SECONDARY MULTIPLICATION IN TATE COHOMOLOGY OF GENERALIZED QUATERNION GROUPS

MARTIN LANGER

(communicated by J. P. C. Greenlees)

Abstract
Let \( k \) be a field, and let \( G \) be a finite group. By a theorem of D. Benson, H. Krause, and S. Schwede, there is a canonical element in the Hochschild cohomology of the Tate cohomology \( \gamma_G \in HH^{3,1}H^*(G) \) with the following property: Given any graded \( H^*(G) \)-module \( X \), the image of \( \gamma_G \) in \( \text{Ext}^{3,1}_H(G)(X,X) \) is zero if and only if \( X \) is isomorphic to a direct summand of \( H^*(G,M) \) for some \( kG \)-module \( M \). In particular, if \( \gamma_G = 0 \) then every module is a direct summand of a realizable \( H^*(G) \)-module.

We prove that the converse of that last statement is not true by studying in detail the case of generalized quaternion groups. Suppose that \( k \) is a field of characteristic 2 and \( G \) is generalized quaternion of order \( 2^n \) with \( n \geq 3 \). We show that \( \gamma_G \) is non-trivial for all \( n \), but there is a \( H^*(G) \)-module detecting this non-triviality if and only if \( n = 3 \).

1. Introduction
Let \( k \) be a field, \( G \) a finite group, and let \( H^*(G) \) denote the graded Tate cohomology algebra of \( G \) over \( k \). The starting point of this paper is the following theorem of D. Benson, H. Krause, and S. Schwede:

**Theorem 1.1.** [2] There exists a canonical element in Hochschild cohomology of \( H^*(G) \)

\[ \gamma_G \in HH^{3,1}H^*(G), \]

such that for any graded \( H^*(G) \)-module \( X \), the following are equivalent:

(i) The image of \( \gamma_G \) in \( \text{Ext}^{3,1}_H(G)(X,X) \) is zero.

(ii) There exists a \( kG \)-module \( M \) such that \( X \) is a direct summand of the graded \( H^*(G) \)-module \( H^*(G,M) \).

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Let us call an \( \hat{H}^*(G) \)-module realizable if it is isomorphic to a module of the form \( \hat{H}^*(G, M) \) for some \( kG \)-module \( M \). As an immediate consequence we get the following.

**Corollary 1.2.** If \( \gamma_G = 0 \), then every \( \hat{H}^*(G) \)-module is a direct summand of a realizable module.

At this point it is natural to ask for the converse of that statement. That is, given the fact that \( G \neq 0 \), is there some \( \hat{H}^*(G) \)-module detecting the non-triviality of \( G \)?

Theorem 1.1 works more generally in the situation of differential graded algebras, and in that setup the converse of the corresponding corollary is known to be false: Benson, Krause, and Schwede provide an example of a dg algebra \( A \) such that the canonical class \( A \in \text{HH}^3,1(\hat{H}^*A) \) is non-trivial, but every \( \hat{H}^*A \)-module is realizable (see [2, Proposition 5.16]). Nevertheless, the author believes that the question whether there is such an example coming from Tate cohomology of groups is still open.

In this paper we will compute \( \gamma_G \) explicitly for the generalized quaternion groups \( G \). In what follows, let \( t \geq 2 \) be a power of 2, and let \( G = Q_{4t} \) be the group of generalized quaternions

\[
Q_{4t} = \langle g, h \mid g^t = h^2, ghg = h \rangle.
\]

Let \( k \) be a field of characteristic 2, and denote by \( L = kG \) the group algebra of \( G \) over \( k \). Then the Tate cohomology ring \( \hat{H}^*(G) \) is well known; it is given by

\[
\hat{H}^*(Q_{4t}) = \text{Ext} \bigg( \frac{k[x, y, s^{\pm 1}]}{(x^2 + y^2 = xy, y^3 = 0)} \bigg) \quad \text{if } t = 2,
\]

\[
\frac{k[x, y, s^{\pm 1}]}{(x^2 = xy, y^3 = 0)} \quad \text{if } t \geq 4,
\]

with degrees \( |x| = |y| = 1, |s| = 4 \) (see, e.g., [4, Chapter XII §11] and [1, IV Lemma 2.10]). Our main goal is to prove the following theorem.

**Theorem 1.3.** The element \( \gamma_{Q_8} \in \text{HH}^3,1(\hat{H}^*(Q_8)) \) is non-trivial, and the cokernel of the map

\[
\hat{H}^*(Q_8)[-1] \oplus \hat{H}^*(Q_8)[-1] \xrightarrow{(y x + y)} \hat{H}^*(Q_8) \oplus \hat{H}^*(Q_8)
\]

is a graded \( \hat{H}^*(Q_8) \)-module which is not a direct summand of a realizable one. For \( t \geq 4 \) the element \( \gamma_{Q_{4t}} \in \text{HH}^3,1(\hat{H}^*(Q_{4t})) \) is non-trivial, but every graded \( \hat{H}^*(Q_{4t}) \)-module is a direct summand of a realizable one.

The plan is as follows: In the first section we will briefly recall the definitions needed in Theorem 1.1; most of this part is taken from [2], and the reader interested in details should consult that source. In the second section we turn to the computation of a Hochschild cocycle \( m \) representing the canonical class \( \gamma_G \). In the third section we prove the statements about realizability of modules. Theorem 1.3 will then follow from Theorems 3.6, 3.8, 4.3, and Propositions 4.7 and 4.8.

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2. Prerequisites

2.1. Notation and conventions

All occurring modules will be right modules. We shall often work over a fixed ground field \( k \); then \( \otimes \) means tensor product over \( k \). Whenever convenient, we write \((a_1, a_2, \ldots, a_n)\) instead of \( a_1 \otimes a_2 \otimes \cdots \otimes a_n \). If \( G \) is a group, then \( k \) is often considered as a trivial \( kG \)-module.

Let \( R \) be a ring with unit, and let \( M \) be a \( \mathbb{Z} \)-graded \( R \)-module. The degree of every (homogeneous) element \( m \in M \) will be denoted by \([m]\). For every integer \( n \) the module \( M[n] \) is defined by \( M[n]^j = M^{n+j} \) for all \( j \). Given two such modules \( M \) and \( L \), a morphism \( f : L \to M \) is a family \( f^j : L^j \to M^j \) of \( R \)-module homomorphisms. The group of all these morphisms is denoted by \( \operatorname{Hom}_R(L, M) \). Furthermore, we have \( \operatorname{Hom}_R^m(L, M) = \operatorname{Hom}_R(L, M[m]) \), the morphisms of degree \( m \). The graded module \( L \otimes M \) is given by \((L \otimes M)^m = \bigoplus_{i+j=m} L^i \otimes M^j \). If \( M \) is a differential graded \( R \)-module with differential \( d \), then the differential of \( M[n] \) is given by \((-1)^n d \).

2.2. Tate Cohomology

Let us recall briefly the definition and basic properties of Tate cohomology. Let \( k \) be a field, and let \( G \) be a finite group. Then \( L = kG \) is a self-injective algebra (i.e., the classes of projective and injective right-\( L \)-modules coincide). For any \( L \)-module \( N \) we get a complete projective resolution \( P_* \) of \( N \) by splicing together a projective and an injective resolution of \( N \):

\[
\cdots \leftarrow P_{-2} \leftarrow P_{-1} \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots
\]

Given another \( L \)-module \( M \), we can apply the functor \( \operatorname{Hom}_L(-, M) \) to \( P_* \); then Tate cohomology is defined to be the cohomology groups of the resulting complex:

\[
\hat{\operatorname{Ext}}^n_L(N, M) = H^n(\operatorname{Hom}_L(P_*, M)) \quad \text{for all} \quad n \in \mathbb{Z}.
\]

For arbitrary \( L \)-modules \( X, Y, \) and \( Z \), we have a cup product

\[
\hat{\operatorname{Ext}}^m_L(X, Z) \otimes \hat{\operatorname{Ext}}^n_L(X, Y) \to \hat{\operatorname{Ext}}^{m+n}_L(X, Z);
\]

see, e.g., [3, \S 6]. Therefore, \( \hat{H}^*(G) = \hat{H}^*(G, k) = \hat{\operatorname{Ext}}^*_L(k, k) \) is a graded algebra, and \( \hat{H}^*(G, M) = \hat{\operatorname{Ext}}^*_L(k, M) \) is a graded \( \hat{H}^*(G) \)-module for every \( kG \)-module \( M \).

We call a graded \( H^*(G) \)-module \( X \) realizable if it is isomorphic to \( H^*(G, M) \) for some \( kG \)-module \( M \).

