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LUSTERNIK-SCHNIRELMANN CATEGORY AND PRODUCTS OF LOCAL SPACES

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Abstract

Let X, Y be finite 1-connected p-local CW-complexes. We show that $\mathrm{cat}(X \times Y) = \mathrm{cat}(X) + \mathrm{cat}(Y)$ holds if p is large and the loop space homology of X, Y satisfies a certain flatness assumption.

1. Introduction

It has been well known for a long time that the L.S. category of a product of two spaces X, Y satisfies the inequality $\mathrm{cat}(X \times Y) \leqslant \mathrm{cat}(X) + \mathrm{cat}(Y)$ and that there are examples of spaces for which the inequality is strict. But for simply connected rational spaces X_0, Y_0 of finite type there is, by a result of Félix, Halperin and Lemaire [**FHL**], always the equality $\mathrm{cat}(X_0 \times Y_0) = \mathrm{cat}(X_0) + \mathrm{cat}(Y_0)$. On the other hand, Iwase constructed recently [**I**], for each prime p, a two-cell complex Q_p with $\mathrm{cat}(Q_p \times S^n) = \mathrm{cat}(Q_p) = 2$ for some n. This equation still holds after localization at p.

The aim of this paper is to prove that a generalization of the theorem of [**FHL**] holds for a class of spaces which are somehow intermediate between rational spaces and Iwase's counterexample to Ganea's conjecture. Denote by $\mathbb{Z}_{(p)}$ the integers localized at the prime p and let $\alpha(n,p) = \min(n + 2p - 3, np - 1)$.

Theorem 1.1. *Let X, Y be p-local n-connected, $n \geqslant 1$ CW-complexes of finite $\mathbb{Z}_{(p)}$-type. Suppose that $\dim X + \dim Y \leqslant \alpha(n,p)$ and that $\widetilde{H}_*(\Omega X, \mathbb{Z}_{(p)})$, $\widetilde{H}_*(\Omega Y, \mathbb{Z}_{(p)})$ are free R-modules for R a quotient ring of $\mathbb{Z}_{(p)}$ and $* \leqslant \dim X + \dim Y - 1$. Then* $\mathrm{cat}(X \times Y) = \mathrm{cat}(X) + \mathrm{cat}(Y)$.

Corollary 1.2. *Let R be a quotient ring of $\mathbb{Z}_{(p)}$ and let $P^\ell(R)$ be the Moore space with top cell in dimension ℓ. Suppose that Y is as in 1.1 and that* $\dim Y + \ell \leqslant \alpha(n,p)$. *Then* $\mathrm{cat}(P^\ell(R) \times Y) = \mathrm{cat}(Y) + 1$.

So in the class of spaces considered above a mod p version of Ganea's conjecture holds. Note that wedges, fat wedges and products of the spaces $P^\ell(R)$ satisfy the condition on the loop space homology in 1.1.

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The behaviour of L.S. category with respect to localization is not fully understood. There are infinite complexes X for which the category of the localization $X_{(p)}$ is less than m for all p but $\mathrm{cat}(X) = m$ [**R**]. For finite complexes no such examples are known; see [**C**]. That a result like 1.1 should hold was conjectured by Hess in [**He**]. The homotopy category of p-local CW-complexes in the range specified in 1.1 is isomorphic to a homotopy category of dg Lie algebras [**A1**]. Moreover, the loop space homology of such a space X is isomorphic to the homology of the universal enveloping algebra of the dg Lie algebra representing the homotopy type of X.

The strategy to the proof of 1.1 which follows the lines of the rational case in [**FHL**] and [**He**] will be described now. The first step is to give, for spaces as in 1.1, an interpretation of cat à la Félix and Halperin in terms of minimal models over $\mathbb{Z}_{(p)}$. Here we rely heavily on the work of Scheerer and Tanré and we just have to interpret their results which they formulated for Lie and coalgebras. In the next main step we establish a local version of a famous rational result due to Hess. It follows from this theorem that one can compute cat in the category of dg modules over the minimal algebra model. It is only here where the assumption on the loop space homology in 1.1 is needed. In the last step we transport the computation of cat in a category of cochain complexes over $\mathbb{Z}_{(p)}$. It is shown that cat equals a generalized Toomer invariant. This is done using duality as in [**FHL**]. Then a direct computation gives the product formula.

The adaptation of the proof of the rational theorem to the local situation faces two main technical obstacles. The first is the need to work with n-type approximations from the very beginning. So the construction of various (minimal) models has to be based on the notion of an n-equivalence. For this reason, we have to investigate the localization, with respect to n-equivalences, of the unbounded derived category and certain homotopy categories of algebras. The second is that there are obstructions, due to the action of the Bockstein operators on the loop space homology, for a proof of a general p-local version of Hess's theorem. These obstructions are not present if one works over the ground field $\mathbb{F}_p$. So we have to use lifting arguments and here the assumptions made in 1.1 come in.

The organization of the paper is as follows. Some homotopy theory of unbounded differential graded modules over a differential graded algebra is developed in Section 2. We study certain localizations of the standard model category, and construct minimal models in various settings. We suggest that the reader skip Section 2 at first reading and passes back to it when needed. The basics of tame and mild homotopy theory in various algebraic categories are recalled in Section 3. In addition, we study n-types in this setting. In Section 4 we show how to compute cat using differential graded Lie and (co)algebras. The aim of Section 5 is to construct and study a certain extension of minimal algebras which represents a generalized Ganea fibration. In Section 6 it is shown that this extension splits as a dg algebra if and only if it splits as a dg module over a minimal algebra. The main theorem is proved in Section 7 using a generalized Toomer invariant and duality.

We assume familiarity with the language of homotopical algebra and especially with the theory of model categories and (co)fibration categories. The classical sources for the former are [**Q1, Q2**]. There are now some more recent books on this subject [**GJ, Hi, Ho**]. For the latter, we refer to Baues' book [**B**].

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2. Homotopy theory for differential graded modules over differential graded algebras and n-types

We work over a commutative ground ring R with unit. A graded R-module is a family $M = (M^i)_{i\in\mathbb{Z}}$ of R-modules and we write $|x| = i$ for the degree of $x \in M^i$. A differential on M is a linear map d of degree 1 such that $d^2 = 0$. The category of cochain complexes, i.e., graded modules with a differential, will be denoted by $\mathrm{Coch}(R)$. The suspension of $M \in \mathrm{Coch}(R)$ is the cochain complex sM defined by $sM_i = M_{i-1}$ and $dsx = -sdx$. A differential graded algebra, or DGA, is a cochain complex A with an associative multiplication and a unit $1 \in A^0$, such that $d_A(xy) = d_A(x)y + (-1)^{|x|}x\, d_A(y)$ holds. A left module over a DGA is a cochain complex M with an A-action $A \otimes M \to M$, $a \otimes m \to am$, such that $d_M(am) = d_A(a)m + (-1)^{|a|}a\, d_M(m)$ holds. Denote the category of A-modules by A-mod. (See [**F1**] for more information about this category.) In particular, most of the proof of the following theorem can be found there. The same result is also stated in [**Ro**] in Théorème 4.6 but without the characterization of the cofibrations. For the case in which A is the ground ring R, the theorem appears in [**Ho**] where it is noted that the cofibrations can be described as below. Since [**F1**] is not published and hard to get we give below a sketch of some of Félix's arguments.

Theorem 2.1. *There is a closed model category on A-*mod *with* we $=$ *quisms, i.e., maps inducing isomorphisms in cohomology,* fib $=$ *surjective morphisms, and the class* cof *is defined via the left lifting property with respect to the class* we $\cap$ fib*.*

The class cof has an internal description as the closure under retracts of semi-free extensions:

Definition 2.2. An extension P of an A-module M which is isomorphic to an extension of the form $M \oplus A \otimes V$ with

$$V = \bigoplus_{k\geqslant 0} V_k$$

R-free and

$$d(V_n) \subseteq M \oplus A \otimes \Big(\bigoplus_{k\leqslant n-1} V_k\Big),$$

is called a semi-free extension of M.

Proof of 2.1. It has been shown in [**F1**] that A-mod with we, fib as above and cof $=$ semi-free extensions constitutes a model category (not closed in general). The theorem follows since we and fib are obviously closed under retracts. This suffices by [**Q1**, Proposition 5.2]. We sketch now the construction of the two factorizations. Let

$$f\colon M \to N$$

be any morphism in A-mod and let U be the free R-module on N and $k\colon U \to N$ the obvious map. Consider the semi-free extension and the extension p of f

$$M \xrightarrow{i} M \oplus A \otimes (U \oplus sU) \xrightarrow{p} N$$

with differential D defined as $D(u) = su$ and p is given as $p(u) = k(u), p(su) = dk(u)$. Since the A-module $A \otimes (U \oplus sU)$ is contractible by an A-linear homotopy and p is clearly surjective, the construction gives the factorization into a trivial cofibration and a fibration.

Next we construct the factorization $f = q \circ j$ with cofibration j and trivial fibration q. For this we use the factorization $f = p \circ i$ from above as a first step. Let V_0 be the free R-module on a set of representatives of the cokernel of $f_*\colon H_*(M) \to H_*(N)$ and $k_0\colon V_0 \to N$ the obvious map. Now construct a semi-free extension and an extension q_0 of p

$$M \xrightarrow{j_0} M \oplus A \otimes (U \oplus sU \oplus V_0) \xrightarrow{q_0} N$$

with $D(v) = 0$ and $q_0(v) = k_0(v)$ for $v \in V_0$. Note that q_0 induces an epimorphism in homology. In the next step, let sV_1 be the free R-module on a set of representatives of the kernel of the map induced by q_0 on homology and

$$k_1\colon sV_1 \to M \oplus A \otimes (U \oplus sU \oplus V_0)$$

the obvious map. Consider

$$M \xrightarrow{j_1} M \oplus A \otimes (U \oplus sU \oplus V_0 \oplus V_1) \xrightarrow{q_1} N$$

with differential acting as $D(v) = k_1(sv)$ and $q_1(v) = w$ with a w such that $d_M(w) = q_0k_1(sv)$. Now proceed by induction to kill the kernel of q_{n*} and take the colimit. Using the fact that homology commutes with direct limits, one sees that $q = q_\infty$ is a weak equivalence. □

In case the ground ring R is a principal ideal domain and $A = R$, the cofibrations simplify (see the theorem below). Recall that an R-module M is called flat if the tensor product with M is an exact functor. Write $\mathrm{Tor}(M)$ for the torsion submodule of M and fl-Coch(R) for the category of flat cochain complexes.

Proposition 2.3. *Suppose that R is a principal ideal domain. Then a closed model category is defined on* Coch(R) *and* fl-Coch(R) *by* we = *quisms,* fib = *surjective morphisms,* cof = *injective morphisms with free cokernel.*

Sketch of proof. First note that cof is closed under retracts since R is hereditary. Next we claim that every $f \in$ cof is a semi-free extension of $\operatorname{im} f$. To see this, decompose $V = \operatorname{cokernel} f$ as follows: $V_0 = \{v \in V \mid dv = 0\}$; then V_0 is a direct summand in V, hence free since R is hereditary. The complement $V_1 = V_0^\perp$ maps to V_0 under d and hence V_0, V_1, define a semi-free extension of $\operatorname{im} f$. The theorem for Coch(R) follows now from 2.1. For fl-Coch(R) note that a flat module over a p.i.d. is the same thing as a torsion free module. The functor Q from $\mathrm{Coch}(R)$ to fl-$\mathrm{Coch}(R)$ which sends C_n to $C_n/\operatorname{Tor}(C_n)$ is adjoint to the inclusion functor. So colimits in fl-$\mathrm{Coch}(R)$ can be defined by taking colimits in $\mathrm{Coch}(R)$ and then applying Q. Note that a cokernel of a cofibration agrees with the usual cokernel in $\mathrm{Coch}(R)$. Finally, the construction of the two factorizations given above can be done in fl-$\mathrm{Coch}(R)$ and the verification of the other axioms requires no new arguments. □

For the applications which we have in mind, it is often sufficient to approximate a given object only up to a certain degree. The class of n-equivalences will induce a theory of n-types on fl-$\mathrm{Coch}(R)$. For the rest of this section R will be a p.i.d.

Definition 2.4. A map in fl-$\mathrm{Coch}(R)$ $f\colon L \to K$ is called a weak n-equivalence, if $H^*(f;\pi)$ is an isomorphism for $* \leqslant n$ and all R-module coefficients π. If, in addition, $H^{n+1}(f;\pi)$ is a monomorphism for all π, then we call f an n-equivalence. We call K n-equivalent to zero if $H^{\leqslant n}(K;\pi) = 0$ for all π. Let $f\colon L \to K$ be a morphism in fl-$\mathrm{Coch}(R)$. An n-model of f consists of a free cochain complex V, concentrated in degrees less than $n+2$ and such that $H^{n+2}(V)$ is torsion, together with a map $t\colon sV \to L$, and an n-equivalence $g\colon C_t \to K$ with $g_{|L} = f$, where C_t is the cone on t. An n-model of $0 \to K$ is an n-model of K.

Observe that a cochain map f is an n-equivalence if and only if the cone on f is n-equivalent to zero. The use of variable coefficients may seem strange at first sight. But note that a chain map between free chain complexes which have homology groups of finite type is n-connected, i.e., it induces an isomorphism in homology up to degree n and an epimorphism in degree $n+1$, if and only if the map induced on the R-linear dual is an n-equivalence. This can be seen by a diagram chase using the universal coefficient theorem.

Theorem 2.5. *For each $n \in \mathbb{Z}$, the category* fl-$\mathrm{Coch}(R)$ *with* $\mathrm{we}_n =$ *weak n-equivalences,* $\mathrm{fib}_n =$ *surjective cochain maps, and* cof_n *defined by LLP with respect to* $\mathrm{fib}_n \cap \mathrm{we}_n$*, defines a closed model category on* fl-$\mathrm{Coch}(R)$*.*

It is possible to derive Theorem 2.5 from general facts on localization in model categories to be found in [**Hi**], but we prefer to give an elementary homemade treatment. For the proof we need some preparations. We start with:

Lemma 2.6.

α) A morphism $f\colon V \to K$ in fl-$\mathrm{Coch}(R)$ *is a weak n-equivalence if and only if $f^*\colon H^{\leqslant n}(V) \to H^{\leqslant n}(K)$ and $f^*\colon \operatorname{Tor} H^{n+1}(V) \to \operatorname{Tor} H^{n+1}(K)$ are isomorphisms.*

β) Suppose that $H^{n+1}(V)$ is torsion and $H^{\geqslant n+2}(V)$ is torsion free. Then

$$f\colon V \to K \quad in \quad \text{fl-}\mathrm{Coch}(R)$$

is an n-equivalence if and only if it is a weak n-equivalence.

