TOWARD EQUIVARIANT IWAUSA THEORY, IV

JÜRGEN RITTER and ALFRED WEISS

(communicated by J.F. Jardine)

Abstract

Let \( l \) be an odd prime number and \( K_\infty/k \) a Galois extension of totally real number fields, with \( k/Q \) and \( K_\infty/k_\infty \) finite, where \( k_\infty \) is the cyclotomic \( \mathbb{Z}_l \)-extension of \( k \). In [RW2] a “main conjecture” of equivariant Iwasawa theory is formulated which for pro-\( l \) groups \( G_\infty \) is reduced in [RW3] to a property of the Iwasawa \( L \)-function of \( K_\infty/k \). In this paper we extend this reduction for arbitrary \( G_\infty \) to \( l \)-elementary groups \( G_\infty = \langle s \rangle \times U \), with \( \langle s \rangle \) a finite cyclic group of order prime to \( l \) and \( U \) a pro-\( l \) group. We also give first nonabelian examples of groups \( G_\infty \) for which the conjecture holds.

Dedicated to Victor Snaith on the occasion of his 60-th birthday.

Let \( l \) be a fixed odd prime number and \( K_\infty/k \) a Galois extension of totally real number fields with \( [k : Q] \) finite and \( k_\infty \), the cyclotomic \( l \)-extension of \( k \), contained in \( K_\infty \) with \( [K_\infty : k_\infty] \) also finite. The respective Galois groups are \( G_\infty = G_{K_\infty/k}, H = G_{K_\infty/k_\infty}, \Gamma_k = G_{k_\infty/k} \). We also fix a finite set \( S \) of primes of \( k \) containing \( l, \infty \) and all primes which ramify in \( K_\infty \).

In [RW2, §4] we formulated an equivariant refinement of the Main Conjecture of (classical) Iwasawa theory [Wi]. The main point of this paper is to reduce this “main conjecture” to a conjectural property of the Iwasawa \( L \)-function of \( K_\infty/k \).

**Theorem (A).** The “main conjecture” of equivariant Iwasawa theory for \( K_\infty/k \) is, up to its uniqueness assertion, equivalent to \( L_{K_\infty/k,S} \) belonging to \( \text{Det} K_1(\Lambda(G_\infty)) \).

The Iwasawa \( L \)-function \( L_{K_\infty/k} (= L_{K_\infty/k,S}) \) incorporates all the \( l \)-adic \((S\text{-truncated})\) Artin \( L \)-functions of \( K_\infty/k \) by assigning to each \( l \)-adic character \( \chi \) of \( G_\infty \) the Iwasawa power series of the corresponding \( L \)-function. This \( L_{K_\infty/k} \) is a homomorphism from the character ring \( R_l(G_\infty) \) to the units of the “Iwasawa algebra” \( \Lambda_c(\Gamma_k) \) of \( k \), which is Galois equivariant, compatible with \( W \)-twisting, and

Received December 14, 2004, revised March 15, 2005; published on November 12, 2005.
2000 Mathematics Subject Classification: 11R23, 11R32, 11R37, 11R42, 11S23, 11S40
Key words and phrases: Iwasawa theory, \( l \)-adic \( L \)-functions
We acknowledge financial support provided by NSERC and the University of Augsburg.

\(^1\)The reference to \( S \) is normally suppressed.
which satisfies the congruences $L_{K_{\infty}/k}(\chi) \equiv \Psi(L_{K_{\infty}/k}(\psi \chi)) \mod \lambda^c(\Gamma_k)$. These properties of $L_{K_{\infty}/k}$ are the foundation of the proof of Theorem A. For the notation we refer to the introductory §1 which also contains the map $\text{Det} : K_1(\Lambda(G_{\infty})) \to \text{HOM}^*(R_t(G_{\infty}), \Lambda^c(\Gamma_k)^\times)$.

The technical core of the proof of Theorem A is

**Theorem (B).** $\text{Det} K_1(\Lambda(G_{\infty})) \cap \text{HOM}^*(R_t(G_{\infty}), \Lambda^c(\Gamma_k)^\times) \subset \text{Det} K_1(\Lambda(G_{\infty}))$

When $G_{\infty}$ is an $l$-group, equivalent theorems are stated in [RW3] with $\bullet$ in place of $\dddot{\cdot}$; for the proofs in [RW3] the $\dddot{\cdot}$-form of Theorem B is however essential (see [RW3, §6]). We have emphasized here the $\dddot{\cdot}$-form because this technical advantage persists (e.g. in Proposition 2).

The proof in [RW3, §1] that Theorem B implies Theorem A works not only for general groups $G_{\infty}$ but also with $\bullet$ replaced by $\dddot{\cdot}$. Therefore it remains to use induction techniques to reduce Theorem B to the $l$-group case. These techniques are generalizations of those in [Ty, Fr] for finite groups to the setting of Iwasawa theory.

In the same way we obtain

**Theorem (C).** $L_{K_{\infty}/k} \in \text{Det} K_1(\Lambda(G_{\infty}))$ if, and only if, $L_{K'/k'} \in \text{Det} K_1(\Lambda(G_{K'}/k'))$ whenever $G_{K'}/k'$ is an $l$-elementary section of $G_{\infty}$.

Here $G_{K'}/k'$ is a section of $G_{\infty}$, if $k \subset k' \subset K' \subset K_{\infty}$ is such that $k'/k$ is finite and $K_{\infty}/K'$ finite Galois; a section $G_{K'}/k'$ is $l$-elementary, if $G_{K'}/k' = \langle s \rangle \times U$ for some finite cyclic subgroup $\langle s \rangle$ of order prime to $l$ and some open $l$-subgroup $U$.

If $G_{\infty}$ is abelian, then the “main conjecture” holds by the Corollary to Theorem 9 in [RW3]. Theorem C provides first nonabelian examples of the “main conjecture”. We expect more such examples to follow from the logarithmic methods of [RW3] for $l$-elementary groups. In more generality we know only that some $l$-power of $L_{K_{\infty}/k}$ is in $\text{Det} K_1(\Lambda(G_{\infty}))$.

The paper is organized as follows. Its first section has some background material. In §2 we discuss $K_1(\Lambda(G_{\infty}))$ for $Q_l$- $l$-elementary groups $G_{\infty}$ and deduce Theorems B and C for them. Then §3 is preliminary material on $Q_l$-$q$-elementary groups $G_{\infty}$, with $q$ a prime number different from $l$, which is used for the proof, in §4, of the full Theorems B and C. In §5 the examples appear.

We remark that because Theorems A and C are based on [RW3] they depend on the vanishing of Iwasawa’s $\mu$-invariant for $k'_{\infty}/k'$, for which we refer to [Ba].

1. **Background**

The Iwasawa $L$-function $L_{K_{\infty}/k,S}$ of $K_{\infty}/k$ is defined as follows (compare [RW2, §4]). Let $\chi$ be a $Q_l$-$S$-character of $G_{\infty}$ with open kernel and write the $l$-adic $S$-truncated
Artin $L$-function $L_{l,S}(1-s,\chi)$, for $s \in \mathbb{Z}$, as the fraction $L_{l,S}(1-s,\chi) = \frac{G_{\chi,S}(\gamma^s) - 1}{H_{\chi}(\gamma^s) - 1}$ of the Deligne-Ribet power series $G_{\chi,S}(T), H_{\chi}(T) \in \mathbb{Q}^c \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[T]]$ associated to a generator $\gamma_k$ of $\Gamma_k$ [DR]. Above, $u \in 1 + l\mathbb{Z}$ describes the action of $\gamma_k$ on the $l$-power roots of unity. Now set

$$L_{K_\infty/k,S}(\chi) = \frac{G_{\chi,S}(\gamma_k) - 1}{H_{\chi}(\gamma_k) - 1}$$

(which is independent of the choice of $\gamma_k$).

Recall that $\mathcal{Q}(G_\infty)$ is the total ring of fractions of the completed group ring $\Lambda(G_\infty) = \mathbb{Z}_l[[G_\infty]]$ of $G_\infty$ over $\mathbb{Z}_l$ (it is enough to invert the nonzero elements of $\Lambda(\Gamma)$ for a central open subgroup $\Gamma \approx \mathbb{Z}_l$). The algebra $\mathcal{Q}(G_\infty)$ is a finite dimensional semisimple algebra over $\mathcal{Q}(\Gamma)$ with $\Gamma$, as before, central open in $G_\infty$.

The map

$$\text{Det} : K_1(\mathcal{Q}(G_\infty)) \to \text{Hom}^*(R_l(G_\infty), \mathcal{Q}^c(\Gamma_k)^{\times})$$

is now defined as follows (compare [RW2, §3]).

If $[P, \alpha]$ represents an element in $K_1(\mathcal{Q}(G_\infty))$, with $P$ a finitely generated projective $\mathcal{Q}(G_\infty)$-module and $\alpha$ an $\mathcal{Q}(G_\infty)$-automorphism of $P$, then

$$\text{Det} [P, \alpha] = \text{det} \mathcal{Q}^c(\Gamma_k)(\alpha \mid \text{Hom}_{\mathcal{Q}^c}(H, V_\chi \otimes \mathcal{Q}_l, P)) \text{ .}$$

Here, $\mathcal{Q}^c(\Gamma_k) = \mathcal{Q}_l^c \otimes_{\mathcal{Q}_l} \mathcal{Q}(\Gamma_k)$, and $V_\chi$ is a $\mathcal{Q}_l^c$-representation of $G_\infty$ with character $\chi$ (always with open kernel). The * on $\text{Hom}$ requires $G_\mathcal{Q}^c$-invariance and compatibility with $W$-twists; these properties are inherited from the representation theory of $\mathcal{Q}(G_\infty)$.

