On the Willmore Conjecture for Surfaces

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In the course of exploring the global behavior of surfaces in $\mathbb{R}^3$ that are invariant under Euclidean motion of the three-space (a subject went back to Euler, Gauss, Riemann and others), differential geometers made the following major achievements:

1. Question of rigidity and infinitesimal rigidity of surfaces in $\mathbb{R}^3$: the major works are for convex surfaces in three-dimensional Euclidean space. This problem was studied by Hilbert, Hadamard, Cohn-Vossen, Herglotz and Pogorelov. In the late 1940's, Pogorelov [41] generalized the famous work of Cohn-Vossen to the effect that any closed convex surfaces in $\mathbb{R}^3$ is rigid, independent of its regularity. Pogorelov was awarded the Lenin prize for this deep work. It is probably one of the deepest works in convex geometry. On the other hand, I am not sure how many people understand the proof of Pogorelov. Note that it is still an outstanding problem whether there is a nontrivial continuous family of isometric closed surfaces in $\mathbb{R}^3$. Connelly [15] gave a counter-example when the manifold is a non-convex polyhedron. But thirty-five years after Connelly, there is still no smooth counter-example in sight. On the other hand, while infinitesimal rigidity of surfaces was studied extensively, no outstanding theorems were proven so far (except the work of Blaschke [7, 8], but it is for convex surfaces.)

2. Existence of isometric embeddings: the proof of the Weyl embedding problem for surfaces with positive curvature into $\mathbb{R}^3$ was established by Lewy [27], Pogorelov [41] and Nirenberg [36]. The above rigidity theorem is needed for this proof. Existence of local isometric embedding for any surfaces with nonnegative curvature was due to C.-S. Lin [30]. Pogorelov also proved that every metric on sphere with curvature greater than -1 can be isometrically embedded into the hyperbolic three-space with curvature equal to -1. Whether this remains true for surfaces of higher genus is not known. Perhaps a weaker statement is true: every metric on a compact surface can be isometrically embedded into the four-dimensional Minkowski spacetime. An "optimal" embeddings of surfaces with genus zero into Minkowski spacetime can be defined using suitable energy which is related to mean curvature. This was studied by PoNing Chen, Mu-Tao Wang and myself [13]. They are related to the definition of quasi-local mass in general relativity.

3. The solution of Minkowski problem was obtained by Minkowski [34], Lewy [28], Pogorelov and Nirenberg [36]. The problem is to determine the surface by the Gaussian curvature defined on the sphere via the Gauss map. So far, the Minkowski problem works well only for convex surfaces.

4. Hilbert's theorem [22] states that hyperbolic surfaces cannot be isometrically immersed into $\mathbb{R}^3$. There was a spectacular generalization by Efimov [16] that any complete surface with curvature less than -1 cannot be isometrically embedded into $\mathbb{R}^3$. Efimov got the Lenin prize for this work.

5. Carathéodory conjectured that there is at least two umbilical points on a closed convex surface, which was proved for real analytic surfaces only ([19], see also [10] and [25]).

6. Questions related to mean curvature: an old question was to classify complete surfaces with constant mean curvature. The first important question was compact surfaces with constant mean curvature: Hopf [23] proved that for any compact surfaces with genus zero, constant mean curvature implies it is the standard sphere. Alexandroff [2] used his famous reflection principle to prove that any embedded compact surfaces with constant mean curvature is a round sphere. In 1986, Wente [47] produced an immersed torus with constant mean curvature. A closely related question is of course the classification of complete minimal surfaces in $\mathbb{R}^3$. Complete classification of complete embedded minimal surfaces in $\mathbb{R}^3$ withe genus zero was done by Meeks and Rosenberg [32]. Higher genus surfaces with finite topology are still not classified. Another problem is to understand the quality behavior of minimal surfaces. Meeks-Yau used the method of Dehn's lemma to prove that any solution of the Plateau problem for extremal curves are embedded.

Let us now review a new development due to Marques and Neves [31]: there are two major invariants for surfaces: Gaussian curvature and mean curvature. The integral of Gaussian curvature gives the Euler number which is the Gauss-Bonnet theorem. The integral of the square of the Gaussian curvature is the Yang-Mills functional.

