THE GLUING PROBLEM DOES NOT FOLLOW FROM HOMOLOGICAL PROPERTIES OF $\Delta_p(G)$

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(communicated by J. P. C. Greenlees)

Abstract

Given a block $b$ in $kG$ where $k$ is an algebraically closed field of characteristic $p$, there are classes $\alpha_Q \in H^2(\text{Aut}_F(Q); k^{\times})$, constructed by Külshammer and Puig, where $F$ is the fusion system associated to $b$ and $Q$ is an $F$-centric subgroup. The gluing problem in $F$ has a solution if these classes are the restriction of a class $\alpha \in H^2(\mathcal{F}^c; k^{\times})$. Linckelmann showed that a solution to the gluing problem gives rise to a reformulation of Alperin's weight conjecture. He then showed that the gluing problem has a solution if for every finite group $G$, the equivariant Bredon cohomology group $H^1_G(|\Delta_p(G)|; \mathcal{A}^1)$ vanishes, where $|\Delta_p(G)|$ is the simplicial complex of the non-trivial $p$-subgroups of $G$ and $\mathcal{A}^1$ is the coefficient functor $G/H \mapsto \text{Hom}(H, k^{\times})$. The purpose of this note is to show that this group does not vanish if $G = \Sigma_{p^2}$ where $p \geq 5$.

1. Introduction

Given a functor $M : \mathcal{C} \to \text{Ab}$, where $\mathcal{C}$ is a small category, we will write $H^* (\mathcal{C}; M)$ for the groups $\lim_{\leftarrow}^* M$. When $\mathcal{C}$ has one object with a group $G$ of automorphisms, a functor $M : \mathcal{C} \to \text{Ab}$ is the same thing as a $G$-module and $H^* (G; M) \cong \lim_{\leftarrow}^* M$.

Let us now fix a prime $p$ and let $\mathcal{F}$ be the fusion system of a block $b$ of a finite group $G$. As usual, we will write $\mathcal{F}^c$ for the full subcategory generated by the $F$-centric subgroups in $\mathcal{F}$. Let $k$ be an algebraically closed field of characteristic $p$. In [8] Külshammer and Puig show that for every $\mathcal{F}$-centric subgroup $Q$ there is a canonically chosen class $\alpha_Q \in H^2(\text{Aut}_F(Q); k^{\times})$. We view $\text{Aut}_F(Q)$ as a full subcategory of $\mathcal{F}^c$ and say that the gluing problem has a solution in $\mathcal{F}$ if there exists a class $\alpha \in H^2(\mathcal{F}^c; k^{\times})$, where $k^{\times}$ is the constant functor, such that the restriction $\alpha|_{\text{Aut}_F(Q)}$ is equal to $\alpha_Q$ for all $Q \in \mathcal{F}^c$. Linckelmann showed in [10] that if the gluing problem has a solution in the fusion systems of all blocks then Alperin’s weight conjecture is logically equivalent to a relation between the number $k(b)$ of complex representations of $G$ associated to $b$ by
Knörr and Robinson [7] and the Euler characteristic of a certain cochain complex built from the fusion system of \( b \) and the cohomology class \( \alpha \).

Let \( G \) be a finite group and \( \mathcal{C} \) a finite \( G \)-poset. The (combinatorial) simplicial complex associated to \( \mathcal{C} \), see [13, Chap. 3], is denoted \( S(\mathcal{C}) \). The \( n \)-simplices are sequences \( c_0 \preceq \cdots \preceq c_n \in \mathcal{C} \) which we denote \( c \). Face maps are inclusion of simplices. We view \( S(\mathcal{C}) \) as a topological space via the geometric realization. Clearly \( G \) acts on \( S(\mathcal{C}) \) whose orbit space is denoted \( [S(\mathcal{C})] \). It is a CW-complex obtained as the geometric realization of the simplicial set \( \text{Nr}(\mathcal{C})/G \) where \( \text{Nr}(\mathcal{C}) \) is the nerve construction [3, XI.2.1]. By abuse of notation, \( [S(\mathcal{C})] \) will also denote the poset of the cells of \( [S(\mathcal{C})] \) ordered by inclusion.