Proof of α). The commuting diagram

$$\begin{tikzcd} H^i(V)\otimes\pi \arrow[r, tail] \arrow[d, "f^*\otimes\pi"] & H^i(V;\pi) \arrow[r, two heads] \arrow[d, "f^*"] & H^{i+1}(V)*\pi \arrow[d, "f^**\pi"] \\ H^i(K)\otimes\pi \arrow[r, tail] & H^i(K;\pi) \arrow[r, two heads] & H^{i+1}(K)*\pi \end{tikzcd}$$

shows $(\leftarrow)$ by $H*\pi = \operatorname{Tor} H*\pi$, and $(\rightarrow)$ by $H*R/\lambda R = \{x\in H \mid \lambda\cdot x = 0\}$, if $\lambda\in R/0$ (for each module H).

Proof of β). Put $i=n+1$ in the diagram above. We get zero on the top right, thus only injectivity of $f^*\otimes\pi$ is to be shown, by the 5-lemma. The last property of f^* in α) and the first of V imply that $f^*\colon H^{n+1}(V)\to H^{n+1}(K)$ is injective and has torsion free cokernel, so that $\operatorname{Ker}(f^*\otimes\pi) = \operatorname{Coker}(f^**\pi) = 0$. □

Proposition 2.7.

a) Every $K\in$ fl-Coch(R) possesses an n-model, for each n.

b) Suppose that K is m-equivalent to zero with $m\leqslant n$, and that $H^{\leqslant n}(K)$ and $\operatorname{Tor} H^{n+1}(K)$ are finitely generated. Then there is an n-model $g\colon V\to K$ such that V is finitely generated and concentrated in dimensions between m and $n+1$.

Proof of a). Choose free resolutions $G^{i-1}\rightarrowtail F^i \twoheadrightarrow H^i(K)$, $i\leqslant n$, and $G^n\rightarrowtail F^{n+1}\twoheadrightarrow \operatorname{Tor} H^{n+1}(K)$. Define the short complex $V_i = (G^{i-1}\rightarrowtail F^i)$, $i\leqslant n+1$, and let $V=\oplus V_i$. There is an obvious cochain map $g\colon V\to K$ which induces isomorphisms in $H^{\leqslant n}$ and maps $H^{n+1}(V)$ isomorphic onto $\operatorname{Tor} H^{n+1}(K)$. Then by 2.6.$\alpha$), g is a weak n-equivalence, and hence by 2.6.β) an n-equivalence.

Proof of b). By assumption, we can choose $F^i=0$ for $i<m$ and finitely generated for $i\leqslant n+1$. Since $H^m(K)$ is torsion free by 2.6.α), applied in the case $V=0$, and finitely generated it is free. It follows that $G^{m-1}=0$. □

Recall that the cone on a cochain map $f\colon L\to K$ is the complex $C_f = K\oplus s^{-1}L$ with differential acting as

$$\partial(k, s^{-1}l) = (\partial_K(k) + f(l), -s^{-1}\partial_L(l)).$$

The following elementary lemma enables us to prove a relative version of 2.7.

Lemma 2.8. *Let $f\colon L\to K$ and $\eta\colon V\to C_f$ be cochain maps.*
Consider

$$\begin{tikzcd} C_t \arrow[dr, "g"] & & V \arrow[d, "\eta"] & \\ L \arrow[u] \arrow[r, "f"] & K \arrow[r, "k"] \arrow[dr, "k'k"'] & C_f \arrow[r, "q"] \arrow[d, "k'"] & s^{-1}L \\ sV \arrow[u, "t"] & & C_\eta & \end{tikzcd}$$

with k, k', q parts of the cofiber sequences of f respectively η. Then there is t such that f extends to a map g from C_t to K with $C_g\cong C_\eta$.

Proof. Write $\eta = (\eta_1, \eta_2)\colon V \to K \oplus s^{-1}L = C_f$. Let $t := s\eta_2 s^{-1}$ and $g := (f, -\eta_1)$. It is easy to check that this defines cochain maps and that indeed $C_g \cong C_\eta$. □

Proposition 2.9.

a) Every $f\colon L \to K$ in fl-Coch(R) *has an n-model, for each n.*

b) Suppose that f is an $(m-1)$-equivalence $m \leqslant n$ and that

$$H^{\leqslant n}(C_f) \quad \text{and} \quad \operatorname{Tor} H^{n+1}(C_f)$$

are finitely generated. Then there is an n-model of f, $t\colon sV \to L$ such that V is finitely generated and concentrated in dimensions between m and $n+1$.

Proof of a). Use 2.7.a) to find an n-model $\eta\colon V \to C_f$. Then C_η is n-equivalent to zero and by 2.8 there are t and g with $C_g \cong C_\eta$. Thus g is an n-equivalence.

Proof of b). The proof is now clear. □

The next lemma gives a sufficient condition, for the assumption in b) above to hold, depending only on L and K.

Lemma 2.10. *Let $f\colon L \to K$ be a cochain map in* fl-Coch(R). *If*

$$H^{\leqslant n+1}(L), \quad \operatorname{Tor} H^{n+2}(L) \quad \text{and} \quad H^{\leqslant n+1}(K)$$

are all finitely generated, then $H^{\leqslant n}(C_f)$ and $\operatorname{Tor} H^{n+1}(C_f)$ *are finitely generated as well.*

Proof. This follows directly from the long exact sequence in cohomology, together with the fact that an exact sequence $0 \to A \to B \to C$ induces an exact sequence $0 \to \operatorname{Tor} A \to \operatorname{Tor} B \to \operatorname{Tor} C$. □

Definition 2.11. A complex V is called n-cofibrant if V is free, $H^{n+1}(V)$ is torsion, and $H^{\geqslant n+2}(V) = 0$. A map i is an n-cofibration if it is injective and $V = \operatorname{Coker} i$ is n-cofibrant.

Proposition 2.12. *An n-cofibration has the left lifting property with respect to any $p \in \mathrm{fib}_n \cap n$-equivalences.*

Proof. The obstruction for a lifting is a homotopy class of a map from V to C_p. So it is enough to prove that all such classes are trivial. Since V is free it is a direct sum of short complexes V_i concentrated in dimensions i, $i+1$. Thus we have to show $[V_i, C_p] = 0$. For $i \geqslant n+2$ we have $H^i(V_i) = H^i(V) = 0$, so that $V_i \simeq 0$. For $i \leqslant n+1$, each $\alpha\colon V_i \to C_p$ is trivial in cohomology since C_p is n-equivalent to zero. Thus $\alpha \simeq 0$ by the following elementary fact: Let V_{i-1} be free and short of degree $i-1$, i and $\alpha\colon V_{i-1} \to Y$ with $H^i(\alpha) = 0$ and $H^{i-1}(Y) = 0$. Then $\alpha \simeq 0$. □

Proposition 2.13. *The class* cof_n *consists of the n-cofibrations $i\colon K \to C_t$ such that $t\colon sV \to K$ lifts to an n-model $U \overset{\psi}{\twoheadrightarrow} K$ with $\psi \in \mathrm{fib}_n \cap n$-equivalences.*

Proof. Suppose that $i \in \mathrm{cof}_n$. Let $U \overset{\psi}{\twoheadrightarrow} K$ be an n-model with $\psi \in \mathrm{fib}_n \cap n$-equivalences. Then the map $C_\psi \to 0$ is a weak n-equivalence.

Consider

$$\begin{tikzcd} U \arrow[r, two heads, "\psi"] & K \arrow[r, tail, "j"] \arrow[d, tail, "i"] & C_\psi \arrow[d] \\ & C_t \arrow[r] \arrow[ur, "g"] & 0. \end{tikzcd}$$

Since $i \in \mathrm{cof}_n$ a lift g exists. Write $g_{|V} \colon V \to K \oplus s^{-1}U$ as (g_1, g_2).

It is now easy to check that g_2 defines a cochain map $\overline{t}$ with $\psi \circ \overline{t} \simeq t$, where the homotopy is given by g_1. Now lift this homotopy to obtain a lift of t.

For the other direction, suppose that $p \in \mathrm{fib}_n \cap \mathrm{we}_n$ and consider:

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, tail, "i"] & N \arrow[d, two heads, "p"] \\ C_t \arrow[r, "f"] \arrow[ur, "\sigma"] & 0. \end{tikzcd}$$

Choose n-models $U \overset{\psi}{\twoheadrightarrow} K$, $W \overset{\varphi}{\twoheadrightarrow} N$ with $\psi, \varphi \in \mathrm{fib}_n \cap n$-equivalences, and lift g to $\overline{g} \colon U \to W$ such that $g \circ \psi = \varphi \circ \overline{g}$. Let $\overline{t} \colon sV \to U$ lift t. Then ψ extends to a cochain map $\overline{\psi} \colon C_{\overline{t}} \to C_t$. The composition $p \circ \varphi$ is an n-equivalence. By 2.12 there is a lift $\overline{\sigma} \colon C_{\overline{t}} \to W$ of $f \circ \overline{\psi}$ to $\overline{g}$. It is then easy to check that

$$\sigma = (\varphi \circ \overline{\sigma}|V, g) \colon C_t = V \oplus K \to N$$

defines a lift of f. □

Proof of 2.5. We verify the axioms in [**Q1**]. The axioms $\mathbf{M_0}$ and $\mathbf{M_5}$ are clear.

$\mathbf{M_3}$: The only point which needs proof is that cof_n is stable under cobase change. To see this we use 2.13. Let $i \colon A \rightarrowtail C_t \in \mathrm{cof}_n$ and $f \colon A \to B$ induce $j \colon A \to C_{f \circ t}$. Choose n-models $U \overset{\psi}{\twoheadrightarrow} A$, $W \overset{\varphi}{\twoheadrightarrow} B$ with $\psi, \varphi \in \mathrm{fib}_n \cap n$-equivalences. Use 2.12 to lift f to $\overline{f} \colon U \to W$ and let $\overline{t}$ lift t.Then $\overline{f} \circ \overline{t}$ lifts $f \circ t$. Hence j is in cof_n by 2.13.

$\mathbf{M_2}$i): Given $f : L \to K$, let CsK be the acyclic complex with differential induced by $\mathrm{id}_{\mathrm{sk}}$, and $CsK \overset{P}{\twoheadrightarrow} K$ the obvious map. Then $L \oplus CsK \overset{(f,P)}{\twoheadrightarrow} K$ is in fib_n. The inclusion $L \rightarrowtail L \oplus CsK$ is in cof_n by 2.13, for example, and clearly in we_n as well.

$\mathbf{M_2}$ii): Let $f \colon L \to K$ be given, and choose an n-model $U \overset{\psi}{\twoheadrightarrow} L$, $\psi \in \mathrm{fib}_n \cap n$-equivalences. Apply 2.9 to $f \circ \psi$ to find $\overline{t} \colon sV \to U$, and a factorization of $f \circ \psi$ into $U \overset{j}{\hookrightarrow} C_{\overline{t}} \overset{q}{\to} K$ with $j \in \mathrm{cof}_n$ by 2.13 and $q \in n$-equivalences. This induces a factorization of f into $L \overset{i}{\hookrightarrow} C_t \overset{p}{\to} K$, where $i \in \mathrm{cof}_n$ by 2.13 and $p \in n$-equivalences by the 5-lemma. At last add the cone on sK to C_t to produce a fibration.

$\mathbf{M_1}$: One part follows by definition of cof_n. For the other, consider

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, tail, "i"] & M \arrow[d, two heads, "p"] \\ K \oplus V \arrow[r, "f"] & N \end{tikzcd}$$

with $i \in \mathrm{cof}_n \cap \mathrm{we}_n$ and $p \in \mathrm{fib}_n$. Factorize i as in M_2i) into a trivial cofibration j and a fibration q. Then q is in we_n and so $\mathrm{id}_{K\oplus V}$ can be lifted; i.e., q has a section. So it is sufficient to show that a lift exists for every free extension with an acyclic complex. But this is obvious.

$\mathbf{M_4}$: Note first that each $i \in \mathrm{cof}_n \cap \mathrm{we}_n$ is an n-equivalence. This can be seen as follows: Consider $i\colon A \rightarrowtail C_t$ and an n-model U, ψ of A as above. Let $\bar{t}$ lift t and write j for the inclusion of U in $C_{\bar{t}}$. Moreover, let $\overline{\psi}$ be the induced map from $C_{\bar{t}}$ to C_t. Since ψ is an n-equivalence, it follows from the 5-lemma that $\overline{\psi} \in \mathrm{we}_n$. So $\overline{\psi} \circ j = i \circ \psi$ is in we_n by M_5. But then j is in fact an n-equivalence by 2.6. It follows that the common cofiber of i and j is n-equivalent to zero. Hence i is an n-equivalence. The second part of M_4 is now clear. For the first, let $p \in \mathrm{fib}_n \cap \mathrm{we}_n$ and q a base change along $f\colon A \to B$. Moreover, we write K for the kernel of p. By consideration of the long exact sequence induced of p, we see that $H^{\leqslant n}(K;\pi) = 0$ and that the boundary operator is trivial in dimension n. Hence the same is true for q, by a diagram chase. So M_4 is now proved.

$\mathbf{M_6}$: It is immediate that $\mathrm{fib}_n, \mathrm{we}_n$ and cof_n are closed under retracts. We now apply [**Q1**, Proposition 5.2]. □

We turn to minimal models for maps in fl-Coch(R) with R local.

Definition 2.14. Suppose R is a local p.i.d. with maximal ideal $\mathcal{M}$ and residue field $\mathbb{K}$, and let $f\colon L \to K$ be a map in fl-Coch(R).

A factorization of f into

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "i"'] & & K \arrow[dl, "q"] \\ & C_t = L \oplus V & \end{tikzcd}$$

such that i is an n-cofibration, q is an n-equivalence, and V is finitely generated is called a minimal n-model of f, if the differential induced on $\mathbb{K} \otimes V$ is trivial.

Proposition 2.15. *Let R be a local p.i.d. and $f\colon L \to K$ a $(m-1)$-equivalence $(m < n)$ such that $H^{\leqslant n}(C_f)$ and $\mathrm{Tor}\, H^{n+1}(C_f)$ are finitely generated. Then f has an n-minimal model which is unique up to isomorphism.*

Proof. The existence part follows from 2.7 and 2.9. Note that by the structure theorem for finitely generated modules over a p.i.d., every free resolution in 2.7.b) can be chosen to be a direct sum of the following two types: i) R, and ii) $R \overset{\xi^j}{\rightarrowtail} R$ with ξ a generator of the maximal ideal $\mathcal{M}$. The resulting factorization is clearly minimal.

