Comparing Commutative and Associative Unbounded Differential Graded Algebras Over $\mathbb{Q}$ from a Homotopical Point of View

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Abstract

In this paper we establish a faithfulness result, in a homotopical sense, between a subcategory of the model category of augmented differential graded commutative algebras $\text{CDGA}$ and a subcategory of the model category of augmented differential graded algebras $\text{DGA}$ over the field of rational numbers $\mathbb{Q}$.

1. Introduction

It is well known that the forgetful functor from the category of commutative $k$-algebras to the category of associative $k$-algebras is fully faithful. We have an analogue result between the category of unbounded differential graded commutative $k$-algebras $\text{dgCAlg}_k$ and the category of unbounded differential graded associative algebras $\text{dgAlg}_k$. The question that we want to explore is the following: Suppose that $k = \mathbb{Q}$; we want to know if it is true that forgetful functor $U : \text{dgCAlg}_k \to \text{dgAlg}_k$ induces a fully faithful functor at the level of homotopy categories

$$\mathbf{R}U : \text{Ho}(\text{dgCAlg}_k) \to \text{Ho}(\text{dgAlg}_k).$$

The answer is no. A nice and easy counterexample was given by Lurie [10]. He has considered $k[x, y]$ the free commutative CDGA in two variables concentrated in degree 0. It follows obviously that

$$\text{Ho}(\text{dgCAlg}_k)(k[x, y], S) \simeq H^0(S) \oplus H^1(S),$$

while

$$\text{Ho}(\text{dgAlg}_k)(k[x, y], S) \simeq H^0(S) \oplus H^0(S) \oplus H^{-1}(S).$$

Something nice happens if we consider the category of augmented CDGA denoted by $\text{dgCAlg}^*_k$ and augmented DGA denoted by $\text{dgAlg}^*_k$.
Theorem 1.1 (Theorem 4.1). For any $R$ and $S$ in $\text{dgCAlg}_k^\ast$, the induced map by the forgetful functor

$$\Omega \text{Map}_{\text{dgCAlg}_k^\ast}(R, S) \to \Omega \text{Map}_{\text{dgAlg}_k^\ast}(R, S)$$

has a retract; in particular

$$\pi_i \text{Map}_{\text{dgCAlg}_k^\ast}(R, S) \to \pi_i \text{Map}_{\text{dgAlg}_k^\ast}(R, S)$$

is injective $\forall i > 0$.

Definition 1.2 ($[8, 7]$). Let $M$ be a model category and let $a, a'$ be cofibrant objects and $b, b'$ be fibrant objects. The (derived) mapping space denoted by $\text{Map}_M$ is a simplicial set having the following properties:

- $\pi_0 \text{Map}(a, b) \cong \text{Ho}(M)(a, b)$, where $\text{Ho}(M)$ is the homotopy category of $M$.
- For any weak equivalence $a \to a'$, we have a weak equivalence of simplicial sets $\text{Map}(a', b) \to \text{Map}(a, b)$.
- For any weak equivalence $b \to b'$, we have a weak equivalence of simplicial sets $\text{Map}(a, b) \to \text{Map}(a, b')$.

Remark 1.3. [4, Theorem 2.12] In our work we use only the formal properties of the derived mapping space in a model category. For any Quillen adjunction between model categories

$$M \xrightarrow{G} N$$

and for any cofibrant object $a \in M$ and any fibrant object $c \in N$, we have the following (zig-zag) equivalence of simplicial sets:

$$\text{Map}_N(Ga, c) \sim \text{Map}_M(a, Uc).$$

Let $S$ be a differential graded commutative algebra which is a “loop” of another CDGA algebra $A$, i.e., $S = \text{Holim}(k \to A \leftarrow k)$, where the homotopy limit is taken in the model category $\text{dgCAlg}_k$. A direct consequence of our theorem is that the right derived functor $RU$ is a faithful functor, i.e., the induced map $\text{Ho}(\text{dgCAlg}_k^\ast)(R, S) \to \text{Ho}(\text{dgAlg}_k^\ast)(R, S)$ is injective.