As a special case consider the poset \( \Delta_p(G) \) of the non-trivial \( p \)-subgroups of a finite group \( G \). Note that the isotropy group of an \( n \)-simplex \( P = (P_0 < \cdots < P_n) \) in \( S(\Delta_p(G)) \) is

\[
N_G(P) = \cap_{i=0}^n N_G(P_i).
\]

The objects of the poset \( [S(\Delta_p(G))] \), viewed as a small category, are the \( G \)-conjugacy classes \( [P] \) of the simplices of \( S(\Delta_p(G)) \) and there is a unique morphism \( [Q] \to [P] \) if the simplex \( Q \) is conjugate in \( G \) to a face of \( P \). There is a functor \( N_G : [S(\Delta_p(G))] \to \text{Ab} \) defined by Linckelmann in [9]

\[
N_G([P]) = \text{Hom}(N_G(P), k^\times) = \text{Hom}(N_G(P)_{ab}, k^\times).
\]

Theorem 1.2 of [9] implies that the gluing problem in \( \mathcal{F} \) has a solution if we can prove that \( H^1([S(\Delta_p(K))]; N_K) = 0 \) for all \( K = \text{Aut}_\mathcal{F}(Q)/\text{Inn}(Q) \) where \( Q \) is an \( \mathcal{F} \)-centric subgroup. Thus, if we can prove that \( H^1([S(\Delta_p(G))]; N_G) = 0 \) for all finite groups \( G \), then the gluing problem has a solution for all fusion systems. The purpose of this note is to show that this programme, suggested by Linckelmann, is not feasible.

**Theorem 1.1.** Set \( G = \Sigma_{p^2} \). If \( p \geq 5 \) then \( H^1([S(\Delta_p(G))]; N_G) \neq 0 \).

We remark that \( \Sigma_{p^2} \) appears as an outer \( \mathcal{F} \)-automorphism group of \( Q = (C_p)^p \) in the fusion system of the principal block of \( C_p \wr \Sigma_{p^2} \). We also remark, without proof, that Theorem 1.1 is valid for \( p = 3 \) but it fails if \( p = 2 \). For \( p = 2 \) one observes that \( H^1_\mathcal{F}([B_p(G)]; \mathcal{H}^1) = 0 \), see equation (1), because \( \mathcal{H}^1 \) vanishes on all the orbits of \( [B_p(G)] \). For \( p = 3 \) one has to examine the exact sequence (3) more carefully than we do in Propositions 4.2–4.4.

## 2. Subdivision categories and higher limits

Let \( G \) be a finite group. As in the introduction, if \( \mathcal{C} \) is a finite \( G \)-poset, let \( S(\mathcal{C}) \) denote the associated \( G \)-simplicial complex and let \( [S(\mathcal{C})] \) denote its orbit space. We will denote the set of \( n \)-simplices of \( S(\mathcal{C}) \) by \( S(\mathcal{C})_n \). It is the set of the non-degenerate \( n \)-simplices of \( \text{Nr}(\mathcal{C}) \). Thus, the \( n \)-simplices of \( S(\mathcal{C}) \) are sequences \( c \) of the form \( c_0 \preceq \cdots \preceq c_n \in \mathcal{C} \). The faces of \( c \) are its non-empty subsequences.

The space \( [S(\mathcal{C})] \) is the geometric realization of the simplicial set \( \text{Nr}(\mathcal{C})/G \) whose set of non-degenerate simplices is \( [S(\mathcal{C})]_n = S(\mathcal{C})_n/G \) which in turn, corresponds to the set of \( n \)-cells of \( [S(\mathcal{C})] \). We obtain a poset, abusively denoted \( [S(\mathcal{C})] \), whose objects
are the $G$-orbits of the simplices of $S(C)$ with an arrow $[c'] \to [c]$ if $c'$ is in the orbit of a face of $c$. The objects of $[S(C)]$ will be referred to as “simplices”.

Given an $n$-simplex $c_0 \succ \cdots \succ c_n$ in $S(C)$ where $n \geq 1$, we will write $\partial_i c$ for the $(n - 1)$-simplex obtained by removing $c_i$ where $0 \leq i \leq n$. We obtain face maps

$$\partial_i : S(C)_n \to S(C)_{n-1}$$

and

$$[\partial_i] : [S(C)]_n \to [S(C)]_{n-1}, \quad (0 \leq i \leq n)$$

denoted $[\partial_i]$.

where $\partial_i$ is $G$-equivariant and an $n$-simplex $[c]$ in $[S(C)]$ gives rise to a map of transitive $G$-sets $[c] \xrightarrow{[\partial_i]} [\partial_i c]$.

**Definition 2.1.** Fix a commutative ring $R$. Let $C$ be a finite $G$-poset and consider a functor $A : [S(C)] \to R$-mod. Define a cochain complex $C^\bullet(A)$ as follows.

$$C^n(A) = \prod_{[c] \in [S(C)]_n} A([c]), \quad \text{and} \quad d : C^{n-1}(A) \xrightarrow{\sum_{j=0}^n (-1)^j d^j} C^n(A).$$

The homomorphisms $d^j : C^{n-1}(A) \to C^n(A)$ are defined on the $[c]$-th component of $C^n(A)$, where $[c] \in [S(C)]_n$, by the composition

$$C^{n-1}(A) \xrightarrow{\text{proj}} A([\partial_j c]) \xrightarrow{A([\partial_j c] \otimes [c])} A([c]).$$

**Lemma 2.2** (cf. [10, Proposition 3.2]). With the notation of Definition 2.1, the cohomology groups of $C^\bullet(A)$ are isomorphic to $H^\bullet([S(C)]; A)$.