Suppose that we are given two n-minimal models of f. Consider

$$\begin{tikzcd} L \arrow[r, tail, "i_2"] \arrow[d, tail, "i_1"] & L \oplus V' \arrow[r] \arrow[d, "q_2"] & L \oplus V' \oplus CsK \arrow[d, two heads, "q_3"] \\ L \oplus V \arrow[r, "q_1"] & K \arrow[r, equal] & K, \end{tikzcd}$$

where the split extension with the acyclic complex CsK makes q_3 into a fibration. By 2.12 there is a lift of q_1 to $L \oplus V' \oplus CsK$. Projection to $L \oplus V'$ defines a weak n-equivalence $L \oplus V \xrightarrow{\eta_1} L \oplus V'$. By symmetry there is also a weak n-equivalence $L \oplus V' \xrightarrow{\eta_2} L \oplus V$. It follows from 2.6 that V and V' with the induced differentials are n-equivalent via the induced maps $\overline{\eta}_1, \overline{\eta}_2$. Take coefficients in $\mathbb{K}$ and note, since the differential is trivial mod $\mathcal{M}$, that

$$V \otimes \mathbb{K} \xrightarrow{\overline{\eta}_1 \otimes \mathbb{K}} V' \otimes \mathbb{K}$$

is an isomorphism up to degree n and a monomorphism in degree $n+1$. But the same is true for $\overline{\eta}_2 \otimes \mathbb{K}$ and since $V \otimes \mathbb{K}$ is of finite dimension in each degree, $V \otimes \mathbb{K}$ is found to be isomorphic to $V' \otimes \mathbb{K}$. To lift this isomorphism to V just note that $\overline{\eta}_1$ is in each degree a map between finite equidimensional free R-modules whose determinant is not divisible by ξ. $\square$

Let A be an augmented dg algebra over the local p.i.d. R.

Definition 2.16. A minimal n-model over A for a map of A-modules $f\colon M \to N$ is given by a factorization of f into

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "i"'] & & N \\ & P \arrow[ur, "\varphi"'] & \end{tikzcd}$$

such that $M \overset{i}{\hookrightarrow} P$ is a semi-free extension with trivial differential on

$$\mathbb{K} \otimes_A P/M \simeq \mathbb{K} \otimes V,$$

and φ is an n-equivalence.

Assume that A is R-free with augmentation ideal I concentrated in dimensions $\geqslant 2$. An n-minimal model with $\mathbb{K} \otimes V$ bounded below and of finite type has a filtration which is quite useful and will be described now. It gives a kind of Postnikov decomposition for f. Since the module P of a given model is semi-free it is of the form $M \oplus A \otimes V$. Suppose that the finite type complex V with the induced differential is concentrated in dimensions $\geqslant s$. Let V_s denote the saturation of the complex V in dimension s, i.e., the smallest pure subcomplex which contains all elements in dimensions $\leqslant s$ (here S is a pure submodule of T means that $aS = aT \cap S$ for all $a \in R$). Note that V_s is concentrated in degrees s and $s+1$ and consists in dimension $s+1$ of all elements which are in the image of the boundary operator up to a power of ξ. The module V_s is a direct factor in V, because V/V_s is flat by purity and of finite type by assumption hence free. Define the filtration of index s by $M \oplus A \otimes V_s$.

It is easy to check that this submodule is closed under the differential. Suppose that $M \oplus A \otimes V_{<k}$ is defined. Let V_k stand for the saturation in dimension k of $V/V_{<k}$ and put

$$M \oplus A \otimes V_{\leqslant k} := M \oplus A \otimes V_{<k} \oplus A \otimes V_k.$$

This defines a differential filtration of P.

Theorem 2.17. *Suppose that A is R-free of cohomological finite type and that the augmentation ideal I of A is concentrated in dimensions > 1. Let $f\colon M \to N$ be a $(m-1)$-equivalence in* A-mod *with M, N R-free of finite type. Then f has an n-minimal model over A which is unique up to isomorphism for each n.*

Proof. By 2.15 there is an m-minimal model of cochain complexes

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "i"'] & & N \\ & M \underset{t}{\oplus} V_m = C_t \arrow[ur, "\varphi"'] & \end{tikzcd}$$

with V_m concentrated in dimensions $m, m+1$.

The map t induces a map of A modules $\widetilde{t}\colon A \otimes V_m \to M$ and a semi-free extension

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N. \\ & C_{\widetilde{t}} = M \oplus A \otimes V_m \arrow[ur, "\varphi"'] & \end{tikzcd}$$

The cochain map ψ extends to an A-module map φ. Because $A \otimes V_m = V_m$ in dimensions $\leqslant m+1$, the map φ is still an m-equivalence. The minimality of the extension is then clear. Now proceed inductively.

For the uniqueness statement suppose that $M \overset{i'}{\rightarrowtail} M \oplus A \otimes V' \overset{\varphi'}{\to} N$ is another minimal n-model. We construct inductively an isomorphism from the n-minimal model above. We may assume that V_m is nontrivial. Let k be the smallest integer such that V'_k is not trivial. We claim that $m = k$ and that $M \oplus A \otimes V_m$ is isomorphic to $M \oplus A \otimes V'_m$. This will start the induction. Suppose that $m \leqslant k$. There is a cochain map g_m which lifts φ_m up to cochain homotopy in:

$$\begin{tikzcd} M \arrow[r, "i'"] \arrow[d, "i"'] & M \oplus A \otimes V \arrow[d, "\varphi'"] \\ M \oplus V_m \arrow[ur, "g_m"] \arrow[r, "\varphi_m"] & N. \end{tikzcd}$$

To see this, factor φ' into a cochain equivalence and a fibration and apply 2.12. Since φ_m, φ' are both m-equivalences the same is true for g_m. Because I is concentrated in dimensions $\geqslant 2$, g_m maps in fact to $M \oplus V'_{\leqslant m}$. Denote the map induced from V_m to V' by $\overline{g}_m$ it is an m-equivalence by the 5-lemma. Reducing mod $\mathcal{M}$ we find that $V \otimes \mathbb{K}$ is isomorphic to $V' \otimes \mathbb{K}$ in dimension m by minimality. But then the same is true over R using the determinant as above. By 2.6 the torsion subgroups of $H^{m+1}(V_m)$ and $H^{m+1}(V'_m)$ are isomorphic via $\overline{g}_m$. At last extend g_m by A-linearity to all of $M \oplus A \otimes V_m$. For the inductive step assume that an isomorphism is found in filtration $< s$ and repeat the argument above to find one in filtration s. $\square$

3. (Co)algebras and Lie algebras in tame homotopy theory

Let $r \geqslant 2$ and let $\overline{\mathrm{Alg}}_r(R)$ and $\overline{\mathrm{Coalg}}_r(R)$ denote the category of r-reduced (co)commutative and (co)associative differential graded (co)algebras over R which are free as R-modules. Let $\mathrm{Lie}_{r-1}(R)$ denote the category of $(r-1)$-reduced $r \geqslant 2$ differential graded Lie algebras over R, and $\overline{\mathrm{Lie}}_{r-1}(R)$ the subcategory of objects which are free as R-modules. Note that the differential in $\mathrm{Lie}_{r-1}(R)$ and $\overline{\mathrm{Coalg}}_r(R)$ has degree -1, but it has degree 1 in $\overline{\mathrm{Alg}}_r(R)$. The subcategories which have objects of finite type will be denoted by a prefix f. We will suppose, if nothing else is mentioned, that $R \subseteq \mathbb{Q}$ and $\frac{1}{2}, \frac{1}{3} \in R$ so that there is no torsion in free Lie algebras.

There is (see [**ST3**]) a pair of adjoint functors

$$\mathcal{L}\colon \overline{\mathrm{Coalg}}_r(R) \rightleftarrows \overline{\mathrm{Lie}}_{r-1}(R) :\mathcal{C}$$

defined as follows:

Definition 3.1. For $L \in \overline{\mathrm{Lie}}_{r-1}(R)$ let $(\mathcal{C}(L), d) := (S(sL), d)$, where S is the symmetric graded coalgebra functor. The differential d is the sum of two coderivations d_i, d_e characterized by:

$$d_i(sx) = -sd_L x$$

$$d_e(sx \otimes sy + (-1)^{|sx|\,|sy|} sy \otimes sx) = (-1)^{|sx|} 2s[x, y].$$

Definition 3.2. For $C \in \overline{\mathrm{Coalg}}_r(R)$ let $(\mathcal{L}(C), d) := (\mathbb{L}(s^{-1}\bar{C}), d)$, where $\bar{C}$ is the submodule of elements of positive degree, $\mathbb{L}$ is the free graded Lie algebra functor and d is the sum of two derivations d_i, d_e characterized by:

$$\begin{aligned} d_i(s^{-1}x) &= -s^{-1} d_C(x), \\ d_e(s^{-1}x) &= \sum_i (-1)^{|x'_i|} [s^{-1}x'_i, s^{-1}x''_i], \end{aligned}$$

where $\overline{\triangle}(x) = \sum_i x'_i \otimes x''_i$ is the reduced diagonal of x.

The authors in [**ST3**] studied only the case $R = \mathbb{Z}$, but there is no problem to have general $R \subseteq \mathbb{Q}$. Taking the R-linear dual in each degree on $\mathcal{C}(L)$ defines a functor to a category of flat algebras, which sends f-$\overline{\mathrm{Lie}}_{r-1}(R)$ to f-$\overline{\mathrm{Alg}}_r(R)$. We will denote this cochain algebra by $\mathcal{C}^*(L)$. It is a graded version of the Chevalley-Eilenberg complex of cochains on a Lie algebra.

It was shown in [**D, ST3**] that on $\mathrm{Lie}_{r-1}(R)$, $\overline{\mathrm{Coalg}}_r(R)$ there are closed model category structures respectively cofibration category structures, which we will explain now. Moreover, it was proved that the adjoint pair $(\mathcal{C}, \mathcal{L})$ induces an equivalence on the homotopy categories.

Let $R_0 = R \subseteq R_1 \subseteq R_2 \cdots$ be a system of subrings of $\mathbb{Q}$. We say that R_* is mild with respect to $r \geqslant 2$, if each prime p, which satisfies $(p-1)(r-1) \leqslant k$ is invertible in R_k. The ring system is called tame for $r \geqslant 3$, if all primes p with $2p-3 \leqslant k$ are invertible in R_k and tame for $r=2$, if the ring system is mild with respect to $r=2$.

A large part of the literature in tame homotopy theory considers only the case $r \geqslant 3$. We will make use in the following of some results which are only published for tame ring systems with $r \geqslant 3$, if the same proof works for $r=2$. The closed model category in $\mathrm{Lie}_{r-1}(R)$, which is given by the mild ring system R_*, is defined as follows:

$$\begin{aligned}
\text{we } &= f \text{ s.t. } H_{r-1+k}(f;R_k) \text{ is an isomorphism for all } k \geqslant 0,\\
\text{fib } &= f \text{ s.t. } f \text{ is surjective in dimensions } > r-1, H_{r-1+k}(\ker f) \text{ is an}\\
&\qquad R_k\text{-module, and Coker } H_{r+k}(f) \text{ is without } p \text{ torsion for } \frac{1}{p} \in R_k,\\
\text{cof } &= f \text{ which have the LLP with respect to fib} \cap \text{we}.
\end{aligned}$$

See [**D**] for a proof.

On $\overline{\mathrm{Coalg}}_r(R)$ the mild ring system R_* induces the structure of a cofibration category (see [**ST3**] for a proof and [**B**] for information on (co)fibration categories) as follows:

$$\begin{aligned}
\text{we } &= f \text{ s.t. } \mathcal{L}(f) \in \text{we in } \mathrm{Lie}_{r-1}(R),\\
\text{cof } &= f \text{ s.t. } f \text{ is injective with free cokernel.}
\end{aligned}$$

The categories f-$\overline{\mathrm{Coalg}}_r(R)$ and f-$\overline{\mathrm{Alg}}_r(R)$ are equivalent by the functor, which takes an object A to the degree wise R-linear dual A^*.

We will use this equivalence to induce a homotopy structure on f-$\overline{\mathrm{Alg}}_r(R)$. Note that a morphism g in f-$\overline{\mathrm{Alg}}_r(R)$ is surjective if and only if $g^* \in \mathrm{cof}$.

In the rest of this section we will study minimal models for morphisms in f-$\overline{\mathrm{Alg}}_r(R)$. Let ΛV denote the free graded commutative algebra over R on the graded module V. We suppose that R is $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p = \mathbb{K}$ from now on.

Definition 3.3. Let $f\colon M \to N$ be a morphism in $\overline{\mathrm{Alg}}_r(R)$. A factorization of f into

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "i"'] & & N \\
& M \otimes \Lambda V \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

is called a minimal n-model for f if the following holds:

a) V is free of finite type and concentrated dimensions between r and $n+1$.
b) $H^{n+1}(V)$ is torsion where the cohomology is taken with respect to the differential $\overline{d}$ induced on the indecomposables $V = Q\Lambda V$, and
$$\overline{d} \otimes \mathbb{K}\colon V \otimes \mathbb{K} \to V \otimes \mathbb{K}$$
is trivial.
c) φ is an n-equivalence.

In cases where there is danger of confusion, we will also speak of a multiplicative minimal model. Exactly as in the case of A-modules, one shows that a minimal model has a Postnikov decomposition. We write $\Lambda V_{\leqslant k}$ for it.

Theorem 3.4. *Suppose that $f\colon M \to N$ is an $(m-1)$-equivalence in $f\text{-}\overline{\mathrm{Alg}}_r(R)$. There is then a minimal n-model for f such that V is concentrated in dimensions between m and $n+1$.*

Sketch of proof. The proof is almost the same as the one give for 2.17. One constructs inductively minimal cochain ℓ-models $m \leqslant \ell \leqslant n$; then adjointness induces a minimal extension in $f\text{-}\overline{\mathrm{Alg}}_r(R)$ and the induced map to N is still an ℓ-equivalence since everything is r-reduced. $\square$

A minimal n-model of the unit map $R \to N$ will be called minimal n-model of N. We turn now to the question of uniqueness of minimal models.

The argument in the theorem below uses the following description of homotopy groups in $\overline{\mathrm{Coalg}}_r(R)$ and $\overline{\mathrm{Lie}}_{r-1}(R)$: The homotopy groups of an object C in the cofibration category $\overline{\mathrm{Coalg}}_r(R)$ are defined as homotopy classes of maps from sphere objects $S(n)$ which have one R-free primitive generator in dimension n and are trivial otherwise to a fibrant model of C. These groups may be identified with the homology of the primitives of any fibrant model. Similarly, in the model category $\overline{\mathrm{Lie}}_{r-1}(R)$, the homotopy groups of any object are isomorphic to the homology groups of any fibrant model. Under the equivalence of homotopy categories induced by $\mathcal{L}$ and $\mathcal{C}$, these groups become identified. Note that there is a shift of dimensions involved.