Restricting $\text{Det}$ to $K_1(\Lambda(G_\infty))$, it takes values in $\text{Hom}^*(R_l(G_\infty), \Lambda^c(\Gamma_k)^{\times})$, with $\Lambda^c(\Gamma_k) = \mathbb{Z}_l^c \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)$, and indeed $\text{Det} x = f$ has values satisfying the congruences

$$f(\chi)^l \equiv \Psi(f(\psi \chi)) \mod l\Lambda^c(\Gamma_k),$$

which define the subgroup $\text{HOM}^*(R_l(G_\infty), \Lambda^c(\Gamma_k)^{\times})$ of $\text{Hom}^*$ (see [RW3, §2]). Above, $\Psi$ is the $\mathbb{Z}_l^c$-algebra endomorphism of $\Lambda^c(\Gamma_k)$ induced by $\gamma \mapsto \gamma^l$ on $\Gamma_k$, and $\psi_l$ is the $l$-th Adams operation on $R_l(G_\infty)$.

However, the values $L_{K_\infty/k}(\chi)$ are not in $\Lambda^c(\Gamma_k)^{\times}$ but in $\Lambda^c_\infty(\Gamma_k)^{\times}$, where $\Lambda^c_\infty(\Gamma_k) = \mathbb{Z}_l^c \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)_\infty$ with $\Lambda(\Gamma_k)_\infty$ the localization of $\Lambda(\Gamma_k)$ at $l$. We work with the completion $\Lambda^c(\Gamma_k)_\infty$ of $\Lambda(\Gamma_k)$ at $l$ because logarithmic methods apply to $K_1(\Lambda(G_\infty)_\infty$ (see [RW3, beginning of §5]). We arrive at

$$\text{Det} : K_1(\Lambda(G_\infty)_\infty) \to \text{HOM}^*(R_l(G_\infty), \Lambda^c(\Gamma_k)^{\times}),$$

with $\Lambda^c(\Gamma_k) = \mathbb{Z}_l^c \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)_\infty$, and now $L_{K_\infty/k} \in \text{HOM}^*(R_l(G_\infty), \Lambda^c(\Gamma_k)^{\times})$.

The induction techniques that we are going to apply will also involve $\Lambda^D(G) = \mathcal{O} \otimes_{\mathbb{Z}_l} \Lambda(G)$ and $\Lambda^D(G)_\infty$, where $\mathcal{O}$ is the ring of integers of a finite unramified...
extension $N/Q_l$. All that has been said so far remains true except that the $G_{Q_l}/Q_l$-invariance on $\text{Hom}^*$ gets replaced by $G_{Q_l}/N$-invariance to define $\text{Hom}^N$ and that the Frobenius automorphism $\text{Fr}$ of $N/Q_l$ appears (see [RW3, Proposition 4]).

2. $Q_l$-l-elementary groups $G_\infty$

In this section the Galois group $G_\infty = G_{K_\infty/k}$ is assumed to be $Q_l$-l-elementary, i.e., a semidirect product $G_\infty = \langle s \rangle \rtimes U$ of a finite cyclic group $\langle s \rangle$ of order prime to $l$ and an open $l$-subgroup $U$ whose action on $\langle s \rangle$ induces a homomorphism $U \rightarrow G_{Q_l(G)/Q_l}$, where $\zeta$ is a root of unity of order $|\langle s \rangle|$.

We fix a set $\{\beta_i\}$ of representatives of $G_{Q_l}/Q_l$-orbits of the $Q_l$-irreducible characters of $\langle s \rangle$ and denote the stabilizer group of $\beta_i$ by $U_i = \{u \in U : \beta_i^u = \beta_i\}$. Note that $U_i \triangleleft U$ and set $A_i = U/U_i \leq G_{N_i/Q_l}$, with $N_i$ the field of character values of $\beta_i$.

**Theorem 1.**
1. There are natural maps $r, r'$ so that

$$
\begin{align*}
K_1(\Lambda(G_\infty)) & \xrightarrow{r} \prod_i K_1(\Lambda^{\mathfrak{O}_i}(U_i)) \\
\text{Det} & \downarrow \quad \text{Det} \\
\text{Hom}^*(R_1(G_\infty), \Lambda^c(\Gamma_k)^\times) & \xrightarrow{r'} \prod_i \text{Hom}^{N_i}(R_1(U_i), \Lambda^c(\Gamma_{k_i})^\times)
\end{align*}
$$

commutes and $r'$ is injective. Here $k_i = K_\infty^{U_i}$ and $\mathfrak{O}_i$ is the ring of integers of $N_i$. Moreover, $r$ induces an isomorphism

$$
\text{Det} K_1(\Lambda(G_\infty)) \rightarrow \prod_i (\text{Det} K_1(\Lambda^{\mathfrak{O}_i}(U_i)))^{A_i}.
$$

2. The same holds in the completed situation, i.e., with $\Lambda$ replaced by $\Lambda_\infty$.

**Proof.** (Compare [Ty, p.67-71] or [Fr, p.89-96].) In order to use subscripts we abbreviate $G_{\infty}$ by $G$.

Set $G_i = \langle s \rangle \rtimes U_i$, $e_i = \frac{1}{|\langle s \rangle|} \sum_j \mod |\langle s \rangle| \text{tr}_{N_i/Q_i}(\beta_i(s^{-j}))s^j \in \mathbb{Z}_l\langle s \rangle$ and let $R_i^{(e_i)}(G) \subset R_i(G)$ be the span of the irreducible $\chi \in R_i(G)$ with $\chi(e_i) \neq 0$. Observe that $e_i$ is a central idempotent of $\Lambda(G_\infty)$.

We first glue the following squares together

$$
\begin{align*}
K_1(\Lambda(G)) & \xrightarrow{\text{res}^G_{\Lambda_i}} K_1(\Lambda(G_i)) \\
\text{Det} & \downarrow \quad \text{Det} \\
\text{Hom}^*(R_i(G), \Lambda^c(\Gamma_k)^\times) & \xrightarrow{\text{res}^G_{\Lambda_i}} \text{Hom}^*(R_i(G_i), \Lambda^c(\Gamma_{k_i})^\times)
\end{align*}
$$

$$
\begin{align*}
K_1(\Lambda(G_i)) & \rightarrow K_1(e_i\Lambda(G_i)) \\
\text{Det} & \downarrow \quad \text{Det} \\
\text{Hom}^*(R_i(G_i), \Lambda^c(\Gamma_{k_i})^\times) & \rightarrow \text{Hom}^*(R_i^{(e_i)}(G_i), \Lambda^c(\Gamma_{k_i})^\times)
\end{align*}
$$

Actually, both diagrams should have the field $k_i' = K_\infty^{G_i}$ in place of $k_i$; however, $\Gamma_{k_i'}$ and $\Gamma_{k_i}$ get identified as subgroups of $\Gamma_k$ since $[k_i : k_i'] = |\langle s \rangle|$ is not divisible by $l$. 


The upper diagram commutes by [RW2, Lemma 9], and \( \Lambda(G_i) = e_i \Lambda(G_i) \times (1 - e_i) \Lambda(G_i) \) implies the commutativity of the bottom one. Note that there is no ambiguity in writing \( \text{Hom}^*(R_i^{(c_i)}(G_i), \Lambda^c(\Gamma_{k_i})^\times) \) because \( \chi(e_i) = (\chi \rho)(e_i) \) for characters \( \rho \) of \( G_i \) of type \( W \).

There are natural actions of \( A_i = G/G_i \) on \( K_1(\Lambda(G_i)) \) and on

\[
\text{Hom}^*(R_i(G_i), \Lambda^c(\Gamma_{k_i})^\times);
\]

moreover,

\[
\text{res}^G_G(K_1(\Lambda(G))) \subset K_1(\Lambda(G_i))^{A_i},
\]

\[
\text{res}^G_G(\text{Hom}^*(R_i(G), \Lambda^c(\Gamma_{k_i})^\times)) \subset (\text{Hom}^*(R_i(G_i), \Lambda^c(\Gamma_{k_i})^\times))^{A_i}.
\]

The maps in the bottom diagram are all \( A_i \)-equivariant. For this we only need to check the \( A_i \)-equivariance of \( \text{Det} : K_1(Q(G_i)) \to \text{Hom}^*(R_i(G_i), Q^c(\Gamma_{k_i})^\times) \):

Set \( H_i = \ker(G_i \to \Gamma_{k_i}) \). Further, let \([P, \alpha]\) represent an element of \( K_1(Q(G_i))\), with \( \alpha \) an automorphism of the projective module \( P \). If \( a \in A_i \) has preimage \( g \in G \), then \([P, \alpha]^g = [P^{[g]}, \alpha^{[g]}]\) where \( P^{[g]} = \{[p] : p \in P\} \) with \( y[p] = [y^{-1}g^{-1}p] \) for \( y \in G_i \) and \( \alpha^{[g]}([p]) = [\alpha(p)] \). Taking \( V = V_{\chi^{\rho^{-1}}} \), so \( V^{[g]} = V_{\chi^g} \), it suffices to show that

\[
\text{Hom}_{Q^c[H_i]}(V, Q^c \otimes Q \ P) \to \text{Hom}_{Q^c[H_i]}(V^{[g]}, Q^c \otimes Q \ P^{[g]}),
\]

\[
\varphi \mapsto [\varphi] \text{ with } [\varphi([v])] = [\varphi(v)]
\]

is a \( Q^c(\Gamma_{k_i}) \)-vector space isomorphism which is natural for the respective actions of \( \alpha \). Now,

\[
(y[\varphi])([v]) = y([\varphi](y^{-1}v)) = y([\varphi](y^{-1}g^{-1}v))
\]

\[
= y[y^{-1} \varphi(y^{-1}v)] = [y^{-1}g^{-1} \varphi(y^{-1}v)] = [y^{-1}g^{-1} \varphi(v)],
\]

and taking \( y \in H_i \) implies that \([\varphi] \in \text{Hom}_{Q^c[H_i]}(V^{[g]}, Q^c \otimes Q \ P^{[g]})\). Reading the above for \( y \in \Gamma_{k_i} \) we see the map is \( Q^c(\Gamma_{k_i}) \)-linear.