Interpretation of the result in the derived algebraic geometry

Rationally, any pointed topological $X$ space can be viewed as an augmented (connective) commutative differential graded algebra via its cochain complex $C^\ast(X, \mathbb{Q})$. In the case where $X$ is a simply connected rational space, the cochain complex $C^\ast(X, \mathbb{Q})$ carries all the homotopical information about $X$, by the Sullivan theorem [6]. Moreover, the bar construction $BC^\ast(X, \mathbb{Q})$ is identified (as $E_{\infty}$-DGA) to $C^\ast(\Omega X, \mathbb{Q})$ and $\Omega C^\ast(X, \mathbb{Q})$ is identified (as $E_{\infty}$-DGA) to $C^\ast(\Sigma X, \mathbb{Q})$; cf. [5]. This interpretation allows us to make the following definition: A generalized rational pointed space is an augmented commutative differential graded $\mathbb{Q}$-algebra (possibly unbounded). In the same spirit, we define a pointed generalized noncommutative rational space as an augmented differential graded $\mathbb{Q}$-algebra (possibly unbounded). Let $A$ be any augmented CDGA (resp. DGA); we will call a CDGA (resp. DGA) of the form $\Omega A$ an op-suspended CDGA (resp. DGA). Our Theorem 4.1 can be interpreted as follows:
The homotopy category of op-suspended generalized commutative rational spaces is a subcategory of the homotopy category of op-suspended generalized noncommutative rational spaces.

2. DGA, CDGA, and $E_\infty$-DGA

We work in the setting of unbounded differential graded $k$-modules $dgMod_k$. This is a symmetric monoidal closed model category ($k$ is a commutative ring). We denote the category of (reduced) operads in $dgMod_k$ by $Op_k$. We follow notations and definitions of [2]; we say that an operad $P$ is admissible if the category of $P-dgAlg_k$ admits a model structure where the fibrations are degree wise surjections and weak equivalences are quasi-isomorphisms. For any map of operads $\phi : P \to Q$ we have an induced adjunction of the corresponding categories of algebras:

$$P - dgAlg_k \overset{\phi_*}{\longrightarrow} Q - dgAlg_k.$$  

A $\Sigma$-cofibrant operad $P$ is an operad such that $P(n)$ is $k[\Sigma_n]$-cofibrant in $dgMod_k[\Sigma_n]$. Any cofibrant operad $P$ is a $\Sigma$-cofibrant operad [2, Proposition 4.3]. We denote the associative operad by $Ass$ and the commutative operad by $Com$. The operad $Ass$ is an admissible operad and $\Sigma$-cofibrant, while the operad $Com$ is not admissible in general. In the rational case, when $k = \mathbb{Q}$ the operad $Com$ is admissible and $\Sigma$-cofibrant. More generally any cofibrant operad $P$ is admissible [2, Proposition 4.1, Remark 4.2]. We define a symmetric tensor product of operads by the formulae

$$[P \otimes Q](n) = P(n) \otimes Q(n), \quad \forall n \in \mathbb{N}.$$  

**Lemma 2.1.** Suppose that $\phi : Ass \to P$ is a cofibration of operads. The operad $P$ is admissible and the functor $\phi^* : P - dgAlg_k \to dgAlg_k$ preserves fibrations, weak equivalences, and cofibrations with cofibrant domain in the underlying category $dgMod_k$.