Theorem 3.5. *Suppose that $(r-1)p-1 \geqslant r+k$ for some k. Then a minimal n-model of $f\colon M \to N$ in $f\text{-}\overline{\mathrm{Alg}}_r(R)$ is unique up to isomorphism if $n \leqslant r+k$.*

Proof. Let CsN denote the cone on sN in $\mathrm{Coch}(R)$, and ΛCsN the free dg algebra on it. By our assumption on $p, H^*(\Lambda CsN;\pi) = 0$ for $* \leqslant n+1$. This can be seen as follows: The dg algebra ΛCsN is isomorphic via the natural inclusion, in dimensions $\leqslant rp-1$, to the free graded algebra with divided powers on CsN. But this algebra is well known to be acyclic. Hence the cohomology of ΛCsN with arbitrary coefficients vanishes in dimensions $\leqslant rp-2$. The assertion follows since we have

$$rp-2 \geqslant (r-1)p \geqslant r+k+1 \geqslant n+1$$

by assumption. Thus we can alter a minimal n-model of f to $M \overset{i_1}{\rightarrowtail} M \otimes \Lambda Y_1 \otimes \Lambda CsN \overset{\widetilde{\varphi}_1}{\to} N$ such that $\widetilde{\varphi}_1$ is an epimorphism and an n-equivalence. So if two minimal n-models are given to us we may use 2.12 to get $\overline{g}$ in

$$\begin{tikzcd}
 & M \otimes \Lambda Y_1 \otimes \Lambda CsN \arrow[dr, "\widetilde{\varphi_1}"] & \\
M \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & N. \\
 & M \otimes \Lambda Y_2 \arrow[uu, "\overline{g}"'] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

The map $\overline{g}$ induces $g\colon \Lambda Y_1 \to \Lambda Y_2$ which is also an n-equivalence. It is enough to show that g is an isomorphism. To see this consider the R-linear dual g^* in $f\text{-}\overline{\mathrm{Coalg}}_r(R)$

with the cofibration category defined by the ring system

$$R = R_0 = R_1 \cdots R_k, R_{k+1} = \mathbb{Q} \cdots .$$

There is a Whitehead theorem in $\overline{\mathrm{Coalg}}_r(R)$ (see 3.6 below) from which it follows that g^* induces an isomorphism on homotopy groups up to dimension n. This argument uses the fact, proved in 3.8, that the dual of a minimal n-model is fibrant. The homotopy groups of $(\Lambda Y_1)^*, (\Lambda Y_2)^*$ up to dimension n are identified with the homology of the primitives $(Y_1^*), (Y_2^*)$ with the induced differential. By minimality this differential is trivial when tensored with $\mathbb{K}$. Using these facts, a diagram chase shows that g^* induces an isomorphism $(Y_2)^* \otimes \mathbb{K} \to (Y_1)^* \otimes \mathbb{K}$ in dimensions $\leqslant n$. Since everything is of finite type the same is true over $\mathbb{Z}_{(p)}$. Furthermore, g^* is a monomorphism in dimension $n+1$ by another look at the relevant diagram. But by symmetry and the finite type of Y_1 and Y_2, g^* is in fact an isomorphism. This proves the theorem. □

Theorem 3.6.

α) Let $f\colon L \to L'$ be a $(n-1)$-connected map in $\overline{\mathrm{Lie}}_{r-1}(\mathbb{Z}_{(p)})$. Suppose that $(r-1)p-1 \geqslant n$ holds. Then $\mathcal{C}(f)$ is n-connected and $\mathcal{C}^*(f)$ is an n-equivalence.

β) Let $f\colon C \to C'$ be a n-connected map in $\overline{\mathrm{Coalg}}_r(\mathbb{Z}_{(p)})$. Suppose that $(r-1)p-1 \geqslant n$ holds. Then $\mathcal{L}(f)$ is $(n-1)$-connected.

Proof. We will prove α and leave β as an exercise. Consider the convergent spectral sequences which are defined by the primitive filtration of $\mathcal{C}(L)$ and $\mathcal{C}(L')$. The E_1 terms are given as the homology with respect to the internal differentials d_i. This is the homology of the abelian Lie algebras underlying L and L'. Thus it is enough to show that the map induced on the E_1 terms is n-connected. So we may assume that L and L' are abelian. In this case the symmetric tensors of length k $S^k(sL)$ form a subcomplex. Moreover, for $k < p$ this complex is a natural retract of the complex of all tensors of length k $T^k(sL)$. That $T^k(sf)$ is n-connected follows from the Künneth theorem. Hence $S^k(sf)$ is also n-connected for $k < p$. Note that the module $S^{\geqslant p}(V)$ is zero in dimensions $\leqslant rp-1$ if V is r-reduced. It follows that $S(sf)$ is n-connected if $n+1 \leqslant rp-2$. But $rp-3 \geqslant (r-1)p-1$ and so the claim on $\mathcal{C}(f)$ is proved. Using the universal coefficient theorem, we see that $\mathcal{C}(f;W)$ induces on cohomology an isomorphism for $* \leqslant n$ and a monomorphism for $* = n+1$ for all modules W. Finally, observe that to detect an n-equivalence between flat $\mathbb{Z}_{(p)}$-complexes it is sufficient to look at finitely generated coefficients or, even more special, at cyclic ones. This follows from the commutation of homology with direct limits and the structure theorem for finitely generated modules over a p.i.d. For W cyclic it is immediate that $\mathrm{Hom}(B,W)$ is isomorphic to $\mathrm{Hom}(B,\mathbb{Z}_{(p)}) \otimes W$ if B is free. This gives us the second claim and the theorem is proved. □

Let $\tau_{\leqslant n}\colon \overline{\mathrm{Lie}}_{r-1}(\mathbb{Z}_{(p)}) \to \overline{\mathrm{Lie}}_{r-1}(\mathbb{Z}_{(p)})$ the n-th truncation functor. The Lie-algebra $\tau_{\leqslant n}(L)$ equals L in degrees $\leqslant n$ and $L/\operatorname{Ker}\partial$ in degree $n+1$. In all other degrees it is defined to be zero.

Corollary 3.7. *Let $f\colon L \to L'$ be a morphism in $\overline{L_{r-1}}(\mathbb{Z}_{(p)})$ which induces an isomorphism in homology in degrees $\leqslant n-1$. Suppose the ring system $R_0 = \mathbb{Z}_{(p)} = \cdots = R_k, R_{k+1} = \mathbb{Q}$ is mild with respect to r and $r+k=n$. Then the map $\mathcal{C}^*(\tau_{\leqslant n-1}f)$ is an n-equivalence in fl-$\mathrm{Coch}(\mathbb{Z}_{(p)})$.*

Proof. The map $\tau_{\leqslant n-1}f$ is $(n-1)$-connected in $\overline{L_{r-1}}(\mathbb{Z}_{(p)})$. $\square$

Lemma 3.8. *Let ΛV be a minimal n-model, and $f\colon \Lambda V \to \mathcal{C}^*\mathcal{L}B$ a morphism of dg algebras with $B \in \overline{\mathrm{Coalg}}_r(\mathbb{Z}_{(p)})$. Then $\operatorname{im} f \subseteq \operatorname{im} \mathcal{C}^*(q_{n-1})$ with q_{n-1} the canonical map $q_{n-1}\colon \mathcal{L}B \to \tau_{\leqslant n-1}(\mathcal{L}B)$.*

Proof. Write $V = \oplus_{i=r}^{n} V_i$ and let $\Lambda V_{<j}$ be the dg algebra generated by $V = \oplus_{i=r}^{j-1} V_i$. For $\Lambda V_{<n}$ there is nothing to prove. Set $V_n = (V_n^0 \xrightarrow{\delta} V_n^1)$. Then again it is clear that $f(V_n^0) \subseteq \operatorname{im} \mathcal{C}^* q_{n-1}$. Recall that $H^{n+1}(V_n, \delta)$ is torsion and is finitely generated. So every element $v_1 \in V_n^1$ is in $\operatorname{im}\delta$ up to a power of p. Let $v_1 \in V_n^1$ and write $p^k \cdot v_1 = \delta v_0$. Then, since $dv_0 - \delta v_0 \in \Lambda V_{<n}$, it follows that $f(p^k v_1) \in \operatorname{im} \mathcal{C}^*(q_{n-1})$. But $\mathcal{C}^*\mathcal{L}B$ is torsion free and $\operatorname{im} \mathcal{C}^*(q_{n-1})$ is a direct factor. So we must have $f(v_1) \in \operatorname{im} \mathcal{C}^*(q_{n-1})$ which proves the lemma. $\square$

Recall that an object X in a cofibration category is fibrant if any trivial cofibration with domain X has a retraction.

Theorem 3.9. *Let ΛV be a minimal n-model and $\mathbb{Z}_{(p)} = R_0 = \cdots = R_k, R_{k+1} = \mathbb{Q}$ mild for r and $r + k = n$. Then $(\Lambda V)^* \cong SV^*$ is fibrant in the cofibration category defined on $\overline{\mathrm{Coalg}}_r(\mathbb{Z}_{(p)})$ by R_*.*

Proof. We show that η exists in

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, tail, "i"'] & SV^* \arrow[d, "\varepsilon^*"] \\ B \arrow[r, "f"'] \arrow[ur, "\eta"] & \mathbb{Z}_{(p)} \end{tikzcd}$$

for $i \in \mathrm{cof} \cap \mathrm{we}$ in $\overline{\mathrm{Coalg}}_r(\mathbb{Z}_{(p)})$ and ε the unit. Dualize the diagram and consider the diagram of dg algebras:

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r, "f^*"] \arrow[d, tail, "i"'] & B^* \arrow[d, "i^*"] \\ \Lambda V \arrow[r, "g^*"'] \arrow[ur, "\overline{\eta}"] & A^*. \end{tikzcd}$$

We will show that $\overline{\eta}$ exists. Suppose this to be true for a moment. Dualize again and consider

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, tail, "i"'] & A^{**} \arrow[r, "g^{**}"] \arrow[d, "i^{**}"] & SV^* \arrow[d, "\varepsilon^*"] \\ B \arrow[r, "j"'] & B^{**} \arrow[r, "f^{**}"'] \arrow[ur, "\overline{\eta}^*"] & \mathbb{Z}_{(p)}, \end{tikzcd}$$

where j is the inclusion in the double dual and we have identified all object of finite type with their double duals. Although A^{**}, B^{**} are not coalgebras in general, it is easy to check that $\overline{\eta}^* \circ j$ is a map of dg coalgebras which lifts f. So we see that it is enough to show that $\overline{\eta}$ exists.

Consider

$$\begin{tikzcd}
\mathbb{Z}_{(p)} \arrow[r, "f^*"] \arrow[d, "\varepsilon"] & B^* \arrow[r, "\psi_B"] \arrow[d, "i^*"] & \mathcal{C}^*\mathcal{L}B \arrow[r, "\mathcal{C}^*(q_{n-1})"] \arrow[d, "\mathcal{C}^*\mathcal{L}(i)"] & \mathcal{C}^*(\tau_{\leqslant n-1}\mathcal{L}B) \arrow[d, "\mathcal{C}^*(\tau_{\leqslant n-1}\mathcal{L}(i))"] \\
\Lambda V \arrow[r, "g^*"] & A^* & \mathcal{C}^*\mathcal{L}A \arrow[l, "\psi_A"] & \mathcal{C}^*(\tau_{\leqslant n-1}\mathcal{L}B), \arrow[l, "\mathcal{C}^*(q_{n-1})"']
\end{tikzcd}$$

where ψ is the dual of the unit of the adjunction. We construct $\overline{\eta}$ by induction on j in $\Lambda V_{<j}$. Note that ψ admits a cochain map σ with $\psi \circ \sigma = \mathrm{id}$.

Suppose $\overline{\eta}$ is found on $\Lambda V_{<j}$, $j \leqslant n$. Consider

$$\begin{tikzcd}
\Lambda V_{<j} \arrow[r, "\widehat{\eta}_{<j}"] \arrow[d, tail] & \mathcal{C}^*(\tau_{\leqslant j-1}\mathcal{L}B) \arrow[d, "\mathcal{C}^*(\tau_{\leqslant j-1}\mathcal{L}(i))"] \\
\Lambda V_{<j} \oplus V_j \arrow[r, "\widehat{g}_j"'] \arrow[ur, "\widehat{\eta}_j"] & \mathcal{C}^*(\tau_{\leqslant j-1}\mathcal{L}A),
\end{tikzcd}$$

where $\hat{g}_j$, respectively $\hat{\eta}_{<j}$, are the maps of dg algebras induced by $\sigma_A \circ g^*$, respectively $\sigma_B \circ \overline{\eta}_{<j}$. By 3.8 $\hat{g}_j$ maps to $\mathrm{im}(C^*(q_{j-1}))$. By 3.7 $\mathcal{C}^*(\tau_{\leqslant j-1}\mathcal{L}(i))$ is a j-equivalence. Since $\mathcal{C}^*(\tau_{\leqslant j-1}\mathcal{L}(i))$ is an epimorphism in degrees $\leqslant j$, the proof of 2.12 shows that the cochain map $\hat{\eta}_j$ exists. The induced map of dg algebras $\hat{\eta}_{\leqslant j}$ gives us $\overline{\eta}$ on $\Lambda V_{\leqslant j}$ by $\overline{\eta}_{\leqslant j} := \psi_B \circ \mathcal{C}^*(q_{j-1}) \circ \hat{\eta}_{\leqslant j}$. Now put $A = SV^*$ to see that SV^* is also fibrant in the sense of a cofibration category. □

We have done most of the work in proving the theorem below. A nonproper fibration category consists of a category C endowed with two distinguished classes of morphisms fib and we, called fibrations and weak equivalences, such that all of Baues axioms for a fibration category except possibly the properness axiom $F2(a)$ are satisfied (see [**B**, p. 7]).