There is another way of describing the product of \( \hat{H}^*(G) \), in terms of \( P_* \). Consider the differential graded algebra \( \mathcal{A} = \operatorname{Hom}_L^*(P_*, P_*) \), which (in degree \( n \)) is given by

\[
\mathcal{A}^n = \prod_{j \in \mathbb{Z}} \operatorname{Hom}_L(P_{j+n}, P_j),
\]

and the differential \( d : \mathcal{A}^n \to \mathcal{A}^{n+1} \) is defined to be

\[
(df)_j = \partial \circ f_{j+1} - (-1)^n f_j \circ \partial.
\]

Here \( \partial \) denotes the differential of \( P_* \). \( \mathcal{A} \) is called the endomorphism dga of \( P \). With this definition, the cocycles of \( \mathcal{A} \) (of degree \( n \)) are exactly the chain transformations
$P[n] \to P$, and two cocycles differ by a coboundary if and only if they are chain homotopic. Using standard arguments from homological algebra, one shows that the following map is an isomorphism of $k$-vector spaces:

$$H^n A \overset{\cong}{\longrightarrow} \text{Ext}^n_L(k, k), \quad [f] \mapsto [\epsilon \circ f_0].$$

(1)

Here $\epsilon: P_0 \to k$ is the augmentation. This isomorphism is compatible with the multiplicative structures. We will often write $\bar{a}$ for elements of the endomorphism $dga$; if $\bar{a}$ is a cocycle, then $a$ denotes the corresponding cohomology class.

2.3. Hochschild Cohomology

We now give a short review of Hochschild cohomology. Let $\Lambda$ be a graded algebra over the field $k$, and let $M$ be a graded $\Lambda$-$\Lambda$-bimodule, the elements of $k$ acting symmetrically. Define a cochain complex $C^{s,t}(\Lambda; M)$ by

$$C^{n,m}(\Lambda; M) = \text{Hom}_k^m(\Lambda^n; M);$$

with a differential $\delta$ of bidegree $(1,0)$ given by

$$(\delta \varphi)(\lambda_1, \ldots, \lambda_{n+1}) = (-1)^{m+1} \varphi(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) + \sum_{i=1}^n (-1)^i \varphi(\lambda_1, \ldots, \lambda_i \lambda_{i+1}, \ldots, \lambda_{n+1}) + (-1)^{n+1} \varphi(\lambda_1, \ldots, \lambda_n) \lambda_{n+1}.$$

The Hochschild cohomology groups $HH^{s,t}(\Lambda; M)$ are defined as the cohomology groups of that complex:

$$HH^{s,t}(\Lambda; M) = H^s(C^{s,t}(\Lambda, M)).$$

In particular, we can regard $M = \Lambda$ as a bimodule over itself; we will then write $HH^{s,t}(\Lambda) = HH^{s,t}(\Lambda, \Lambda)$. For example, an element of $HH^{3,-1}(\Lambda)$ is represented by a family of $k$-linear maps

$$m = \{m_{i,j,l}: \Lambda^i \otimes \Lambda^j \otimes \Lambda^l \to \Lambda^{i+j+l-1}\}_{i,j,l \in \mathbb{Z}}$$

satisfying the cocycle relation

$$(-1)^{|a|} a \cdot m(b, c, d) - m(ab, c, d) + m(a, bc, d) - m(a, b, cd) + m(a, b, c) \cdot d = 0$$

for all $a, b, c, d \in \Lambda$.

Whenever $X$ and $Y$ are $\Lambda$-$\Lambda$-bimodules, one has a cup product pairing

$$\cup: \text{Hom}_\Lambda(X, Y) \otimes HH^{s,t}(\Lambda, \Lambda) \to \text{Ext}^{s,t}_\Lambda(X, Y).$$

Here $\text{Ext}^{s,t}_\Lambda(X, Y)$ is defined to be $\text{Ext}^{s,t}_\Lambda(X, Y[\ell])$. In particular, we have the map

$$HH^{3,-1} \hat{H}^*(G) \to \text{Ext}^{3,-1}_{\hat{H}^*(G)}(X, X)$$

$$\phi \mapsto \text{id}_X \cup \phi$$

for every $\hat{H}^*(G)$-module $X$. This is the map occurring in Theorem 1.1.

2.4. The canonical element $\gamma$

We are now going to describe the construction of the element $\gamma$ mentioned in Theorem 1.1. More generally, we will construct an element $\gamma_A \in HH^{3,-1} H^* A$ for
every differential graded algebra $A$ over $k$; then we can take $A$ to be the endomorphism algebra of a complete projective resolution of $k$ as a trivial $kG$-module to get $\gamma_G \in HH^{3,-1}(G)$.

For a dg-algebra $A$, consider $H^*A$ as a differential graded $k$-module with trivial differential. Then choose a morphism of dg-$k$-modules $f_1: H^*A \to A$ of degree 0 which induces the identity in cohomology. This is the same as choosing a representative in $A$ for every class in $H^*A$ in a $k$-linear way. For every two elements $x, y \in H^*A$, $f_1(xy) - f_1(x)f_1(y)$ is null-homotopic; therefore, we can choose a morphism of graded modules

$$f_2: H^*A \otimes H^*A \to A$$

of degree $-1$ such that for all $x, y \in H^*A$, we have

$$df_2(x, y) = f_1(xy) - f_1(x)f_1(y).$$

Then for all $a, b, c \in H^*A$,

$$f_2(a, b)f_1(c) - f_2(a, bc) + f_2(ab, c) - (-1)^{|a|}f_1(a)f_2(b, c)$$

is a cocycle in $A$, the cohomology class of which will be denoted by $m(a, b, c)$. This defines a map $m: (H^*A)^{\otimes 3} \to H^*A$ of degree $-1$. An explicit computation shows that $m$ is a Hochschild cocycle, thereby representing a class $\gamma_A \in HH^{3,-1}(H^*A)$. This class is independent of the choices made.

3. Computation of the canonical element

From now on, let $k$ be a field of characteristic 2. Let $t \geq 2$ be a power of 2, and let $G = Q_{4t}$ be the group of generalized quaternions

$$Q_{4t} = \langle g, h \mid g^2 = h^2, ghg = h \rangle.$$

We denote by $kG$ the group algebra of $G$ over $k$, and $F = kG$ denotes the free module of rank 1 over that algebra. In this section, we are going to explicitly compute a Hochschild cochain $m$ representing the canonical class $\gamma_G$.

3.1. The class of a map

We begin with an observation that will reduce the subsequent computations somewhat. Let us recall the construction of a representative of $\gamma_G$. First of all, we have to construct a projective resolution $P$, and we will actually find a minimal projective resolution. Then we have to choose a cycle selection homomorphism $f_1: \hat{H}^*(G) \to \text{Hom}_{kG}(P, P)$ such that any class $a$ is mapped to a representative $f_1(a)$. We can find a $k$-linear map $f_2: \hat{H}^*(G) \otimes \hat{H}^*(G) \to \text{Hom}_{kG}(P, P)$ of degree $-1$ satisfying $df_2(a, b) = f_1(a)f_1(b) - f_1(ab)$ for all $a, b$. Finally, we are interested in terms of the form

$$f_2(a, b)f_1(c) + f_2(a, bc) + f_2(ab, c) + f_1(a)f_2(b, c);$$

this is a cocycle in $\text{Hom}_{kG}(P, P)$. In order to determine the class of this cocycle, it is enough to know the degree 0 map of it (cf. (1)). This observation leads to the following definition.
Definition 3.1. For every $f \in \text{Hom}_{kG}^n(P, P)$, i.e., a family of maps $f_j : P_{j+n} \to P_j$ ($j \in \mathbb{Z}$), not necessarily commuting with the differential, we denote by $C(f)$ the class of the map $\epsilon \circ f_0 : P_n \to k$ in $H^n \text{Hom}_{kG}(P, k) = \hat{H}^n(G)$.

Note that the complex $\text{Hom}_{kG}(P, k)$ has trivial differential; thus, every element in $\text{Hom}_{kG}(P, k)$ and in particular $\epsilon \circ f_0$ is a cocycle. The definition above gives a map

$$C : \text{Hom}_{kG}^n(P, P) \to \hat{H}^n(G)$$

$$f \mapsto [\epsilon \circ f_0].$$

Proposition 3.2. The map $C$ has the following properties:

(i) If $f \in \text{Hom}_{kG}^n(P, P)$ is a cocycle, then $C(f)$ is the cohomology class of $f$; in particular, $C \circ f_1 = \text{id}$.

(ii) The map $C$ is $k$-linear.

(iii) If $C(f_1) = C(f_2)$ for some $f_1, f_2 \in \text{Hom}_{kG}^n(P, P)$, then $C(f_1 g) = C(f_2 g)$ for all $g \in \text{Hom}_{kG}^n(P, P)$.

(iv) If $a \in \text{Hom}_{kG}^n(P, P)$ is a cocycle and $f \in \text{Hom}_{kG}^n(P, P)$ is an arbitrary element, then $C(f a) = C(f) C(a)$.