By composing the above two squares we arrive at

\[
\begin{array}{c}
K_1(\Lambda(G)) \\
\downarrow \text{Det} \\
\text{Hom}^*(R_i(G), \Lambda^c(\Gamma_{k_i})^\times) \\
\downarrow \text{Det}
\end{array}
\]

\[
\begin{array}{c}
\prod_i K_1(e_i \Lambda(G_i))^{A_i} \\
\prod_i \text{Hom}^*(R_i^{(c_i)}(G_i), \Lambda^c(\Gamma_{k_i})^\times)^{A_i}.
\end{array}
\]

We claim that the lower horizontal map in (D1) is injective. To see this we first observe that it is also the composite

\[
\text{Hom}^*(R_i(G), \Lambda^c(\Gamma_{k_i})^\times) \to \prod_i \text{Hom}^*(R_i^{(c_i)}(G), \Lambda^c(\Gamma_{k_i})^\times)
\]

\[
\to \prod_i \text{Hom}^*(R_i^{(c_i)}(G_i), \Lambda^c(\Gamma_{k_i})^\times)
\]

and that \( R_i(G) = \bigoplus_i R_i^{(c_i)}(G) \). Hence, as induction on characters is restriction on \( \text{Hom}^* \), we are done once we know \( \text{ind}_G^G(\bigoplus_i R_i^{(c_i)}(G_i)) = R_i^{(c_i)}(G) \). However, if \( \chi \in R_i(G) \) is irreducible, then Clifford theory [CR I, 11.8, p.265] implies \( \chi = \text{ind}_G^G(\beta \xi) \)
for some irreducible $\xi \in R_i(U_i)$ and the $i$ and $\sigma \in G_{N_i/Q_i}$ so that $\beta_i^{\sigma}$ appears in $\text{res}_G^{(s)}(\chi)$; here $\beta_i \in R_i(G_i)$ is defined by $\beta_i(s^j u) = \beta_i(s^j)$.

Note that $e_i\Lambda(G_i) = e_i\mathbb{Z}_l(s) \otimes_{\mathbb{Z}_l} \Lambda(U_i)$ is, via $\beta_i$, isomorphic to $\mathcal{D}_i \otimes_{\mathbb{Z}_l} \Lambda(U_i) = \Lambda^{\mathcal{D}_i}(U_i)$. We next show that the square

\begin{equation}
\begin{array}{ccc}
K_i(e_i\Lambda(G_i)) & \xrightarrow{\beta_i^{*}} & K_i(\Lambda^{\mathcal{D}_i}(U_i)) \\
\text{Det} & \downarrow & \text{Det} \\
\text{Hom}^\ast(R_i(e_i)(G_i), \Lambda^c(\Gamma_k)_i)^\times & \xrightarrow{\beta_i^{*}} & \text{Hom}^N_i(R_i(U_i), \Lambda^c(\Gamma_k)_i)^\times
\end{array}
\end{equation}

(D2)

commutes, with the top horizontal map induced by $\beta_i$ and $\beta_i^{*}$ defined by $f \mapsto f^{'}$, $f^{'}(\xi) = f(\beta_i\xi)$. The map $\beta_i^{*}$ is injective because $R_i(e_i)(G_i)$ is spanned by the $\beta_i^{*}\xi$.

Turning to the commutativity of (D2), it suffices to show that $(\text{Det}(\alpha))' = \text{Det}(\beta_i(\alpha))$ for units $\alpha \in e_i\Lambda(G_i)$, by [CR II, p.76]. Now, with $V_\xi$ denoting a $\mathbb{Q}^{\xi^c}$-realization of $\xi \in R_i(G_i)$,

\[
\text{Det}(\beta_i(\alpha))(\xi) = \text{det}_{\mathbb{Q}^{\xi^c}[H_i]}(\beta_i(\alpha) | \text{Hom}_{\mathbb{Q}^{\xi^c}[H_i]}(V_\xi, \mathbb{Q}^{\xi^c} \otimes_{\mathbb{Q}_i} \mathcal{Q}^N_i(U_i)))
\]

and

\[
\text{Det}(\alpha)(\beta_i^{*}\xi) = \text{det}_{\mathbb{Q}^{\xi^c}[H_i]}(\alpha | \text{Hom}_{\mathbb{Q}^{\xi^c}[H_i]}(V_{\beta_i^{*}\xi}, \mathbb{Q}^{\xi^c} \otimes_{\mathbb{Q}_i} (e_i\mathbb{Q}_i(s) \otimes_{\mathbb{Q}_i} \mathcal{Q}(G_i))))
\]

where $H_i$, as before, equals $\ker(G_i \rightarrow \Gamma_k)$ and $H'_i = H_i/\langle s \rangle$; see [RW2, §3]. Hence it suffices to exhibit a $\mathbb{Q}^{\xi^c}\text{-isomorphism}$

\[
\text{Hom}_{\mathbb{Q}^{\xi^c}[H_i]}(V_\xi, \mathbb{Q}^{\xi^c}(U_i)) \rightarrow \text{Hom}_{\mathbb{Q}^{\xi^c}[H_i]}(V_{\beta_i^{*}\xi}, (\mathbb{Q}^{\xi^c} \otimes_{\mathbb{Q}_i} e_i\mathbb{Q}_i(s)) \otimes_{\mathbb{Q}_i} \mathbb{Q}^{c}(U_i))
\]

which is natural for the respective actions of $\alpha$. Such a map is given by multiplying $\varphi' \in \text{Hom}_{\mathbb{Q}^{\xi^c}[H_i]}$ by the idempotent $\varepsilon_i = \frac{1}{(s)} \sum_{j \mod \langle s \rangle} \beta_i(s^{-j}) \otimes e_i\mathbb{Q}_i(s)$ to every $\varphi \in \text{Hom}_{\mathbb{Q}^{\xi^c}[H_i]}$ has image in $\varepsilon_i(\mathbb{Q}^{\xi^c} \otimes_{\mathbb{Q}_i} e_i\mathbb{Q}_i(s)) \otimes_{\mathbb{Q}_i} \mathbb{Q}^c(U_i) = \varepsilon_i \otimes_{\mathbb{Q}_i} \mathbb{Q}^c(U_i)$.

Combining (D1) and (D2) gives the commutative square in 1. of the theorem. To complete the proof we are left with showing

\[
\text{Det} K_i(\Lambda(G)) \simeq \prod_i (\text{Det} K_i(\Lambda^{\mathcal{D}_i}(U_i)))^{A_i}.
\]

We first check that the maps in (D2) are all $A_i$-equivariant. The left Det has already been dealt with. The right Det will follow since $\beta_i$ is an isomorphism.

1. The natural embedding $a \mapsto \sigma_a : A_i \rightarrow G_{N_i/Q_i}$ is determined by $\beta_i(s^a) = \beta_i(s^a \otimes_{\mathbb{Q}_i} \Lambda(U_i))$ by $\beta_i$, hence $\beta_i : K_i(e_i\mathbb{Z}_l(s) \otimes_{\mathbb{Z}_l} \Lambda(U_i)) \rightarrow K_i(\Lambda^{\mathcal{D}_i}(U_i))$ is $A_i$-equivariant.

2. We show that $\beta_i^{*}$ is $A_i$-equivariant, with the action of $A_i$ on $\varphi \in \text{Hom}_{\mathbb{Q}_i}(R_i(U_i), \Lambda^c(\Gamma_k)_i)^\times$ defined by $\varphi^{*}(\xi) = \varphi(\xi^{-1})^{\sigma_a}$, where $\sigma_a \in G_{N_i/Q_i}$ is extended to $\mathbb{Q}_i^c$ so that it is the identity on $l$-power roots of unity; this is possible since $N_i/Q_i$ is unramified. Note that $\varphi^{*}$ is well-defined since changing $\sigma_a$ to $\sigma_a$, with $\sigma \in G_{Q_i^c/Q_i}$, the identity on $N_i(\mathbb{Q}_i)$, gives $\varphi(\xi^{-1})^{\sigma_a} = \varphi(\xi^{-1})^{\sigma_a} = \varphi(\xi^{-1})^{\sigma_a}$.
\[ \varphi(\xi^{a-1})^{\sigma_a} \text{ as } \xi^{a-1} \text{ is a character of the l-group } U_i. \text{ Moreover, } \varphi^a \in \text{Hom}_{\mathbb{N}}: \]

If \( \sigma \in G_{\mathbb{N}^i} \), then \( \varphi^a(\xi) = \varphi(\xi^{a-1})^{\sigma_a} = \varphi(\xi^{a-1})^{\sigma_a} = (\varphi(\xi^{a-1})^{\sigma_a})^{\sigma_a} = \varphi(\xi)^{\sigma_a} \), because \( \sigma\sigma_a^{-1} \) is also an admissible extension of \( \sigma_a \).

The \( A_i \)-equivariance of the map \( \beta_i^* \) now follows from \( \beta_i^a = \beta_i^{a-1} \) (which is a reformulation of \( \beta_i(s^{a-1}) = \beta_i(s)^{\sigma_a} \)). Namely, let \( f' \in \text{Hom}_{\mathbb{N}} \), be the image of \( f \in \text{Hom}^* \) and let \( f'' \in \text{Hom}_{\mathbb{N}}^* \) be that of \( f^a \). Then \( f''(\xi) = f^a(\beta_i^* \xi) = f(\beta_i^{a-1} \xi^{a-1}) = f(\beta_i^* \xi^{a-1})^{a-1} = f^a(\xi^{a-1})^{a-1} = f(\xi^{a-1})^{a} = (f')^a(\xi) \).