**Proof.** First of all, the operad $P$ is admissible; indeed we use the cofibrant resolution $r : E_\infty \to Com$ and consider the following pushout in $Op_k$ given by:

$$\begin{array}{ccc}
Ass_\infty & \overset{f}{\longrightarrow} & E_\infty \\
\downarrow{\sim} & & \downarrow{\alpha} \\
Ass & \overset{I}{\longrightarrow} & E' \end{array}$$

where $Ass_\infty$ is a cofibrant replacement of $Ass$ in $Op_k$ and $Ass_\infty \to E_\infty$ is a cofibration which factors $Ass_\infty \to Ass \to Com$. Since the category $Op_k$ is left proper in the sense of [13, Theorem 3], we have that $\alpha : E_\infty \to E'_\infty$ is an equivalence. We denote by $I$ the unit interval in the category $dgMod_k$ which is strictly coassociative [12, p. 503]; i.e., there is a map of operads $Ass \to End^{op}(I)$. Moreover, there is a map of operads $E_\infty \to End^{op}(I)$ which endows $I$ with the structure of $E_\infty$-coalgebra [2, Remark 4.2]. Since $I$ is a strict coalgebra (in particular an $Ass_\infty$-coalgebra), the operad map $Ass_\infty \to$
$\text{End}^{op}(I)$ factors through $\text{Ass}$; i.e., we have two compatible maps of operads:

$$
\begin{align*}
\begin{array}{c}
\text{Ass}_\infty \xrightarrow{f} E_\infty \\
\downarrow \sim \\
\text{Ass} \xrightarrow{\Delta} \text{Ass} \otimes \text{Ass}
\end{array}
\end{align*}
$$

By the universality of the pushout, we have a map of operads $E'_\infty \to \text{End}^{op}(I)$. This means that the unit interval $I$ has a structure of $E'_\infty$-coalgebra [2, p. 4]. Moreover, we have a commutative diagram in $\text{Op}_k$ given by

$$
\begin{align*}
\begin{array}{c}
\text{Ass} \xrightarrow{\Delta} \text{Ass} \otimes \text{Ass} \xrightarrow{\phi \otimes f} P \otimes E'_\infty \\
\downarrow \phi \\
P \xrightarrow{id} P \otimes \text{Com} = P
\end{array}
\end{align*}
$$

where the operad map $r : E'_\infty \to \text{Com}$ is obtained by the universal property of the pushout (1) and the diagonal map $\Delta : \text{Ass} \to \text{Ass} \otimes \text{Ass}$ is induced by the diagonals $\Sigma_n \to \Sigma_n \times \Sigma_n$; therefore the commutativity of the diagram is a consequence of the co-unit property of the diagonal map $\Delta$ and universal choice of $r$. Hence, the map $P \otimes E'_\infty \to P$ admits a section. It implies by [2, Proposition 4.1] that $P$ is admissible and $\Sigma$-cofibrant. Since all objects in $P - \text{dgAlg}_k$ are fibrant and $\phi^*$ is a right Quillen adjoint, it preserves fibrations and weak equivalences.

Since $P$ is an admissible operad, we have a Quillen adjunction

$$
\text{dgAlg}_k \xrightarrow{\phi} P - \text{dgAlg}_k,
$$

where the functor $\phi^*$ is identified to the forgetful functor. Moreover, the model structure on $P - \text{dgAlg}_k$ is the transferred model structure from the cofibrantly generated model structure $\text{dgAlg}_k$ via the adjunction $\phi, \phi^*$. Suppose that $f : A \to B$ is a cofibration in $P - \text{dgAlg}_k$ such that $A$ is cofibrant in $\text{dgMod}_k$. We factor this map as a cofibration followed by a trivial fibration

$$
A \xrightarrow{i} P \xrightarrow{p} B
$$

in the category $\text{dgAlg}_k$; therefore $i$ is a cofibration [14, Proposition 2.3 (3)] (Toën’s initial argument is for cofibrant objects, but it works for cofibrations, i.e., the forgetful functor $\text{dgAlg}_k \to \text{dgMod}_k$ preserves cofibrations) and $p$ is obviously a trivial fibration in $\text{dgMod}_k$. By [11, Lemma 4.1.16], we have an induced map of endomorphism operads of diagrams [11, Section 4.1.1]:

$$
\text{End}_{\{A \to P \to B\}} \to \text{End}_{\{A \to B\}}
$$
which is a trivial fibration of operads since \(p\) is a trivial fibration. Notice that Rezk’s arguments are performed in the category of simplicial \(k\)-modules and are formally transposable in the context of differential graded \(k\)-modules. By definition of our endomorphism operads of diagrams, we have the following commutative diagram in \(\mathcal{O}p_k\):

\[
\begin{array}{ccc}
\text{Ass} & \longrightarrow & \text{End}_{\{A \to P \to B\}} \\
\downarrow & & \downarrow \sim \\
\longrightarrow & \longrightarrow & \text{End}_{\{A \to B\}}
\end{array}
\]

where the first horizontal map of operads translates the fact that \(A \to P \to B\) are maps in \(\text{dgAlg}_k\); respectively, the second horizontal map translates the fact that \(A \to B\) is a map of \(P\)-algebra. Since \(\mathcal{O}p_k\) is a model category, it implies that we have a lifting map of operads \(P \to \text{End}_{\{A \to P \to B\}}\); hence \(i\) and \(p\) are maps of \(P - \text{dgAlg}_k\). Therefore, we consider the following commutative square in the category \(P - \text{dgAlg}_k\):

\[
\begin{array}{ccc}
A & \longrightarrow & P \\
\downarrow f & & \downarrow p \\
B & \longrightarrow & B
\end{array}
\]

The lifting map \(r\) exists since \(P - \text{dgAlg}_k\) is a model category. We conclude that \(p \circ r = \text{id}\) and \(r \circ f = i\), which means that \(f\) is a retract of \(i\); hence \(f\) is a cofibration in \(\text{dgAlg}_k\).

Remark 2.2. With the same notation as in Lemma 2.1 if \(A\) is a cofibrant object in \(P - \text{dgAlg}_k\) then \(A\) is a cofibrant object in \(\text{dgMod}_k\). Indeed \(k \to A\) is a cofibration in \(P - \text{dgAlg}_k\); by the previous lemma \(k \to A\) is a cofibration in \(\text{dgAlg}_k\). Therefore, \(k \to A\) is a cofibration in \(\text{dgMod}_k\).

3. Suspension in CDGA and DGA

We denote the operad \(E'_\infty\) of the previous section by \(E_\infty\), and \(k = \mathbb{Q}\).

3.1. \(E_\infty\)-DGA

We have a map of operads \(\text{Ass} \to \text{Com}\), which we factor as cofibration followed by a trivial fibration:

\[
\begin{array}{ccc}
\text{Ass} & \longrightarrow & E_\infty \\
\longrightarrow & \longrightarrow & \sim \text{Com}
\end{array}
\]

As a consequence, we have the following Quillen adjunctions:

\[
\text{dgAlg}_k \xrightarrow{\text{Ab}_U} E_\infty \text{dgAlg}_k \xrightarrow{\text{str}} U' \text{dgCAAlg}_k.
\]

These adjunctions have the following properties:

- The functors \(U'\) and \(U \circ U'\) are the forgetful functors; they are fully faithful (cf. Propositions 3.3 and 3.2).
The functors \( str, U' \) form a Quillen equivalence since \( k = \mathbb{Q} \) (cf. [9], Corollary 1.5, Part II). The functor \( str \) is the strictification functor.

The functors \( Ab_\infty, U \) form a Quillen pair.

The composition \( str \circ Ab_\infty \) is the abelianization functor \( Ab : \text{dgAlg}_k \to \text{dgCAlg}_k \).

The functors \( str \) and \( Ab \) are idempotent functors (cf. Propositions 3.3 and 3.2). The model categories \( \text{dgCAlg}_k^*, \text{dgAlg}_k^*, \) and \( E_\infty \text{dgAlg}_k^* \) are pointed model categories. It is natural to introduce the suspension functors in these categories.

**Definition 3.1.** Let \( C \) be any pointed model category. We denote the point by 1, and let \( A \in C \); a suspension \( \Sigma A \) is defined as \( \text{hocolim}(1 \leftarrow A \to 1) \).