**Proof.** For every $n \geq 0$ consider the projective functors $P_n : [S(C)] \to \textbf{Ab}$ defined by

$$P_n = \bigoplus_{[c] \in [S(C)]_n} \mathbb{Z} \otimes \text{Mor}_{[S(C)]}([c], -).$$

For every $0 \leq j \leq n$ there are morphisms $d^j_{n-1} : P_n \to P_{n-1}$ which are induced by Yoneda’s lemma via the morphisms $[\partial_j c] \to [c]$ for every $[c] \in [S(C)]_n$. Define morphisms $d_{n-1} : P_n \to P_{n-1}$ by $d_{n-1} = \sum_{j=0}^n (-1)^j d^j_{n-1}$. We claim that the resulting

$$\cdots \xrightarrow{d_{n-1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \to \mathbb{Z}$$

is a projective resolution of the constant functor $\mathbb{Z}$. Indeed, the evaluation of $P_\bullet$ at every object $[x] \in [S(C)]_n$ yields the chain complex $C_\bullet([\Delta^n]; \mathbb{Z})$ because the faces of $[x]$ in $[S(C)]$ generate the standard simplex $\Delta^n$. Finally, by Yoneda’s Lemma $\text{Hom}(P_\bullet, A) = C^\bullet(A)$ and its homology groups are isomorphic to $\lim_{\leftarrow} \text{proj}^\bullet$. 

\end{proof}

### 3. Bredon cohomology

Throughout this section a space means a simplicial set. Let $G$ be a finite group. A **coefficient functor** $\mathcal{M}$ for $G$ is a contravariant functor $\{G\text{-sets}\} \to \textbf{Ab}$ which turns coproducts of $G$-sets into products of abelian groups. By applying $\mathcal{M}$ to the sets of simplices of a $G$-space $X$, one obtains a cosimplicial abelian group $\mathcal{M}(X)$. The cochain complex associated to $\mathcal{M}(X)$ is denoted $C^\bullet(X; \mathcal{M})$, see [15, 8.2]. Its homology groups are called the Bredon cohomology groups $H^n_C(X; \mathcal{M})$, see e.g., [5, §4]. Note that $C^n(X; \mathcal{M}) = \prod_{[x] \in X} \mathcal{M}([x])$ where the product runs through the orbits of the $n$-simplices of $X$. 

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If $Y$ is a $G$-subspace of $X$ then there is a canonical short exact sequence of cochain complexes

$$0 \to C^*_G(X, Y; \mathcal{M}) \to C^*_G(X; \mathcal{M}) \to C^*_G(Y; \mathcal{M}) \to 0$$

which defines the relative cohomology groups $H^*_G(X, Y; \mathcal{M})$ together with the usual long exact sequences. Bredon cohomology is an equivariant cohomology theory, cf. [4]. In particular it turns $G$-homotopy equivalences into isomorphisms and if $X$ is a union of subspaces $Y_1 \cup Y_2$, one has the usual Mayer Vietoris sequence.

The normalized cochain complex $NC^*(X; \mathcal{M})$ is a sub-complex of $C^*(X; \mathcal{M})$ defined by

$$NC^n(X; \mathcal{M}) = \bigcap_{i=0}^{n-1} \left( \ker(C^n(X; \mathcal{M}) \xrightarrow{s^i} C^{n-1}(X; \mathcal{M})) \right),$$

where $s^i$ are the codegeneracy maps of the cosimplicial group $\mathcal{M}(X)$. If $[x]$ is the orbit of a simplex in $X$ and $s_i$ is a degeneracy operator of $X$, it is easy to check that $s_i: [x] \to [s_i x]$ is an isomorphism of transitive $G$-sets and in particular $\mathcal{M}([x]) = \mathcal{M}([s_i x])$. It easily follows that $NC^n(X; \mathcal{M}) = \prod_{[x] \subseteq X} \mathcal{M}([x])$ where $[x]$ runs through the orbits of the non-degenerate $n$-simplices of $X$.

It is well known that the inclusion of $NC^*(X; \mathcal{M})$ in $C^*(X; \mathcal{M})$ is a homology equivalence. See [15, 8.3].

Recall that the Borel construction of a $G$-space $U$ is $U \times_G EG$ where $EG$ is a contractible space on which $G$ acts freely. If $U = G/K$ then $U \times_G EG = BK$ is the classifying space of $K$.