For 1. of Theorem 1 it now remains to show that \( r' \) induces an epimorphism \( K_1(\Lambda(G)) \to \prod_i (\text{Det } K_1(\Lambda^{\text{D}_i}(U_i)))^{A_i} \). From

\[
\begin{align*}
K_1(\Lambda(G)) & \xrightarrow{\text{res}_{G_i}^G} K_1(\Lambda(G_i)) \\
\downarrow & \downarrow \\
K_1(e_i \Lambda(G)) & \xrightarrow{\text{res}_{G_i}^G} K_1(e_i \Lambda(G_i)) \\
& \xrightarrow{\beta_i \cong} K_1(\Lambda^{\text{D}_i}(U_i))
\end{align*}
\]

and the surjectivity of the left vertical arrow we deduce

\[ \text{im}(r) \supset \prod_i \beta_i \text{res}_{G_i}^G(K_1(e_i \Lambda(G))) \supset \prod_i \beta_i \text{res}_{G_i}^G \text{ind}_{G_i}^G(K_1(e_i \Lambda(G))). \]

Hence, by [RW2, Lemma 9] and [RW3, Lemma 1],

\[ r'(\text{Det } K_1(\Lambda(G))) \supset \prod_i \beta_i \text{res}_{G_i}^G \text{ind}_{G_i}^G(\text{Det } K_1(e_i \Lambda(G))) \]

\[ \cong \prod_i \beta_i \text{N}_{A_i}(\text{Det } K_1(e_i \Lambda(G))) = \prod_i \text{N}_{A_i}(\text{Det } K_1(\Lambda^{\text{D}_i}(U_i))) \]

where \( \cong \) is due to Mackey’s subgroup theorem and \( G_i/G_i = A_i \):

\[ \text{res}_{G_i}^G \text{ind}_{G_i}^G(f_i)(\beta_i^* \xi) = f_i(\text{res}_{G_i}^G \text{ind}_{G_i}^G(\beta_i^* \xi)) = (\prod_{a \in A_i} f_i^a(\beta_i^* \xi) = (N_{A_i} f_i)(\beta_i^* \xi). \]

All arguments above apply to 2. of Theorem 1 without changes.

The proposition below now finishes the proof of Theorem 1.

**Proposition 2.** \( N_{A_i}(\text{Det } K_1(\Lambda^{\text{D}_i}(U_i))) = (\text{Det } K_1(\Lambda^{\text{D}_i}(U_i)))^{A_i} \text{ and the same with } \Lambda \text{ replaced by } \Lambda_. \)

Since the \( U \) in \( G_\infty = \langle s \rangle \times U \) will not occur in the proof of the proposition, we drop the index \( i \) throughout, so \( U = U_i \) is now a pro-\( l \) group and we need to consider the \( A \)-module \( \text{Det } K_1(\Lambda'(U)) \). Recall that \( A \) acts on \( U \) by group automorphisms and on \( \mathcal{D} \) by \( A \to G_{\mathbb{N}/Q_i} \).

Let \( a \) denote the kernel of \( \Lambda(U) \to \Lambda(U_{ab}) \) and set \( \mathfrak{A} = \mathcal{D} \otimes_{\mathbb{Z}_l} a. \)

By surjectivity of \( (\Lambda'(U))^{\times} \to K_1(\Lambda'(U)) \) (see [CR II, p.76]) we have

\[ \text{Det } (\Lambda'(U)^{\times}) = \text{Det } K_1(\Lambda'(U)). \]

We start out the proof of the proposition from the diagram

\[
\begin{align*}
\text{Det } & \quad \longrightarrow \quad \Lambda^{\text{D}_i}(U)^{\times} \quad \longrightarrow \quad \Lambda^{\text{D}_i}(U_{ab})^{\times} \\
\downarrow & \quad \quad \downarrow \text{Det} \quad \downarrow \text{Det} \\
\text{Det } (1 + \mathfrak{A}) & \quad \longrightarrow \quad \text{Det } (\Lambda^{\text{D}_i}(U)^{\times}) \quad \longrightarrow \quad \text{Det } (\Lambda^{\text{D}_i}(U_{ab})^{\times})
\end{align*}
\]
with the top row exact because \( a \) is contained in the radical of \( \Lambda(U) \). The right square of the diagram commutes [RW2, Lemma 9] and the right \( \text{Det} \) is an isomorphism (see [CRII, 45.12, p.142]). Therefore the whole diagram commutes and its bottom sequence is exact.

We claim that \( \text{Det}(1 + \mathfrak{A}) \simeq \tau(\mathfrak{A}) \) with \( \tau(\mathfrak{A}) \) the image of \( \mathfrak{A} \subset \Lambda^D(G_\infty) \) in \( T(\Lambda^D(G_\infty)) = \Lambda^D(G_\infty)/[\Lambda^D(G_\infty), \Lambda^D(G_\infty)] \) (see [RW3,§3]) . Since \( L: \text{Det}(1 + \mathfrak{A}) \to \text{Tr}(\tau(\mathfrak{A})) \) is an isomorphism by the Corollary to Theorem B, in [RW3], it remains to see that \( L \) and \( \text{Tr} \) are \( A \)-equivariant. For \( L \) this follows as \( \Psi \) is induced by \( \gamma \mapsto \gamma^l \) for \( \gamma \in \Gamma_k \). For \( \text{Tr} \) it follows from Lemma 6 and Proposition 3 of [RW3]: Let \( a \in A, \omega \in \mathcal{O}, \) and \( u \in U \). Then

\[
\text{Tr}(\omega u)^a(\chi) = \text{Tr}(\omega u)(\chi^{-1})^{\sigma_a} = \text{trace}(\omega u | \mathfrak{M}_{\chi^{-1}})^{\sigma_a} = (\omega \chi^{-1}(u)\pi)^{\sigma_a} \\
= \omega^{\sigma_a} \chi(u^a)\pi \text{ trace}(\omega^{\sigma_a} u^a | \mathfrak{M}_{\chi}) = \text{Tr}(\omega^{\sigma_a} u^a)(\chi).
\]

Collecting everything so far, the starting diagram gives the exact \( A \)-module sequence

\[
\tau(\mathfrak{A}) \xrightarrow{\text{Det}} (\Lambda^D(U)^\times) \rightarrow (\Lambda^D(U^{ab})^\times).
\]

So the proof of the proposition will be finished once we have shown that

\[
\tau(\mathfrak{A}) \text{ and } \Lambda^D(U^{ab})^\times \text{ are } A\text{-cohomologically trivial.}
\]

For \( \tau(\mathfrak{A}) \) this holds because \( \tau(\mathfrak{A}) = \mathcal{O} \otimes_{\mathbb{Z}_l} \tau(a) \) has diagonal \( A \)-action and \( \mathcal{O} \) is \( \mathbb{Z}_l[A]\)-cohomologically trivial, as \( \mathcal{O}/\mathbb{Z}_l \) is unramified. By [Se1, Theorem 9, p.152] then the tensor product is cohomologically trivial as well.

The proof of the cohomological triviality of \( \Lambda^D(U^{ab})^\times \) uses the following fact:

If \( (X_n, f_n : X_n \to X_{n-1}) \) is a projective system of \( A \)-modules with surjective maps \( f_n \), then \( X = \lim X_n \) is cohomologically trivial if all the \( X_n \) are. This holds because of the exact sequence \( X : \prod X_n \to \prod X_n \) in which \( (\cdots, x_n, \cdots) \mapsto (\cdots, f_{n+1}(x_{n+1})-x_n, \cdots) \) is the second map. Note that the \( X_n \) are cohomologically trivial, if \( X_1 \) and all \( \ker(X_{n+1} \to X_n) \) are so.

Set \( g = \ker(\Lambda(U^{ab}) \to \Lambda(\Gamma_k)) \) and \( \mathfrak{G} = \mathcal{O} \otimes_{\mathbb{Z}_l} g \). Since some power of \( g \) is contained in \( l\Lambda(U^{ab}) \) (compare the beginning of the proof of [RW3, Theorem 8]), \( \Lambda(U^{ab}) \) is complete with respect to its \( g \)-adic topology. Also, \( 1 + g \subset \Lambda(U^{ab})^\times \), and thus the short exact sequence \( 1 + \mathfrak{G} \to \Lambda^D(U^{ab})^\times \to \Lambda^D(\Gamma_k)^\times \) implies the cohomological triviality of \( \Lambda^D(U^{ab})^\times \), if \( 1 + \mathfrak{G} \) and \( \Lambda^D(\Gamma_k)^\times \) are \( A \)-cohomologically trivial.

Setting \( X_n = \frac{1+\mathfrak{G}}{1+\mathfrak{G}^n}, \ker(X_{n+1} \to X_n) \simeq \mathcal{O} \otimes_{\mathbb{Z}_l} \frac{g^n}{g^{n+1}}, \) which is cohomologically trivial by [Se1, loc.cit.].

For the right term of the above short exact sequence we identify \( \Lambda^D(\Gamma_k) \) and \( \mathcal{O}[T] \), as usual, and set \( X_n = \frac{\mathcal{O}[T]^\times}{1+T^{n} \mathcal{O}[T]} \); so \( X_1 = \mathcal{O}^\times \) and \( \ker(X_{n+1} \to X_n) = \mathcal{O} \), which both are cohomologically trivial.

Adding \( \Lambda \) at the appropriate places, Proposition 2 is established.

