \( \text{ACT}_{OBST,n} \) for which \( \varphi[1 \to n-1] \otimes P_n \) is split into the hands \( \varphi[1 \to i] \) and \( \varphi[i+1 \to n-1] \otimes P_n \) are
\[
\pm m_j(\varphi[1 \to i] \wedge \varphi[i+1 \to n-1] \otimes P_n) = \pm m_{j+1}(\varphi[1 \to i] \otimes m_j m_{i-1}(\varphi[i+1 \to n-1] \otimes P_n)).
\]

We will show that when \( \Psi \) is applied to each of the terms above, their sum becomes
\[
\pm l_j(\varphi[1 \to i] \wedge \varphi[i+1 \to n-1] \wedge \varphi_n) = \pm l_{i+1}(\varphi[1 \to i] \wedge l_j(\varphi[i+1 \to n-1] \wedge \varphi_n)).
\]
This fact completes the proof because we can expand $\mathcal{J}_{\text{OBST}}(\varphi_{[1 \to n]})$ so that $\varphi_n$ is never moved. For any $(i, j - 1)$-unshuffle which moves $\varphi_n$, we will use the graded symmetry of the shLie structure maps to move $\varphi_n$ back to the last position. With this approach,

$$
\mathcal{J}_{\text{OBST}}(\varphi_{[1 \to n]}) = \sum_{i+j=n+1 \atop i \geq 0 \atop j > 0 \atop i, j \geq -2} \left( l_j (l_i (\varphi_{[1 \to i]} \wedge \varphi_{[i+1 \to n]}) \pm l_{i+1} (\varphi_{[1 \to i]} \wedge l_j - 1 (\varphi_{[i+1 \to n]})) \right).
$$

Without loss of generality, we shall show that the desired equality holds for the identity $(i, j - 2)$-unshuffle $c$. The $c$-terms of $\mathcal{A} \mathcal{T}_{\text{OBST}}(\varphi_{[1 \to n-1]} \otimes \mathcal{P}_n)$ are

$$m_j (l_i (\varphi_{[1 \to i]} \wedge \varphi_{[i+1 \to n-1]} \otimes \mathcal{P}_n))
= \pm Z_{[1 \to i]}^A P_A e_m j (\varphi_{A_n} \wedge \varphi_{[i+1 \to n-1]} \otimes \mathcal{P}_n)
= (\mathcal{M}_{i+j}, \mathcal{L}_j)
$$

and

$$m_{i+1} (\varphi_{[1 \to i]} \otimes m_{j-1} (\varphi_{[i+1 \to n-1]} \otimes \mathcal{P}_n))
= \pm \{Z_{[1 \to i]}^A, Z_{[i+1 \to n]}^B \} P_A P_B
= (\{i, j-2\})
$$

$$\pm Z_{[i+1 \to n]}^B P_B e_{m_{i+1}} (\varphi_{[1 \to i]} \otimes \mathcal{P}_B).
= (\mathcal{M}_{i+1} \mathcal{M}_{j-1})
$$

Applying $\Psi$ to $\mathcal{M}_{i+j} \mathcal{L}_j$, produces

$$\pm Z_{[1 \to i]}^A P_A e_l (\varphi_{A_n} \wedge \varphi_{[i+1 \to n]} \otimes Z_{[1 \to i]}^A) \varphi_{A_n}.
= \mathcal{A} \mathcal{T}_{\text{OBST}}(\varphi_{[1 \to n-1]} \otimes \mathcal{P}_n)
$$

where the map $\Psi$ acts on $P_A P_B$ in the $\{i, j-2\}$ term to produce two terms $\pm P_B e_p A \varphi_{A_n} \pm P_A P_B e_r B$. Using the derivational property of the bracket and the definition of $m_j$, the first term of $\{i, j-2\}$ equals $m_j (\varphi_{[i+1 \to n]} \otimes Z_{[1 \to i]}^A) P_A e_m a n$. Adding this term to $\Psi (\mathcal{M}_{i+j} \mathcal{L}_j)$, we have

