ERRATUM TO ‘CATEGORY OF $A_{\infty}$-CATEGORIES’

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Abstract

The erroneous statement (HHA 5 (2003), no. 1, 1–48) that the collection of unital $A_{\infty}$-categories, all $A_{\infty}$-functors, and all $A_{\infty}$-transformations (resp. equivalence classes of natural $A_{\infty}$-transformations) form a $\mathcal{K}$-2-category $\mathcal{K}^{A_{\infty}}$ (resp. ordinary 2-category $^*A_{\infty}$) is corrected as follows. All 2-category axioms are satisfied, except that $1_e \cdot f$ does not necessarily equal $1_{ef}$ for all composable 1-morphisms $e, f$. The axiom $e \cdot 1_f = 1_{ef}$ does hold. The mistake does not affect results on invertible 2-morphisms and quasi-invertible 1-morphisms in $^*A_{\infty}$.

Let $\mathcal{V} = (V, \otimes, c, 1)$ be a symmetric monoidal category. Besides the notions of a 1-unital 2-unital $\mathcal{V}$-2-category (Definition A.1) and a 1-unital non-2-unital $\mathcal{V}$-2-category (a 2-category enriched in $\mathcal{V}$ which has unit 1-morphisms, but does not have unit 2-morphisms) (Definition A.2) the article [Lyu03] should contain the following intermediate notion:

**Definition A.3** (1-unital left-2-unital $\mathcal{V}$-2-category). A 1-unital left-2-unital $\mathcal{V}$-2-category consists of a 1-unital non-2-unital $\mathcal{V}$-2-category $\mathfrak{A}$ plus a morphism $1_f : 1 \to \mathfrak{A}(A, B)(f, f)$ for any 1-morphism $f : A \to B$, which is a two-sided unit with respect to vertical composition of 2-morphisms $m_2$, such that

$$
eq 1_{ef} \quad (1)$$

for all composable 1-morphisms $e, f$. Moreover, if

$$1_f \cdot k \equiv \left( A \xrightarrow{f} B \xrightarrow{k} C \right) = 1_{fk} \quad (2)$$

for all composable 1-morphisms $f, k$, such $\mathfrak{A}$ is the same as a 1-unital 2-unital $\mathcal{V}$-2-category.

Let $\mathcal{K}$ denote the homotopy category of the differential graded category of complexes of $k$-modules, $k$ being a commutative ring with a unit. Morphisms of $\mathcal{K}$ are chain maps modulo homotopy. It is correctly stated in [Lyu03] that the collection

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of all $A_\infty$-categories, all $A_\infty$-functors and all $A_\infty$-transformations (resp. equivalence classes of natural $A_\infty$-transformations) is a 1-unital non-2-unital $\mathcal{K}$-2-category $\mathcal{K}A_{\infty}$ (resp. 1-unital non-2-unital 2-category $A_{\infty}$). It is correctly stated there that the collection of unital $A_\infty$-categories, unital $A_\infty$-functors and all $A_\infty$-transformations (resp. equivalence classes of natural $A_\infty$-transformations) is a 1-unital 2-unital $\mathcal{K}$-2-category $\mathcal{K}uA_{\infty}$ (resp. ordinary 2-category $A_{\infty}^u$). However, it is claimed incorrectly in Corollaries 7.11, 7.12 [ibid.] that the latter property holds also for the collection of unital $A_\infty$-categories, all $A_\infty$-functors, and all $A_\infty$-transformations (resp. equivalence classes of natural $A_\infty$-transformations). The correct statement is that the stated collection constitutes a 1-unital left-2-unital $\mathcal{K}$-2-category $\mathcal{K}uA_{\infty}$ (resp. 1-unital left-2-unital 2-category $A_{\infty}^u$). Fortunately, the notions of an invertible 2-morphism, of a 1-morphism which is an equivalence, etc. make sense in $uA_{\infty}$.

All other results of [Lyu03] which concern $uA_{\infty}$ remain valid. For instance, if $B, C$ are unital $A_\infty$-categories, $r: f \to g: B \to C$ is an isomorphism of $A_\infty$-functors and $f$ is unital, then $g$ is unital as well.

The proof of property (1) for all $A_\infty$-functors $e: D \to A$, $f: A \to B$ with unital $A_\infty$-category $B$ consists of the line $e \cdot 1_f = e \cdot (fi_B)s^{-1} = (efi_B)s^{-1} = 1_e f$, where $i_B: id_B \to id_B: B \to B$ is the unit $A_\infty$-transformation. For any $A_\infty$-functor $f: A \to B$ and a unital $A_\infty$-functor $k: B \to C$, property (2) follows from the chain maps

$$
1_f \cdot k = (f^B s^{-1}) \cdot k = (f^B k)s^{-1}: k \to (A_\infty(A, C)(fk, fk), m_1),
$$

$$
1_{fk} = (fk \ell^C)s^{-1}: k \to (A_\infty(A, C)(fk, fk), m_1)
$$

being equal in $\mathcal{K}$. In fact, these cycles are homologous, since $i^B k \equiv ki^C$ implies $f^B k \equiv fk^C$.

The erroneous statement was also referred to (but not used in any reasoning) after Corollary 5.6 of [LO06]. Other articles on the subject are not influenced by the mistake described here.

References


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