**Corollary** (to Theorem 1). Let \( G_\infty \) be \( \mathbb{Q}_l \)-l-elementary. Then

\[
\text{Det} K_1(\Lambda(G_\infty)) \cap \text{Hom}^\times(R_1(G_\infty), \Lambda^\times(\Gamma_k)^\times) \subset \text{Det} K_1(\Lambda(G_\infty)).
\]
Namely, by Theorem 1,
\[
\det K_1(\Lambda(G_{\infty})) \cap \text{Hom}^* (R_l(G_{\infty}), \Lambda^c(\Gamma_k)^{\times}) \\
\subset \prod_i (\det K_1(\Lambda^D(U_i)))^{A_i} \cap \prod_i \text{Hom}^{N_i} (R_l(U_i), \Lambda^c(\Gamma_k)^{\times})^{A_i} \\
\subset \prod_i (\det K_1(\Lambda^D(U_i)) \cap \text{Hom}^{N_i} (R_l(U_i), \Lambda^c(\Gamma_k)^{\times}))^{A_i} \\
\subset \prod_i (\det K_1(\Lambda^D(U_i)))^{A_i} \subset \det K_1(\Lambda(G_{\infty}))
\]
with \(\tilde{\cdot}\) by [RW3, Theorem B.].

**Proposition 3.** Let \(G_{\infty}\) be \(\mathbb{Q}_{\ell}\)-l-elementary. Then \(L_{K_{\infty}/k} \in \det K_1(\Lambda(G_{\infty}))\) if, and only if, \(L_{K'/k'} \in \det K_1(\Lambda(G_{K'/k'}))\) whenever \(G_{K'/k'}\) is an l-elementary section of \(G_{\infty}\).

If \(L_{K_{\infty}/k} \in \det K_1(\Lambda(G_{\infty}))\) and if \(G_{K'/k'} = G_{K_{\infty}/k'}/G_{K_{\infty}/k'}\) is an l-elementary section of \(G_{\infty}\) with \(k \subset k' \subset K' \subset K_{\infty}\), then \(\text{defl}_{G_{K_{\infty}/k'}}^{G_{K_{\infty}/k'}} \text{res}_{G_{K_{\infty}/k'}}^{G_{K_{\infty}/k'}} L_{K_{\infty}/k} = L_{K'/k'}\) (see [RW2, §4]). And by [RW2, Lemma 9], \(L_{K'/k'} \in \det K_1(\Lambda(G_{K'/k'}))\).

For the converse it may help to review the notation of that part of the proof of Theorem 1 where (D2) appears. The point is that \(\overline{G}_i \overset{\text{def}}{=} G_i/\ker \beta_i = \langle \overline{s}_i \rangle \times U_i\), with \(\langle \overline{s}_i \rangle = \langle s \rangle/\ker \beta_i\), is an l-elementary section. And as \(G_i = \langle s \rangle \times U_i\),
\[
\text{Hom}^* (R_l(G_{\infty}), \Lambda^c(\Gamma_k)^{\times}) \overset{\text{res}}{\to} \prod_i \text{Hom}^* (R_l(G_i), \Lambda^c(\Gamma_k)^{\times})^{A_i} \overset{\text{defl}}{\to} \\
\prod_i \text{Hom}^* (R_l(U_i), \Lambda^c(\Gamma_k)^{\times})^{A_i}
\]
takes \(L_{K_{\infty}/k}\) to \(\prod_i L_{K_i'/k_i'}\) where \(k_i' = K_{\infty}^{G_i}\) and \(K_i' = K_{\infty}^{\ker \beta_i}\). Note here that the \(i\)th deflation map is \(A_i\)-equivariant since \(\langle s \rangle \to \langle \overline{s}_i \rangle\) is so.

By assumption, \(L_{K_i'/k_i'} = \det y_i\) where \(y_i \in K_1(\Lambda(\overline{G}_i))\) and so \(\det y_i \in (\det K_1(\Lambda(\overline{G}_i)))^{A_i}\). Projecting to \(e_i(\Lambda(\overline{G}_i))\), \(L_{K_i'/k_i'}\) induces a function in \(\text{Hom}^* (R_l^{(e_i)}(\overline{G}_i), \Lambda^c(\Gamma_k)^{\times})^{A_i}\). But \(e_i(\Lambda(\overline{G}_i)) = \pi_i(\Lambda(\overline{G}_i)) = e_i Z_i(s) \otimes Z_i \Lambda(U_i)_s\), so \(\pi_i y_i \in K_1(e_i Z_i(s) \otimes Z_i \Lambda(U_i)_s)\) and \(\det (\pi_i y_i) \in (\det K_1(e_i Z_i(s) \otimes Z_i \Lambda(U_i)_s))^{A_i}\). Now \(\prod_i (\det K_1(e_i Z_i(s) \otimes Z_i \Lambda(U_i)_s))^{A_i} = \det K_1(\Lambda(G_{\infty}))\), by Theorem 1, and the proof is finished.

**Remark.** In Proposition 3, the Iwasawa L-function \(L_{K_{\infty}/k}\) may be replaced by any function \(f \in \text{Hom}^* (R_l(G_{\infty}), \Lambda^c(\Gamma_k)^{\times})\) on setting \(f_{K'/k'} = \text{defl}_{G_{K_{\infty}/k'}}^{G_{K_{\infty}/k'}} \text{res}_{G_{K_{\infty}/k'}}^{G_{K_{\infty}/k'}} f\) for all l-elementary sections \(G_{K'/k'}\) of \(G_{\infty}\).
3. $\mathbb{Q}_l \cdot q$-elementary groups $G_\infty$

In this section $q$ is a prime number $\neq l$.

We say that the Galois group $G_\infty = G_{K_\infty/k}$ is a $\mathbb{Q}_l \cdot q$-elementary group, if $G_\infty = H \times \Gamma$ for some central open $\Gamma \subseteq G_\infty$ and a finite $\mathbb{Q}_l \cdot q$-elementary group $H$. Recall that a finite group $H$ is called $\mathbb{Q}_l \cdot q$-elementary if it is a semidirect product $\langle s \rangle \rtimes H_q$ of a cyclic normal subgroup $\langle s \rangle$ of order prime to $q$ and a $q$-group $H_q$ whose action on $\langle s \rangle$ induces a homomorphism $H_q \to G_{\mathbb{Q}_l(\zeta)/\mathbb{Q}_l}$, where $\zeta$ is a root of unity of order $|\langle s \rangle|$.

**Lemma 4.**

1. If $\Gamma$ is a central open subgroup of $G_\infty$ so that (the finite group) $G_\infty/\Gamma$ is a $\mathbb{Q}_l \cdot q$-elementary group, then $G_\infty$ is $\mathbb{Q}_l \cdot q$-elementary.

2. Let $G_\infty$ be $\mathbb{Q}_l \cdot q$-elementary, $G_\infty = H \times \Gamma$, $H = \langle s \rangle \rtimes H_q$. Then each irreducible character $\chi \in R_l(G_\infty)$ can be written as $\chi = \rho \cdot \text{ind}^G_\Gamma(\xi)$ with an abelian character $\rho$ of $G_\infty$ of type $W$ and an abelian character $\xi$ of a subgroup $G' \supset \langle s \rangle \rtimes \Gamma$ of $G_\infty$ so that $\xi = 1$ on $\Gamma$.

In order to see 1, we pick a Sylow-$l$ subgroup $U$ of $G_\infty$ containing the central open $\Gamma$. Then $U/\Gamma$ is an $l$-subgroup of the finite $\mathbb{Q}_l \cdot q$-elementary group $G_\infty/\Gamma$, hence cyclic and normal in $G_\infty/\Gamma$. We conclude that $U$ is an abelian normal subgroup of $G_\infty$, and, moreover, that $G_\infty = U \times H'$ with a finite $\mathbb{Q}_l \cdot q$-elementary group $H'$ of order prime to $l$. Writing the abelian $U$ as $U = H \times H_1$ with $H_1$ finite (cyclic) and $\Gamma_1 \simeq \mathbb{Z}_l$, so $H_1 < G_\infty$, the usual Maschke argument provides a $\mathbb{Z}_l[H']$-decomposition $U = H_1 \times \Gamma_2$ with $\Gamma_2 \simeq \mathbb{Z}_l$, by $|H'| \in \mathbb{Z}_l^\times$. We infer from $\Gamma'' \subset \Gamma_2$ for some $n$ that $H'$ acts trivially on $\Gamma_2$. Thus $G_\infty = H \times \Gamma_2$ with $H = H_1 \times H'$ a finite $\mathbb{Q}_l \cdot q$-elementary group and $\Gamma_2$ central open in $G_\infty$.

For 2. we first restrict $\chi$ to $\Gamma$ and obtain $\text{res}^\Gamma_{G_\infty} \chi = \chi(1) \cdot \rho_1$ for some abelian character $\rho_1$ of $\Gamma$. Via $G_\infty/H = \Gamma_k$, $\rho_1$ is the restriction of a type $W$ character $\rho$ of $G_\infty$. Since $\rho^{-1}$ is trivial on $\Gamma$, we may henceforth assume that $\chi$ is trivial on $\Gamma$, whence is inflated from an irreducible $\mathbb{Q}_l \cdot q$-character of $H$. By Clifford theory [CR I, p.265] the $\mathbb{Q}_l \cdot q$-irreducible characters of $H$ are of the form $\text{ind}^H_\Gamma(\xi \cdot \omega)$ with an abelian character $\xi$ of some subgroup $\tilde{H} \supset \langle s \rangle$ and an irreducible character $\omega$ of $H/\langle s \rangle$ (inflated to $H$). The group $H/\langle s \rangle$ is a $q$-group, so monomial, from which we deduce an equality $\text{ind}^H_\Gamma(\xi \cdot \omega) = \text{ind}^H_\Gamma(\xi)$ with $\langle s \rangle \leq H' \leq \tilde{H}$ and an abelian character $\xi$ of $H'$. Setting $G' = H' \times \Gamma$ finishes the proof of 2. and of the lemma.

**Lemma 5.** Assume that $G_\infty = H \times \Gamma$ with $H$ of order prime to $l$. Then $\mathbb{Q}(G_\infty)$ is the group algebra of the finite group $H$ over the field $\mathbb{Q}(\Gamma)$ and each $f \in \text{Hom}^*(R_l(G_\infty), \Lambda(\Gamma_k)^\times)$ is a $\text{Det} z$ for some $z \in \Lambda(G_\infty)^\times$.