Proof. (i) follows from (1). (ii) holds by definition. (iii): If $C(f_1) = 0$, then $\epsilon \circ f_1 = 0$. This implies $\epsilon \circ f_1 \circ g = 0$; hence $C(f_1 g) = 0$. For general $f_1, f_2$, note $C(f_1 - f_2) = 0$; by what we just proved, $C((f_1 - f_2) g) = 0$ and therefore $C(f_1 g) = C(f_2 g)$. (iv): Choose a cocycle $h \in \text{Hom}_{kG}^n(P, P)$ satisfying $C(h) = C(f)$. Then by (iii)

$$C(f a) = C(h a) = C(h) C(a) = C(f) C(a).$$

The following corollary will simplify computations later on.

Proposition 3.3. The map $f_2$ can be chosen in such a way that $C \circ f_2 = 0$.

Proof. Choose any $f_2$ (satisfying $d f_2(a,b) = f_1(a) f_1(b) - f_1(ab)$). Put $f_2 = \hat{f}_2 - f_1 \circ C \circ \hat{f}_2$. Since $df_1 = 0$, we get

$$d f_2(a,b) = df_2(a,b) = f_1(a) f_1(b) - f_1(ab),$$

and from $C \circ f_1 = \text{id}$, it follows that

$$C \circ f_2 = C \circ \hat{f}_2 - C \circ f_1 \circ C \circ \hat{f}_2 = 0.$$

Consider (2) with this simplified version of $f_2$. By applying $C$, we get the term

$$C(f_2(a,b) f_1(c)) + C(f_2(a,bc)) + C(f_2(ab,c)) + C(f_1(a)f_2(b,c)).$$

This is the cohomology class of (2). Note that the individual terms $f_2(a,b) f_1(c)$, $f_2(a,bc) \ldots$ will not be cocycles in general, but the map $C$ assigns cohomology classes to them in such a way that the sum will be the class we are looking for.

By our choice of $f_2$ (such that $C \circ f_2 = 0$), the first three terms in the sum vanish (note that $C(f_2(a,b) f_1(c)) = C(f_2(a,b)c)$ by Proposition 3.2 (iv)). Thus we are interested in terms of the form $C(f_1(a)f_2(b,c))$, where $a, b, c$ run through all elements of a $k$-basis of $\hat{H}^*(G)$. 


3.2. Generating cocycles and homotopies

Now we start the actual computation of $\gamma$. We begin with the construction of a minimal projective resolution $P$ and some cocycles in the endomorphism $dga$ of $P$. Let us define some elements of the group algebra $kG$ as follows. Put $a = g + 1$, $b = h + 1$ and $c = hg + 1$. Furthermore, we write $N = \sum_{j \in G} j$ for the norm element. Here are some formulae we will frequently use:

\[
\begin{align*}
& a^t = b^2 = c^2, & a^{2t} = b^4 = 0, \\
& ba = ac = a + b + c, & N = a^{2t-1}b, \\
& c = a + by, & gc = a + b, \\
& N = ca^{2t-2}b = ca^{2t-1}, & N = a^{2t-1} + a^{2t-2}b + ca^{2t-2}, \\
& ca^{t-1}b = ca^{t-1} + a^{t-1}b.
\end{align*}
\]

Also note that $a^{2t-1}$, $a^{2t-2}$, and $a^{2t-4}$ lie in the center of $kQ_4$. Now a 4-periodic complete projective resolution of the trivial $kG$-module $k$ is given as follows (see [4, Chapter XII §7]):

\[
\cdots \to P_0 = F \xleftarrow{(a,b)} P_1 \xrightarrow{(a)} P_2 = F^2 \xleftarrow{(a^t c)} P_3 = F^2 \xrightarrow{(c)} P_4 = F \xleftarrow{(a)} \cdots
\]

Since the resolution is minimal, the differential of the complex $\text{Hom}_{kG}(P, k)$ vanishes; therefore, we immediately get the well-known additive structure of $H^*(G)$:

\[
\hat{H}^{4n}(G) \cong \hat{H}^{4n+3}(G) \cong k, \quad \hat{H}^{4n+1}(G) \cong \hat{H}^{4n+2}(G) \cong k^2.
\]

Let us write $\tilde{s}: P \to P[4]$ for the shift map, given by the identity map in every degree. This is an invertible cocycle; thus, multiplication by a suitable power of $s$ yields an isomorphism $\hat{H}^{4n+u}(G) \cong \hat{H}^u(G)$ for $u = 0, 1, 2, 3$ and $n \in \mathbb{Z}$. Now we are heading for explicit generators $x, y$ of $\hat{H}^1(G) \cong H^1 k_0G(P, P)$, which are represented by chain maps $\tilde{x}, \tilde{y}: P[1] \to P$. By construction, we have $P_1 = F^2$ and $P_0 = F$. We extend the two projections $P_1 \to P_0$ to chain transformations $P[1] \to P$ as follows: For $\tilde{x}: P \to P[1]$ we take

\[
\begin{align*}
& \cdots \to F \xleftarrow{(a,b)} F^2 \xrightarrow{(a^t c)} F^2 \xrightarrow{(a)} F \xrightarrow{N} F \xleftarrow{\cdots} \\
& \downarrow_{a^{2t-2}b} \quad \downarrow_{(1\ 0)} \quad \downarrow_{(a^{t-2} \ 1)} \quad \downarrow_{(1)} \quad \downarrow_{a^{2t-2}b} \\
& \cdots \to F \xleftarrow{(a,b)} F \xrightarrow{(a^t c)} F^2 \xrightarrow{(a)} F \xleftarrow{\cdots} \\
& \end{align*}
\]

and extend this 4-periodically. The 4-periodic chain map $\tilde{y}: P \to P[1]$ is defined as follows:

\[
\begin{align*}
& \cdots \to F \xleftarrow{(a,b)} F^2 \xrightarrow{(a^t c)} F^2 \xrightarrow{(a)} F \xrightarrow{N} F \xleftarrow{\cdots} \\
& \downarrow_{a^{2t-1}} \quad \downarrow_{(0\ 1)} \quad \downarrow_{(0\ 1)} \quad \downarrow_{(0\ 1)} \quad \downarrow_{a^{2t-1}} \\
& \cdots \to F \xleftarrow{(a,b)} F \xrightarrow{(a^t c)} F^2 \xrightarrow{(a)} F \xleftarrow{\cdots} \\
& \end{align*}
\]
Since these cocycles are 4-periodic, they commute with \( \bar{s} \). Let us determine the pairwise products of these maps. We start with \( \bar{y}\bar{x} \):

\[
\begin{array}{c}
\cdots \longleftrightarrow F \leftrightarrow (a \ b) \leftrightarrow F^2 \leftrightarrow (a_{b}^{-1} \ c_{a}) \leftrightarrow F^2 \leftrightarrow (a_{c}) \leftrightarrow F \leftrightarrow N \leftrightarrow F \leftrightarrow \cdots \\
\cdots \longleftrightarrow F^2 \leftrightarrow (a_{c}) \leftrightarrow F \leftrightarrow N \leftrightarrow F \leftrightarrow (a \ b) \leftrightarrow F^2 \leftrightarrow (a_{b}^{-1} \ c_{a}) \leftrightarrow \cdots \\
\end{array}
\]

The product \( \bar{y}\bar{x} \) is given as follows:

\[
\begin{array}{c}
\cdots \longleftrightarrow F \leftrightarrow (a \ b) \leftrightarrow F^2 \leftrightarrow (a_{b}^{-1} \ c_{a}) \leftrightarrow F^2 \leftrightarrow (a_{c}) \leftrightarrow F \leftrightarrow N \leftrightarrow F \leftrightarrow \cdots \\
\cdots \longleftrightarrow F^2 \leftrightarrow (a_{c}) \leftrightarrow F \leftrightarrow N \leftrightarrow F \leftrightarrow (a \ b) \leftrightarrow F^2 \leftrightarrow (a_{b}^{-1} \ c_{a}) \leftrightarrow \cdots \\
\end{array}
\]

Next, we compute \( \bar{x}^2 \):

\[
\begin{array}{c}
\cdots \longleftrightarrow F \leftrightarrow (a \ b) \leftrightarrow F^2 \leftrightarrow (a_{b}^{-1} \ c_{a}) \leftrightarrow F^2 \leftrightarrow (a_{c}) \leftrightarrow F \leftrightarrow N \leftrightarrow F \leftrightarrow \cdots \\
\cdots \longleftrightarrow F^2 \leftrightarrow (a_{c}) \leftrightarrow F \leftrightarrow N \leftrightarrow F \leftrightarrow (a \ b) \leftrightarrow F^2 \leftrightarrow (a_{b}^{-1} \ c_{a}) \leftrightarrow \cdots \\
\end{array}
\]