$$\pm Z_{[1 \to i]}^A P_A e_l (\varphi_{A_n} \wedge \varphi_{[i+1 \to n]} \otimes Z_{[1 \to i]}^A) \varphi_{A_n},
= \mathcal{A} \mathcal{T}_{\text{OBST}}(\varphi_{[1 \to n-1]} \otimes \mathcal{P}_n)
$$

which equals $l_j (l_i (\varphi_{[1 \to i]} \wedge \varphi_{[i+1 \to n]}))$. Similarly, the sum of $\Psi (\mathcal{M}_{i+1} \mathcal{M}_{j-1})$ and the second term of $\Psi (\{i, j-2\})$ is the second desired term

$$\pm l_{i+1} (\varphi_{[1 \to i]} \wedge l_j - 1 (\varphi_{[i+1 \to n]})).
$$

We produce all the terms in $\mathcal{J}_{\text{OBST}}(\varphi_{[1 \to n]})$ as we run through all $(i, j - 2)$-unshuffles with $2 < j < n$. □

Return to $n = 2$:
We can define $m_2 (\varphi_{\alpha_n} \otimes \mathcal{P}_{\beta_n})$ inductively on $\varphi_{\alpha_n} \otimes \mathcal{P}_{\beta_n}$ with $\varphi_{\alpha_n} \leq \varphi_{\beta_n}$, just as in proposition 3.6, except for $\varphi_{\alpha_1} \otimes \mathcal{P}_{\beta_1}$ with $\varphi_{\alpha_1} \leq \varphi_{\beta_1}$. Since $m_2$ must be a chain map, $m_1 (m_2 (\varphi_{\alpha_1} \otimes \mathcal{P}_{\beta_1}))$ should equal $m_2 (\varphi_{\alpha_1} \otimes m_1 (\mathcal{P}_{\beta_1}))$. Having chosen structure functions $C_{\alpha_1 \beta_1}$ in $A$, we find that $m_2 (\varphi_{\alpha_1} \otimes m_1 (\mathcal{P}_{\beta_1})) = \{y_{\alpha_1}, y_{\beta_1}\} = C_{\alpha_1 \beta_1} y_{\gamma_1}$, which is in $\mathcal{I}$. Therefore a pre-image exists under $m_1$; we set $m_2 (\varphi_{\alpha_1} \otimes \mathcal{P}_{\beta_1}) = C_{\alpha_1 \beta_1} \mathcal{P}_{\gamma_1}$. 


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We have completed the definition of $m_2$ and in doing so, have finished the definition of $l_2$ on $K_{I/I^2}$ as well.

**STEP 4:**
To complete the construction of the homotopy Lie-Rinehart resolution $(K_{A/I}, K_{I/I^2})$, we must verify that $H_{m_1}(m_2) = \hat{m}_2$ on $I/I^2 \otimes A/I$ and that $H_{l_1}(l_2) = \hat{l}_2$ on $I/I^2 \wedge I/I^2$, i.e., conditions (ii) and (i) in definition 2.24 are satisfied.

Let $\lambda_1$ be a splitting from $A/I$ into $K_{A/I}$ and let $\lambda_2$ be a splitting of $I/I^2$ into $K_{I/I^2}$. Then for $\sigma \otimes f \in I/I^2 \otimes A/I$, the map $m_2$ maps $\lambda_2(I/I^2) \otimes \lambda_1(A/I)$ into $A$. It follows that $H_{m_1}(\sigma \otimes f) = m_2(\lambda_2(\sigma) \otimes \lambda_1(f))$. Suppose $\lambda_2(\sigma) = g^{\alpha_1}_{\alpha_2} \varphi_{\alpha_1}$, which implies that $\sigma = \hat{g}^{\alpha_1}_{\alpha_2} y_{\alpha_1}$. Then

$$m_2(g^{\alpha_1}_{\alpha_2} \varphi_{\alpha_1} \otimes \lambda_1(f)) = g^{\alpha_1}_{\alpha_2} [y_{\alpha_1}, \lambda_1(f)]$$
$$= \{g^{\alpha_1}_{\alpha_2} y_{\alpha_1}, \lambda_1(f)\} - \{g^{\alpha_1}_{\alpha_2}, \lambda_1(f)\} y_{\alpha_1}$$
$$= \{g^{\alpha_1}_{\alpha_2} y_{\alpha_1}, \lambda_1(f)\}$$
$$= \hat{m}_2(\sigma \otimes f).$$