This is straightforward: $\mathbb{Q}(G_\infty) = \mathbb{Q}(\Gamma)[H] = \mathbb{Q}(\Gamma) \otimes_{\mathbb{Q}_l}[H]$ is isomorphic to a product of matrix rings over the character fields $\mathbb{Q}(\Gamma)(\chi)$ (see [CR II, 74.11, p.740]), where $\chi$ runs through the $\mathbb{Q}_l \cdot q$-irreducible characters of $H$ modulo $G_{\mathbb{Q}_l^\times/q}$-action. By $l \mid |H|$, $\Lambda(\Gamma)[H] = \Lambda(\Gamma) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[H]$ is a maximal order in $\mathbb{Q}(\Gamma)[H]$, hence a product of matrix rings over the integral closures of $\Lambda(\Gamma)$ in the centre fields $\mathbb{Q}(\Gamma)(\chi)$. 

Proposition 6. Assume that \( G_\infty \) is \( \mathbb{Q}_l \)-\( q \)-elementary. Let \( f \in \text{Hom}^*(R_l(G_\infty),\Lambda^c(\Gamma_k)') \) satisfy \( (\mathrm{res}^G_{\mathbb{Q}_l^c} f)^\iota(\chi)^t = \Psi((\mathrm{res}^G_{\mathbb{Q}_l^c} f)(\psi \chi')) \mod l\Lambda^c(\Gamma_{k'}) \) for all open subgroups \( G' \) of \( G_\infty \) (with \( k' = K_\infty^{G'} \)) and all \( \chi' \in R_l(G') \). Then there exists a \( z \in \text{Det} K_1(\Lambda(G_\infty)) \) such that \( ((\text{Det} z)^{-1} f)^m \in \text{Hom}^*(R_l(G_\infty), 1+l\Lambda^c(\Gamma_k)) \) for some power \( l^m \). The same holds with \( \Lambda \) replaced by \( \Lambda_\iota \).

For the proof (compare also [Ty, p.94/95]) we set \( \overline{G} = G_\infty / H_l = \overline{H} \times \Gamma \) with \( \overline{H} \) finite of order prime to \( l \). In particular, \( \Lambda(\overline{G}) = \Lambda(\Gamma)[\overline{H}] \). We proceed from the commutative square (see [RW2, Lemma 9])

\[
\begin{array}{ccc}
K_1(\Lambda(G_\infty)) & \stackrel{\text{defl}}{\longrightarrow} & K_1(\Lambda(\overline{G})) \\
\text{Det} \downarrow & & \text{Det} \downarrow \\
\text{HOM}^*(R_l(G_\infty), \Lambda^c(\Gamma_k)') & \stackrel{\text{defl}}{\longrightarrow} & \text{HOM}^*(R_l(\overline{G}), \Lambda^c(\Gamma_k)')
\end{array}
\]

and consider \( \text{defl} f \). By Lemma 5, \( \text{defl} f = \text{Det} \varpi \) is solvable for some \( \varpi \in \Lambda(\overline{G})^\times \). Lift \( \varpi \) to a unit \( z \in \Lambda(G_\infty)^\times \), which is possible as \( H_l = \ker(G_\infty \to \overline{G}) \) is an \( l \)-group, and read this \( z \) in \( K_1(\Lambda(G_\infty)) \) (via \( \Lambda(G_\infty)^\times \to K_1(\Lambda(G_\infty)) \)). Then \( f' \overset{\text{defl}}{=} (\text{Det} z)^{-1} f \in \text{Hom}^*(R_l(G_\infty), \Lambda^c(\Gamma_k)') \) and \( \text{defl}(f') = 1 \).

Next, pick an irreducible \( \chi \in R_l(G_\infty) \) which is trivial on \( \Gamma \). So \( \chi = \text{ind}^G_{\mathbb{Q}_l^c} (\xi) \), with a \( \mathbb{Q}_l^c \)-irreducible character \( \xi \) of \( G' \) which is trivial on \( \Gamma' \), by 2. of Lemma 4. We define \( \overline{\chi} = \text{ind}^G_{\mathbb{Q}_l^c} (\overline{\xi}) \) where \( \xi = \xi_1 \cdot \xi_2 \) has been decomposed into its \( l \)-singular and \( l \)-regular components \( \xi_1, \xi_2 \), respectively. As \( \overline{\xi} \) is trivial on \( H_l \), \( \overline{\chi} \) is inflated from \( G \).

Now, \( f'(\chi - \overline{\chi}) = f'(\text{ind}^G_{\mathbb{Q}_l^c} (\xi - \overline{\xi})) = (\text{res}^G_{\mathbb{Q}_l^c} f')(\xi - \overline{\xi}) \). The assumption on \( f \) and the above Remark imply that

\[
f'(\chi - \overline{\chi})^m \equiv 1 \mod l\Lambda^c(\Gamma_{k'})
\]

if \( m \) is big enough so that \( \psi_m^\iota(\xi) = \psi_m^\iota(\overline{\xi}) \):

\[
f'(\chi - \overline{\chi})^m = (\text{res}^G_{\mathbb{Q}_l^c} f')(\xi - \overline{\xi})^m \equiv \Psi^m((\text{res}^G_{\mathbb{Q}_l^c} f')(\psi_m^\iota(\xi - \overline{\xi})) \mod l\Lambda^c(\Gamma_{k'})
\]

And since \( \text{defl}(f') = 1 \) and \( \overline{\chi} \) is inflated from \( \overline{G} \), \( f'(\overline{\chi}) = 1 \), we arrive at \( (f')^m(\chi) \equiv 1 \mod l\Lambda^c(\Gamma_{k'}) \) (see [RW2, Definition in §2]).

Remark. Observe that the above hypothesis is satisfied by \( f = L_{K_\infty/k} \) (see [RW3, 2. of Corollary to Theorem 9; RW2, 2. of Proposition 12]) and by every \( f \in \text{Det} K_1(\Lambda(G_\infty)) \) (see [RW2, Lemma 9; RW3, Proposition 4, 1. of Proposition 11]).

4. Proofs of Theorem B and C

In this section we prove Theorems B and C in full generality. This is done by using character actions on \( K_1 \) and \( \text{Hom}^* \) (as well as the Corollary to Theorem 1 and Proposition 3).
For an open subgroup $U$ of $G_{\infty}$, we denote by $R_{Q_l}(U)$ the ring of all characters of finite dimensional $\mathbb{Q}_l$-representations of $U$ with open kernel. We view $R_{Q_l}$ as a Frobenius functor of the open subgroups of $G_{\infty}$ in the sense of [CR II, 38.1].

We make $\text{Hom}^*(R_l(U), \Lambda^c(\Gamma_{k_l})^\times)$, with $k_U = K_{\infty}^{U}$, into an $R_{Q_l}(U)$-module by

$$(\kappa f)(\chi) = f(\kappa \chi) \quad \text{for} \quad f \in \text{Hom}^*, \kappa \in R_{Q_l}(U), \chi \in R_l(U),$$

with $\kappa$ the contragredient of $\kappa$.

We make $K_1(\Lambda(U))$ into an $R_{Q_l}(U)$-module as follows. If $\kappa$ is a character in $R_{Q_l}(U)$, and if $[P, \alpha]$ represents an element in $K_1(\Lambda(U))$, then choosing $U' \subset \ker \kappa$, an open subgroup of $U$, and a $\mathbb{Z}_{l}[U/U']$-lattice with character $\kappa$, we define

$$(\star) \quad \kappa \cdot [P, \alpha] = [M \otimes_{\mathbb{Z}_{l}} P, \text{id}_M \otimes_{\mathbb{Z}_{l}} \alpha]$$

(compare [CR II, p.175]).

**Lemma 7.** $\text{Det} : K_1(\Lambda(-)) \rightarrow \text{Hom}^*(R_l(-), \Lambda^c(\Gamma_{k_l})^\times)$ is a morphism of Frobenius modules over the Frobenius functor $U \mapsto R_{Q_l}(U)$.

The lemma is shown in the same way as its analogue in the case of group rings of finite groups. We only need to observe that the $\Lambda(U)$-module structure of $M \otimes_{\mathbb{Z}_{l}} P$ is derived from the diagonal action of $U$ on $M \otimes_{\mathbb{Z}_{l}} P$:

First, the $\Lambda(U')$-module structure on $P$ gives $M \otimes_{\mathbb{Z}_{l}} P$ a $\mathbb{Z}_{l}[U'] \rightarrow \mathbb{Z}_{l}[U]$ $\Lambda(U')$-structure. The pushout diagram then determines a $\Lambda(U')$-module structure.

In order to check $\Lambda(U)$-projectivity of $M \otimes_{\mathbb{Z}_{l}} P$, it suffices to take $P = \Lambda(U)$ and then Frobenius reciprocity $M \otimes_{\mathbb{Z}_{l}} \text{ind}_{U'}^U(\Lambda(U')) = \text{ind}_{U'}^U(\text{res}_{U'}^U(M) \otimes_{\mathbb{Z}_{l}} \Lambda(U'))$ takes care of this, since $M$ is $\mathbb{Z}_{l}$-free.

We next recall Swan’s theorem (see [CR II, 39.10, p.47]) which implies the independence of $(\star)$ from the choice of the lattice $M$. Indeed, given $\kappa$ and $U' \subset \ker \kappa$ as above, then two $\mathbb{Z}_{l}[U/U']$-lattices $M_1, M_2$ with character $\kappa$ induce the same element in the Grothendieck group $G_{\mathbb{Z}_{l}}^\times(\mathbb{Z}_{l}[U/U'])$ of finitely generated $\mathbb{Z}_{l}[U/U']$-lattices (see [CR I, §16B]). Moreover, it is readily checked from [CR II, 38.20, 38.24, p.14.16] that $[M_1 \otimes_{\mathbb{Z}_{l}} P, \text{id}_{M_1} \otimes_{\mathbb{Z}_{l}} \alpha] = [M_2 \otimes_{\mathbb{Z}_{l}} P, \text{id}_{M_2} \otimes_{\mathbb{Z}_{l}} \alpha]$ in $K_1(\Lambda(U))$.

