THE ISOMORPHISM BETWEEN MOTIVIC COHOMOLOGY AND $K$-GROUPS FOR EQUI-CHARACTERISTIC REGULAR LOCAL RINGS

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(communicated by J.F. Jardine)

Abstract
One of the well-known problems in algebraic $K$-theory is the comparison of higher Chow groups and $K$-groups. In this paper, using the motivic complex defined by Voevodsky–Suslin–Friedlander, we prove the comparison theorem for equi-characteristic regular local rings.

1. Introduction
Voevodsky–Suslin–Friedlander [8] defined the motivic cohomology $CH^r_{\text{Zar}}(X,n)$ by using equi-dimensional cycle groups $Z_{\text{equi}}(X \times \Delta^r \times A^r/X \times \Delta^r,0)$ for smooth noetherian schemes $X$ over a field and showed the contravariant functoriality for morphisms of schemes. Friedlander–Suslin [2] proved that $CH^r_{\text{Zar}}(X,n) = CH^r(X,n)$ for smooth quasi-projective schemes $X$ over a field, where $CH^r(X,n)$ is the higher Chow group of $X$ defined by Bloch [1]. For smooth quasi-projective schemes $X$ over a field, Bloch [1] proved that $\bigoplus_{r \geq 0} CH^r(X,n)$ coincides with the $n$-th algebraic $K$-group $K_n(X)$ after tensoring with $\mathbb{Q}$. We use the subscript $-\mathbb{Q}$ to mean $-\otimes_{\mathbb{Z}} \mathbb{Q}$.

In this paper, we consider the motivic cohomology groups $CH^r_{\text{Zar}}(X,n)$ for regular schemes by using an equi-dimensional cycle group [8] and prove that there is an isomorphism between the $K$-group $K_n(X)$ and the motivic cohomology group $CH^r_{\text{Zar}}(X,n)$ for the spectrum of an arbitrary regular local ring containing a field after tensoring with $\mathbb{Q}$.

Theorem 1.1. Let $R$ be a regular local ring containing a prime field. Then the cycle class map
$$\text{cl}^{(r)}: K_n(R)^{(r)}_{\mathbb{Q}} \to CH^r_{\text{Zar}}(R,n)_{\mathbb{Q}}$$
is an isomorphism for any $n,r \geq 0$, where $\text{cl}^{(r)}$ is the cycle-class map constructed in Section 3.1 and $K_n(R)^{(r)}_{\mathbb{Q}}$ is the eigenspace of the Adams operation $\Psi^k: K_n(R)_{\mathbb{Q}} \to K_n(R)_{\mathbb{Q}}$ with the eigenvalue $k^r$ for $k = 2, 3, \ldots$.

This theorem is proved by using Popescu’s result [6, Corollary 2.7] that says that any equi-characteristic regular local ring $R$ is a directed inductive limit of smooth
sub-algebras $R_{\alpha}$ over a field $F$. Since we may assume that $F$ is perfect $R = \lim_{\to} R_{\alpha}$ and $K_n(R) = \lim_{\to} K_n(R_{\alpha})$, we can reduce Theorem 1.1 to the case of a smooth $F$-algebra $R$. Then we have to prove that the functor $\text{CH}^r_{\text{Zar}}(-, n)_\mathbb{Q}$ commutes with directed inductive limits of algebras, and this is proved by Proposition 2.2.

Acknowledgements

I would like to express thanks to Professor Masaki Hanamura for his valuable advice. I wish to deeply thank the anonymous referee for very valuable suggestions and comments.

2. Motivic cohomology of equi-dimensional cycles

In this section, we always assume that all schemes are regular noetherian and separated. A morphism $p: X \to S$ of schemes of finite type is said to be *equi-dimensional of dimension $r$*, if $\dim p^{-1}(p(x)) = r$ for any $x \in X$ and any irreducible component of $X$ dominates an irreducible component of $S$. In particular, any equi-dimensional morphism of dimension zero is a quasi-finite morphism and dominates an irreducible component.

Let $\mathcal{Z}_{\text{equi}}(X/S, r)$ be the free abelian group generated by closed integral subschemes of $X$ which are equi-dimensional of dimension $r$ over $S$. We call $\mathcal{Z}_{\text{equi}}(X/S, r)$ the *equi-dimensional cycle group of dimension $r$*.