**Proposition 3.2.** Any map \( f : A \to S \) in \( E_\infty \text{dgAlg}_k \), where \( S \) is in \( \text{dgCAlg}_k \), factors in a unique way as \( A \to str(A) \to S \) and the forgetful functor \( U' : \text{dgCAlg}_k \to E_\infty \text{dgAlg}_k \) is fully faithful. Moreover, the unit of the adjunction \( \nu_A : A \to str(A) \) is a fibration.

**Proof.** Suppose that we have a map \( h : R \to S \) in \( E_\infty \text{dgAlg}_k \) such that \( R \) and \( S \) are objects in \( \text{dgCAlg}_k \). By definition of the operad \( E_\infty \) the map \( h \) respects the multiplication; therefore \( h \) is a morphism in \( \text{dgCAlg}_k \) since \( R \) and \( S \) are commutative differential graded algebras. The forgetful functor \( U' : \text{dgCAlg}_k \to E_\infty \text{dgAlg}_k \) is fully faithful: this implies that \( str(S) = S \) for any \( S \in \text{dgCAlg}_k \). We have a commutative diagram induced by the unit \( \nu \) of the adjunction \((U', str)\):

\[
\begin{array}{ccc}
A & \xrightarrow{f} & S \\
\downarrow{\nu_A} & & \downarrow{\nu_S = \text{id}} \\
str(A) & \xrightarrow{str(f)} & str(S) = S \\
\end{array}
\]

We conclude that \( f = str(f) \circ \nu_A \). The surjectivity of the \( \nu_A \) follows from the universal property of \( str(A) \). Hence, \( \nu_A \) is a fibration in \( E_\infty \text{dgAlg}_k \).

**Proposition 3.3.** Any map \( f : A \to S \) in \( \text{dgAlg}_k \), where \( S \) is in \( \text{dgCAlg}_k \), factors in a unique way as \( A \to Ab(A) \to S \), and the forgetful functor \( U \circ U' : \text{dgCAlg}_k \to \text{dgAlg}_k \) is fully faithful. Moreover, the unit of the adjunction \( \nu_A : A \to Ab(A) \) is a fibration.

**Proof.** The proof is the same as in Proposition 3.2.

**Proposition 3.4.** Suppose that we have a trivial cofibration \( k \to \bar{k} \) in \( E_\infty \text{dgAlg}_k \). Then the universal map \( \pi : Ab(\bar{k}) \to str(\bar{k}) \) is a trivial fibration and admits a section in the category \( \text{dgCAlg}_k \).

**Proof.** We consider the following commutative diagram in \( E_\infty \text{dgAlg}_k \):

\[
\begin{array}{ccc}
k & \xrightarrow{\sim} & \bar{k} \\
\downarrow{id} & & \downarrow{id} \\
k = str(k) & \xrightarrow{\sim} & str(\bar{k}) \\
\end{array}
\]

The map \( \bar{k} \to str(\bar{k}) \) is an equivalence since \( str \) is a left Quillen functor; the same thing holds for the abelianization functor. More precisely, the forgetful functor \( E_\infty \text{dgAlg}_k \to \)
dgAlg \_k\ preserves cofibration (Lemma 2.1) (P = E\_\infty); therefore the map Ab(k) \to Ab(\bar{k}) is a weak equivalence in dgCAlg \_k. It follows that we have a commutative diagram in dgAlg \_k:

\[
\begin{array}{ccc}
  k & \sim & k \\
  \downarrow^{id} & & \downarrow \\
  k = Ab(k) & \sim & Ab(\bar{k})
\end{array}
\]
i.e., \(k \to Ab(\bar{k})\) is a trivial fibration, since \(k \to Ab(\bar{k})\) is surjective by definition of the abelianization functor. On the other hand the map \(k \to str(\bar{k})\) is a trivial fibration in \(E\_\infty dgAlg_\bar{k}\) (Proposition 3.2) and hence in \(dgAlg_\bar{k}\); therefore it can be factored (cf. Proposition 3.3) as \(k \to Ab(k) \to str(\bar{k})\), where \(Ab(k) \to str(k)\) is a trivial fibration between cofibrant objects in \(dgCAlg_k\). It follows that we have a retract \(l : str(k) \to Ab(k)\). 