It remains to show that $\text{Det}$ is a Frobenius module homomorphism. Let $\chi \in R_l(G_{\infty})$ and let $[P, \alpha] \in K_1(\Lambda(G_{\infty}))$, $[M] \in G_{\mathbb{Z}_{l}}^\times(\mathbb{Z}_{l}[U/U'])$ as in $(\star)$; set $\mathbb{Q}_l \otimes_{\mathbb{Z}_{l}} M = V_{\kappa}$. Then

$$(\det [M \otimes_{\mathbb{Z}_{l}} P, 1 \otimes_{\mathbb{Z}_{l}} \alpha])(\chi) = \frac{1}{\det \mathbb{Q}_l(\Gamma_{k_l})(1 \otimes_{\mathbb{Z}_{l}} \alpha | \text{Hom}_{\mathbb{Q}_l}(V_{\chi}, \mathbb{Q}_l \otimes_{\mathbb{Z}_{l}} (M \otimes_{\mathbb{Z}_{l}} P)))}$$

$$(\text{det} \mathbb{Q}_l(\Gamma_{k_l})(1 \otimes_{\mathbb{Z}_{l}} \alpha | \text{Hom}_{\mathbb{Q}_l}(V_{\chi}, \mathbb{Q}_l \otimes_{\mathbb{Z}_{l}} (M \otimes_{\mathbb{Z}_{l}} P)))$$

$$(\text{det} \mathbb{Q}_l(\Gamma_{k_l})(\alpha | \text{Hom}_{\mathbb{Q}_l}(V_{\chi}, \mathbb{Q}_l \otimes_{\mathbb{Z}_{l}} (M \otimes_{\mathbb{Z}_{l}} P)))$$

$$(\det [P, \alpha])(\kappa \chi) = (\kappa \det [P, \alpha])(\chi),$$

with $\frac{1}{\text{det}}$ and $\kappa$ due to the naturality on $H$-fixed points of the isomorphisms [CR I, 10.30, 2.19], respectively.
Corollary. $SK_1(Q(G_\infty)) = 0$ if $SK_1(Q(G') = 0$ for all open $Q_l$-elementary subgroups $G'$ of $G_\infty$.

This follows because $SK_1(Q(-))$ is a Frobenius module over $R_{Q_l}(-)$, by Lemma 7 with $\Lambda$ replaced by $Q$. Now apply the Witt-Berman induction theorem (see [CR I, 21.6, p.459]) to the finite group $G_\infty/\Gamma$ where $\Gamma$ is a central open subgroup: There exist $Q_l$-elementary subgroups $\overline{G}_i \subseteq G_\infty/\Gamma$ and (virtual) $Q_l$-characters $\overline{\xi}_i$ of $\overline{G}_i$ such that $1_{G_\infty} = \sum_i \text{ind}^G_{G_i}(\overline{\xi}_i)$, with $G_i$ the full preimage of $\overline{G}_i$ in $G_\infty$ and $\xi_i = \text{infl}_{G_i}^G(\overline{\xi}_i)$. By Lemma 4 the groups $G_i$ are $Q_l$-elementary (this is trivial for the prime number $l$). Now let $z \in SK_1(Q(G_\infty))$ and apply the above character relation to get from $\text{res}^G_{G_i} z = 0$

$$z = 1_{G_\infty} \cdot z = \sum_i \text{ind}^G_{G_i}(\xi_i) \cdot z = \sum_i \text{ind}^G_{G_i}(\xi_i \cdot \text{res}^G_{G_i} z) = 0.$$

Lemma 8. $\text{Det} K_1(\Lambda(G_\infty)) \cap \text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k))$ is a $\mathbb{Z}_l$-module, and the same with $\Lambda$ replaced by $\Lambda_\infty$.

It suffices to show $(\text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k)))^m \subset \text{Det} K_1(\Lambda(G_\infty))$ for some non-zero integer $m$, as this implies that $\text{Det} K_1(\Lambda(G_\infty)) \cap \text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k))$ is a $\mathbb{Z}_l$-submodule of the $\mathbb{Z}_l$-module $\text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k))$: For $f \in \text{Det} K_1(\Lambda(G_\infty)) \cap \text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k))$ and $c \in \mathbb{Z}_l$, then, writing $c = a + mb$ with $a, b \in \mathbb{Z}_l$, $c^f = f^a(f^b)^m$, and $f^a \in \text{Det} K_1(\Lambda(G_\infty)) \cap \text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k))$, $f^b \in \text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k))$, so $(f^b)^m \in \text{Det} K_1(\Lambda(G_\infty)) \cap \text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k))$.

We next prove the containment claimed above when $G_\infty = H \times \Gamma$ is abelian. Let $f \in \text{Hom}^*(R_l(G_\infty), 1 + l\Lambda^c(\Gamma_k))$, whence $f^{[H]} \in \text{Hom}^*(R_l(G_\infty), 1 + |H|\Lambda^c(\Gamma_k))$. Moreover, by (\ast) in the proof of [RW2, Theorem 8] and [CR II, 45.12, p.142],

$$f^{[H]} = \text{Det} q \quad \text{with} \quad q = \sum_{h \in H} q_h h \quad \text{in} \quad Q(G_\infty) = Q(\Gamma)[H].$$

Hence, by [RW3, Proposition 3], $f^{[H]}(\chi) = \sum_{h \in H} \overline{q}_h \chi(h)$ for every irreducible character $\chi \in \text{R}_l(G_\infty)$ which is trivial on $\Gamma$, where $\overline{q}$ is the isomorphism $\Gamma \rightarrow \Gamma_k$. It follows that

$$|H|\overline{q}_h = \sum_{\chi} f^{[H]}(\chi) \chi(h^{-1}) \equiv \sum_{\chi} \chi(h^{-1}) \equiv 0 \quad \text{mod} \quad |H|\Lambda^c(\Gamma_k),$$

i.e., $q_h \in \Lambda^c(\Gamma) \cap Q(\Gamma) = \Lambda(\Gamma)$. By [RW3, Lemma 10], $q \in \Lambda(G_\infty)^\times$.

For the general case we apply Artin induction: If $\Gamma$ is central open of index $n$ in $G_\infty$, then there exist subgroups $\Gamma \subset A_1 \subset G_\infty$ with $A_1/\Gamma$ cyclic so that $n \cdot 1_{G_\infty} = \sum_i \text{ind}^A_{A_i}(1_{A_i})$. It follows that the $A_i$ are abelian, and whence, with $k_i = \text{K}_\infty A_i$, $\text{Hom}^*(R_l(A_i), 1 + \Lambda(\Gamma_k))^m_i \subset \text{Det} K_1(\Lambda(A_i))$ for suitable integers $m_i$. Setting $m = \prod m_i$, we get $\text{Hom}^*(R_l(A_i), 1 + \Lambda(\Gamma_k))^m \subset \text{Det} K_1(\Lambda(A_i))$. Thus, if $f^m \in \text{Hom}^*(R_l(G_\infty), 1 + \Lambda(\Gamma_k))^m$, then the above character relation yields
By Lemma 8 the group on the right is a
obtain
for some power
following the proposition) and so there exist
(3)
and an integer
(2)
ξ
so, by the Corollary to Theorem 1, res
But res
Hom
2
n
U
to
G
∞
Proof of Theorem B.
Choose a central open subgroup \( \Gamma \) and apply the Witt-Berman induction theorem to \( G_\infty / \Gamma \). By [Se2, Theorem 28, p.98] there are \( \mathbb{Q}_l \)-\( l \)-elementary open subgroups \( U_i \subseteq G_\infty \) containing \( \Gamma \) together with characters \( \xi_i \in R_{Q_l}(U_i) \) so that we have
\[
(1) \quad n \cdot 1_{G_\infty} = \sum_i \text{ind}^{G_\infty}_{U_i}(\xi_i)
\]
for an integer \( n \mid [G_\infty : \Gamma] \) prime to \( l \). Now, let \( d \in \text{Det}_1(M(U_i)) \cap \text{Hom}^*(R_l(U_i), \Lambda^e(\Gamma_{k_i})^\times) \) and apply this character relation to it:
\[
d^n = \prod_i \text{ind}^{G_\infty}_{U_i}(\xi_i)d = \prod_i \text{ind}^{G_\infty}_{U_i}(\xi_i, \text{res}_{G_\infty}^{U_i}d). \tag{2}
\]
But \( \text{res}_{G_\infty}^{U_i}d \in \text{Det}_1(M(U_i)) \cap \text{Hom}^*(R_l(U_i), \Lambda^e(\Gamma_{k_i})^\times) \), with \( k_i = K_\infty U_i \), and so, by the Corollary to Theorem 1, \( \text{res}_{G_\infty}^{U_i}d \in \text{Det}_1(M(U_i)) \). It follows first that \( \xi_i, \text{res}_{G_\infty}^{U_i}d \in \text{Det}_1(M(U_i)) \) and then, from [RW3, Lemma 1], that
\[
(2) \quad d^n \in \text{Det}_1(M(G_\infty)).
\]
On the other hand, by 1. of Lemma 4 we find, for each prime number \( q \) dividing \( n \), \( \mathbb{Q}_l \)-\( q \)-elementary subgroups \( U'_j \) of \( G_\infty \) containing \( \Gamma \), characters \( \xi'_j \in R_{Q_l}(U'_j) \) and an integer \( n' \mid [G_\infty : \Gamma] \) prime to \( q \) such that
\[
(3) \quad n' \cdot 1_{G_\infty} = \sum_j \text{ind}^{G_\infty}_{U'_j}(\xi'_j).
\]
And, setting \( f_j = \text{res}_{G_\infty}^{U'_j}d \in \text{Det}_1(M(U'_j)) \cap \text{Hom}^*(R_l(U'_j), \Lambda^e(\Gamma_{k'_j})^\times) \), with \( k'_j = K_\infty U'_j \), then \( f_j \) is a function \( f \) as in Proposition 6 (compare the Remark following the proposition) and so there exist \( z_j \in K_1(M(U'_j)) \) such that
\[
((\text{Det} z_j)^{-1}f_j)^{m'_j} \in \text{Hom}^*(R_l(U'_j), 1 + l\Lambda^e(\Gamma_{k'_j})),
\]
for some power \( l^{m'_j} \). Combining this with (2), and setting \( m' = \max_j \{m'_j\} \), we obtain
\[
((\text{Det} z_j)^{-1}f_j)^{nl^{m'}} \in \text{Det}_1(M(U'_j)) \cap \text{Hom}^*(R_l(U'_j), 1 + l\Lambda^e(\Gamma_{k'_j})).
\]
By Lemma 8 the group on the right is a \( \mathbb{Z}_l \)-module, hence, as \( l \nmid n \),
\[
((\text{Det} z_j)^{-1}f_j)^{l^{m'}} \in \text{Det}_1(M(U'_j)).
\]
\[\text{The notation is an additive-multiplicative compromise.}\]
and consequently \( f_j^{\nu^r} = (\text{res}_{G_{\infty}} U_f) d^{\nu^r} \in \text{Det} K_1(\Lambda(U_f')) \). Now (3) yields \( d^{\nu^r} \in \text{Det} K_1(\Lambda(G_{\infty})) \) and then, by (2), \( d^{\nu^r} \in \text{Det} K_1(\Lambda(G_{\infty})) \). Letting \( q \) vary we obtain Theorem B.