Let $X$ be an $S$-scheme of finite type. According to [8, Chapter 2, Theorem 3.3.1, Lemma 3.3.6 and Corollary 3.4.3], for any morphism of regular noetherian schemes $f: T \to S$, we have a homomorphism $f^*: \mathcal{Z}_{\text{equi}}(X/S, r) \to \mathcal{Z}_{\text{equi}}(X \times_S T/T, r)$ and $\mathcal{Z}_{\text{equi}}(X \times_S -/-, r)$ is a contravariant functor for morphisms of regular noetherian schemes. Furthermore, the functor $\mathcal{Z}_{\text{equi}}(X \times_S -/-, r)$ is an étale-sheaf [2, p. 816] on $S$, hence this is a Zariski-sheaf on $S$. We define the motivic cohomology $\text{CH}^r_{\text{Zar}}(X, n)$ for finite dimensional regular noetherian schemes $X$:

**Definition 2.1.** Let $X$ be a regular noetherian scheme of finite dimension. Write $\Delta^n = \text{Spec} \mathbb{Z}[t_0, \ldots, t_n]/(t_0 + \cdots + t_n - 1)$. Then $X \times \Delta^\bullet$ is a regular, noetherian cosimplicial scheme in the usual sense, and $\mathcal{Z}_{\text{equi}}(- \times \Delta^\bullet \times \mathbb{A}^r/ - \times \Delta^\bullet, 0)$ is a simplicial sheaf on $X$. We define the motivic cohomology to be the Zariski-hypercohomology:

$$\text{CH}^r_{\text{Zar}}(X, n) = \mathbb{H}^{-n}_{\text{Zar}}(X, \mathcal{Z}_{\text{equi}}(- \times \Delta^\bullet \times \mathbb{A}^r/ - \times \Delta^\bullet, 0)).$$

Let $(T_\alpha, f_{\alpha\beta})$ be a directed inverse system of smooth schemes over a regular noetherian scheme $S$ with a directed ordered index set $I$, where each transition map $f_{\alpha\beta}: T_\beta \to T_\alpha$ is affine and dominant ($\beta \geq \alpha$). Assume that $T = \lim_{\to} T_\alpha$ is regular and noetherian. Then we have the following:

**Proposition 2.2.** Let $X$ be a scheme of finite type over $T$ and assume that there exists a scheme $X_0$ of finite type over $S$ such that $X = X_0 \times_S T$. Then the canonical morphism of Zariski sheaves on $T$

$$\lim_{\to} f_\alpha^* \mathcal{Z}_{\text{equi}}(X_\alpha \times_{T_\alpha} -/-, 0)_{\mathbb{Q}} \to \mathcal{Z}_{\text{equi}}(X \times_T -/-, 0)_{\mathbb{Q}}$$

is an isomorphism, where $f_\alpha: X \to X_\alpha = X_0 \times_S T_\alpha$ denotes the canonical morphism
induced by $T \to T_\alpha$ and $f_\alpha^* \mathcal{Z}_{\text{equi}}(X_\alpha \times_{T_\alpha} -/\alpha, 0)_\mathbb{Q}$ is the inverse image of the Zariski sheaf $\mathcal{Z}_{\text{equi}}(X_\alpha \times_{T_\alpha} -/\alpha, 0)_\mathbb{Q}$ on $T_\alpha$.

Proof. Let $T_\alpha$ be the category of Zariski-open subschemes of $T_\alpha$. Note that the family of inverse images
\[
\{ f_\alpha^{-1}(U_\alpha) \mid U_\alpha \in T_\alpha, f_\alpha^{-1}(U_\alpha) = U_\beta \text{ for } \beta \geq \alpha, \alpha \in I \}
\]
is an open basis of $X \times_T U$. We prove that the canonical morphism
\[
\lim_{\beta \geq \alpha} \mathcal{Z}_{\text{equi}}(X_\beta \times_S U_\beta/U_\beta, 0)_\mathbb{Q} \to \mathcal{Z}_{\text{equi}}(X \times_S U/U, 0)_\mathbb{Q}
\]
is bijective. The injectivity is obvious. We prove its surjectivity. Let $[W] \in \mathcal{Z}_{\text{equi}}(X \times_T U/U, 0)_\mathbb{Q}$ be the cycle of an integral scheme $W \subset X \times_T U$. Since $W \to U$ is quasi-finite, there exists an index $\gamma$ and a closed integral subscheme $W_\gamma \subset X_\gamma \times_T U_\gamma$ such that $W = W_\gamma \times_U U$ and each $W' \times_T T_\gamma' \to U_\gamma \times_T T_\gamma$ is quasi-finite for $\gamma' \geq \gamma$ by [4, Theorem 8.10.5]. Since $W \to U$ and $U \to U_\gamma$ are dominant, $W_\gamma \to U_\gamma$ is dominant. Hence the cycle $[W_\gamma]$ is in $\mathcal{Z}_{\text{equi}}(X_\gamma/U_\gamma, 0)$. By [8, Chapter 2, Lemma 3.3.6], $f_\gamma[W_\gamma]$ is a formal linear combination of irreducible components of $W = W_\gamma \times_U U$. Furthermore, for a composition $Z \xrightarrow{g} Y \xrightarrow{f} X$ of morphisms of $X$-schemes, one has $(g \circ f)^* = f^* \circ g^*$ if $f^*$, $g^*$ and $(g \circ f)^*$ are defined.