**Definition 3.5.** The suspension functor in the pointed model categories \(dgCAlg_\ast^k\), \(dgAlg_\ast^k\), and \(E\_\infty dgAlg_\ast^k\) are denoted by \(B\) (resp. \(\Sigma\) and resp. \(B\_\infty\)).

**Remark 3.6.** The notation \(\Sigma\) is a generic notation for the suspension functor in a pointed model category. In the case of \(dgCAlg_\ast^k\) and \(dgAlg_\ast^k\) we have used the notation \(B\) and \(B\_\infty\) to make a link with the Bar construction for commutative \((E\_\infty)\) differential graded algebra; this coincides with the generic suspension functor.

**Lemma 3.7.** Suppose that \(A\) is a cofibrant object in \(E\_\infty dgAlg_\ast^k\). Then \(str(B\_\infty A)\) is a retract of \(Ab(\Sigma A)\) in the category \(dgCAlg_k\).

**Proof.** First of all, if a map \(f\) is associative (or commutative, or an \(E\_\infty\) map) we put index \(f_a\) (or \(f_c\), or \(f\_\infty\), respectively). Notice that by definition of the operad \(E\_\infty\) any \(E\_\infty\)-map is a strictly associative map. Suppose that \(A\) is a cofibrant object in \(E\_\infty dgAlg_k\). Consider the following commutative square:

\[
\begin{array}{ccc}
  A & \xrightarrow{i\_\infty} & k \\
  \downarrow^{i\_\infty} & & \downarrow\
  \Sigma A & \xrightarrow{h_a} & B\_\infty A
\end{array}
\]

where \(\Sigma A\) is the (homotopy in Lemma 2.1) pushout in \(dgAlg_k\) and \(B\_\infty A\) is the (homotopy) pushout in \(E\_\infty dgAlg_\ast^k\). By Proposition 5.2 and Proposition 3.3 we have the
following commutative square in $\text{dgAlg}_k$:

\[
\begin{array}{ccc}
\Sigma A & \xrightarrow{u_a} & B_\infty A \\
\downarrow & & \downarrow \\
Ab(\Sigma A) & \xrightarrow{x_c} & \text{str}[B_\infty A] = B[\text{str}(A)].
\end{array}
\]

By Proposition 3.4 we have an inclusion of commutative differential graded algebras $l_c : \text{str}(k) \to Ab(k)$ and after strictification we obtain another homotopy pushout (inner) square in $\text{dgCAlg}_k$ given by

\[
\begin{array}{ccc}
\text{str}(A) & \xleftarrow{i_c} & \text{str}(k) & \xleftarrow{l_c} & Ab(k) \\
\downarrow{i_c} & & \downarrow{f_c} & & \downarrow{h_c} \\
\text{str}(k) & \xleftarrow{f_c} & B[\text{str}(A)] & \xrightarrow{h_c} & \text{str}[B_\infty A] \\
\downarrow{l_c} & & \downarrow{!} & & \downarrow{u_c} \\
Ab(k) & \xleftarrow{h_c} & Ab(\Sigma A) & \xleftarrow{x_c} & \text{str}[B_\infty A].
\end{array}
\]

In order to prove that $B[\text{str}(A)]$ is a retract of $Ab(\Sigma(A))$ it is sufficient to prove that $x_c \circ h_c \circ l_c = f_c$.