Theorem 3.10. *The category $f\text{-}\overline{\mathrm{Alg}}_r(\mathbb{Z}_{(p)})$ with p mild with respect to $n = r + k$ is a nonproper fibration category with the following classes:* fib = *epimorphisms,* we = f *such that $\mathcal{L}(f^*)$ induces an isomorphism in $H_{\leqslant n-1}$.*

Proof. It is immediate that F_1, which is the usual two-out-of-three property for we, holds. Clearly, pullbacks exist in $f\text{-}\overline{\mathrm{Alg}}_r(\mathbb{Z}_{(p)})$ and if we pullback a fibration q along a map f, then the induced map $\overline{q}$ is also a fibration. For the rest of F_2, we must show that if $q \in \mathrm{we} \cap \mathrm{fib}$ then also $\overline{q} \in \mathrm{we} \cap \mathrm{fib}$. To see this, note that taking the $\mathbb{Z}_{(p)}$-linear dual turns pullbacks into pushouts and fibrations into cofibrations in $f\text{-}\overline{\mathrm{Coalg}}_r(\mathbb{Z}_{(p)})$. Moreover, the functor $\mathcal{L}$ preserves pushouts since it is a left adjoint. An easy 5-lemma argument shows that $H_{\leqslant n-1}(\mathcal{L}(\overline{q}^*))$ is an isomorphism if this is true for q. For F_3 we must factor a given map f into $q \circ g$ with $q \in \mathrm{fib}$ and $g \in \mathrm{we}$. It follows from 3.6.β) that an n-equivalence is a weak equivalence. A factorization of a map f into an n-equivalence i followed by a fibration can be constructed by tensoring with $C(s(\mathrm{target}(f)))$ as was shown in the proof of 3.5. For F_4 we must show the existence of a cofibrant model $M(X) \xrightarrow{q} X$ with $q \in \mathrm{we} \cap \mathrm{fib}$. The proof of 3.9 shows that a minimal n-model, for a given object X, is cofibrant. Finally, the same operation which was used in F_3 turns the model map into a fibration. □

4. Computation of cat using minimal models

Let $L_{r-1}^{m-1}(R)$ denote the subcategory of $\overline{\mathrm{Lie}}_{r-1}(R)$ whose objects are free Lie algebras and generated in dimensions $\leqslant m-1$. Let $\mathrm{CW}_r^m(R)$ stand for the category of R-local CW-complexes with cells in dimensions between r and m. The main result of Anick in [**A1**] is that the homotopy categories of $L_{r-1}^{m-1}(R)$ and $\mathrm{CW}_r^m(R)$ are equivalent, if $R \subseteq \mathbb{Q}$ and the least prime p which is not invertible in R satisfies $m \leqslant \alpha(r-1,p)$. Recall that $\alpha(n,p) = \min(n+2p-3, np-1)$. The universal enveloping algebra $U\mathbb{L}V$ of the dg Lie algebra, which corresponds to a given CW-complex X, is a valid Adams-Hilton model over R of X. But more is true. It was shown by Anick that the diagonal map on $U\mathbb{L}V$ for which $\mathbb{L}V$ are the primitives is homotopic to the geometric diagonal. The homotopy theory on $L_{r-1}^{m-1}(R)$, used by Anick, is given by an explicit cylinder object. In [**ST2**] the authors reproduce the main theorem of [**A1**], and show that the homotopy category of $L_{r-1}^{m-1}(R)$ injects into the tame homotopy category of $\mathrm{Lie}_{r-1}(R)$ for suitable ring systems. In [**ST1**], this is used to deduce a theorem which shows how to compute cat for $Z \in \mathrm{CW}_r^m(R)$ in $\overline{\mathrm{Coalg}}_r(R)$ and $L_{r-1}^{m-1}(R)$. Let us remark that what we said above about the case $r=2$ still holds here.

Let R_*, r be a tame ring system such that $R \supseteq R_i$ for $i \leqslant k$ and write $r+k=m$. The following theorem is implicit in [**ST1**], because it was proved in Theorem 4 that the fibration corresponding to j (see below) is an n-LS-fibration in the sense of [**ST4**]. By definition, such fibrations $\tilde{p}_n$ come with maps s,t over the base from and to the n-th Ganea fibration p_n such that $p_n \circ t \simeq \tilde{p}_n$ and $\tilde{p}_n \circ s \simeq p_n$. Thus n-LS-fibrations over Z detect $\mathrm{cat}(Z)$ in the same way as the Ganea fibrations.

Theorem 4.1. *Suppose that the least prime $p \in R \subseteq \mathbb{Q}$, which is not invertible in R satisfies $m \leqslant \alpha(r-1,p)$ for $m \in \mathbb{N}$. If $Z \in \mathrm{CW}_r^m(R)$ is represented in homotopy by $\mathbb{L}V \in L_{r-1}^{m-1}(R)$ then $\mathrm{cat}(Z) \leqslant n$ if and only if $j\colon P_n\mathcal{C}(\mathbb{L}V) \to \mathcal{C}(\mathbb{L}V)$ has a section up to homotopy in the tame homotopy category defined by R_* on $\overline{\mathrm{Coalg}}_r(R)$.*

Here P_n stands for the n-th term in the primitive filtration of a coalgebra. To make this theorem useful for us, it is convenient to relate cat and n-types. The following elementary lemma will do the job.

Lemma 4.2. *Let Z be a 1-connected CW-complex of dimension m and $E \overset{p}{\twoheadrightarrow} Z$ a fibration with fibre F. Denote the Postnikov systems of E, Z by E^m, Z^m, and let p^m be the map induced on them. Further, let p_{m-1}^m denote the $(m-1)$-th Postnikov section of the map p^m. Then p has a section if and only if p_{m-1}^m admits a section up to homotopy.*

Proof. One direction follows from naturality up to homotopy of Postnikov sections [**W**]. For the other, note that every section up to homotopy of p_{m-1}^m lifts to a homotopy section of p^m since the only obstruction sits in $H^{m+1}(Z^m;\pi_m(F))$. This group is zero because Z^m is m-equivalent to Z. Since Z^m, E^m are obtained up to homotopy by attaching cells of dimensions $\geqslant m+2$ to Z, E every homotopy section of p^m induces a section of p. □

We will apply 4.2 to the Ganea fibration or more generally to an n-LS-fibration.

Now let $\mathbb{L}V \in L^{m-1}_{r-1}(\mathbb{Z}_{(p)})$ be of finite type and represent $Z \in \mathrm{CW}^m_r(\mathbb{Z}_{(p)})$ in homotopy. We assume that $\alpha(r-1,p) \geqslant m$. Consider $P_n\mathcal{C}(\mathbb{L}V) \xrightarrow{j} \mathcal{C}(\mathbb{L}V)$ and the dual $\mathcal{C}^*(\mathbb{L}V) \overset{j^*}{\twoheadrightarrow} \mathcal{C}^*(\mathbb{L}V)/I^{n+1}$. The symbol I^{n+1} denotes the $n+1$ power of the augmentation ideal I. An m-minimal model

$$\psi\colon \Lambda X \longrightarrow \mathcal{C}^*(\mathbb{L}V)$$

induces

$$\psi_n\colon \Lambda X/I^{n+1} \longrightarrow \mathcal{C}^*(\mathbb{L}V)/I^{n+1}.$$

Proposition 4.3. *The morphism ψ_n is an m-equivalence.*

Proof. First, note that the homotopy groups, computed in the fibration category on $f\text{-}\overline{\mathrm{Alg}}_r(\mathbb{Z}_{(p)})$, or equivalently in the cofibration category $\overline{\mathrm{Coalg}}_r(\mathbb{Z}_{(p)})$, with respect to $\mathbb{Z}_{(p)} = R_0 = \cdots = R_m, R_{m+1} = \mathbb{Q}\cdots$, of $\Lambda X, \mathcal{C}^*(\mathbb{L}V)$, are isomorphic in dimensions $\leqslant m$ to $H_*(X^*)$, respectively $H_*(s\mathbb{L}V)$. To see this, observe that SX^* and $\mathcal{C}(\tau_{\leqslant m-1}\mathbb{L}V)$ are fibrant, the former by 3.8 the latter by essentially the same proof. Represent the t-sphere by the trivial coalgebra $S(t)$ with one generator in dimension t, and note that homotopy classes of maps from $S(t)$, $t \leqslant m$, to SX^*, respectively $\mathcal{C}(\mathbb{L}V)$, may be identified with homotopy classes of chain maps. From this the claim follows. By the Whitehead theorem, ψ^* induces an isomorphism on these homotopy groups. To finish the argument, look at the spectral sequences obtained by filtering $\Lambda X/I^{n+1}$ and $\mathcal{C}^*(\mathbb{L}V)/I^{n+1}$ by the images of I^j and $j \leqslant n$. The differential on E_0 of these spectral sequences is given by the tensor product of the differentials induced on X respectively $(s\mathbb{L}V)^*$. The theorems of Künneth and of universal coefficients show that $E_0(\psi_n)$ is an m-equivalence. Hence the same holds for $E_\infty(\psi_n)$ and the proposition is proved. $\square$

Let Z be an object in $\mathrm{CW}^m_r(\mathbb{Z}_{(p)})$ that is represented by $\mathbb{L}V \in L^{m-1}_{r-1}(\mathbb{Z}_{(p)})$. Moreover, let (i,φ) in

$$\begin{tikzcd}
 & \Lambda X \arrow[dl, "i"'] \arrow[dd, two heads, "p_{n+1}"] \arrow[r, "\psi"] & \mathcal{C}^*(\mathbb{L}V) \arrow[dd, "j^*"] \\
\Lambda X \otimes \Lambda Y \arrow[dr, "\varphi"'] & & \\
 & \Lambda X/I^{n+1} \arrow[r, "\psi_n"] & \mathcal{C}^*(\mathbb{L}V)/I^{n+1}
\end{tikzcd}$$

be an $(m-1)$-minimal model of the quotient map p_{n+1}.

Proposition 4.4. *Using the conventions above,* $\mathrm{cat}(Z) \leqslant n$ *if and only if* $\Lambda X \rightarrowtail \Lambda X \otimes \Lambda Y$ *admits a retraction of dg algebras.*

Proof. Both SX^* and $SX^* \otimes SY^*$ are fibrant in the cofibration category defined on $\overline{\mathrm{Coalg}}_r(\mathbb{Z}_{(p)})$ by

$$R_0 = \mathbb{Z}_{(p)} = \cdots = R_{m-r}, R_{m-r+1} = \mathbb{Q}.$$

The homotopy category of this cofibration category is isomorphic to the homotopy category of tame spaces [**ST3**]. The coalgebra SX^* represents the m-th Postnikov section of $\mathcal{C}(\mathbb{L}V)$. Moreover, the map i^* represents p^m_{m-1} for p the fibration associated with $P_n\mathcal{C}(\mathbb{L}V) \xrightarrow{j} \mathcal{C}(\mathbb{L}V)$. Now apply 4.1 and 4.2. $\square$

5. A model for $\Lambda X \to \Lambda X/I^{n+1}$

Let $(\Lambda X, d)$ be a minimal m-model of $\mathcal{C}^*(\mathbb{L}V)$ with $\mathbb{L}V \in L^{m-1}_{r-1}(\mathbb{Z}_{(p)})$ of finite type and $m \leqslant \alpha(r-1,p)$.

Furthermore, let

$$\begin{tikzcd} \Lambda X \arrow[r, "p_{n+1}"] \arrow[d, tail, "i"] & \Lambda X/I^{n+1} \\ \Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n] \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

be a minimal ΛX-module m-model for p_{n+1}, and suppose that $n \geqslant 2$.

Define a bigrading on the objects in the diagram above by setting $|\Lambda^i X| = i$ and $|Y^n| = n$. From now on we will call the usual cohomological degree the dimension and the degree defined above the degree.

Write $D_{|Y^n} = \delta + \tau$ with

$$\tau \colon Y^n \longrightarrow \Lambda^+ X \otimes [\mathbb{Z}_{(p)} \oplus Y^n]$$

and

$$\delta \colon Y^n \longrightarrow Y^n.$$

Lemma 5.1. *The m-model over ΛX can be chosen such that $\tau(Y^n) \subseteq I^{n+1} \oplus I \otimes Y^n$ and $\overline{\varphi}(Y^n) = 0$ holds.*

Proof. Recall the construction of the minimal m-model over ΛX from Section 2 and note that if the map f which is modeled is surjective one can replace (up to shift of dimension) C_f by $\operatorname{Ker} f$ in 2.8 and 2.9. Hence the maps of cochain complexes $t_{r+k} \colon sV_{r+k} \to \Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n_{<r+k}]$, which define the minimal model, can be chosen to map to $\operatorname{Ker} \overline{\varphi}_{<r+k}$. So $\overline{\varphi}(Y^n) = 0$ is now clear. The first assertion follows by induction on k, using the fact that $\operatorname{Ker} \overline{\varphi}_{<r+k}$ in dimensions $\geqslant r+k+1$ equals $I^{n+1} \oplus I \otimes Y^n_{<r+k}$ by inductive hypothesis and because $Y_{<r+k}$ is concentrated in dimensions $\leqslant r+k$. So t_{r+k} maps to $I^{n+1} \oplus I \otimes Y^n_{<r+k}$ and since $\tau_{|Y_{r+k}}$ is induced by t_{r+k} the first assertion is also clear. $\square$

We are going to construct a multiplicative minimal m-model in $f\text{-}\overline{\mathrm{Alg}}_r(\mathbb{Z}_{(p)})$ out of this module model. For Y^n, δ as above, let $\mathbb{L}(s^{-1}Y^n)^*, \delta^*$ be the free dg Lie algebra on the desuspension of the dual complex. Define a second grading on $\mathbb{L}(s^{-1}Y^n)^*$ and on $\mathcal{C}^*(\mathbb{L}(s^{-1}Y^n)^*)$ as follows: Let $\Gamma_1 \supseteq \Gamma_2 \cdots$ be the lower central series of $\mathbb{L}(s^{-1}Y^n)^*$, and $\oplus_{k\geqslant 1}\Gamma_k/\Gamma_{k+1}$ the associated graded object which is well known to be isomorphic to $\mathbb{L}(s^{-1}Y^n)^*$ as a $\mathbb{Z}_{(p)}$-module. This defines a decomposition of $(s\mathbb{L}(s^{-1}Y^n)^*)^* \cong \oplus_{k\geqslant 1} s(\Gamma_k/\Gamma_{k+1})^*$. Now put $|s(\Gamma_k/\Gamma_{k+1})^*| = n + (n-1)\,(k-1) = M_k$ and extend this grading to $\mathcal{C}^*(\mathbb{L}(s^{-1}Y^n)^*)$ by setting $|x_1 \wedge \cdots \wedge x_j| = \sum_{t=1}^{j} |x_t|$ for $x_t \in s(\mathbb{L}(s^{-1}Y^n)^*)^*$.

The linear, respectively quadratic, part of the differential $\partial = \partial_1 + \partial_2$ in $\Lambda Y := \mathcal{C}^*(\mathbb{L}(s^{-1}Y^n)^*)$ then satisfies

$$\partial_1 \colon Y^{M_k} \longrightarrow Y^{M_k},$$
$$\partial_2 \colon Y^{M_k} \longrightarrow \bigoplus_{i+j=k} Y^{M_i} \wedge Y^{M_j}.$$

Since ∂_1 is induced by δ the differential graded algebra ΛY is minimal because $\delta \otimes \mathbb{Z}/p = 0$.

Lemma 5.2. *The inclusion of cochain complexes $Y^n, \delta \to \Lambda Y, \partial$ induces an isomorphism in cohomology $H^*(Y) \xrightarrow{\cong} H^*(\Lambda Y)$.*

Proof. If the ground ring is the field $\mathbb{Z}/p$, then the differential on $\mathbb{L}(s^{-1}Y^n)^*$ is trivial and the lemma is well known. From this the case $R = \mathbb{Z}_{(p)}$ follows easily. $\square$

We will need the technical lemmas below:

Lemma 5.3. *Let ΛY be as above.*

α) If $\partial_2 y = 0 \mod p^s$ for $y \in Y$ then $y \in Y^n \mod p^s$.

β) If $\partial_2 v = 0 \mod p^s$ for $v \in \Lambda^2 Y$ then there is $y \in Y$ with $\partial_2 y = v \mod p^s$.