**Definition 3.1.** Fix a finite group $G$, an abelian group $A$ and an integer $n \geq 0$. Define a coefficient functor $\mathcal{H}^n$ for $G$ by $\mathcal{H}^n(U) = H^n(U \times_G EG; A)$. Observe that $\mathcal{H}^n(G/K) = H^n(K; A)$ where $A$ has the trivial action of $K$.

**Definition 3.2.** Let $\mathcal{C}$ be a finite $G$-poset and let $\mathcal{M}$ be a coefficient system. The underlying set of every object $[c]$ of $[S(\mathcal{C})]$ is a transitive $G$-set and we define a functor $\mathcal{A}_{\mathcal{M}}: [S(\mathcal{C})] \to \text{Ab}$ by

$$\mathcal{A}_{\mathcal{M}}([c]) = \mathcal{M}([c]).$$

If $[c']$ is a face of $[c]$, we define $\mathcal{A}_{\mathcal{M}}([c'] \to [c])$ by applying $\mathcal{M}$ to the map $[c] \to [c']$ of transitive $G$-sets.

By inspection of Definition 2.1, $C^*(\mathcal{A}_{\mathcal{M}}) \cong NC^*_G([\mathcal{C}]; \mathcal{M})$ and the next result follows from Lemma 2.2. It has been observed by Skolinska [12, p. 116] and by others e.g., Grodal in [6, Theorem 7.3], Linckelmann [10, Proposition 3.5] and Dwyer in [5].

**Lemma 3.3.** Let $\mathcal{C}$ be a finite $G$-poset and let $\mathcal{M}$ be a coefficient functor for $G$. With the notation of Definition 3.2, $H^*([S(\mathcal{C})]; \mathcal{A}_{\mathcal{M}}) \cong H^*_G([\mathcal{C}]; \mathcal{M})$.

**4. Proof of Theorem 1.1**

Set $G = \Sigma_p^2$ and let $\mathcal{C}$ denote the poset $\Delta_p(G)$ of the non-trivial $p$-subgroups of $G$. First we observe that $\text{Hom}(K, A) = H^1(K; A)$ for any finite group $K$ and any abelian group $A$. Thus, the functor $N_G: [S(\mathcal{C})] \to \text{Ab}$ defined in the introduction is
canonically isomorphic to $A_{p^3}$ as defined in 3.2 and in 3.1 with $A = k^\times$ where $k$ is an algebraically closed field of characteristic $p$. In light of Lemma 3.3 we need to prove that $H^1_G(\{\Delta_p(G); \mathcal{H}^1\}) \neq 0$. Consider the $G$-subposet $B_p(G)$ of the non-trivial radical $p$-subgroups of $G$, namely the non-trivial $p$-subgroup $P \leq G$ such that $N_G(P)/P$ contains no non-trivial normal $p$-subgroup. It is well known that the inclusion $|B_p(G)| \subseteq |\Delta_p(G)|$ is a $G$-homotopy equivalence, see e.g., [2, Proposition 6.6.5]. Therefore, it remains to prove that

$$H^1_G(|B_p(G); \mathcal{H}^1|) \neq 0. \tag{1}$$

The radical $p$-subgroups of the symmetric groups were classified by Alperin and Fong in [1]. In $G = \Sigma_{p^2}$ they form the following conjugacy classes:

(R1) The conjugacy class of the Sylow $p$-subgroup $V_{1,1} \overset{\text{def}}{=} C_p \triangleleft C_p \leq \Sigma_{p^2}$. Its normalizer is $V_{1,1} \rtimes (\text{GL}_1(p) \times \text{GL}_1(p))$ with the diagonal action of $\text{GL}_1(p)$ on the base group $(C_p)^p$ and the usual action of the second $\text{GL}_1(p)$ on $C_p$ at the top.

(R2) The conjugacy class of the subgroup $V_2 = C_p \times C_p$ embedded in $\Sigma_{p^2}$ via its action on itself by translation. Its normalizer is $V_2 \rtimes \text{GL}_2(p)$.

(R3) For every $k = 1, \ldots, p$ the conjugacy class of the subgroup $V_1 \times_k$ which is isomorphic to $C_p \times_k$ as a subgroup of $\Sigma_{p^2} \triangleleft \Sigma_{p^2}$. The normalizer of $V_1 \times_k$ is

$$\left((V_1 \rtimes \text{GL}_1(p)) \triangleright \Sigma_k\right) \times \Sigma_{p(p-k)}.$$

**Definition 4.1.** Consider the following subposets of $B_p(G)$.

1. Let $D_1$ be the subposet consisting of the conjugacy class of $V_{1,1}$ and the conjugacy classes of $V_1, V_1 \times 2, \ldots, V_1 \times p$.
2. Let $D_2$ be the subposet consisting of the conjugacy classes of $V_1, V_1 \times 2, \ldots, V_1 \times p$.
3. Let $D_2$ be the subposet consisting of the conjugacy classes of $V_1$ and $V_2$.
4. Let $D_4$ be the subposet consisting of the conjugacy class of $V_{1,1}$.