Similarly, for any $\sigma \wedge \overline{\sigma}$, there exists an element $g^{\alpha_1}_{\alpha_2} \varphi_{\alpha_1} \wedge h^{\beta_1}_{\beta_2} \varphi_{\beta_1}$ in $K_{I/I^2} \wedge K_{I/I^2}$ whose class is $\sigma \wedge \overline{\sigma}$. Then

$$H_{l_1}(l_2)(\sigma \wedge \overline{\sigma}) = [l_2(g^{\alpha_1}_{\alpha_2} \varphi_{\alpha_1} \wedge h^{\beta_1}_{\beta_2} \varphi_{\beta_1})]$$
$$= \{g^{\alpha_1}_{\alpha_2} h^{\beta_1}_{\beta_2} C^{\gamma_1}_{\alpha_1, \beta_1} \varphi_{\gamma_1} + g^{\alpha_1}_{\alpha_2} m_2(\varphi_{\alpha_1} \otimes h^{\beta_1}_{\beta_2}) \varphi_{\beta_1} - h^{\beta_1}_{\beta_2} m_2(\varphi_{\beta_1} \otimes g^{\alpha_1}_{\alpha_2}) \varphi_{\alpha_1}\}$$
$$= \{g^{\alpha_1}_{\alpha_2} h^{\beta_1}_{\beta_2} C^{\gamma_1}_{\alpha_1, \beta_1} y_{\gamma_1} + g^{\alpha_1}_{\alpha_2} [y_{\alpha_1}, h^{\beta_1}_{\beta_2}] y_{\beta_1} - h^{\beta_1}_{\beta_2} [y_{\beta_1}, g^{\alpha_1}_{\alpha_2}] y_{\alpha_1}\}$$
$$= \{g^{\alpha_1}_{\alpha_2} y_{\alpha_1}, h^{\beta_1}_{\beta_2} \overline{y}_{\beta_1}\}$$
$$= \hat{l}_2(\sigma \wedge \overline{\sigma}).$$

We conclude that the homotopy Rinehart algebra $R$ for the homotopy Lie-Rinehart resolution $(K_{A/I}, K_{I/I^2})$ is a model for the Rinehart algebra $R$ for the Lie-Rinehart pair $(A/I, I/I^2)$.

4. Comparing the homotopy Rinehart algebra $R$ with the BRST algebra $A$

Following [FV75], [BF83] and [BV85], the classical BRST algebra $A$ for the Poisson reduction of the Poisson algebra $A$ by a finitely presented coisotropic ideal $\mathcal{I}$ is a differential graded Poisson algebra, built from a specific choice of Koszul-Tate resolution $K_{A/I}$ of $A/I$. The BRST algebra $A$ is formed by tensoring the graded commutative algebra $\Lambda N$ with $K_{A/I}$, where $N$ and $P$ are isomorphic as vector spaces over $k$. The basis elements of $N$ are denoted by $\{\eta^{\alpha_1}, \eta^{\alpha_2}, \eta^{\alpha_3}, \ldots\}$, and are called *ghosts*. The result is

$$A \approx \Lambda N \otimes A \otimes \Lambda P$$
(see [FHST89], [HT88], [Sta92], [HT92], [Sta96] etc.). Each $\eta^n$ has the same degree as the corresponding $\mathcal{P}_n$, namely $n$. If we let $N^I$ denote a string of ghosts $\eta^{\alpha_1} \wedge \cdots \wedge \eta^{\alpha_d} = \eta^{\alpha_{d+1} \cdots d}$, the degree of $N^I$ is the sum $\sum_1^d n_p$, which we shall denote by $I$ without confusion. There are two significant gradings on the BRST algebra $\mathcal{A}$. Let $\mathbf{N}^I f_j^I \mathbf{P}_j$ be an element of $\mathcal{A}$. The first grading is the internal degree, which is the difference between the ghost degree $gh(N^I f_j^I \mathbf{P}_j) = I$ and the antighost degree $\text{antigh}(N^I f_j^I \mathbf{P}_j) = J$. The second grading is by ghost number. The ghost number $gh\#(N^I f_j^I \mathbf{P}_j) = i$ if $N^I = \eta^{\alpha_{d+1} \cdots d}$. The multiplication on $\mathcal{A}$ is given by the multiplication in each of its three pieces and obeys the Koszul sign convention, that is to say,