Proof of α). We have $H^*(\Lambda Y; \mathbb{Z}/p) \cong Y^n \otimes \mathbb{Z}/p$ by 5.2. So $y = y_1 + p\overline{x}_1 + \partial \overline{z}_1$ with $y_1 \in Y^n$ and $\overline{x}_1, \overline{z}_1 \in \Lambda Y$. Since $\partial_1 = 0 \mod p$, we find, by comparing coefficients, that in fact $y = y_1 + pz_1$ with $z_1 \in Y$. Now $\partial_2 y_1 + p\partial_2 z_1 = \partial_2 y = 0 \mod p^s$. Since $\partial_2 y_1 = 0$, we see that $\partial_2 z_1 = 0 \mod p^{s-1}$. So there is $y_2 \in Y^n$ and $z_2 \in Y$ with $z_1 = y_2 + pz_2$ and so on. We arrive at $y = \sum_{i=1}^{s} p^{i-1} y_i \mod p^s$.

Proof of β). Again using 5.2, we find $y_0 \in Y$ and $z_1 \in \Lambda^2 Y$ with $\partial_2 y + pz_1 = v$. Since $\partial_2^2 = 0$ on ΛY, we find that $\partial_2 z_1 = 0 \mod p^{s-1}$. So there are $y_1 \in Y$ and $z_2 \in \Lambda^2 Y$ with $\partial_2 y_1 + pz_2 = z_1$ and so on. This gives us $v = \partial_2(\sum_{i=0}^{s-1} p^i y_i) \mod p^s$. $\square$

Filtering $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n]$ and $\Lambda X / I^{n+1}$ by degree defines convergent spectral sequences. The part of a differential d which raises degree by $i - 1$ will be denoted by d_i.

Lemma 5.4. *The map $\overline{\varphi} \otimes \mathbb{Z}/p$ induces an isomorphism respectively monomorphism on $E_2^{p,q}$ of the spectral sequences with $\mathbb{Z}/p$ coefficients for $p + q \leqslant m$ respectively $p + q \leqslant m + 1$.*

Proof. Let $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n \oplus V^n_{>m}]$ be a ∞ minimal ΛX-model for $\Lambda X / I^{n+1}$ with $V^n_{>m}$ concentrated in dimensions $\geqslant m + 1$. It is shown in [**FHJLT**, pp. 312–315] that there is a minimal $\Lambda X \otimes \mathbb{Z}/p$-module model for $\Lambda X / I^{n+1} \otimes \mathbb{Z}/p$, such that the differential of this model is of the form $D = \partial_2 + \omega$, where ∂_2 is of bidegree $(1, 0)$ and ω raises degree by more than two. Moreover, the differential ∂_2 defines a minimal model for the module $\Lambda X / I^{n+1} \otimes \mathbb{Z}/p$ with differential $\overline{d}_2$. Here $\overline{d}_2$ denotes the reduction mod p of the quadratic part of the differential induced on $\Lambda X / I^{n+1}$. So this model induces an isomorphism on E_2 in each bidegree. By uniqueness up to isomorphism the same is true for $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n \oplus V^n_{>m}] \otimes \mathbb{Z}/p$. But since $V^n_{>m}$ is concentrated in dimensions $\geqslant m + 1$ the assertion of the lemma follows. $\square$

Lemma 5.5. *Let $\sigma \in \Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n] \otimes \mathbb{Z}/p$ be a cocycle of degree $\geqslant j$ and dimension i with $j \geqslant n+1$ and $i \leqslant m+1$. Then there is $\tau \in \Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n] \otimes \mathbb{Z}/p$ with $|\tau| =$ degree $\tau \geqslant j-1$ and $D\tau = \sigma$. If $j \geqslant n+2$ then τ can be chosen as an element of the $(m-1)$-model $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n_{\leqslant m-1}] \otimes \mathbb{Z}/p$.*

Proof. Write $\sigma = \sum_{s \geqslant j} \sigma^s$ with $|\sigma^s| = s$. So σ^j is a D_2 cocycle where D_2 stands for the quadratic part of the differential. Since $j > n$, σ^j is dead in the E_2 term. By 5.4, there is $\tau_1 \in \Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n] \otimes \mathbb{Z}/p$ with $|\tau_1| \geqslant j-1$ and $D_2\tau_1 = \sigma^j$ ($E_0 = E_1$ by minimality). Next consider $\sigma - D\tau_1$ which is of degree $\geqslant j+1$ and continue. The process stops since the degree function is bounded above in each dimension. We end up with the element $\tau = \sum_{i=1}^{s} \tau_i$. Suppose that $j \geqslant n+2$. Since $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n]$ and $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n_{\leqslant m-1}]$ differ in dimensions $\leqslant m+1$ only by V_m which is of degree n the element τ is in $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n_{\leqslant m-1}] \otimes \mathbb{Z}/p$. □

Let R be a quotient ring of $\mathbb{Z}_{(p)}$. So R is $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p^t$. We define $o(R)$ to be 0 or p^t respectively. The next lemma is a lifted version of 5.5.

Lemma 5.6. *Assume that the cohomology groups of dimension $\leqslant m$ of (Y^n, ∂_1) and (X, d_1) are both R-free. Let $\sigma \in (\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n], D)$ be a cocycle* mod $o(R)$ *of degree $\geqslant j$ and dimension i. Suppose that $j \geqslant n+1$ and $i \leqslant m+1$. Then there is τ with $|\tau| \geqslant j-1$ and $D\tau = \sigma$* mod $o(R)$*. If $j \geqslant n+2$ then τ can be chosen in $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n_{\leqslant m-1}]$.*

Proof. We write α^i for the degree i part of $\alpha \in \Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n]$. Suppose first that $o(R) = p^t$ for some t. By 5.5 there are τ_1, η_1 with $|\tau_1| \geqslant j-1$ and $D\tau_1 = \sigma + p\eta_1$. Comparing coefficients, we see $|\eta_1| \geqslant j-1$. In case $o(R) = p$ there is nothing left to do. By our assumption on ∂_1 and d_1, we have

$$p\eta_1^{j-1} = D_1(\tau_1^{j-1}) = 0 \mod p^t.$$

It follows that

$$0 = D^2(\tau_1) = D(\sigma + p\eta_1) = D(p\eta_1) \mod p^t = D(p\eta_1^{\geqslant j}) \mod p^t.$$

Hence we find

$$D(\eta_1^{\geqslant j}) = 0 \mod p^{t-1}.$$

Apply 5.5 to $\eta_1^{\geqslant j}$ and repeat the argument to find τ_2, η_2 and so on. Then

$$D\tau = D(\textstyle\sum_{i=0}^{t-1} (-1)^i p^i \tau_{i+1}) = \sigma \mod p^t$$

and $|\tau| \geqslant j-1$. Next suppose $o(R) = 0$. In this case $D_1 = 0$, coming from the filtration by the degree. We claim that $E_2^{p,q} \otimes \mathbb{Z}/p = 0$ for $p > n$ and $p+q \leqslant m+1$. This is a consequence of 5.4 and the universal coefficient theorem. But $E_2^{p,q}$ is of finite type so the claim holds without tensoring by $\mathbb{Z}/p$. Now repeat the argument given in 5.5 to find τ. Also the last assertion can be seen as in 5.5 above. □

We remark that if the last lemma could be proved without the assumption on d_1, ∂_1, then the main theorem could be proved without the assumption on $\widetilde{H}_*(\Omega Z; \mathbb{Z}_{(p)})$. A sufficient condition for this (which however is not always satisfied) is that the map $\overline{\varphi}$ induced on the E_1 terms of the spectral sequences above is an m-equivalence. See also the work of Halperin and Tanré in [**HT**], where the authors consider the situation over a field. We have been unable to find a translation of this condition into terms of more common topological invariants, so we settle with our assumption on $\widetilde{H}_*(\Omega Z;\mathbb{Z}_{(p)})$.

Since the inclusion of Y^n, ∂_1 into ΛY induces an isomorphism in cohomology, it is not hard to show that the differential on $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n]$ and the map $\overline{\varphi}$ extend to $\Lambda X \otimes \Lambda Y$. The next theorem says that this extension can be chosen to be quite simple. Moreover, if the condition on the loop space homology in 5.6 is satisfied, then the behaviour of this differential with respect to the filtration by degree can be estimated. This will be the key for the proof that cat equals M-cat. The proof of the next theorem is a generalization of the proof of Lemma 3 in [**He**].

Theorem 5.7. *Let (X, d_1), (Y^n, ∂_1) be the complexes introduced at the beginning of the section. Suppose that cohomology groups of these complexes are R-free for a fixed quotient R of $\mathbb{Z}_{(p)}$. Then there exists a differential D on $\Lambda X \otimes \Lambda Y_{\leqslant m-1}$ with the following properties:*

α) D extends the differential on $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n_{\leqslant m-1}]$.

β) The extension φ of $\overline{\varphi}$ which is defined by setting $\varphi_{|Y^{M_k}_{\leqslant m-1}} = 0$ for $k > 1$, is a morphism of differential graded algebras.

γ) $D_{|Y^{M_k}_{\leqslant m-1}} = 1 \otimes \partial + \omega$ with $\omega(Y^{M_k}) \subseteq [I \otimes [\mathbb{Z}_{(p)} \oplus Y^{\leqslant M_k}]^{\geqslant M_k+1}$ for $k > 1$.

Proof. We will do the construction inductively on $Y^{\leqslant M_k}_{\leqslant r+\ell-1}$. Suppose that for some $k > 1$, $\ell \geqslant 1$, D is defined with all properties. Note that $Y^{M_k}_r = 0$ for $k > 1$, which gives the start for the induction. One easily reduces to the case of an extension with a complex $\mathbb{Z}_{(p)}(y \xrightarrow{p^t} \overline{y})$, respectively $\mathbb{Z}_{(p)}(y)$, concentrated in dimensions $r + \ell$ and $r + \ell + 1$, respectively $r + \ell$, and of degree M_k. We do the proof for the first case and the other is similar.

Write $\partial y = \partial_1 y + \partial_2 y = p^t \overline{y} + \partial_2 y$ with $\partial_2 y \in [\Lambda^2 Y^{<M_k}]^{M_k+1}$. Let $D\partial_2 y = \partial\partial_2 y + \omega\partial_2 y = \partial_1\partial_2 + \omega\partial_2 y = -\partial_2\partial_1 y + \omega\partial_2 y = \omega\partial_2 y \mod p^t$. Write $\omega\partial_2 y = \tau_1 + \tau_2$ with $\tau_i \in I \otimes \Lambda^i Y$ and $|\tau_i| > M_k + 1$. This can be done by induction. We have $0 = D^2\partial_2 y = D\tau_1 + D\tau_2 \mathrm{mod}\ p^t$.

Note that

$$
\begin{aligned}
D\tau_1 &\in \Lambda X \otimes \Lambda^2 Y, \\
D\tau_2 - \partial_2\tau_2 &\in \Lambda X \otimes \Lambda^{\leqslant 2} Y, \\
\partial_2\tau_2 &\in \Lambda X \otimes \Lambda^3 Y.
\end{aligned}
$$

It follows that $\partial_2\tau_2 = 0 \mod p^t$.

Next write $\tau_2 = \sum_i \mu_i v_i$ with $v_i \in \Lambda^2 Y$ and where μ_i runs through a base of I. Then we get $\partial_2 v_i = 0 \mod p^t$ for all i. By 5.3 there are $y_i \in Y^{2 \leqslant j_i \leqslant M_k}$ with $v_i = \partial_2 y_i$ mod p^t. Since the dimension of the y_i is less than $r + \ell$, we can apply our inductive hypothesis on them. We find

$$\begin{aligned}
&D\,\partial_2 y - D\textstyle\sum_i(-1)^{||\mu_i||}\mu_i y_i\\
&\quad= \omega\partial_2 y - \textstyle\sum_i(-1)^{||\mu_i||}d\mu_i y_i - \sum_i \mu_i\ D\,y_i \mod p^t\\
&\quad= \tau_1+\tau_2 - \textstyle\sum_i(-1)^{||\mu_i||}d\mu_i y_i - \tau_2 - \sum_i \mu_i\omega y_i - \sum_i \mu_i\partial_1 y_i \mod p^t\\
&\quad= \sigma_0+\sigma_1 \mod p^t
\end{aligned}$$

with $\sigma_i \in \Lambda X\otimes\Lambda^i Y^{\leqslant M_k}$. ($||x||$ denotes the dimension of x.)

We claim that $|\sigma_i| > M_k+1$. Since $|\tau_2| > M_k+1$, we have $|\mu_i\cdot v_i| > M_k+1$. So $|\mu_i\cdot y_i| \geqslant M_k+1$. But since $\partial_1 y = d_1 y_i = 0 \mod p^t$, $|\omega(y_i)| > |y_i|$ and $|\tau_1| > M_k+1$ all summands of σ_0,σ_1 have degree $> M_k+1$.

Next write $\sigma_1 = \sum_i \mu_j\widetilde{y}_j$ with μ_j again a base of I. From $D\sigma_0 + D\sigma_1 = 0 \mod p^t$ it follows that $\partial_2\sigma_1 = 0 \mod p^t$ and this shows us that $\widetilde{y}_j\in Y^n \mod p^t$ by 5.3. So we can choose $\sigma_0+\sigma_1\in[I\otimes[\mathbb{Z}_{(p)}\oplus Y^n]^{>M_k+1}$. Then $\overline{\varphi}(\sigma_i)=0$ and so there is $\eta\in\operatorname{Ker}\overline{\varphi}$ with $|\eta|\geqslant M_k+1$ and $D\eta = \sigma_0+\sigma_1 \mod p^t$ by 5.6.

Putting everything together we find $D(\partial_2 y - \sum_i \mu_i y_i - \eta) = 0] \mod p^t$ with the whole argument of degree $\geqslant M_k+1$.

So we can define

$$\begin{aligned}
Dy &= \partial_2 y - \sum_i \mu_i y_i - \eta + p^t\overline{y},\\
D\overline{y} &= -D(\partial_2 y - \sum_i \mu_i y_i - \eta)/p^t.
\end{aligned}$$

Since all terms in $Dy - p^t\overline{y}$ map to zero under φ, setting $\varphi(y) := 0$, $\varphi(\overline{y}) := 0$ defines a map of differential graded algebras. □

Proposition 5.8. *The map $\varphi\colon \Lambda X\otimes\Lambda Y_{\leqslant m-1}\to\Lambda X/I^{n+1}$ is an $(m-1)$-equivalence.*

Proof. Consider

$$\begin{tikzcd}
Y^n \arrow[r, tail] & \Lambda Y \\
Y^n_{\leqslant m-1} \arrow[r, tail] \arrow[u, tail] & \Lambda Y_{\leqslant m-1}. \arrow[u, tail]
\end{tikzcd}$$

Since $Y^n\to\Lambda Y$ induces an isomorphism in cohomology and $Y^n_{\leqslant m-1},\Lambda Y_{\leqslant m-1}$ differ from $Y^n,\Lambda Y$ only in dimensions $\geqslant m$ it is clear that $Y^n_{\leqslant m-1}\to\Lambda Y_{\leqslant m-1}$ is a weak $(m-1)$-equivalence. But by 2.6 it is also an $(m-1)$-equivalence.