By Proposition 3.2 and Proposition 3.3 the composition $E_\infty$-maps

\[
\begin{array}{ccc}
k & \xrightarrow{f_\infty} & B_\infty A & \xrightarrow{\pi} & \text{str}[B_\infty A] \\
\downarrow{id} & & \downarrow{h_a} & & \downarrow{=h_a} \\
k & \xrightarrow{pr} & Ab(\Sigma A) & \xrightarrow{x_c} & \text{str}[B_\infty A].
\end{array}
\]

By unicity, $\alpha_c = f_c$. On the other hand, using the first pushout in $E_\infty\text{dgAlg}_k$, the previous composition $k \to \text{str}[B_\infty A]$ is factored as

\[
\begin{array}{ccc}
k & \xrightarrow{h_a} & \Sigma A & \xrightarrow{x_c} & \text{str}[B_\infty A].
\end{array}
\]

We summarize the previous remarks in the following commutative diagram:

\[
\begin{array}{ccc}
k & \xrightarrow{pr} & Ab(k) & \xrightarrow{\pi} & \text{str}(k) & \xrightarrow{f_c} & \text{str}[B_\infty A] \\
\downarrow{id} & & \downarrow{h_a = Ab(h_a)} & & \downarrow{id} & & \downarrow{id} \\
k & \xrightarrow{pr} & Ab(\Sigma A) & \xrightarrow{x_c} & \text{str}[B_\infty A].
\end{array}
\]

By definition of $h_a$, the dotted map $h_c$ makes the left square commutative. Since the whole square is commutative and the map $pr$ is surjective, we conclude that
\[x_c \circ h_c = f_c \circ \pi.\] Since the map \(l_c : \text{Str}(k) \to \text{Ab}(k)\) is a retract of \(\pi\) (cf. [3,4]), i.e., \(\pi \circ l_c = \text{id}\), we conclude that \(x_c \circ h_c \circ l_c = f_c\). Finally, by unicity of the pushout, we deduce that the following composition

\[B[\text{str}(A)] \xrightarrow{uc} \text{Ab}(\Sigma A) \xrightarrow{x_c} B[\text{str}(A)]\]

is identity.

4. Main result and applications

Theorem 4.1. For any \(R\) and \(S\) in \(\text{dgCAlg}_{\mathbb{Q}}\), the induced map by the forgetful functor

\[\Omega \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(R, S) \to \Omega \text{Map}_{\text{dgAlg}_{\mathbb{Q}}}(R, S)\]

has a retract; in particular

\[\pi_i \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(R, S) \to \pi_i \text{Map}_{\text{dgAlg}_{\mathbb{Q}}}(R, S)\]

is injective \(\forall \, i > 0\).

Proof. Suppose that \(R\) is a cofibrant object in \(E_\infty \text{dgAlg}_{\mathbb{Q}}\) and \(S\) any object in \(\text{dgCAlg}_{\mathbb{Q}}\). By adjunction, we have that

\[\Omega \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(\text{str}(R), S) \sim \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(B[\text{str}(R)], S)\]

(2)

\[\sim \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(\text{str}[B_\infty R], S)\]

(3)

\[\sim \text{Map}_{E_\infty \text{dgAlg}_{\mathbb{Q}}}(B_\infty R, S)\]

(4)

\[\sim \Omega \text{Map}_{E_\infty \text{dgAlg}_{\mathbb{Q}}}(R, S)\].

(5)

By Lemma 3.7, we have a retract

\[\text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(B[\text{str}(R)], S) \to \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(\text{str}[B_\infty R], S) \to \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(B[\text{str}(R)], S)\].

Again by adjunction:

\[\text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(\text{str}[B_\infty R], S) \sim \text{Map}_{\text{dgAlg}_{\mathbb{Q}}}(\Sigma R, S) \sim \Omega \text{Map}_{\text{dgAlg}_{\mathbb{Q}}}(R, S)\].