And now \( \bar{y}^2 \):

\[
\begin{array}{c}
\cdots \longleftrightarrow F \leftrightarrow (a \ b) \leftrightarrow F^2 \leftrightarrow (a_{b}^{-1} \ c_{a}) \leftrightarrow F^2 \leftrightarrow (a_{c}) \leftrightarrow F \leftrightarrow N \leftrightarrow F \leftrightarrow \cdots \\
\cdots \longleftrightarrow F^2 \leftrightarrow (a_{c}) \leftrightarrow F \leftrightarrow N \leftrightarrow F \leftrightarrow (a \ b) \leftrightarrow F^2 \leftrightarrow (a_{b}^{-1} \ c_{a}) \leftrightarrow \cdots \\
\end{array}
\]

In each of these cocycles, the map \( P_2 \rightarrow P_0 \) determines the cohomology class by the isomorphism (1); in \( k^2 \), they correspond to \((0 \ 1), (0 \ 1), (\epsilon(a^{t-2}) \ 1), \) and \((1 \ 0)\), respectively. Hence, \( \hat{H}^2(G) \) is generated by \( x^2 \) and \( y^2 \), and we have \( xy = yx \). Furthermore, we also see from this description that

\[
xy = \begin{cases} 
  x^2 + y^2 & \text{if } t = 2, \\
  x^2 & \text{otherwise.}
\end{cases}
\]

But we will need explicit chain homotopies for all these relations later on, so let us start with the commutator relation \( xy = yx \). Let \( \bar{p} \) be the 4-periodic null-homotopy
for \(\bar{x}\bar{y} + \bar{y}\bar{x}\) defined as follows:

\[
\cdots \leftarrow F \xleftarrow{(a)} F^2 \xleftarrow{(a)} F \xrightarrow{(b)} F^2 \xrightarrow{(a)} F \xrightarrow{(b)} \cdots
\]

\[
\cdots \leftarrow F^2 \xleftarrow{(a)} F \xrightarrow{(b)} F \xrightarrow{(a)} F \xrightarrow{(b)} \cdots
\]

Now let us compute \(\bar{y}^3\):

\[
\cdots \leftarrow F \xleftarrow{(a)} F \xrightarrow{(a)} F^2 \xrightarrow{(a)} F \xrightarrow{(a)} \cdots
\]

\[
\cdots \leftarrow F \xrightarrow{(a)} F \xrightarrow{(a)} F \xrightarrow{(a)} F \xrightarrow{(a)} \cdots
\]

Then we find a null-homotopy for that map in two steps: First, consider the 4-periodic extension of the map

\[
\cdots \leftarrow F \xleftarrow{(a)} F^2 \xrightarrow{(a)} F \xrightarrow{(b)} F \xrightarrow{(a)} \cdots
\]

and call it \(\tilde{w}\). Note that this will not quite be a homotopy for \(\bar{y}^3\), because it yields the wrong result in degrees \(P_{4n+2} \to P_{4n-1}\) for all \(n \in \mathbb{Z}\). But if we put

\[P_{8n+j+3} \to P_{8n+j}: \bar{w}\bar{s}_{8n+j} = \begin{cases} 
\bar{w}^j_{8n+j} & \text{if } j = 0, 1, 2, 3, \\
(\bar{w}^j + \bar{y}^2)_{8n+j} & \text{if } j = 4, 5, 6, 7,
\end{cases}\]

then we get an 8-periodic null-homotopy for \(\bar{y}^3\) which will be called \(\tilde{w}\) and satisfies \(\bar{s}\bar{w} + \tilde{w}\bar{s} = \bar{y}^2\bar{s}\).

### 3.3. Computation for the quaternion group

Due to the different multiplicative relation in \(\tilde{H}^*(G)\), we need to consider the cases \(t = 2\) and \(t \geq 4\) separately. We start with \(t = 2\). In this case, the map

\[
\cdots \leftarrow F^2 \xleftarrow{(a)} F^2 \xrightarrow{(a)} F \xrightarrow{(b)} F \xrightarrow{(a)} F^2 \xrightarrow{(a)} \cdots
\]

\[
\cdots \leftarrow F \xleftarrow{(a)} F \xrightarrow{(b)} F \xrightarrow{(a)} F \xrightarrow{(b)} \cdots
\]

can be extended (as we did with \(\tilde{w}\) above) to an 8-periodic null-homotopy \(\tilde{r}\) for \(\bar{x}^2 + \bar{x}\bar{y} + \bar{y}^2\) satisfying \(\bar{s}\tilde{r} + \tilde{r}\bar{s} = (\bar{x} + \bar{y})\bar{s}\). Notice that \(\bar{x}\bar{y}^2: P_3 \to P_0\) is the identity
map, which implies that \( xy^2 \neq 0 \in \hat{H}^3(G) \). Gathering the results we obtained so far, we recover the known fact that
\[
\hat{H}^*(G) \cong k[x, y, s^\pm 1]/(x^2 + y^2 = xy, y^3 = 0).
\]

Let us remark here that all monomials in \( x \) and \( y \) of degree bigger than 3 vanish in this ring.

**Proposition 3.4.** Let \( \alpha, \beta, \gamma \) be monomials in the (non-commutative) variables \( \bar{x}, \bar{y} \), and assume that the degree \(|\beta| \geq 3\). Then we have the following formulae:
\[
\begin{align*}
C(\bar{p}\alpha) &= 0, & C(\bar{r}\alpha) &= 0, & C(\bar{w}\alpha) &= 0, \\
C(\bar{x}\bar{p}\alpha) &= xyC(\alpha), & C(\bar{y}\beta\alpha) &= 0, & C(\bar{\gamma}\bar{w}\alpha) &= 0, \\
C(\bar{y}\bar{p}\alpha) &= 0, & C(\bar{x}^2\bar{p}\alpha) &= x^2yC(\alpha), \\
C(\bar{y}^2\bar{p}\alpha) &= 0, & C(\bar{\beta}\bar{p}\alpha) &= 0.
\end{align*}
\]

**Proof.** By Proposition 3.2.(iii) we can assume that the degree of \( \beta \) is at most 3. Furthermore, we can assume \( \alpha = 1 \) by Proposition 3.2.(iv). In order to determine \( C(\bar{a}\bar{w}) \) for any given cocycle \( \bar{a} \) of degree \( n \), we consider the composition
\[
P_{n+2} \xrightarrow{\tilde{w}_n} P_n \xrightarrow{\tilde{a}_0} P_0 \xrightarrow{\epsilon} k
\]
as an element of \( H^{n+2} \text{Hom}_{kG}(P_0, k) \). Notice \( \text{im}(\tilde{w}_n) \subset \ker(\epsilon) \cdot P_n \). Therefore, \( \text{im}(a_0 \circ \tilde{w}_n) \subset \ker(\epsilon) \cdot P_0 = \ker(\epsilon) \), and hence \( \epsilon \circ a_0 \circ \tilde{w}_n = 0 \). The same proof works for \( \bar{r} \) instead of \( \bar{w} \), so we are left with \( \tilde{p} \). For \( C(\bar{x}\bar{p}) \), consider \( \bar{x}\bar{p} \) in degree 0; i.e.,
\[
P_2 \xrightarrow{\bar{\rho}_1} P_1 \xrightarrow{\bar{\delta}_0} P_0.
\]
This equals \( (0 1) : P_2 \longrightarrow P_0 \), which corresponds to \( xy \). The remaining cases can be shown analogously.

**Remark 3.5.** Using \( C \), we can prove that there is no 4-periodic null-homotopy for \( \bar{x}^2 + \bar{x}\bar{y} + \bar{y}^2 \) as follows: Suppose there is a 4-periodic null-homotopy; call it \( \bar{r} \). Since \( d(\bar{r} - \bar{r}) = 0 \), \( \bar{q} = \bar{r} - \bar{r} \) is a cocycle, representing some class \( q \). By construction, \( \bar{r} = (\bar{r} + \bar{x} + \bar{y})s \). Since \( \bar{r} \) is 4-periodic, we have \( C(\bar{s}\bar{q}) = C(\bar{q}s) - C((\bar{x} + \bar{y})s) = qs - (x + y)s \) by Proposition 3.2. On the other hand, \( C(\bar{s}\bar{q}) = sq \), and hence \( (x + y)s = 0 \), a contradiction. In a similar way, one shows that there is no 4-periodic null-homotopy for \( \bar{x}^3 \).