Consider now:

$$\begin{tikzcd}
\Lambda X\otimes[\mathbb{Z}_{(p)}\oplus Y^n_{\leqslant m-1}] \arrow[r, "\overline{\varphi}"] \arrow[d, tail, "i"] & \Lambda X/I^{n+1}. \\
\Lambda X\otimes\Lambda Y_{\leqslant m-1} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

The map $\overline{\varphi}$ is an $(m-1)$-equivalence and also i by an argument with algebraic Serre spectral sequences. In general, if g and $f\circ g$ are $(m-1)$-equivalences it does not follow that f is an $(m-1)$-equivalence too. But there is a way out since ΛX and $\Lambda X \otimes \Lambda Y_{\leqslant m-1}$ are cofibrant in the fibration category $f\text{-}\overline{\mathrm{Alg}}_r(\mathbb{Z}_{(p)})$ for a ring system with $R_i = \mathbb{Z}_{(p)}$, $i \leqslant m$. Let $\hat{\varphi}\colon \Lambda X \otimes \Lambda Y_{\leqslant m-1} \otimes \Lambda Z_m \to \Lambda X/I^{n+1}$ be a cofibrant minimal m-model of $p_{n+1}\colon \Lambda X \to \Lambda X/I^{n+1}$. (We could take $\Lambda X \otimes \Lambda Y_{\leqslant m}$, but we will not need this fact.) Note that φ is a weak $(m-1)$-equivalence. Consider the map of differential graded algebras

$$\Lambda Y_{\leqslant m-1} \overset{j}{\hookrightarrow} \Lambda Y_{\leqslant m-1} \otimes \Lambda Z_m$$

with the induced differential. We claim that it is enough to show that this map is an $(m-1)$-equivalence. To see this, consider

$$\begin{tikzcd}
\Lambda X \otimes \Lambda Y_{\leqslant m-1} \arrow[r, "\varphi"] \arrow[d, tail, "\bar{j}"] & \Lambda X/I^{n+1} \arrow[d, equal] \\
\Lambda X \otimes \Lambda Y_{\leqslant m-1} \otimes \Lambda Z_m \arrow[r, "\widehat{\varphi}"] & \Lambda X/I^{n+1}
\end{tikzcd}$$

and observe that φ is an $(m-1)$-equivalence if $\bar{j}$ is an $(m-1)$-equivalence. An application of the easy part of the Zeeman comparison theorem to the algebraic Serre spectral sequences of the extensions of ΛX in the diagram above shows that $\bar{j}$ is an $(m-1)$-equivalence if j is an $(m-1)$-equivalence. This proves the claim. Dualize and recall the Whitehead theorem 3.6 and the identification of the homotopy groups with the homology of the primitives of a fibrant model. Since there are no homotopy groups in dimension m for $(\Lambda Y_{\leqslant m-1})^*$, the map j^* is $(m-1)$-connected in homotopy hence in homology, and φ is an $(m-1)$-equivalence. $\square$

Remark 5.9. Under the assumption that the cohomology groups of (X, d_1) and (Y^n, ∂_1) are R-free, it can be shown that there is an isomorphism

$$H^{*,*}(\Lambda\bar{X} \otimes [R \oplus \bar{Y}^n], \bar{D}_2) \cong \mathrm{Ext}^{*,*}_{U\bar{L}}(R, \bar{Y}^n).$$

Here the bar stands for reduction mod $o(R)$, and $\bar{L} = (s^{-1}X)^*$ with bracket given by $\bar{d}_2$, and the representation of $\bar{L}$ in $\bar{Y}^n$ is induced by $\bar{D}_2$. Moreover, this representation corresponds to the holonomy representation, with R coefficients, of the fibration defined by $\Lambda X \to \Lambda X/I^{n+1}$. The proof for the first assertion is almost the same as in [**FHJLT**] with some modifications to be found in [**H**]. Since the proof of 5.7 is central for the whole paper, this shows again the intimate relation between cat and holonomy. In the same direction, in [**ST4**] certain fibrations which are characterized by means of their holonomy representation were found to detect cat in the same way as the Ganea fibration.

6. Module retractions versus algebra retractions

We keep the notions of the last section and suppose $\alpha(r-1, p) \geqslant m$. The main result of this section is

Theorem 6.1. *Suppose that the inclusion $\Lambda X \to \Lambda X \otimes \Lambda Y_{\leqslant m-1}$ admits a ΛX-module retraction r, and $(X, d_1), (Y_{\leqslant m-1}, \partial_1)$ both have R-free cohomology. Then the inclusion admits a retraction of dg algebras.*

The proof, which is an integral version of the one given by Hess in [**He**], needs some preparations. First note that $Y_{\leqslant m-1}$ is concentrated in degrees $< M_p$ since $\alpha(r-1,p) \geqslant m$.

Now let $k < p$, $\phi\colon Y^{\leqslant M_k} \to \Lambda X \otimes Y_{\leqslant m-1}$ a $\mathbb{Z}_{(p)}$-module map, and

$$r\colon \Lambda X \otimes \Lambda Y_{\leqslant m-1} \to \Lambda X$$

a ΛX-module retraction. Define an algebra map (not commuting with the differential in general)

$$\rho\colon \Lambda X \otimes \Lambda Y^{\leqslant M_k}_{\leqslant m-1} \to \Lambda X$$

and a ρ-derivation

$$\theta\colon \Lambda X \otimes \Lambda Y^{\leqslant M_k}_{\leqslant m-1} \to \Lambda X$$

as follows:

α) $\rho_{|X} := \mathrm{id}$ and $\rho_{|Y^{M_j}} = \frac{1}{j} \cdot r \circ \phi$;

β) $\theta_{|X} := 0$ and $\theta_{|Y^{M_j}} := j \cdot \rho - \phi$.

Note that j is a unit in $\mathbb{Z}_{(p)}$ since $j \leqslant k < p$ was assumed.

The proof of the following lemma is identical to the one given in [**He**] over $\mathbb{Q}$ and is therefore omitted.

Lemma 6.2. *Suppose that $D\phi = j \cdot \rho D - \theta D$ holds on $Y^{M_j}_{\leqslant m-1}$ for $j \leqslant k$. Then the following holds on $\Lambda X \otimes \Lambda Y^{\leqslant M_k}_{\leqslant m-1}$:*

i) $r\theta = 0$,

ii) $\rho D = d\rho$,

iii) $\theta D = D\theta$.

In particular, 6.2 gives an equation, in terms of ϕ and r which makes ρ a retraction of dg algebras. The next proposition generalizes Theorem 2. in [**He**].

Proposition 6.3. *If $\Lambda X \otimes \Lambda Y_{\leqslant m-1}$ is as above and r is given, then there is*

$$\phi\colon Y^{\leqslant M_{p-1}}_{\leqslant m-1} \to [\Lambda X \otimes \Lambda Y_{\leqslant m-1}]^{>n}$$

such that $D\phi = j \cdot \rho D - \theta D$ holds on $Y^{M_j}_{\leqslant m-1}$ for $j \leqslant p-1$.

Proof. Again we argue inductively on the degree and the dimension of the extensions in $Y_{\leqslant m-1}$. The case $Y^n_{\leqslant m-1}$ can be done in one step: Define $\phi_{|Y^n_{\leqslant m-1}} := \mathrm{id}$. Then $\rho y = ry$ and $\theta y = ry - y$. Since θ and ρ are ΛX linear, we have that on $A := \Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n_{\leqslant m-1}]$,

$$\rho = r,$$
$$\theta = r - \mathrm{id}_A\,.$$

Since A is invariant under D, it follows that:

$$D\phi y Z = \rho D y - \theta D y, \qquad y \in Y^n_{\leqslant m-1}.$$

Now suppose ϕ is constructed on $Y^{M_k}_{<r+\ell}$, $k > 1$, $\ell \geqslant 1$. We assume that $R = \mathbb{Z}/p^t$, again the case $R = \mathbb{Z}_{(p)}$ is easier. Let $y \in Y^{M_k}_{r+\ell}$ with $||y|| = r + \ell$. So $\partial_1 y = p^t \overline{y}$. Write $D = \partial_1 + \tau$, then $\tau y \in \Lambda X \otimes \Lambda Y^{\leqslant M_k}_{<r+\ell}$. Furthermore, $d(k \cdot \rho\tau y - \theta\tau y) = 0 \bmod p^t$.

Claim. $\varphi(k \cdot \rho\tau y - \theta\tau y) = 0$.

Write $\tau y = \psi_0 + \psi_1 + \psi_2$, with $\psi_0 \in I$, $\psi_1 \in I \otimes Y^{\leqslant M_k}_{<r+\ell}$ and $\psi_2 \in \Lambda^2 Y^{<M_k}_{\leqslant r+\ell}$, and all three terms have degree $\geqslant M_k + 1$ by 5.7. The rest of the argument showing the claim is now exactly as in the proof of Theorem 2 in [**He**].

It follows that there are $\alpha_1, \alpha_2 \in \operatorname{Ker}\varphi$ of degree $\geqslant n$ with

$$D\alpha_1 = k\rho\tau y - \theta\tau y + p^t \alpha_2.$$

Define $\phi(y) := \alpha_1$, $\phi(\overline{y}) := \alpha_2$.

Then

$$\begin{aligned} D\phi y &= k\rho\tau y - \theta\tau y + p^t\alpha_2 \\ &= k\rho(Dy - \partial_1 y) - \theta(Dy - \partial_1 y) + p^t\alpha_2 \\ &= k\rho Dy - k\rho p^t \overline{y} - \theta Dy + \theta p^t \overline{y} + p^t\alpha_2 \\ &= k\rho Dy - p^t r\alpha_2 - \theta Dy + p^t\theta\overline{y} + p^t\alpha_2 \\ &= k\rho Dy - p^t r\alpha_2 - \theta Dy + p^t r\alpha_2 - p^t\alpha_2 + p^t\alpha_2 \\ &= k\rho Dy - \theta Dy. \end{aligned}$$

The same holds for $\overline{y}$ by a similar calculation using 6.2. □

Proof of 6.1. The theorem is a direct consequence of 6.2 and 6.3. □

Proposition 6.4. *Suppose that the cohomology of (X, d_1) R-free. Then the same holds for $(Y_{\leqslant m}, \partial_1)$.*

We have to rely on certain results of Scheerer and Tanré in [**ST1, ST4**]. The proof would be straightforward, if $\Lambda X \to \Lambda X \otimes \Lambda Y_{\leqslant m-1}$ represented the $(m-1)$-Postnikov section of the n-th Ganea fibration over Z. But this is already wrong over $\mathbb{Q}$. There is an additional wedge product of rational spheres (see [**F2**]). Let R_* be a ring system which is tame with respect to $r \geqslant 2$. Recall from [**D**] that an $(r-1)$-connected space X is tame with respect to R_*, r if $\pi_{r+k}(X)$ is a module over R_k. Tame spaces are the fibrant objects in the model category constructed by Dwyer. So every space admits a universal arrow to a tame space. Note that the finite Postnikov sections which we have used so far, to compute $\operatorname{cat}(X)$ for a finite p-local complex, are all tame with respect to a suitable ring system.

Proof of 6.4. Recall that ΛX represents the m-th Postnikov section of Z, and $\Lambda X \to \Lambda X \otimes \Lambda Y_{\leqslant m}$ the $(m-1)$-th Postnikov section of the map representing $\mathcal{L}P_n\mathcal{C}\mathbb{L}V \to \mathbb{L}V$. Using the identification of $H_*(X^*, d_1^*)$ for $* \leqslant m$ with the homotopy groups of Z in dimensions $\leqslant m$, we see that it is enough to show the following: If the homotopy groups of Z are R-free for $* \leqslant m$, then the same is true for the space W which represents $\mathcal{L}P_n\mathcal{C}\mathbb{L}V$. To see this consider the exact sequence in $\mathrm{Lie}_{r-1}(\mathbb{Z}_{(p)})$. The map q represents an n-LS-fibration by [**ST1**, Proposition 3.1]. The looping of an n-LS-fibration splits. Hence it is enough to show that the tame space representing K_n has R-free homotopy groups for $* \leqslant m$. It is proved in [**ST1**, Proof of 3.1] that the space representing K_n in tame homotopy is a retract of $\Sigma((\Omega Z \times A)/\Omega Z)$. Here A is the tame space represented by $\mathbb{L}(s^{-1}P_n\mathcal{C}\mathbb{L}V/P_{n-1}\mathcal{C}\mathbb{L}V)$ with linear differential $s^{-1}\overline{\partial}$, and $\overline{\partial}$ is induced by the differential ∂ of $\mathcal{C}\mathbb{L}V$. We claim that $\Sigma\Omega Z$ and A both decompose into a wedge product of Moore spaces and all factors $P^{s+1}(G)$ for $s \leqslant m$ have type $G = R$. The decomposition of tame suspensions into Moore spaces is a general fact of tame homotopy theory. So only the assertion of the type needs proof. For $\Sigma\Omega Z$ this follows from our assumption on the homotopy groups of Z. For A, note that the $\mathbb{Z}_{(p)}$-dual of $P_n\mathcal{C}\mathbb{L}V/P_{n-1}\mathcal{C}\mathbb{L}V$ is m-equivalent to $\Lambda^n X, d_1$. For $n < p$ the reduced cohomology of $\Lambda^n X, d_1$ is R-free in dimension $\leqslant m$ by our assumption on d_1. But this suffices because $m \leqslant \alpha(r-1, p)$ was assumed. Next, recall that

$$\Sigma(\Omega Z \times A/\Omega Z) \simeq \Sigma\Omega Z \wedge A \vee \Sigma A$$

and that the property of being a wedge product of Moore spaces of a given type is invariant under the operations of taking smash products and retracts by [**CMN**] (we use $p > 2$ here). Furthermore, if $\Sigma\Omega Z$ and A have only type R factors in dimensions $\leqslant m$, the same is true for smash products and retracts. So there are only type R Moore spaces in dimensions $\leqslant m$ in the decomposition of the space representing K_n. $\square$

Remark 6.5. The key of the proof of 6.1 is to gain some control on the behaviour of the differential and the map φ with respect to the filtration by degree. If we want to relate our results to cat in $\mathrm{CW}_r^m(\mathbb{Z}_{(p)})$, then this filtration is forced on us by 4.1. Over $\mathbb{Z}/p$ there is no linear part in the differential, and consequently $(d_2)^2 = 0$ which makes it possible to do perturbation theory on d_2. Since this is not true over $\mathbb{Z}_{(p)}$ ($(d_2)^2 + d_1d_3 + d_3d_1 = 0$), we need an assumption on d_1 to lift these perturbation arguments. But in general the bigrading does not induce a bigrading on cohomology with respect to any finite approximation of d. To put it in another way, in contrast to the situation over $\mathbb{Q}$, the filtrations which determine coformality [**N**] and cat differ. To make this point clearer, let us define another grading on a minimal m-model which we call the L-grading. For any minimal extension, induced by an extension with the short complex $X_k = X'_k \xrightarrow{\delta} X''_k$, define $|X'_k| = 1$ and $|X''_k| = 2$. This induces a grading on ΛX, and the part of the differential $d_{(2)}$ which is quadratic with respect to it satisfies $(d_{(2)})^2 = 0$. So the cohomology with respect to $d_{(2)}$ is bigraded. The differential $d_{(2)}$ determines the Lie-bracket on the homotopy Lie algebra $H_*(X^*, d_1^*)$. In contrast, d_2 does determine the Lie-bracket on homotopy with various coefficients and the multiplicative extensions which show up in the universal coefficient sequences for these Lie algebras. So d_2 determines the whole coherent sequence of Lie algebras in the sense of Anick [**A2**]. Denote the differential ideal in ΛX which is given by elements of L-degree $\geqslant n$ by J^n.