3. The proof of main theorem

3.1. The cycle class maps

In this section, we assume that all schemes are noetherian and separated. Let $\mathcal{CP}(X)$ be the category of bounded complexes of big vector bundles on $X$. Let $F$ be a family of closed subschemes of $X$ and $\mathcal{CP}^F(X)$ the full subcategory of $\mathcal{CP}(X)$ consisting of complexes acyclic outside of $\bigcup_{W \in F} W$. We make $\mathcal{CP}^F(X)$ into a Waldhausen category by cofibrations and weak equivalences to be degree-wise split monomorphisms and quasi-isomorphisms, respectively. (See [7] and [9].)

Assume further that $f: Y \to X$ is a morphism of schemes and $F'$ is a family of closed subschemes of $Y$. The functor $f^*$ takes $\mathcal{CP}^F(X)$ to $\mathcal{CP}^{F'}(Y)$ provided that $f^{-1}(W) \subset \bigcup_{W' \in F} W'$ for all $W \in F$. Furthermore, for a composition $Z \xrightarrow{g} Y \xrightarrow{f} X$ of morphisms of $X$-schemes, one has $(g \circ f)^* = f^* \circ g^*$ if $f^*$, $g^*$ and $(g \circ f)^*$ are defined.

Let $S$ be a regular noetherian scheme. For any regular noetherian schemes $X$, $S \mathcal{CP}^{S,S}(X)$ denotes the Waldhausen’s $S$-construction (cf. [9]) of $\mathcal{CP}^{S,S}(X) := \mathcal{CP}(X \times_S S)/(X \times S)$ with the family of supports $\mathcal{Q}_X(X \times_S S)$ consisting of all closed subschemes quasi-finite over $X$. Further, $K_n^{S,S}(X)$ denotes the $n$-th $K$-group of $\mathcal{CP}^{S,S}(X)$.

For any abelian group $A$, $B_\bullet(A)$ denotes the classifying space of $A$. For any small category $\mathcal{C}$, $N_\bullet(\mathcal{C})$ denotes the nerve of $\mathcal{C}$. If $S = \kappa^r$, we define a map $\text{cl}_0: B_\bullet(K_0^{\kappa^r,N}(X)) \to B_\bullet(\mathcal{Z}_{\text{equi}}(X \times \kappa^r/X, 0))$ of simplicial sets by the formula
\[
\text{cl}_0(F) = \sum_{W \subset X \times \kappa^r} \text{length}(F_W)[W],
\]
where the sum is over all closed integral subschemes $W$ of $X \times \mathbb{A}^r$ which are quasi-finite and dominant over a component of $X$. We consider the composition

$$\cl' : N_\bullet wS_\bullet \mathcal{CP}^{\mathbb{A}^r}(X) \to B_\bullet (K_0^{\mathbb{A}^r}(X)) \xrightarrow{B_\bullet (\cl'_0)} B_\bullet (Z_{\text{equi}}(X \times \mathbb{A}^r/X, 0)),$$

where $wS_\bullet \mathcal{CP}^{\mathbb{A}^r}(X)$ is the subcategory of weak-equivalences in $S_\bullet \mathcal{CP}^{\mathbb{A}^r}(X)$, and $wS_\bullet \mathcal{CP}^{\mathbb{A}^r}(X) \to (K_0^{\mathbb{A}^r}(X))^n$ is the canonical map of bisimplicial sets. (See [7, Section 1].)