As a next step, we are going to define the functions \( f_1 \) and \( f_2 \). A \( k \)-basis of \( \hat{H}^*(G) \) is given by \( \mathcal{C} = \{ s^i, xs^i, ys^i, x^2s^i, y^2s^i, x^2ys^i | i \in \mathbb{Z} \} \). Define the \( k \)-linear map \( f_1 \) on the basis \( \mathcal{C} \) by
\[
f_1: \hat{H}^*(G) \to \text{Hom}_{kG}^*(P, P)
\]
\[
x^i \bar{y}^j \bar{s}^i \mapsto x^i \bar{y}^j \bar{s}^i
\]
for all \( i, \varepsilon, \delta \in \mathbb{Z} \) for which the expression on the left-hand side lies in \( \mathcal{C} \). Let us define the set \( \mathcal{B} = \{ 1, x, y, x^2, y^2, x^2y \} \). For all \( b, c \in \mathcal{B} \) and \( i, j, k \in \mathbb{Z} \), we have \( f_1(bs^ic^j \bar{s}^k) = \).
$f_1(bc)\tilde{s}^{i+j}$ and $f_1(bs^i)f_1(cs^j) = f_1(b)f_1(c)\tilde{s}^{i+j}$, since $\tilde{s}$ commutes with both $\tilde{x}$ and $\tilde{y}$. This implies that we can define $f_2$ on $B \times B$ and then extend it to $C \times C$ via $f_2(bs^i, cs^j) = f_2(b, c)\tilde{s}^{i+j}$. Now define $f_2$ on $B \times B$ as follows:

$$f_2(b, c) = \begin{array}{ccc}
1 & x & y \\
0 & 0 & 0 \\
0 & \tilde{p} + \tilde{r} & 0 \\
0 & \tilde{x}^2 + \tilde{y} + \tilde{w} & 0 \\
0 & \tilde{y} + \tilde{x} \tilde{y} + \tilde{w} + \tilde{w} + \tilde{x} \tilde{y} & 0 \\
0 & \tilde{x} \tilde{y} + \tilde{y}^2 + \tilde{x} \tilde{w} + \tilde{y} \tilde{w} & 0 \\
\end{array}$$

Direct verification shows that $dC \circ f_2 = 0$, as one can check using Proposition 3.4.

As a final step, we need to investigate the term

$$m(a, b, c) = C(f_1(a)f_2(b, c))$$

for all $a, b, c \in C$. Since $f_2(b, c)$ is 8-periodic, we have

$$m(as^{2h}, bs^i, cs^j) = m(a, b, c)s^{2h+i+j}$$

for all integers $h, i, j$ and $a, b, c \in C$. Therefore, it is enough to consider all triples $(a, b, c) \in (B \cup B) \times B \times B$.

Consider the case $a \in B$. If $a = 1$, then $C(f_1(a)f_2(b, c)) = C(f_2(b, c)) = 0$. If $a \in \{y^2, x^2y\}$, then $f_1(a)f_2(b, c)$ is a sum of terms $\beta \tilde{p}a, \beta \tilde{r}a, \beta \tilde{w}a$, and $\beta \tilde{x}\tilde{y}a$, where $\alpha$ and $\beta$ are monomials in $\tilde{x}$ and $\tilde{y}$, the degree of $\beta$ is at least 2, and $\beta \neq \tilde{x}^2$. Hence, $C(f_1(a)f_2(b, c)) = 0$ by Proposition 3.4.

Next, consider $a = x$. By Proposition 3.4 we get $C(\tilde{x}f_2(b, c))$ from $f_2(b, c)$ by the following rule: Put an $\tilde{x}$ in front of all monomials in $\tilde{x}$ and $\tilde{y}$. Then remove all summands containing $\tilde{p}, \tilde{r}$, or $\tilde{w}$, except those beginning with $\tilde{p}, \tilde{x}\tilde{p}$, or $\tilde{y}\tilde{p}$, where we replace the $\tilde{p}$ by $x\tilde{p}$ and $\tilde{x}\tilde{p}$ by $x^2y$. Finally, replace all $\tilde{x}$ and $\tilde{y}$ by $x$ and $y$, respectively. Using this procedure, we get the following table for $C(\tilde{x}f_2(b, c))$: 

$$\begin{array}{|c|c|c|}
\hline
x & y & z \\
\hline
1 & 0 & 0 \\
0 & \tilde{x}^2 + \tilde{y} + \tilde{w} & 0 \\
0 & \tilde{y}^2 + \tilde{x} \tilde{y} + \tilde{w} + \tilde{y} \tilde{w} & 0 \\
\hline
\end{array}$$
Here each \( \ast \) stands for some homogeneous polynomial in \( x, y \) of degree at least 4.

Almost all these expressions vanish, and the only remaining terms are

\[
m(x, y, x) = xy, 
m(x, y, x^2) = x^2y.
\]

For the case \( a = y \) we use a similar method resulting from Proposition 3.4, and we end up with \( m(y, b, c) = 0 \) for all \( b, c \in B \). Finally, for \( a = x^2 \) we find that the only non-zero term is \( m(x^2, y, x) = x^2y \).

The case \( a \in B_s \) is slightly more difficult. Consider the map

\[
h(b, c) = \bar{s}f_2(b, c)\bar{s}^{-1} - f_2(b, c),
\]

measuring how far away \( f_2 \) is from 4-periodicity. From the equations

\[
\bar{s}\bar{p}\bar{s}^{-1} = \bar{p}, 
\bar{s}\bar{r}\bar{s}^{-1} = \bar{r} + \bar{x} + \bar{y}, 
\bar{s}\bar{w}\bar{s}^{-1} = w + \bar{y}^2,
\]

we get the following table for \( h \):

\[
\begin{array}{c|cccccc}
\bar{h}(b, c) & 1 & x & y & x^2 & y^2 & x^2y \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & x+y & \bar{x}^2 & \bar{x}\bar{y} & \bar{x}^2\bar{y} \\
y & 0 & x+y & 0 & \bar{y}^2 & \bar{y}\bar{x} + \bar{x}\bar{y} & \bar{y}^2 \\
x^2 & 0 & \bar{x}^2 & 0 & \bar{x}^3 & 0 & * \\
y^2 & 0 & \bar{y}x & \bar{y}^2 & 0 & \bar{y}^3 & * \\
x^2y & 0 & \bar{x}^2\bar{y} & 0 & * & * & * \\
\end{array}
\]

where \( * \) denotes certain homogeneous polynomials in \( \bar{x} \) and \( \bar{y} \) of degree at least 4.

Applying \( C \) to this table and using relations in \( \check{H}^*(G) \), we get

\[
\begin{array}{c|cccccc}
\bar{C}(h(b, c)) & 1 & x & y & x^2 & y^2 & x^2y \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & x+y & x^2 & x^2 + y^2 & x^2y \\
y & 0 & x+y & 0 & 0 & y^2 & 0 \\
x^2 & 0 & x^2 & 0 & 0 & 0 & 0 \\
y^2 & 0 & x^2 + y^2 & y^2 & 0 & 0 & 0 \\
x^2y & 0 & x^2y & 0 & 0 & 0 & 0 \\
\end{array}
\]
By definition of $h$, we have $h(b, c)s = \tilde{s}f_2(b, c) - f_2(b, c)s$; hence
\[ C(h(b, c))s = C(\tilde{s}f_2(b, c)) - C(f_2(b, c))s = m(s, b, c). \]

Therefore, this table shows the values $m(s, b, c)$ with $b, c \in B$. On the other hand, we know that $m$ is a Hochschild cocycle; in particular, for all $a, b, c \in B$,
\[ a m(s, b, c) + m(as, b, c) + m(a, sb, c) + m(a, s, bc) + m(a, s, b)c = 0. \]

Using $m(a, s, b)c = m(a, 1, b)s = 0$, $m(a, s, bc) = m(a, 1, bc)s = 0$, and $m(a, sb, c) = m(a, b, c)s$, we get
\[ m(as, b, c) = a m(s, b, c) + m(a, b, c)s. \]

We know the right-hand side for all $a, b, c \in B$. Gathering all results, we get the following theorem.

**Theorem 3.6.** The canonical element $\gamma_G$ is represented by the Hochschild cocycle $m$ which is given by the formulae
\[
\begin{align*}
  m(x, y, x) &= xy, \\
m(x, y, x^2) &= x^2y, \\
m(x^2, y, x) &= x^2y, \\
m(a, b, c) &= 0 \quad \text{for all other } a, b, c \in B, \\
m(sa, b, c) &= sm(a, b, c) + saC(h(b, c)), \quad \text{where } C(h(b, c)) \text{ is given by (3),} \\
m(s^{2i}a, s^j b, s^l c) &= s^{2i+j+l}m(a, b, c).
\end{align*}
\]

The element $\gamma \in HH^{3, -1}_B(\hat{H}^*(G))$ represented by $m$ is non-trivial.