Define the invariant L-cat to be the least integer for which a ΛX retraction r exists in

$$\begin{tikzcd} \Lambda X \arrow[r, "\bar{p}_{n+1}"] \arrow[d, shift right] & \Lambda X/J^{n+1}. \\ \Lambda X \otimes \Lambda Y \arrow[u, shift right, "r"'] \arrow[ur, "\varphi"'] & \end{tikzcd} \qquad (*)$$

Note that the minimal model of a $\mathbb{Z}_{(p)}$-local Moore space, if it has torsion in homology, has L-cat $= 2$. It can be shown that the invariant L-cat is closely related to spherical cat as defined by Scheerer and Tanré in [**ST5**].

7. Duality, the Toomer invariant and M-cat

Let R be a commutative ring and A a differential graded algebra over R.

Suppose that A is filtered by a sequence of differential graded ideals

$$A = I^0 \supseteq I^1 \supseteq \cdots$$

with $I^i \cdot I^j \subseteq I^{i+j}$. Recall that every dg A-module M has a semi-free cofibrant A-module model $P \to M$.

Definition 7.1. Let M be a dg A-module and P a cofibrant model of M. Let $q_k \colon P \to P/I^{k+1}$ be the quotient morphism.

- The Toomer invariant $e_A(M)$ is the least number k such that q_k has a retraction up to homotopy in $\mathrm{Coch}(R)$ and it is ∞ if no such number exists.
- The module category M-$\mathrm{cat}_A(M)$ of M is the least number k such that q_k has a retraction up to homotopy in A-mod and it is ∞ if no such number exists.

Let $\mathbb{L}V \in L^{m-1}_{r-1}(\mathbb{Z}_{(p)})$ with $m \leqslant \alpha(r-1,p)$ be of finite type, and $\Lambda X \to \mathcal{C}^*(\mathbb{L}V)$ a multiplicative minimal m-model. Extend ΛX to a ∞-minimal model $\Lambda\overline{X} = \Lambda X \otimes \Lambda W$. Note that W is concentrated in dimensions $> m$. Let $\Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n] \xrightarrow{\overline{\varphi}} \Lambda X/\Lambda^{>n}X$ be a $(m-1)$-minimal model of $\Lambda X \to \Lambda X/\Lambda^{>n}X$, and $\Lambda\overline{X} \otimes [\mathbb{Z}_{(p)} \oplus \overline{Y}^n] \xrightarrow{\varphi} \Lambda\overline{X}/\Lambda^{>n}\overline{X}$ an extension which is a minimal model of $\Lambda\overline{X} \to \Lambda\overline{X}/\Lambda^{>n}\overline{X}$, such that $\overline{Y}^n = Y^n \oplus Z$ and Z concentrated in dimensions $\geqslant m$. Filter $\Lambda\overline{X}$ by the ideals $I^k = \Lambda^{\geqslant k}\overline{X}$ and consider M-cat for $A = \Lambda\overline{X}$ with respect to this filtration for $R = \mathbb{Z}_{(p)}$. From now on we suppress the A in M-cat_A, e_A.

Proposition 7.2. *M-$\mathrm{cat}(\Lambda\overline{X}) = n$ holds if and only if $\Lambda X \to \Lambda X \otimes [\mathbb{Z}_{(p)} \oplus Y^n]$ admits a ΛX-module retraction r.*

Proof. Suppose r is given. By [**H**, 10.3], $H^*(\mathcal{C}^*(\mathbb{L}V);\pi) = 0$ for $* > m$ and all coefficients. Hence the obstructions to extend r to the free extensions in Z are zero since they sit in $H^{*>m}(\Lambda\overline{X};\pi)$. On the other hand, given a retraction $\overline{r}$ of $\Lambda\overline{X} \to \Lambda\overline{X} \otimes [\mathbb{Z}_{(p)} \oplus \overline{Y}^n]$ then $\overline{r}$ maps Y^n to ΛX and hence defines a retraction r. □

Note that the dual M^* of a $\Lambda\overline{X}$-module M is also a $\Lambda\overline{X}$-module. The $\Lambda\overline{X}$ action on M^* is defined by $a\psi = \psi_a$ with $\psi_a(m) = (-1)^{|\psi||a|}\psi(am)$. In particular, $(\Lambda\overline{X})^*$ is a $\Lambda\overline{X}$-module.

Theorem 7.3. *M-$\mathrm{cat}(\Lambda\overline{X}) = e((\Lambda\overline{X})^*)$.*

Proof. Suppose $e((\Lambda\overline{X})^*) = n$ and let $P \xrightarrow{\rho} (\Lambda\overline{X})^*$ be a semi-free cofibrant model. The dual $\Lambda\overline{X} \cong (\Lambda\overline{X})^{**} \xrightarrow{\rho^*} P^*$ is a $\Lambda\overline{X}$-module map and a weak equivalence. As such it is determined by $\rho^*(1) = z$. Since $P \xrightarrow{p_{n+1}} P/(\Lambda^{>n}\overline{X} = I^{n+1})$ has a cochain retraction up to homotopy, the cohomology class of z is in the image p^*_{n+1} in cohomology, i.e., $z + \partial\alpha = p^*_{n+1}(w)$. So the $\Lambda\overline{X}$-module morphism from $\Lambda\overline{X}$ to P^* which sends 1 to

$z + \partial\alpha$ is a weak equivalence and factors over (P^*/I^{n+1}). By [**FHL**, Prop. 2iii)], it follows that $M\text{-cat}(\Lambda\overline{X}) \leqslant n$. Now suppose that $M\text{-cat}(\Lambda\overline{X}) \leqslant n$. By [**FHL**, Prop. 3], one knows that $M\text{-cat}(N) \leqslant M\text{-cat}(\Lambda\overline{X})$ for every $N \in \Lambda\overline{X} - \text{mod}$. This gives us $M\text{-cat}((\Lambda\overline{X})^*) \leqslant n$, and since always $e(N) \leqslant M\text{-cat}(N)$ the result. $\square$

Before we deduce our first product formula, we need a lemma which tells us how to detect a cochain retraction.

Lemma 7.4. *Let $f\colon C \to D$ be a map in $\mathrm{Coch}(\mathbb{Z}_{(p)})$, such that C, D are $\mathbb{Z}_{(p)}$-free with $H^*(C), H^*(D)$ of finite type and bounded below. Then the following conditions are equivalent:*

α) f has a retraction up to homotopy in $\mathrm{Coch}(\mathbb{Z}_{(p)})$;

β) $f^*\colon H^*(C;\pi) \to H^*(D;\pi)$ is a monomorphism for all finitely generated $\mathbb{Z}_{(p)}$-coefficients π.

Proof. Of course, one direction is clear. So suppose that β) holds. Since $H^*(C)$ and $H^*(D)$ are bounded below we can use a minimal extension for the factorization of f into cofibration and weak equivalence in $\mathrm{Coch}(\mathbb{Z}_{(p)})$. The free extension can be chosen to be of finite type, since $H^*(C)$ and $H^*(D)$ are of finite type. Suppose that we have found a retraction on all complexes $V_{\ell<k}$ concentrated in dimensions $(\ell, \ell+1)$. As usual, we can reduce to $V_k = \mathbb{Z}_{(p)}(y \xrightarrow{p^t} \overline{y})$ or $\mathbb{Z}_{(p)}(y)$, and we deal with the first case. Write $Dy = p^t\overline{y} + \tau y$, $\tau y \in C$. Then $[\tau y]$ is in the kernel of

$$f^*\colon H^*(C;\mathbb{Z}/p^t) \to H^*(D;\mathbb{Z}/p^t).$$

Hence there are $\alpha, \overline{\alpha} \in C$ with $d\alpha = \tau y + p^t\overline{\alpha}$ in C. Define $r(y) = \alpha$ and $r(\overline{y}) = \overline{\alpha}$. $\square$

Let P be a semi-free $\Lambda\overline{X}$-module model of $\mathcal{C}(\mathbb{L}V)$.

Lemma 7.5. *Both P and P/I^{n+1} have all properties attributed to C and D in 7.4.*

Proof. It is clear that $P, P/I^{n+1}$ are $\mathbb{Z}_{(p)}$ free. Since $H^*(\mathcal{C}(\mathbb{L}V)) = 0$ for $* < -m$ by [**H**, 10.2], we can construct a minimal model P'. The module P' is then concentrated in dimensions $\geqslant -m$ and $H^*(\mathcal{C}(\mathbb{L}V))$ is of finite type. So the same holds for P'/I^{n+1}. Since P, P' and P/I^{n+1}, P'/I^{n+1} have isomorphic cohomology, the lemma is proved. $\square$

Theorem 7.6. *Let $\mathbb{L}V_i \in L^{m_i-1}_{r-1}(\mathbb{Z}_{(p)})$, $i \in \{1,2\}$, be of finite type and $m_1 + m_2 \leqslant \alpha(r-1,p)$. Let $\Lambda\overline{X}_i$ be minimal models of $\mathcal{C}^*(\mathbb{L}V_i)$. Then*

$$\begin{aligned} M\text{-}\mathrm{cat}_{\Lambda\overline{X}_1\otimes\Lambda\overline{X}_2}(\mathcal{C}^*(\mathbb{L}V_1 \times \mathbb{L}V_2)) &= M\text{-}\mathrm{cat}_{\Lambda\overline{X}_1\otimes\Lambda\overline{X}_2}(\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2) \\ &= M\text{-}\mathrm{cat}_{\Lambda\overline{X}_1}(\Lambda\overline{X}_1) + M\text{-}\mathrm{cat}_{\Lambda\overline{X}_2}(\Lambda\overline{X}_2). \end{aligned}$$

Proof. The first equality is clear. By 7.3 it is enough to dualize and to compute e. So let $P_i \to (\Lambda\overline{X}_i)^*$ be a semi-free $\Lambda\overline{X}_i$-model. Then $P_3 = P_1 \otimes P_2$ is a semi-free $\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2$-model of $(\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2)^*$. Let us denote the ideals which define M-cat for $\Lambda\overline{X}_1, \Lambda\overline{X}_2$ and $\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2$ by I_1, I_2, I_3. Then $I_3^m = \sum_{k+\ell=m} I_1^k \otimes I_2^\ell$. Suppose that $e_{\Lambda\overline{X}_1}((\Lambda\overline{X}_1)^*) = k$ and $e_{\Lambda\overline{X}_2}((\Lambda\overline{X}_2)^*) = \ell$. By 7.4 and 7.5, there are finitely generated $\mathbb{Z}_{(p)}$-modules π_1, π_2 and nontrivial classes $[z_i] \in H^*(P_i;\pi_i)$ with representatives in

$P_1 \cdot I_1^k \otimes \pi_1$, respectively $P_2 \cdot I_2^\ell \otimes \pi_2$. The class $[z_1 \otimes z_2]$ is then nontrivial in

$$H^*(P_1 \otimes P_2; \pi_1 \otimes \pi_2)$$

and has a representative in $P_1 \otimes P_2 \cdot I_1^k \otimes I_2^\ell \subseteq P_3 \cdot I_3^{k+\ell}$. It follows that

$$M - \mathrm{cat}_{\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2}(\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2) \geqslant k + \ell.$$

Conversely, suppose that $P_1 \overset{p_{k+1}}{\to} P_1/I^{k+1}$ and $P_2 \overset{p_{\ell+1}}{\to} P_2/I^{\ell+1}$ admit retractions up to homotopy. Consider

$$\begin{tikzcd} P_1 \otimes P_2 \arrow[r, two heads, "p_{k+\ell+1}"] \arrow[d, "p_{k+1}\otimes p_{\ell+1}"'] & P_1 \otimes P_2/I_3^{k+\ell+1}. \arrow[dl, "q"] \\ P_1/I_1^{k+1} \otimes P_2/I_2^{\ell+1} & \end{tikzcd}$$

Since $I_3^{k+\ell+1}$ acts trivially on $P_1/I_1^{k+1} \otimes P_2/I_2^{\ell+1}$, the arrow q exists in $\mathrm{Coch}(\mathbb{Z}_{(p)})$. It follows that $p_{k+\ell+1}$ is injective in cohomology with all coefficients. By 7.4 and 7.5, we find $e_{\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2}((\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2)^*) \leqslant k + \ell$. This gives us

$$M\text{-}\mathrm{cat}_{\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2}(\Lambda\overline{X}_1 \otimes \Lambda\overline{X}_2) \leqslant k + \ell. \qquad \square$$

by 7.3.

Proof of Theorem 1.1. Let $\mathbb{L}V_1, \mathbb{L}V_2$ represent the homotopy types of X, Y. Since the $(m-1)$-th Postnikov section of $\Omega X, \Omega Y$ splits as a product of Eilenberg-Mac Lane spaces (a basic fact of tame homotopy theory), and the freeness assumption on the loop space homology implies that $\pi_*(X), \pi_*(Y)$ are free R-modules for $* \leqslant m$. The same holds for $\Omega(X \times Y)$. By 4.4, we can use m-minimal models of $\mathcal{C}^*(\mathbb{L}V_1), \mathcal{C}^*(\mathbb{L}V_2)$ and $\mathcal{C}^*(\mathbb{L}V_1 \times \mathbb{L}V_2)$ to compute the L.S. category of X, Y, and $X \times Y$. By 4.4, 6.1, 6.4 and 7.2, the L.S. category equals $M\text{-cat}_{\Lambda\overline{W}}$. By 7.3, $M\text{-cat}_{\Lambda\overline{W}}$ equals the Toomer invariant which is additive by 7.6. $\square$

Remark 7.7. For 1-connected rational spaces of finite type X, there is a result, due to Scheerer and the author [**SS**], which states that $M\text{-cat}(X) \leqslant n$ if and only if the fibration over X obtained by application of a fibrewise version of the infinite symmetric product functor to the n-th Ganea fibration has a section. But $\mathrm{cat}(X) = M\text{-cat}(X)$ by Hess's theorem. So the fibrewise linearized Ganea fibration admits a section if and only if the Ganea fibration admits one. It is very likely that a similar interpretation can be shown for $M\text{-cat}(X)$ and p-local n-connected spaces X if $\dim(X) \leqslant \alpha(n, p)$. Such a result may help to decide if the condition in 1.1 is necessary.

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