Observe that $V_2$ is a transitive subgroup of $\Sigma_{p^2}$ so it cannot be conjugate to a subgroup of $V_1 \times_k$ whose orbits have cardinality $p$. Also, $V_2$ acts freely so it cannot contain a conjugate of $V_1 \times_k$ since the latter do not act freely on the underlying set of $p^2$ elements. We see that up to conjugacy $B_p(G)$ has the form

$$[V_1] < [V_1 \times 2] < \ldots < [V_1 \times p] < [V_{1,1}] > [V_2]$$

and it follows that

$$|B_p(G)| = |D_1| \cup |D_2|, \quad \text{and} \quad |D_3| = |D_1| \cap |D_2|. \tag{2}$$

The Mayer Vietoris sequence gives an exact sequence

$$\cdots \rightarrow H^0_G(|D_3|; \mathcal{H}^1) \oplus H^0_G(|D_2|; \mathcal{H}^1) \rightarrow H^0_G(|D_3|; \mathcal{H}^1) \rightarrow H^1_G(|B_p(G)|; \mathcal{H}^1) \rightarrow \cdots \tag{3}$$

For what follows, it will be convenient to denote

$$L = \text{Hom}(\text{GL}_1(p), k^\times) \cong \mathbb{F}_p^\times.$$

**Proposition 4.2.** $H^0_G(|D_3|; \mathcal{H}^1) \cong L \times L$ and $H^1_G(|D_3|; \mathcal{H}^1) = 0.$
Proposition 4.3. \(H^0_G([D_2]; H^1) \cong L\) and \(H^*_G([D_2]; H^1) = 0\).

Proposition 4.4. \(H^0_G([D_1]; H^1) \cong C_2\) and \(H^*_G([D_1]; H^1) = 0\).

Propositions 4.2–4.4 together with the exact sequence (3) immediately imply (1) because by hypothesis \(p \geq 5\), whence \(|L| \geq 4\).

Recall that \(k\) has characteristic \(p\). Therefore the kernel of any group homomorphism \(H \to k^\times\) contains the commutator subgroup of \(H\) and any \(p\)-subgroup of \(H\). We will use this fact repeatedly.

Proof of Proposition 4.2. Since \(D_3\) is a single orbit of \(G\) with isotropy group \(N_G(V_{1,1})\) it follows from (R1) that \(H^*_G([D_3]; H^1) = \text{Hom}(N_G(V_{1,1}), k^\times) = L \times L\).

Proof of Proposition 4.3. Since \([B_p(G)]\) is \(G\)-equivalent to \([\Delta_p(G)]\), Symond’s resolution of Webb’s conjecture in [14] shows that the orbit space \([B_p(G)]/G\) is contractible. But (2) shows that \([B_p(G)]/G = ([D_1]/G) \vee ([D_2]/G)\). It follows that the CW-complex \([D_2]/G\), namely \([S(D_2)]\), is contractible and since it is 1-dimensional with two 0-simplices \([V_2]\) and \([V_{1,1}]\), the poset \([S(D_2)]\) must have the form

\[
[V_2] \to [V_2 < V_{1,1}] \leftarrow [V_{1,1}].
\]

Now, \(V_2 \leq V_{1,1} = C_p \lhd C_p\) is generated by the copy of \(C_p\) at the top and the diagonal copy of \(C_p\) in the base group \(C_p \times \cdots \times C_p\) which is the centre of \(V_{1,1}\). One easily deduces from (R1) and (R2) that \(N_G(V_2 < V_{1,1})/N_{V_{1,1}}(V_2) \cong GL_1(p)^2\) as a diagonal subgroup of \(GL_2(p)\). With the notation of Definition 3.2 we have

\[
\mathcal{A}_{\mathcal{H}_1}([V_2 < V_{1,1}]) \cong \text{Hom}(GL_1(p)^2, k^\times) \cong \mathcal{A}_{\mathcal{H}_1}([V_{1,1}]),
\]

and \(\mathcal{A}_{\mathcal{H}_1}([V_2]) = \text{Hom}(GL_2(p), k^\times) \cong L\) because \(GL_2(p)_{ab} = F^*_p\). By Lemma 3.3, the groups \(H^*_G([D_2]; H^1)\) are isomorphic to \(H^*([S(D_2)]; \mathcal{A}_{\mathcal{H}_1})\), namely to the derived functors of the diagram \(L \xrightarrow{\Delta} L \times L \xleftarrow{id} L \times L\). This completes the proof.

Lemma 4.5. The inclusion \(V_1 \subseteq D_1\), see Definition 4.1, induces a \(G\)-equivariant homotopy equivalence \(|V_1| \to |D_1|\).