**Proof.** It remains to prove the non-triviality of $\gamma$. Assume $m = \delta g$ for some Hochschild $(2, -1)$-cochain $g$. Then,
\[ m(a, b, c) = (\delta g)(a, b, c) = a g(b, c) + g(ab, c) + g(a, bc) + g(a, b)c \]

for all $a, b, c$. In particular,
\[
\begin{align*}
0 &= m(y, x, y) = yg(x, y) + g(yx, y) + g(y, xy) + g(y, x)y, \\
0 &= m(x, y, y) = xg(y, y) + g(xy, y) + g(x, y^2) + g(x, y)y, \\
0 &= m(y, y, x) = yg(y, x) + g(y^2, x) + g(y, yx) + g(y, y)x, \\
0 &= m(x, x, x) = xg(x, x) + g(x^2, x) + g(x, x^2) + g(x, x)x, \\
x = m(x, y, x) = xg(y, x) + g(yx, x) + g(x, yx) + g(x, y)x.
\end{align*}
\]

Adding up these equations, we get (using $x^2 + y^2 = xy$)
\[ xy = x \cdot (g(x, y) + g(y, x)). \]

This implies $g(x, y) + g(y, x) = y$. On the other hand, interchanging the roles of $x$ and $y$ we get $g(x, y) + g(y, x) = x$, a contradiction. \qed
3.4. Computation for the generalized quaternion group

From now on, we assume that \( t \geq 4 \). Then there is an 8-periodic null-homotopy \( \bar{v} \) for \( \bar{x}^2 + \bar{x} \bar{y} \), partially given by

\[
\cdots \leftarrow F^2 \leftarrow F^2 \leftarrow F^2 \leftarrow F^2 \leftarrow F^2 \leftarrow F^2 \leftarrow \cdots
\]

\[
\cdots \leftarrow F \leftarrow F \leftarrow F \leftarrow F \leftarrow F \leftarrow F \leftarrow \cdots
\]

satisfying \( \bar{s} \bar{v} + \bar{v} \bar{s} = \bar{x} \). Here we write \( u = ca^{2t-2} + ba^{2t-3} \) and need to prove

\[
au = a^{2t-2}b + a^{2t-1}, \\
cu = a^{2t-2}b + a^{2t-1}, \\
ua = a^{2t-2}b + N, \\
ub = a^{2t-2}b.
\]

For instance, to prove the first formula, note that

\[
au + aca^{2t-2} = a^{2t-2}b + a^{2t-1} - a^{2t-2}b + a^{2t-1} = 0.
\]

The other formulae can be proved similarly.

Again one verifies that \( x^2y \neq 0 \), so that we recover the well-known structure of \( \hat{H}^*(G) \) to be

\[
\hat{H}^*(G) \cong k[x, y, s^{\pm 1}]/(y^3, x^2 + xy).
\]

Using the variable \( z = x + y \), we obtain the isomorphism

\[
\hat{H}^*(G) \cong k[x, z, s^{\pm 1}]/(xz, x^3 + z^3).
\]

In the following, we will frequently switch between these two descriptions.

**Proposition 3.7.** We have the following formulae:

\[
\begin{align*}
\mathcal{C}(p \alpha) &= 0, \\
\mathcal{C}(\bar{x} \alpha) &= 0, \\
\mathcal{C}(\bar{y} \alpha) &= 0, \\
\mathcal{C}(\bar{x}^2 \bar{p} \alpha) &= x^2 \mathcal{C}(\alpha), \\
\mathcal{C}(y \bar{p} \alpha) &= 0, \\
\mathcal{C}(\bar{y} \bar{p} \alpha) &= 0, \\
\mathcal{C}(\bar{y}^2 \bar{p} \alpha) &= 0,
\end{align*}
\]

for any \( \alpha, \beta, \gamma \) monomials in \( \bar{x}, \bar{y} \) with \( |\beta| \geq 3 \).

We omit the straightforward proof and turn to the definition of the maps \( f_1 \) and \( f_2 \). As before, let \( \mathcal{B} = \{1, x, y, x^2, y^2, x^2y\} \); we define \( f_1 \) as

\[
f_1(s^i x^a y^b) = \bar{s}^i \bar{x}^a \bar{y}^b
\]

for all \( a, b, i \in \mathbb{Z} \) for which \( x^a y^b \) lies in \( \mathcal{B} \). Now we define \( f_2 \) on \( \mathcal{B} \times \mathcal{B} \) as follows:
Also put $f_2(s_i a, s_j b) = f_2(a, b) s_i s_j$ for all $i, j \in \mathbb{Z}$ and $a, b \in B$. This function is chosen in such a way that $C(f_2(a, b)) = 0$ for all $a, b \in B$. One verifies that

$$m(x, y, x) = x^2,$$

$$m(x^2, y, x) = x^2 y,$$

$$m(x, y^2, x) = x^2 y,$$

and $m$ vanishes on all other triples $(a, b, c) \in B^3$. Let us define $m'$ as follows:

$$m'(s^i a, s^j b, s^k c) = s^i s^j s^k m(a, b, c)$$

for all $a, b, c \in B$,

and define $h(a, b) = s f_2(a, b) s^{-1} - f_2(a, b)$. Then $C(h(b, c))$ is given by the following table:

<table>
<thead>
<tr>
<th>$f_2(b, c)$</th>
<th>1</th>
<th>$x$</th>
<th>$y$</th>
<th>$x^2 y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\bar{v}$</td>
</tr>
<tr>
<td>$y$</td>
<td>0</td>
<td>$\bar{\bar{p}} + \bar{v}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$x^2$</td>
<td>0</td>
<td>$\bar{x} \bar{v}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$y^2$</td>
<td>0</td>
<td>$\bar{y} \bar{p} + \bar{p} \bar{y} + \bar{v} \bar{y}$</td>
<td>$\bar{w}$</td>
<td></td>
</tr>
<tr>
<td>$x^2 y$</td>
<td>0</td>
<td>$\bar{x}^2 \bar{\bar{p}} + \bar{x} \bar{v} \bar{y} + \bar{v} \bar{y}^2 + \bar{x} \bar{w} + \bar{x}^2 \bar{y}$</td>
<td>$\bar{y}^2 + \bar{x} \bar{w}$</td>
<td></td>
</tr>
</tbody>
</table>

Also put $f_2(s^i a, s^j b) = f_2(a, b) s^{i+j}$ for all $i, j \in \mathbb{Z}$ and $a, b \in B$. This function is chosen in such a way that $C(f_2(a, b)) = 0$ for all $a, b \in B$. One verifies that

$$m(x, y, x) = x^2,$$

$$m(x^2, y, x) = x^2 y,$$

and $m$ vanishes on all other triples $(a, b, c) \in B^3$. Let us define $m'$ as follows:

$$m'(s^i a, s^j b, s^k c) = s^i s^j s^k m(a, b, c)$$

for all $a, b, c \in B$,

and define $h(a, b) = s f_2(a, b) s^{-1} - f_2(a, b)$. Then $C(h(b, c))$ is given by the following table:

<table>
<thead>
<tr>
<th>$C(h(b, c))$</th>
<th>1</th>
<th>$x$</th>
<th>$y$</th>
<th>$x^2$</th>
<th>$y^2$</th>
<th>$x^2 y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y$</td>
<td>0</td>
<td>$x$</td>
<td>0</td>
<td>0</td>
<td>$y^2$</td>
<td>0</td>
</tr>
<tr>
<td>$x^2$</td>
<td>0</td>
<td>$x^2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y^2$</td>
<td>0</td>
<td>$x^2$</td>
<td>$y^2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x^2 y$</td>
<td>0</td>
<td>$x^2 y$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

So we get the following explicit description of $m$:

**Theorem 3.8.** The canonical element $\gamma_G$ is represented by the Hochschild cocycle $m$.
which is given by the formulae:

\[
m(x, y, x) = x^2,
\]

\[
m(x^2, y, x) = x^2 y,
\]

\[
m(x, y, x^2) = x^2 y,
\]

\[
m(a, b, c) = 0 \quad \text{for all other } a, b, c \in B,
\]

\[
m(sa, b, c) = sm(a, b, c) + saC(h(b, c)), \quad \text{where } C(h(b, c)) \text{ is given by } (6),
\]

\[
m(s^{2i}a, s^j b, s^l c) = s^{2i+j+l}m(a, b, c).
\]

The element \( \gamma \in HH^{3-1}H^*(G) \) represented by \( m \) is non-trivial.