$$(N^I f_j^I \mathbf{P}_j)(\mathbf{N}^K f_k^I \mathbf{P}_k) = (-1)^{IJK} \mathbf{N}^I \mathbf{N}^K f_j^I f_k^I \mathbf{P}_j \mathbf{P}_k.$$  

The Poisson bracket on $\mathbf{K}_{A/\mathcal{I}}$ is extended to all of $\mathcal{A}$ by setting $\{N, N\} = \{N, A\} = 0$ and $\{\eta^n, \mathcal{P}_{\beta_n}\} = \delta^\alpha_{\beta_n}$ (where $\delta^\alpha_{\beta_n}$ is the Kronecker delta). If we view $N$ as the dual of $P$, this bracket formula is the usual symplectic structure on $N \otimes P$ [Sta92].

When $\mathcal{I}$ is irreducible, Batalin, Fradkin and Vilkovisky ([BF83] and [BV85]) were the first to define a differential $D$ on $\mathcal{A}$ which is a Poisson derivation and whose zeroth cohomology $H^0(\mathcal{A}, D) \cong (A/\mathcal{I})^2$ [BF83]. Later it was shown that the BRST algebra $(\mathcal{A}, D)$ is a model for the entire Rinehart complex in the irreducible case (see [HT92]).

A $\mathbf{K}_{A/\mathcal{I}}$-linear map $F_k : \bigwedge^k \mathbf{K}_{\mathcal{I}/\mathcal{I}^2} \to \mathbf{K}_{A/\mathcal{I}}$ is completely determined by where it sends each element of the ordered basis $\varphi_{\alpha_{n_1} \cdots \alpha_{n_k}}$. Once we set $\eta^n = \sum_{\alpha_1}^{\alpha_n} F_k(\varphi_{\alpha_{n_1} \cdots \alpha_{n_k}})$.

It follows that the homotopy Rinehart algebra $\mathcal{R} \cong \text{Hom}_A(\bigwedge(\Phi), \mathbf{K}_{A/\mathcal{I}})$. This fact allows us to compare the homotopy Rinehart algebra with the BRST algebra $\mathcal{A}$.

**Theorem 4.1.** Given a Lie-Rinehart pair $(A/\mathcal{I}, \mathcal{I}/\mathcal{I}^2)$ and a specific Koszul-Tate resolution $\mathbf{K}_{A/\mathcal{I}}$ of $A/\mathcal{I}$, the BRST algebra $\mathcal{A} = \bigwedge N \otimes \mathbf{K}_{A/\mathcal{I}}$ is isomorphic to the homotopy Rinehart algebra $\mathcal{R} = \text{Hom}_{\mathbf{K}_{A/\mathcal{I}}}(\bigwedge(\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}), \bigwedge(\mathbf{K}_{\mathcal{I}/\mathcal{I}^2}) \otimes \mathbf{K}_{A/\mathcal{I}})$ as $\mathbf{K}_{A/\mathcal{I}}$-modules and as algebras (but not necessarily as differential graded algebras).

**Proof of Theorem 4.1.** It is straightforward to show that the map sending

$$F_k \mapsto \sum_{\alpha_1}^{\alpha_{n_1} \cdots \alpha_{n_k}} F_k(\varphi_{\alpha_{n_1} \cdots \alpha_{n_k}})$$

is bijective and respects the $\mathbf{K}_{A/\mathcal{I}}$-module structures on both $\mathcal{R}$ and $\mathcal{A}$. We need to show that it is a map of algebras. Let $F_i : \bigwedge^i(\Phi) \to \mathbf{K}_{A/\mathcal{I}}$ and $G_j : \bigwedge^j(\Phi) \to \mathbf{K}_{A/\mathcal{I}}$.