**Proof of Theorem C.**

We only check the nontrivial implication and proceed as above. We start with \( L_{K_{\infty}/k} \in \text{HOM}^* (R_{U}(G_{\infty}), \Lambda_c^0(\Gamma_k)^{\infty}) \) and first use (1). Because \( \text{res}_{G_{\infty}} L_{K_{\infty}/k} \), it follows from the hypothesis and Proposition 3 that \( L_{K_{\infty}/k}^n \in \text{Det} K_1(\Lambda(G_{\infty}))) \). For each \( q \mid n \) we next turn to (3) and use that \( L_{K_{\infty}/k_j} \in \text{Hom}^*(R_{U}(U_j'), \Lambda_c^0(\Gamma_{k_j})^{\infty}) \) is a function \( f \) as in Proposition 6. Thus there is a \( z_j \in K_1(\Lambda(U_j')) \) with \( ((\text{Det} z_j)^{-1} L_{K_{\infty}/k_j'})^{m^j} \in \text{Det} K_1(\Lambda(U_j'))) \). Combining as before, we see that \( ((\text{Det} z_j)^{-1} L_{K_{\infty}/k_j'})^{m^j} \in \text{Det} K_1(\Lambda(U_j'))) \), whence already \( L_{K_{\infty}/k_j}^{m^j} \in \text{Det} K_1(\Lambda(U_j'))) \), by \( l \mid n \). Now apply (3) and get first \( L_{K_{\infty}/k}^{m^j} \in \text{Det} K_1(\Lambda(G_{\infty})) \) and then, from (2), \( L_{K_{\infty}/k}^{m^j} \in \text{Det} K_1(\Lambda(G_{\infty})) \). Varying \( q \), this finishes the proof of Theorem C.

**Remark 1.** The proof shows that the definition of a section of \( G_{\infty} \) could be strengthened to require \( K_{\infty}/K' \) to be finite cyclic of order prime to \( l \).

**Remark 2.** As before we may generalize Theorem C by replacing the Iwasawa \( L \)-functions \( L_{K'/k'} \) by the functions \( f_{K'/k'} \) of the Remark after Proposition 3.

**5. Complements**

We begin this section by presenting some examples:

**Example 1.** If the Sylow-\( l \) subgroups of \( G_{\infty} \) are abelian, then \( L_{K_{\infty}/k} \in \text{Det} K_1(\Lambda(G_{\infty})) \).

Indeed, Theorem C requires us to check whether \( L_{K_{\infty}/k}^{c} \in \text{Det} K_1(\Lambda(E)) \) whenever \( E = G_{\infty}^{c/c} \) is an \( l \)-elementary section of \( G_{\infty} \). But the assumption on the Sylow-\( l \) subgroups of \( G_{\infty} \) implies that the Sylow-\( l \) subgroup of \( E \) is abelian, whence \( E \) itself. Now apply 1. of the Corollary to Theorem 9 in [RW3].

Concerning the full “main conjecture” we have

**Example 2.** If \( G_{\infty} = H \times \Gamma \) satisfies \( l \mid |H| \), then \( SK_1(Q(G_{\infty})) = 1 \). In particular, the “main conjecture” is true for these groups.

The second assertion holds as the Sylow-\( l \) subgroup \( \Gamma \) of \( G_{\infty} \) is abelian; moreover, the first assertion now guaranties uniqueness of \( \Theta_S \) (see [RW2,§3, especially Remark E]).
For the proof of this first assertion, \( SK_1(Q(G_\infty)) = 1 \), we may assume that \( G_\infty \) is \( \mathbb{Q}_l \)-elementary, by the Corollary to Lemma 7.

If \( G_\infty \) is \( \mathbb{Q}_l \)-\( q \)-elementary with \( q \neq l \), then \( G_\infty = H \times \Gamma \) with \( H \) a finite \( \mathbb{Q}_l \)-\( q \)-elementary group. Since \( l \nmid |H| \), Lemma 5 implies that \( Q(G_\infty) \) is totally split.

Next, let \( G_\infty \) be \( \mathbb{Q}_l \)-\( l \)-elementary, so \( G_\infty = \langle s \rangle \times \Gamma \) by \( l \nmid |H| \), whence \( U = \Gamma \) in the notation of Theorem 1 which we continue to use (in particular, \( \beta_i \) is a \( \mathbb{Q}_l \)-irreducible character of \( \langle s \rangle \) with stabilizer subgroup \( \Gamma_i = U_i \leq \Gamma \), \( G_i = \langle s \rangle \times \Gamma_i \), and \( e_i \) is the idempotent associated to the \( G_{\mathbb{Q}_l} / \mathbb{Q}_l \)-orbit of \( \beta_i \)).

Because \( SK_1(Q(G_\infty)) = \prod_i SK_1(e_iQ(G_\infty)) \), it suffices to show that each \( e_iQ(G_\infty) \) is a (full) ring of matrices over a (commutative) field. Recall first that \( e_i\Lambda(G_i) = \Lambda_{\mathcal{D}_i}(\Gamma_i) \).

Finally we give a bound on the order of \( L_{K_\infty/k} \mod K_1(\Lambda(G_\infty)) \).

**Proposition 9.** Set \( l^a = [G' : Z(G')] \), where \( G' \) is a Sylow-\( l \) subgroup of \( G_\infty \) and \( Z(G') \) is its centre. Then \( L_{K_\infty/k}^o \in \det K_1(\Lambda(G_\infty)) \).

We first note that obviously \( a = a(G_\infty) \) is an invariant of \( G_\infty \) and that \( a(G_\infty) \geq a(G_{K'/k'}) \) for all sections \( K'/k' \) of \( K_\infty/k \). Hence, if we can show that \( L_{K'/k'}^o \in \det K_1(\Lambda(G_{K'/k'})) \) for all \( l \)-elementary sections \( K'/k' \) of \( K_\infty/k \), with \( a' = a(G_{K'/k'}) \), then, by Remark 2 following the proof of Theorem C, we have also verified Proposition 9. Hence, from now on, \( G_\infty \) is \( l \)-elementary.

In this case \( l^a = [G_\infty : Z(G_\infty)] \) and we proceed by induction on \( a \). If \( a = 0 \), then \( G_\infty \) is abelian and 1. of Corollary to Theorem 9 in [RW3] gives what we want. If \( a > 0 \), then \( G_\infty \) is nonabelian and consequently \( G_\infty / Z(G_\infty) \) noncyclic. We infer the existence of a normal subgroup \( G' \) of \( G_\infty \) containing \( Z(G_\infty) \) so that \( G' = G_\infty / G' \) is noncyclic of order \( l^2 \). From it we obtain the character relation

\[
l \cdot 1_{G_\infty} \cong \sum M_j \cdot \text{ind} \gamma(M_j) \cdot \text{ind} \gamma(1_T)
\]

with \( \gamma \) running through the maximal subgroups of \( G \). Inflation yields \( l \cdot 1_{G_\infty} = \sum j n_j \cdot \text{ind} \gamma(M_j) \cdot \text{ind} \gamma(1_M) \) with proper open subgroups \( M_j \leq G_\infty \) containing \( Z(G_\infty) \) and with integers \( n_j \). Because \( a(M_j) < a \), induction implies that \( L_{K_\infty/k_j}^{l-1} \in \det K_1(\Lambda(G_{K_\infty/k_j})) \) for all \( j \) (with \( k_j = K_\infty^{M_j} \)), and then the last character relation gives \( L_{K_\infty/k}^{l^a} \in \det K_1(\Lambda(G_\infty)) \).

Proposition 9 is established.
References


This article is available at http://intlpress.com/HHA/v7/n3/a8/

Jürgen Ritter  ritter@math.uni-augsburg.de

Mathematics Department
University of Augsburg
Germany
86135 Augsburg

Alfred Weiss  weissa@ualberta.ca

Mathematics Department
University of Alberta
Canada T6G 2G1
Edmonton