**Proof.** It remains to prove the non-triviality of \( \gamma \). Suppose that \( m \) is a Hochschild coboundary; then \( m = \delta g \) for some \( g : \Lambda \otimes^2 \to \Lambda[-1] \). Adding up the equations

\[
x^3 = m(x, z, x^2) = xg(z, x^2) + g(x, z)x^2
\]

\[
0 = m(x^2, x, z) = x^2 g(x, z) + g(x^2, z)x
\]

\[
0 = m(z, x^2, x) = zg(x^2, x) + g(z, x^3) + g(z, x^2)x
\]

\[
0 = m(z, z^2, z) = zg(z^2, z) + g(z^3, z) + g(z, z^2)z + g(z, z^2)z
\]

\[
0 = zm(z, z, z) = z^2 g(z, z) + zg(z^2, z) + zg(z, z^2) + zg(z, z)z
\]

and simplifying, we get the contradiction \( x^3 = 0 \). \( \square \)

4. Realizability of modules

4.1. Massey products

There is a strong connection between the canonical class \( \gamma \) and triple Massey products over \( H^*(G) \). This has already been noted in [2, Lemma 5.14], and we will generalize this fact to Massey products of matrices (as introduced by May [5]). We start with some notation. Let \( \Lambda \) be a graded \( k \)-algebra, and suppose that \( I \) is a graded set; i.e., a set together with a function \( | \cdot | : I \to \mathbb{Z} \). For every such set, we define \( I[n] \) to be the shifted graded set given by the same set with new grading \( |i|_n = |i| + n \) for all \( i \in I \). We denote by \( \Lambda^I \) the shifted free \( \Lambda \)-module

\[
\Lambda^I = \bigoplus_{i \in I} \Lambda[|i|].
\]

Then \( \Lambda^I[n] = \Lambda^I[n] \). If \( J \) is another graded set, we can consider morphisms \( f : \Lambda^J \to \Lambda^I \). Every such map can be represented by a (possibly infinite) matrix \( (f_{i,j})_{i \in I, j \in J} \) with \( |f_{i,j}| = |i| - |j| \). Such a matrix is column-finite; i.e., for every \( j \) there are only finitely many non-zero \( f_{i,j} \)'s. Let us denote by \( \Lambda^I,J \) the set of such matrices. Every such yields a map \( f : \Lambda^J \to \Lambda^I \).

A triple of matrices \((A, B, C)\) will be called *composable* if there are graded sets \( I, J, K, L \) with \( A \in \Lambda^I,J, B \in \Lambda^J,K, C \in \Lambda^K,L \). Every morphism \( m : \Lambda^{03} \to \Lambda[-1] \) can be extended to the module of all composable triples by putting

\[
m(A, B, C) \in \Lambda^{[I-1],L} : \quad m(A, B, C)_{[i-1],l} = \sum_{j \in J} \sum_{k \in K} m(a_{ij}, b_{jk}, c_{kl}).
\]
From now on we assume $\Lambda = H^*A \cong \hat{H}^*(G)$, where $A$ is the endomorphism-dgA of some projective resolution of the trivial $kG$-module $k$. Also, let $m: \Lambda \otimes \Lambda \to \Lambda[-1]$ be some Hochschild cocycle representing the canonical element $\gamma \in HH^3[-1] H^*(G)$. Recall that (see, e.g., [5]) for every composable triple of matrices $(A, B, C)$ with $AB = 0$ and $BC = 0$ the triple matric Massey product $(A, B, C)$ is defined and a coset of $A \cdot \Lambda^{[1], L} \cdot \Lambda^{[1], K} \cdot C$. Notice that there is no obstruction to generalizing May’s definition to infinite matrices.

**Proposition 4.1.** For every composable triple $(A, B, C)$ with $AB = 0$ and $BC = 0$, we have that $m(A, B, C) \in \langle A, B, C \rangle$.

**Proof.** We have

$$m(A, B, C) = f_1(A)f_2(B, C) + f_2(AB, C) + f_2(AB, BC) + f_2(A, B)f_1(C)$$

and the last term represents one element of the Massey product.

A triple $(A, B, C)$ will be called *exact* if it is composable and the sequence

$$\Lambda^I \xleftarrow{A} \Lambda^J \overset{B}{\xrightarrow{}} \Lambda^K \overset{C}{\xrightarrow{}} \Lambda^L$$

is exact.

**Proposition 4.2.** Let $A \in \Lambda^{I, J}$ be any matrix, and define $M = \text{coker} A$. Then the following are equivalent:

(i) The module $M$ is a direct summand of a realizable module.

(ii) For every composable triple $(A, B, C)$ with $AB = 0$ and $BC = 0$, we have that $0 \in \langle A, B, C \rangle$.

(iii) For some exact triple $(A, B, C)$, we have $0 \in \langle A, B, C \rangle$.

**Proof.** For (ii) $\Rightarrow$ (i), let $M$ be a direct summand of $H^*N$, where $N$ is some dg-$A$-module. Then there are maps $M \xrightarrow{i} H^*N \overset{\pi}{\rightarrow} M$ with $ri = \text{id}_M$. Let $\pi: \Lambda^I \to M$ be the projection map, and put $W = i\pi$. Then $WA = 0$, so that $(W, A, B)$ is defined, and the juggling formula (see, e.g., [5, Corollary 3.2.(iii)]) yields $W \langle A, B, C \rangle = (W, A, B)C$ as cosets of $WA\Lambda^{[1], K}$. Let $E: \Lambda^K \to H^*N[-1]$ be some element in $\langle W, A, B \rangle$. Since $\Lambda^K$ is free, we know that the composition $r \circ E$ lifts as $\Lambda^K \xrightarrow{g} \Lambda^{[1]} \xrightarrow{\pi} M[-1]$ for some matrix $S$. But then

$$\pi SC = rEC \in r \langle W, A, B \rangle C = rW \langle A, B, C \rangle = \pi \langle A, B, C \rangle .$$

This means that there is some matrix $T$ such that $AT + SC \in \langle A, B, C \rangle$, which implies $0 \in \langle A, B, C \rangle$.

The implication (ii) $\Rightarrow$ (iii) is obvious. For (iii) $\Rightarrow$ (i), note that

$$M \leftarrow \Lambda^I \overset{A}{\xleftarrow{}} \Lambda^J \overset{B}{\xrightarrow{}} \Lambda^K \overset{C}{\xrightarrow{}} \Lambda^L$$

is the beginning of a (shifted) free resolution of $M$. We have $m(A, B, C) \in \Lambda^{[1], L}$, and a representative of $\gamma \cup \text{id}_M \in \text{Ext}_{\Lambda}^{3,-1}(M, M)$ is given by the composition

$$g: \Lambda^L \xrightarrow{m(A, B, C)} \Lambda^{[1]} \xrightarrow{\pi} (\text{coker} A)[-1] = M[-1].$$

By assumption and Proposition 4.1, $m(A, B, C) = AX + YC$ for some matrices $X$.
and $Y$, so that this composition equals

$$\Lambda^L \xrightarrow{C} \Lambda^K \xrightarrow{Y} \Lambda^{[-1]} \rightarrow M[-1],$$

which in turn says that $g$ is the coboundary of $\Lambda^K \xrightarrow{Y} \Lambda^{[-1]} \rightarrow M[-1]$; hence $\gamma \cup \text{id}_M = 0$. By Theorem 1.1 of [2], $M$ is a direct summand of some realizable module. \qed

4.2. The group of quaternions

Let $G = Q_8$. We shall make use of one of the implications of Proposition 4.2 to prove the existence of a $H^*G$-module which detects the non-triviality of $\gamma_G$:

**Theorem 4.3.** The cokernel of the map

$$\Lambda^{[-1]} \oplus \Lambda^{[-1]} \xrightarrow{\begin{pmatrix} y & x + y \\ x & y \end{pmatrix}} \Lambda \oplus \Lambda$$

is not a direct summand of a realizable $H^*G$-module.

**Proof.** Let $A = \begin{pmatrix} y & x + y \\ x & y \end{pmatrix}$; then $A^2 = 0$ and therefore the Massey product $\langle A, A, A \rangle$ is defined. We claim that it does not contain 0. An explicit calculation using the description of $m$ given in Theorem 3.6 yields

$$m(A, A, A) = \begin{pmatrix} x^2 & 0 \\ x^2 & x^2 \end{pmatrix}.$$  

Let us denote the latter matrix by $B$; then by Proposition 4.2 we need to prove that $B$ is not of the form $B = AQ + RA$ for some $2 \times 2$-matrices $Q$ and $R$. To do so, define $D = \begin{pmatrix} x & y \\ x + y & x \end{pmatrix}$; then $AD = DA = 0$. If we denote by $\text{tr}$ the trace of a matrix, then we have

$$\text{tr}(BD) = \text{tr}(AQD) + \text{tr}(RAD) = \text{tr}(QDA) + \text{tr}(RAD) = 0$$

(note that these computations take place in a commutative ring). But

$$\text{tr}(BD) = \text{tr} \begin{pmatrix} 0 & x \\ x^2 & y \end{pmatrix} = x^2y \neq 0,$$

a contradiction. \qed

**Remark 4.4.** The triple $(A, A, A)$ is actually exact, but we do not need this.

In order to construct a module which is not a direct summand of a realizable one, it is often enough to consider “ordinary” Massey products, i.e., the case of $1 \times 1$-matrices; this is true for example in the cases $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ([2, Example 7.7]) and $G = \mathbb{Z}/3\mathbb{Z}$ (characteristic 3, [2, Example 7.6]). In our present case, it is not that easy:

**Proposition 4.5.** Let $k = \mathbb{F}_2$ be the field with 2 elements. For all $a, b, c \in \hat{H}^*(Q_8)$ satisfying $ab = 0$ and $bc = 0$, we have $0 \in \langle a, b, c \rangle$. 
Proof. By [2, Lemma 5.14], the class \( m(a, b, c) \) is contained in the Massey product \( \langle a, b, c \rangle \). Therefore, it is enough to show that \( m(a, b, c) \) is an element of the indeterminacy

\[
a \cdot \hat{H}^{[|b|+|c|]-1}(G) + \hat{H}^{[a]+[|b|]-1}(G) \cdot c
\]

for all \( a, b, c \). By construction of \( m \) it is enough to do so for those triples \( (a, b, c) \) and \( (sa, b, c) \) with \( a, b, c \in \{ 1, x, y, x + y, x^2, y^2, x^2 + y^2, x^2y \} \) which satisfy \( ab = 0 \) and \( bc = 0 \).