Recall that the product $F_i \circ G_j : \bigwedge^{i+j}(\Phi) \to \mathbf{K}_{A/\mathcal{I}}$, evaluated on $\varphi_{\alpha_{n_1} \cdots \alpha_{n_{i+j}}}$ is

$$\sum_{(i,j)} \mathbf{K}(\sigma) F_i(\varphi_{\alpha_{n_1} \cdots \alpha_{n_i}}) G_j(\varphi_{\alpha_{n_{i+1}} \cdots \alpha_{n_{i+j}}}).$$
This product can be represented in $\mathcal{A}$ by
\[
\sum_{\alpha_{p_1} \leq \cdots \leq \alpha_{p_{i+j}}} \eta_{\alpha_{p_1} \to \cdots \to \alpha_{p_{i+j}}}(F_i \sim G_j)(\varphi_{\alpha_{p_1} \to \cdots \to \alpha_{p_{i+j}}}).
\]
Similarly, $F_i = \sum_{\alpha_{r_1} \leq \cdots \leq \alpha_{r_i}} \eta_{\alpha_{r_1} \to \cdots \to \alpha_{r_i}} F_i(\varphi_{\alpha_{r_1} \to \cdots \to \alpha_{r_i}})$ and $G_j = \sum_{\alpha_{q_1} \leq \cdots \leq \alpha_{q_j}} \eta_{\alpha_{q_1} \to \cdots \to \alpha_{q_j}} G_j(\varphi_{\alpha_{q_1} \to \cdots \to \alpha_{q_j}})$.

The product of $F_i$ and $G_j$ in $\mathcal{A}$ is
\[
\sum_{\alpha_{r_1} \leq \cdots \leq \alpha_{r_i}, \alpha_{q_1} \leq \cdots \leq \alpha_{q_j}} K(G_j; \varphi_{\alpha_{r_1}; \cdots; \alpha_{r_i}}) \eta_{\alpha_{r_1} \to \cdots \to \alpha_{r_i}} \eta_{\alpha_{q_1} \to \cdots \to \alpha_{q_j}} F_i(\varphi_{\alpha_{r_1} \to \cdots \to \alpha_{r_i}}) G_j(\varphi_{\alpha_{q_1} \to \cdots \to \alpha_{q_j}}),
\]
but the terms $\eta_{\alpha_{r_1} \to \cdots \to \alpha_{r_i}} \eta_{\alpha_{q_1} \to \cdots \to \alpha_{q_j}}$ are no longer in the proper order. Returning the $\eta$’s in each term to the proper order produces the sign $K(\sigma)$ corresponding to the $(i, j)$-unshuffle $\sigma$ which places $n_{[1 \to i+j]}$ into the hands $r_{[1 \to i]}$ and $q_{[1 \to j]}$. The sum above becomes
\[
\sum_{\alpha_{p_1} \leq \cdots \leq \alpha_{p_{i+j}}} \eta_{\alpha_{p_1} \to \cdots \to \alpha_{p_{i+j}}} \left( \sum_{\{i, j\}} K(\sigma) K(G_j; \varphi_{\alpha_{p_1}; \cdots; \alpha_{p_{i+j}}} \varphi_{\sigma_{p_1}; \cdots; \sigma_{p_{i+j}}} \varphi_{\sigma_{p_{i+j+1}; \cdots; \sigma_{p_{i+j}}}) \right) F_i(\varphi_{\alpha_{p_1}; \cdots; \alpha_{p_{i+j}}}) G_j(\varphi_{\alpha_{p_{i+j+1}} \cdots \alpha_{p_{i+j+1}}}),
\]
which equals $\sum_{\alpha_{p_1} \leq \cdots \leq \alpha_{p_{i+j}}} \eta_{\alpha_{p_1} \to \cdots \to \alpha_{p_{i+j}}} (F_i \sim G_j)(\varphi_{\alpha_{p_1} \to \cdots \to \alpha_{p_{i+j}}})$. \(\square\)

The bidegree (ghost number, internal degree) on $\mathcal{A}$ agrees with the bidegree (external degree, suspended internal degree = external degree) on $\mathcal{R}$.