We conclude that

\[\Omega \text{Map}_{E_\infty \text{dgAlg}_{\mathbb{Q}}}(R, S) \xrightarrow{U} \Omega \text{Map}_{\text{dgAlg}_{\mathbb{Q}}}(R, S) \xrightarrow{\pi_i} \Omega \text{Map}_{E_\infty \text{dgAlg}_{\mathbb{Q}}}(R, S)\]

is a retract. Hence, the forgetful functor \(U\) induces an injective map on homotopy groups, i.e.,

\[\pi_i \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(\text{str}(R), S) \simeq \pi_i \text{Map}_{E_\infty \text{dgAlg}_{\mathbb{Q}}}(R, S) \to \pi_i \text{Map}_{\text{dgAlg}_{\mathbb{Q}}}(R, S)\]

is injective \(\forall \, i > 0\).

4.1. Rational homotopy theory

We give an application of our Theorem 4.1 in the context of rational homotopy theory. Let \(X\) be a simply connected rational space such that \(\pi_i X\) is finite dimensional \(\mathbb{Q}\)-vector space for each \(i > 0\). Let \(C^*(X)\) be the differential graded \(\mathbb{Q}\)-algebra cochain associated to \(X\) which is a connective \(E_\infty \text{dgAlg}_{\mathbb{Q}}\). If \(R = C^*(X)\) and \(S = \mathbb{Q}\) then by the Sullivan theorem \(\pi_i X \simeq \pi_i \text{Map}_{\text{dgCAlg}_{\mathbb{Q}}}(C^*(X), \mathbb{Q})\). By Theorem 4.1 we have that
\( \pi_i X \) is a sub \( \mathbb{Q} \)-vector space of \( \pi_i \text{Map}_{\text{dgAlg}_k}(R, S) \). On the other hand \([1]\), since \( C^*(X) \) is connective, we have that for any \( i > 1 \)

\[
\pi_i \text{Map}_{\text{dgAlg}_k}(C^*(X), \mathbb{Q}) \simeq \text{HH}^{1-i}(C^*(X), \mathbb{Q}),
\]

where \( \text{HH}^* \) is the Hochschild cohomology. Since we have assumed finiteness condition on \( X \), we have that

\[
\text{HH}^{1-i}(C^*(X), \mathbb{Q}) \simeq \text{HH}_{i-1}(C^*(X), \mathbb{Q}).
\]

The functor \( C^*(-, \mathbb{Q}) : \text{Top}^{op} \to \text{E}_\infty \text{dgAlg}_k \) commutes with finite homotopy limits, where \( \text{Top} \) is the category of simply connected spaces. Hence,

\[
\text{HH}_{i+1}(C^*(X), \mathbb{Q}) = H^{i-1}[C^*(X) \otimes_{C^*(X \times X)} \mathbb{Q}] \simeq H^{i-1}(\Omega X, \mathbb{Q}).
\]

We conclude that \( \pi_i X \) is a sub \( \mathbb{Q} \)-vector space of \( H^{i-1}(\Omega X, \mathbb{Q}) \).

More generally by the Block-Lazarev result \([3]\) on rational homotopy theory and \([1]\), we have an injective map of \( \mathbb{Q} \)-vector spaces

\[
\text{AQ}^{-i}(C^*(X), C^*(Y)) \to \text{HH}^{i+1}(C^*(X), C^*(Y)),
\]

where the \( C^*(X) \)-(bi)modules structure on \( C^*(Y) \) is given by \( C^*(X) \to \mathbb{Q} \to C^*(Y) \), and \( \text{AQ}^* \) is the André-Quillen cohomology. We also assume that \( X \) and \( Y \) are simply connected and \( i > 1 \).

More generally,

\[
\pi_i \text{Map}_{\text{E}_\infty \text{dgAlg}_k}(R, S) = \text{AQ}^{-i}(R, S) \to \text{HH}^{-i+1}(R, S) = \pi_i \text{Map}_{\text{dgAlg}_k}(R, S)
\]

is an injective map of \( \mathbb{Q} \)-vector spaces for all \( i > 1 \) and any augmented \( \text{E}_\infty \)-differential graded connective \( \mathbb{Q} \)-algebras \( R \) and \( S \), where the action of \( S \) on \( R \) is given by \( S \to \mathbb{Q} \to R \).

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