If \( |a|, |b| \leq 1 \), then \( ab = 0 \) implies \( a = 0 \) or \( b = 0 \) (here we use that \( k = \mathbb{F}_2 \)). If \( |b| \geq 2 \), then \( m(a, b, c) = 0 \) unless \( b \in \{ y^2, y^2 + x^2 \} \) and \( a, c \in \{ x, x + y \} \), in which case \( m(a, b, c) = x^2y \) is divisible by \( a \). So we can assume that \( |a| \geq 2 \) and \( |c| \geq 2 \), which implies \( m(a, b, c) = 0 \) by Theorem 3.6.

For \( m(sa, b, c) \), we have by (4)

\[
m(sa, b, c) = am(s, b, c) + m(a, b, c)s.
\]

We have already seen that the second summand lies in the indeterminacy; the first summand is contained in

\[
a \cdot \hat{H}^{[s]+[|b|]+|c|}-1(G) = sa \cdot \hat{H}^{[|b|]+|c|}-1(G)
\]

and therefore in the indeterminacy. \( \square \)

Remark 4.6. Note that the proposition is not true for arbitrary fields of characteristic 2: If the field \( k \) contains an element \( \alpha \in k \) satisfying \( \alpha^2 + \alpha + 1 = 0 \), then the Massey product

\[
\langle ax + y, \alpha^2x + y, ax + y \rangle
\]

is defined and does not contain 0.

4.3. Generalized quaternions

The picture changes as soon as we consider generalized quaternion groups \( G = Q_{4t} \) with \( t \geq 4 \). It turns out that there is no module detecting the non-triviality of the canonical element \( \gamma_G \).

Let \( m \) be as in Theorem 3.8, and write \( m = m' + m'' \), where \( m' \) is defined in (5). Notice that \( m' \) is a Hochschild cocycle, because it is defined to be \( s \)-periodic, so it is enough to check the cocycle condition on elements in \( \mathcal{B} \). But on these elements, \( m' \) agrees with \( m \). Hence, \( m' \) is a cocycle, and so is \( m'' \). Let \( \gamma' \) and \( \gamma'' \) be the corresponding elements in \( HH^{3-1}G \). In the next two propositions we will show that, for every module \( M \), \( \gamma' \cup \text{id}_M = 0 \) and \( \gamma'' \cup \text{id}_M = 0 \) in \( \text{Ext}^{3-1}(M, M) \), respectively. It will then follow that \( M \) is a direct summand of a realizable module.

Proposition 4.7. For every \( \Lambda \)-module \( M \) we have \( \gamma' \cup \text{id}_M = 0 \).

Proof. Notice that every matrix \( A \in \Lambda^{I,J} \) can be uniquely written as a sum

\[
A = A_1 + A_2x + A_2x^2 + A_y^2 + A_2y^2 + A_2y^2x^2,
\]

where the six matrices on the right-hand side lie in \( k[s^\pm 1]^{I,J}[?] \). The first step in our proof will be to find a suitable free resolution

\[
M \leftarrow \Lambda^I \leftarrow \Lambda^{I,J} \leftarrow \Lambda^K \leftarrow \Lambda^L
\]

of \( M \). We begin with the definition of \( A \). Let \( I \) be a minimal set of generators of the
We start with a slight modification of the representative $\{a, b, c\}$. For every $m \in M$ we have that $\{a, b, c\}$ generates $M$, and by definition of $\gamma$, we get a map $\Lambda^{K} \to \Lambda^{L}$. Therefore, $m(A, B, C) = AV + WC$, and by Proposition 4.2 we get $\gamma' \cup \text{id}_{M} = 0$.

**Proposition 4.8.** For every $\Lambda$-module $M$, we have $\gamma'' \cup \text{id}_{M} = 0$.

**Proof.** We start with a slight modification of the representative $m''$. Let us put $B = \{1, x, z, x^{2}, z^{2}, x^{3}\}$, and define the function $g$ as follows: For all integers $i$, put

\[
g(s^{-1}x^{2}, s^{i}x) = s^{i-1}z^{2},
\]

\[
g(s^{-1}x^{2}, s^{i}z) = s^{i-1}x^{2},
\]

and $g(a, b) = 0$ on all other elements $a, b$ in $\{s^{c}c \mid c \in B\}$. Then $\tilde{m} = m'' + \partial g$ defines a new representative for the element $\gamma''$. For all $a, b, c \in B$ and $i, j \geq 1$, we have

\[
\tilde{m}(a, s^{i}b, s^{j}c) = m''(a, s^{i}b, s^{j}c) + ag(s^{i}b, s^{j}c) + g(s^{i}ab, s^{j}c)
\]

\[
+ g(a, s^{i+j}bc) + g(a, s^{i}b)s^{j}c,
\]

and by definition of $m''$ and $g$ each summand on the right-hand side vanishes. We also have that

\[
\tilde{m}(s^{-1}a, s^{i}b, s^{j}c) = m''(s^{-1}a, s^{i}b, s^{j}c) + s^{-1}a \underbrace{g(s^{i}b, s^{j}c)}_{0} + \underbrace{g(s^{-1}ab, s^{j}c)}_{0}
\]

\[
+ g(s^{-1}a, s^{i+j}bc) + g(s^{-1}a, s^{i}b)s^{j}c.
\]

We claim that this is zero if $|a| \geq 2, |b| \geq 1, \text{ and } |c| \geq 1$. In that case, we have $|bc| \geq 2$ and therefore $g(s^{-1}a, s^{i+j}bc) = 0$, so that it remains to show $m''(s^{-1}a, s^{i}b, s^{j}c) =$
g(s^{-1}a, s^ib)s^jc, or equivalently,
\[ m''(s^{-1}a, b, c) = g(s^{-1}a, b)c. \]
To see this, we consider the several cases for \( a \) separately. If \( a = x^3 \), then
\[ m''(s^{-1}a, b, c) = s^{-1}x^3C(h(b, c)), \]
where \( h \) is as in Theorem 3.8. But \( |h(b, c)| \geq 1 \), so the last expression vanishes, as does \( g(s^{-1}a, b)c \). For \( a = z^2 \) we get
\[ m''(s^{-1}a, b, c) = s^{-1}z^2C(h(b, c)), \]
but \( |h(b, c)| \geq 2 \) or \( C(h(b, c)) \) is divisible by \( x \), and therefore again the right-hand side vanishes. The last case is \( a = x^2 \), where we need to show
\[ s^{-1}x^2C(h(b, c)) = g(s^{-1}x^2, b)c. \]
Both sides vanish for degree reasons unless \( |b| = |c| = 1 \), and in that case both sides will equal \( s^{-1}x^3 \) if \( b \neq c \), and 0 otherwise.

The rest is easy. We start with a free resolution of \( M \) as in the proof of Proposition 4.7. We can (and do) assume that the degree \( |i| \) of every element \( i \in I \) lies in \( \{0, 1, 2, 3\} \). Also, we assume that the degree of every element of \( J \) lies in \( \{-1, 0, 1, 2\} \), the degree of every element of \( K \) belongs to \( \{-8, -7, -6, -5\} \), and the degree of every element of \( L \) is in \( \{-15, -14, -13, -12\} \). Then we know that every non-zero entry of \( B \) and \( C \) is a linear combination of terms of the form \( s^ib \) with \( i \geq 1 \) and \( b \in \mathcal{B} \), \( |b| \geq 1 \). Furthermore, every non-zero entry of \( A \) is a linear combination of elements in \( \mathcal{B} \cup \{s^{-1}x^2, s^{-1}z^2, s^{-1}x^3\} \). By what we have shown above, \( \tilde{m}(A, B, C) = 0 \), and we are done.

**References**


Martin Langer  martinlanger@yahoo.com

Westfälische Wilhelms-Universität Münster, Institut für Mathematik, Einsteinstr. 62, 48149 Münster, Germany