For any morphism $f : Y \to X$ of regular noetherian schemes, $f^* : K_0^{\mathbb{A}^r}(X) \to K_0^{\mathbb{A}^r}(Y)$ coincides with the map $F \mapsto \sum_{i>0} (-1)^i \mathbb{L}_if^*(F)$, where each $\mathbb{L}_if^*$ is the $i$-th left derived functor of the inverse image $f^*$. Using [8, Theorem 3.3.1 and Lemma 3.5.9], we have that the map $B_\bullet (\cl'_0)$ is functorial for any morphism of regular noetherian schemes by the direct calculation. Hence $\cl'$ is functorial for any regular noetherian schemes. In particular, $\cl'$ commutes with all coface maps and codegeneracy maps of the regular noetherian cosimplicial scheme $X \times \Delta^r$. Thus we obtain the map

$$\cl' : N_\bullet wS_\bullet \mathcal{CP}^{\mathbb{A}^r}(X \times \Delta^r) \to B_\bullet (K_0^{\mathbb{A}^r}(X \times \Delta^r)) \xrightarrow{B_\bullet (\cl'_0)} B_\bullet (Z_{\text{equi}}(X \times \Delta^r \times \mathbb{A}^r/X \times \Delta^r, 0))$$

called the cycle-class map. Here $B_\bullet (A_\bullet)$ is the classifying space of a simplicial abelian group $A_\bullet$, and $B_\bullet (A_\bullet)$ is a bisimplicial set.

### 3.2. Friedlander–Suslin’s spectral sequence

In this section, we consider the case where $X$ is smooth over a field $F$. Let $K_n^{\mathbb{A}^r}(X \times \Delta^r)$ denote the $n + 1$-th homotopy group of the diagonal of a 3-fold simplicial set $N_\bullet wS_\bullet \mathcal{CP}^{\mathbb{A}^r}(X \times \Delta^r)$. In the case that $X$ is a smooth scheme over a field, Friedlander–Suslin [2] proved that there exists a strongly convergent spectral sequence:

$$E_2^{pq} = \text{CH}_{\text{Zar}}^{-q}(X, -p - q) \Rightarrow K_{-p-q}(X)$$

by an exact couple $(D_2^{p,q}, E_2^{p,q}, i, j, k)$ defined by the following:

$$D_2^{p,q} = K_2^{\mathbb{A}^r,-q+1}(X \times \Delta^r), \quad E_2^{p,q} = \text{CH}_{\text{Zar}}^{-q}(X, -p - q),$$

where $j$ is the cycle-class map. (See [2, Section 13].) We have that Friedlander–Suslin’s spectral sequence admits Adams operations:

**Proposition 3.1** (cf. [3, Theorem 7]). Let $X$ be a smooth scheme over a field $F$. Then the spectral sequence

$$E_2^{pq} = \text{CH}_{\text{Zar}}^{-q}(X, -p - q) \Rightarrow K_{-p-q}(X)$$

admits Adams operations $\Psi^k$ with the following properties:

1. The $\Psi^k$ are natural in $\text{Sm}_F$.

2. The $\Psi^k : K_n^{\mathbb{A}^r}(X \times \Delta^r) \to K_n^{\mathbb{A}^r}(X \times \Delta^r)$ are compatible with the Adams operations $\Psi^k$ on $K_*(X) = K_*^{\mathbb{A}^0}(X)$.

3. On the $E_2$-term $\text{CH}_{\text{Zar}}^{-q}(X, -p - q)$, $\Psi^k$ acts by multiplication by $k^{-q}$.

**Proof.** The proof is similar to [3, Theorem 7].
Corollary 3.2. Let $X$ be a smooth scheme over a field $F$. The cycle-class map $\text{cl}^r : K^r_n(X \times \Delta^r) \to CH^r_{\text{Zar}}(X, n) \mathbb{Q}$ induces an isomorphism $\text{cl}^{(r)} : K^n_n(X) \mathbb{Q} \to CH^r_{\text{Zar}}(X, n) \mathbb{Q}$ for any $n, r \geq 0$.

3.3. The proof of Theorem 1.1

By Popescu’s result [6, Corollary 2.7], there exist a prime field $F$ and a directed inductive system $(R_\alpha, \psi_{\beta \alpha})$ of smooth $F$-algebras of $R$ such that its inductive limit is $R$. Since each $\psi_{\beta \alpha} : \text{Spec } R_\beta \to \text{Spec } R_\alpha$ is affine, we have that $\lim_{\longleftarrow} CH^r_{\text{Zar}}(R_\alpha, n) \mathbb{Q} = CH^r_{\text{Zar}}(R, n) \mathbb{Q}$ follows from [5, Theorem 5.7] and Proposition 2.2. By the functoriality of cycle-class maps and Corollary 3.2, we obtain

$$K^n_n(R) \mathbb{Q} = \lim_{\longleftarrow} K^n_n(R_\alpha) \mathbb{Q} = \lim_{\longleftarrow} CH^r_{\text{Zar}}(R_\alpha, n) \mathbb{Q} = CH^r_{\text{Zar}}(R, n) \mathbb{Q}. $$

References


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