Proof. Given a subgroup \(P\) of \(G\) let \(\delta_1(P)\) denote the subgroup of \(P\) generated by all the permutations \(g \in P\) whose support contains at most \(p\) elements. Observe that \(\delta_1\) is invariant under conjugation, namely \(\delta_1(gPg^{-1}) = g\delta_1(P)g^{-1}\). By inspection \(\delta_1(V_{1,1}^{\times k}) = V_{1,1}^{\times k}\) and \(\delta_1(V_{1,1}) = V_{1,1}^{\times p}\). We obtain a \(G\)-equivariant morphism of posets \(\delta_1 : D_1 \to V_1\). Clearly, \(|\delta_1| \circ |D_1|_{V_1} = \text{Id}_{|V_1|}\). The inclusions \(\delta_1(P) \leq P\) give a \(G\)-equivariant homotopy \(|\delta_1| \circ |D_1|_{V_1} \simeq \text{Id}_{|D_1|}\), cf. [11, 1.3]. The result follows.

We leave the following result as an easy exercise for the reader.

Lemma 4.6. Let \(K\) be a finite group, fix an integer \(n \geq 1\) and set \(G_n = K \wr \Sigma_n\). Then \((G_n)_{ab} \cong K_{ab} \times (\Sigma_n)_{ab}\). The restriction of \(G_n \to (G_n)_{ab}\) to any one of the factors \(K\) of \(K^n \leq G_n\) is the canonical projection \(K \to K_{ab}\) and the restriction of \(G_n \to (G_n)_{ab}\) to \(\Sigma_n\) is the projection onto \((\Sigma_n)_{ab}\).

If \(n, m \geq 1\) then \(G_n \times G_m \leq G_{n+m}\). The resulting \((G_n)_{ab} \times (G_m)_{ab} \to (G_{n+m})_{ab}\) is induced by the fold map \(K_{ab} \times K_{ab} \to K_{ab}\) and by \((\Sigma_n)_{ab} \times (\Sigma_m)_{ab} \to (\Sigma_{n+m})_{ab}\).
Notation 4.7. The following non-standard description of the \((n - 1)\)-simplex \(\Delta^{n-1}\) will be used throughout. The \(r\)-simplices of \(\Delta^{n-1}\) are sequences \(i_0 < \cdots < i_r\) where \(1 \leq i_0, \ldots, i_r \leq n\). Face maps are obtained by inclusion of sequences. (The usual convention is \(0 \leq i_0, \ldots, i_r \leq n - 1\).)

Proof of Proposition 4.4. In light of Lemma 4.5 and Lemma 3.3, we must prove that \(H^*_G([S(V_1)]; \mathcal{A}_{H^1}) \cong C_2\).

The high transitivity of the symmetric groups and the description of \(N_G(V_1^{\times k})\) in (R3) imply that every \(r\)-simplex of \(S(V_1)\) is conjugate in \(G\) to a simplex of the form \(V_1^{\times i_0} \cdots V_1^{\times i_r}\) where \(1 \leq i_0 < \cdots < i_r \leq p\). With the notation of 4.7 we see that \([S(V_1)] = \Delta^{p-1}\).

For any group \(K\) let \(\hat{K}\) denote the abelian group \(\text{Hom}(K, k^\times)\). Let \(N\) denote the normalizer of \(C_p\) in \(\Sigma_p\). Thus, \(N = C_p \rtimes \text{GL}_1(p)\) and observe that \(\text{GL}_1(p) \leq \Sigma_p\) is generated by an odd permutation, in fact a cycle of even length \((p\) is odd). Set

\[
L = \hat{N} = \text{Hom}(N, k^\times) = \text{Hom}(\text{GL}_1(p), k^\times) \cong C_{p-1}.
\]

Consider the following functor \(\Phi: \text{(\(\Delta^{p-1}\))}^{\text{op}} \to \text{Groups}\). On objects

\[
\Phi(i_0 < \cdots < i_r) = \left( \prod_{t=0}^r N \wr \Sigma_{i_t-i_{t-1}} \right) \times \Sigma_{p^2-i_r, p}, \quad \text{(by convention)} \quad i_{-1} = 0.
\]

For an \(r\)-simplex \(i\) and for \(0 \leq j \leq r\), the effect of \(\Phi(i) \to \Psi(\partial_j i)\) is induced by the inclusions

\[
\begin{align*}
(N \wr \Sigma_{i_{j+1}-i_j}) \times (N \wr \Sigma_{i_{j+1}-i_j}) &\leq (N \wr \Sigma_{i_{j+1}-i_j}) \quad \text{if } 0 \leq j < r \\
(N \wr \Sigma_{i_r-i_{r-1}}) \times \Sigma_{p(p-i_r)} &\leq \Sigma_{p(p-i_r)} \quad \text{if } j = r.
\end{align*}
\]