**The differentials** \(\{Q, \quad \} \) and \(\langle D_M, \quad \rangle \) **when** $\mathcal{I}$ **is irreducible**

The differential $D$ is an inner-derivation on $\mathcal{A}$, i.e. $D = \{Q, \quad \}$, where the element $Q \in \mathcal{A}$ has total degree $+1$ and $\{Q, Q\} = 0$. The element $Q$ (called the BRST charge) is a sum $\sum_{n=0}^{\infty} Q_n$, where $Q_n$ has ghost number $n + 1$. The Jacobi identity for the Poisson bracket guarantees that $D$ has square zero. We construct the BRST charge $Q$ using methods from homological perturbation theory. The BRST algebra $\mathcal{A}$ is filtered by ghost degree and $Q_n$ is defined by induction on $n$ once we select a set of generators $\{y_{\alpha}\}$ for $\mathcal{I}$ and structure functions $C_{\alpha\beta}^\gamma$ [Sta92]. We set $Q_0 = \eta^\alpha y_\alpha$ and find that $Q_1$ must equal $-\frac{1}{2} \eta^\alpha \eta^\beta C_{\alpha\beta}^\gamma P_\gamma$ in order to kill the nonzero piece of $\{Q_0, Q_0\}$ with ghost degree $2$. This process continues: for every $i > 1$, the terms of $\{\sum_{n=0}^{i-1} Q_n, \sum_{n=0}^{i-1} Q_n\}$ have ghost degree at least $i + 1$. An element $Q_i$ with ghost degree $i + 1$ is selected to kill the terms with ghost degree $i + 1$.

With respect to the filtration of $\mathcal{A}$ by ghost number, $\{Q, \quad \}$ decomposes as $1 \otimes m_1 + \hat{\delta}_R + \hat{\delta}_2 + \ldots$. In terms of the BRST differential, $1 \otimes m_1$ is $\{Q_0, \quad \}\{Q_0, \quad \}$ and $\hat{\delta}_R = \{Q_1, \quad \}\{Q_1, \quad \}$ is, loosely speaking, a “lifting” of the Rinehart differential $\delta_R$ on $\mathcal{R}$. The $E_1$ term of the associated spectral sequence is isomorphic to $\wedge N \otimes \mathcal{A} / \mathcal{I}$ and the differential $d_1$ for the $E_1$ term which $\delta_R$ induces on $H_1(\mathcal{A})$ is the map $\delta_R$. 

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The filtration of $\mathcal{R}$ by external degree is the same as the filtration of $\mathcal{A}$ by ghost number. The differential $(\mathcal{D}_M, \ )$ breaks up into the sum $\sum_{i=1}^{\infty} \langle m_i, \rangle$, where $\langle m_i, \rangle$ increases external degree by $i - 1$. For any map $F_k \in \mathcal{R}$, the map $\langle m_1, F_k \rangle$ is $m_1 F_k$ because $l_1 = 0$ on $\wedge(\Phi)$. Suppressing the sum over the ordered basis (it is henceforth understood), we write $F_k$ as $\eta^\alpha_{\alpha' \to \gamma} F_k(\varphi_{\alpha'_{\gamma_{\alpha' \to \gamma}}})$ and the map $m_1 F_k$ is represented in $\mathcal{A}$ by $\eta^\alpha_{\alpha' \to \gamma} m_1 F_k(\varphi_{\alpha'_{\gamma_{\alpha' \to \gamma}}})$, which equals $\langle 1 \otimes m_1 \rangle(\eta^\alpha_{\alpha' \to \gamma} F_k(\varphi_{\alpha'_{\gamma_{\alpha' \to \gamma}}}))$. So $\langle m_1, \rangle$, when realized on $\mathcal{A}$, is $\langle 1 \otimes m_1 \rangle$. We conclude that in the irreducible case, the $E_1$ terms of the spectral sequences associated with $(\mathcal{Q}, -)$ and $(\mathcal{D}_M, \ )$ are isomorphic. The differentials for the $E_1$ terms are induced by $\delta_R$ and $\langle m_2, \rangle$, respectively; we claim that $\delta_R = \langle m_2, \rangle$. Because both $\delta_R$ and $\langle m_2, \rangle$ are derivations, a quick check on the ghosts, antighosts and the elements of $\mathcal{A}$ suffices. For $\eta^\alpha$, we see that $\langle m_2, \eta^\alpha \rangle (\varphi_{\beta} \wedge \varphi_{\gamma}) = \eta^\alpha I_2(\varphi_{\beta} \wedge \varphi_{\gamma}) = \eta^\alpha (C_{\delta \gamma}^\varphi) = C_{\delta \gamma}^\varphi$, and