Inspection of (R3) shows that \(A_{\Phi} = \hat{\Phi}\), namely \(A_{\Phi} = \text{Hom}(\Phi, k^\times)\). Having identified \([S(V_1)]\) with \(\Delta^{p-1}\), it remains to prove that

\[
H^*(\Delta^{p-1}; \hat{\Phi}) \cong C_2. \quad (4)
\]

Consider the following functor \(\Psi: \text{\Delta}^{p-1} \to \text{Ab}\) defined by

\[
\Psi(i_0 < \cdots < i_r) = \left( \prod_{t=0}^r N \wr \Sigma_{i_t-i_{t-1}} \right) \times (N \wr \Sigma_{p^2-i_r}), \quad \text{(by convention)} \quad i_{-1} = 0.
\]

It is a subfunctor of \(\Phi\) via the inclusions \(N \wr \Sigma_{p^2-i_r} \leq \Sigma_{p(p-i_r)}\). We obtain a morphism of functors \(\hat{\Phi} \to \hat{\Psi}\) of abelian groups. Our goal now is to prove that it is a monomorphism and to calculate its cokernel. Fix an \(r\)-simplex \(i = (i_0 < \cdots < i_r)\) in \(\Delta^{p-1}\) and consider \(\hat{\Phi}(i) \to \hat{\Psi}(i)\). Note that \((\Sigma_n)_{ab} = C_2\) if \(n \geq 2\) and that if \(H \leq \Sigma_n\) contains an odd permutation then \(H_{ab} \to (\Sigma_n)_{ab}\) is surjective.

Case (a). If \(i_r = p\) then \(\Sigma_{p^2-i_r} = N \wr \Sigma_{p^2-i_r}\) are the trivial group and therefore \(\hat{\Phi}(i) \to \hat{\Psi}(i)\) is an isomorphism.

Case (b). If \(i_r = p - 1\) then \(N \wr \Sigma_{p^2-i_r} = N\). Since \(N = C_p \leq C_{p-1}\) contains an odd permutation, by applying \(\text{Hom}(\Sigma, k^\times)\) to the inclusion \(N \leq \Sigma_p\) we obtain the monomorphism \(C_2 \to L\) and therefore \(\hat{\Phi}(i) \to \hat{\Psi}(i)\) is injective with cokernel \(L/C_2\).

Case (c). Assume that \(i_r \leq p - 2\). The inclusion of \(N^{p-i_r} \leq \Sigma_{p(p-i_r)}\) contains odd permutations. Since \(p\) is odd, also the diagonal inclusion \(\Sigma_{p^2-i_r} \leq \Sigma_{p(p-i_r)}\) contains
odd permutations. By Lemma 4.6 the induced map \( \Sigma_{p-i_r} \to N^r \Sigma_{p-i_r} \) is the diagonal inclusion \( C_2 \to L \oplus C_2 \) into \( C_2 \oplus C_2 \). It follows that \( \Phi(i) \to \Psi(i) \) is injective with cokernel \( L \).

We obtain a short exact sequence of functors \( \Delta^{p-1} \to \text{Ab} \)
\[
0 \to \hat{\Phi} \to \hat{\Psi} \to \Gamma \to 0,
\]
where the functor \( \Gamma \) has the form
\[
\Gamma(i) = \begin{cases} 
0 & \text{if } i_r = p \\
L/C_2 & \text{if } i_r = p - 1 \\
L & \text{if } i_r \leq p - 2.
\end{cases}
\]

By Lemma 4.6, \( \Gamma(j) \to \Gamma(i) \) are induced by the quotient maps \( L \to L/C_2 \to 0 \).

Let \( \Gamma', \Gamma'' \colon \Delta^{p-1} \to \text{Ab} \) be the functors defined for objects \( i = (i_0 < \cdots < i_r) \) by
\[
\Gamma'(i) = \begin{cases} 
L & \text{if } 1 \leq i_r \leq p - 1 \\
0 & \text{if } i_r = p
\end{cases}
\]
\[
\Gamma''(i) = \begin{cases} 
C_2 & \text{if } i_r = p - 1 \\
0 & \text{if } i_r \neq p - 1.
\end{cases}
\]

Face maps \( i \subseteq j \) induce either the identity or the trivial homomorphisms \( \Gamma'(i) \to \Gamma'(j) \) and \( \Gamma''(i) \to \Gamma''(j) \). We get a short exact sequence of functors
\[
0 \to \Gamma'' \to \Gamma' \to \Gamma \to 0.
\]

We view \( \Delta^{p-2} \) as the \((p-1)\)st face of \( \Delta^{p-1} \), that is, \( \Delta^{p-2} \) consist of the simplices \( i = (i_0 < \cdots < i_r) \) of \( \Delta^{p-1} \) with \( i_r \leq p - 1 \). Similarly \( \Delta^{p-3} \) is the \((p-2)\)nd face of \( \Delta^{p-2} \). Thus, \( \Delta^{p-3} \) is the subcomplex of \( \Delta^{p-1} \) of the simplices \( i \) with \( i_r \leq p - 2 \). At this point we should recall that \( p \geq 5 \).

By inspection of Definition 2.1 we see that \( C^*(\Gamma'') \) is isomorphic to the cochain complex \( C^*(\Delta^{p-2}, \Delta^{p-3}; C_2) \) of the relative simplicial complex \( (\Delta^{p-2}, \Delta^{p-3}) \). Since \( p \geq 5 \), the contractibility of the standard simplices and Lemma 2.2 imply that
\[
H^*(\Delta^{p-1}; \Gamma'') \cong H^*(\Delta^{p-2}, \Delta^{p-3}; C_2) = 0.
\]

The acyclicity of \( \Gamma'' \) now shows that \( \Gamma' \to \Gamma \) induces an isomorphism
\[
H^*(\Delta^{p-1}; \Gamma') \cong H^*(\Delta^{p-1}; \Gamma).
\]

By Lemma 4.6 we see that \( \hat{\Psi} \colon \Delta^{p-1} \to \text{Ab} \) has the following form
\[
\hat{\Psi}(i_0 < \cdots < i_r) = ( \prod_{i_{t}=0}^{r} L \times \Sigma_{i_{t+1}=i_{t}} ) \times \begin{cases} 
0 & \text{if } i_r = p \\
L \times \Sigma_{i_{t}=i_r} & \text{if } i_r < p.
\end{cases}
\]

We obtain a constant subfunctor \( \Psi'(i) = L \) of \( \hat{\Psi} \) via the diagonal inclusion and it is easy to check that the following square commutes
\[
\begin{array}{ccc}
\Psi' & \longrightarrow & \hat{\Psi} \\
\downarrow & & \downarrow \\
\Gamma' & \longrightarrow & \Gamma.
\end{array}
\]

By inspection of Definition 2.1, there are isomorphisms \( C^*(\Psi') \cong C^*(\Delta^{p-1}; L) \) and \( C^*(\Gamma') \cong C^*(\Delta^{p-2}; L) \). The map \( \Psi' \to \Gamma' \) gives rise to the map of cochain complexes
induced by $\Delta^{p-2} \subseteq \Delta^{p-1}$. We deduce from Lemma 2.2 and the contractibility of the standard simplices that $\Psi' \rightarrow \Gamma'$ induces an isomorphism

$$H^*(\Delta^{p-1}; \Psi') \cong H^*(\Delta^{p-1}; \Gamma') \cong \begin{cases} L & \text{if } * = 0 \\ 0 & \text{if } * = 0. \end{cases}$$

(7)

The commutative square above, together with (6) and (7) imply that $\hat{\Psi} \rightarrow \Gamma$ induces an epimorphism $H^*(\Delta^{p-1}; \hat{\Psi}) \rightarrow H^*(\Delta^{p-1}; \Gamma)$. By (6) and (7) and the long exact sequence associated to (5), the proof of (4), whence the proof of this proposition, will be complete if we prove that $H^*(\Delta^{p-1}; \hat{\Psi}) \cong L \oplus C_2$ (cohomology concentrated in degree 0).

Set $K = N \wr \Sigma_p$ and let it act on the poset $\Omega$ of the non-empty subsets of $\{1, \ldots, p\}$ via the projection onto $\Sigma_p$. One easily checks that $[S(\Omega)] = \Delta^{p-1}$ and that, by choosing appropriate representatives, the isotropy groups of the $r$-simplices of $S(\Omega)$ are

$$\text{Iso}_K(i_0 < \cdots < i_r) = \Psi(i_0 < \cdots < i_r).$$

Thus, if $\mathcal{H}_K$ is the coefficient functor for $K$ defined in 3.1 with $A = k^x$, we see that $C^*(\hat{\Psi}) \cong C^*(\mathcal{A}_{\mathcal{H}_K})$, whence by Lemma 3.3,

$$H^*(\Delta^{p-1}; \hat{\Psi}) \cong H^*([S(\Omega)]; \mathcal{A}_{\mathcal{H}_K}) \cong H^*([\Omega]; \mathcal{H}_K).$$

Now, $|\Omega|$ is $K$-equivalent to a point because $\{1, \ldots, p\}$ is a maximal element of $\Omega$ fixed by $K$. Therefore $H^*_K([\Omega]; \mathcal{H}_K) \cong H^1_K(\mathcal{H}_K; \mathcal{H}_K)$.

This completes the proof. \qed

References


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