\[
\left\{ \frac{1}{2} \eta^\alpha \eta^\beta C_{ab}^\delta P_\delta, \eta^\alpha \right\} (\varphi_{\beta} \wedge \varphi_{\gamma}) = \frac{1}{2} \eta^\alpha \eta^\beta C_{ab}^\alpha (\varphi_{\beta} \wedge \varphi_{\gamma}) = \frac{1}{2} C_{\beta \gamma}^\alpha - \frac{1}{2} C_{\alpha \beta}^\gamma = C_{\beta \gamma}^\varphi.
\]

For $\mathcal{P}_\alpha$, we find that $\langle m_2, \mathcal{P}_\alpha \rangle (\varphi_{\beta}) = m_2 \mathcal{P}_\alpha (\varphi_{\beta}) = -m_2 (\varphi_{\beta} \otimes \mathcal{P}_\alpha) = C_{\alpha \beta}^\delta \mathcal{P}_\delta$ and

\[
\left\{ -\frac{1}{2} \eta^\alpha \eta^\beta C_{ab}^\delta P_\delta, \mathcal{P}_\alpha \right\} (\varphi_{\beta}) = \frac{1}{2} \eta^\alpha C_{ab}^\delta \mathcal{P}_\delta (\varphi_{\beta}) - \frac{1}{2} \eta^\beta C_{ab}^\delta \mathcal{P}_\delta (\varphi_{\beta}) = -\frac{1}{2} C_{\beta \alpha}^\delta \mathcal{P}_\delta + \frac{1}{2} C_{\alpha \beta}^\delta \mathcal{P}_\delta = C_{\alpha \beta}^\delta \mathcal{P}_\delta.
\]

And for $f \in \mathcal{A}$, it is straightforward to compute that $\langle m_2, f \rangle (\varphi_{\beta}) = m_2 f (\varphi_{\beta}) = m_2 (\varphi_{\beta} \otimes f) = \{ y_{\beta}, f \}$ and $\{ \eta^\alpha y_{\alpha}, f \} (\varphi_{\beta}) = \eta^\alpha \{ y_{\alpha}, f \} (\varphi_{\beta}) = \{ y_{\beta}, f \}$. We conclude that $(\mathcal{A}, (Q, \ ))$ and $(\mathcal{R}, (\mathcal{D}_M, \ ))$ are equivalent as models for the Rinehart cohomology of the Lie-Rinehart pair $(\mathcal{A}, \mathcal{I}, \mathcal{I}/\mathcal{I}^2)$ when the ideal $\mathcal{I}$ is irreducible.

When the ideal is reducible, we know that $(\mathcal{R}, (\mathcal{D}_M, \ ))$ is also a model for the Rinehart cohomology, but we do not know, except in cases arising from particularly nice symplectic settings (see [FHST89]), whether $(\mathcal{A}, (Q, \ ))$ is a model for the Rinehart cohomology. This problem will be addressed in a future paper.

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Lars Kjeseth  
kjeseth@elcamino.cc.ca.us

Mathematical Sciences Division  
El Camino College  
U.S.A.  
Torrance  
16007 Crenshaw Blvd, Torrance, CA 90506