Minkowski has written a few important integral identities that involves mean curvature, Gaussian curvature and the support functions. They are important for various purposes. But a very interesting invariant is the integral of the square of the mean curvature. This functional was proposed by Poisson [40] in 1812 and later by S. Germain [18] in 1821 to describe elastic energy. It was used by Helfrich [21] to describe the energy of cell membranes. One likes to minimize the energy for surfaces in three-space with a fixed topology and see what we can achieve. One can fix the boundary and the normal vector field along the boundary and minimize the energy for surfaces with such boundary. This is useful to understand the theory of elasticity and the construction in computer graphics. If the surface has genus zero and has no boundary, it is easy to prove that the surface with lowest energy is the round sphere and the energy is 4π. This fact motivates Hawking [20] to define a concept of quasi-local mass for a surface in a three-dimensional time-symmetric space slice of a space-time. His definition is called Hawking mass and is defined to be the square root of area multiplied by ($4\pi-$energy of the surface). This mass has been useful to study the famous Penrose conjecture. A part of it was solved by Huisken and Ilmanen [24]. In view of the proof of the Willmore conjecture, it may be interesting to know whether it makes sense to change the 4π in the definition of Hawking mass when the surface is a torus instead of a sphere.

In 1965, T. J. Willmore [48] observed that if the genus of the surface is one, it is difficult to find examples with energy close to 4π. Instead, after computations, one finds that all the known examples of tori embedded in $\mathbb{R}^3$ have energies greater than the torus obtained by rotating a circle along another circle. The energies of such tori are always greater than $2\pi^2$. This is the Willmore conjecture, which is among the most famous problems in global geometry. (I did make a related conjecture that the first eigenvalue of any embedded minimal surface in the three-sphere is equal to 2. A universal lower bound is proved by Choi and Wang [14], but the optimal constant is not done yet.)

The simplicity of the Willmore conjecture has attracted many differential geometers to look into this problem. A simple idea is to look at the critical points of this energy. In contrast to the minimal surface equation, this is a nonlinear fourth order elliptic equation. In fact, the Euler-Lagrange equation, attributed by Thomsen [45] to Schadow, is given by

$$\Delta H + 2(H^2 - K)H = 0.$$

The equation is a higher order nonlinear elliptic equation and is more complicated than the minimal surface equation. Nevertheless, in 1983, Leon Simon [44] proved the existence of a smooth torus with global minimum of Willmore energy. This work was generalized by Bauer and Kuwert [6] to higher genus. But we do not know much about the value of the energy. It is known that the minimum of the Willmore energy among all surfaces with genus g is less than 8π and approaches 8π when g approaches infinity. On the other hand, in 1982, Li and Yau [29] found that for any immersed surfaces that are not embedded, the Willmore energy is always greater than 8π. Hence in the process of proving the Willmore conjecture, we can always assume the surface with minimum Willmore energy is embedded. This avoids much complexity of the topology of the surface with minimum energy.

In most of the attempts to prove the Willmore conjecture, we use the knowledge that this energy is invariant under conformal changes of the ambient space. This was noticed by Blaschke [9] and Thomsen [45] in the 1920's.

This observation allows us to consider the problem to be surfaces in the three-sphere, as conformally $\mathbb{R}^3$ is part of the sphere. And immediately we see that the minimal surfaces in the three-sphere play an important role as they are automatically critical points of the Willmore functional. It may be a wishful thinking that the converse is true. But Pinkall [37] has constructed infinitely many embedded Willmore torus that are not conformal to a minimal surface in the sphere. Bryant [12] also constructed infinitely many immersed real projective surfaces which are Willmore surfaces and not minimal.

In the above mentioned work of Li and Yau, a concept of conformal area was introduced. Based on this, it was proved that the Willmore conjecture is indeed true for a region of conformal structures on the moduli space, and the region was extended by Montiel and Ros [35]. Ros [42] also proved the Willmore conjecture holds for torus invariant under antipodal map, or symmetric with respect to a point in $\mathbb{R}^3$.

Willmore problem has drawn attentions from mathematicians from different disciplines. This includes people using integrable system to study the Willmore conjecture. But so far the task has not been accomplished in this direction.

Marques and Neves finally achieved the proof of the Willmore conjecture in 2012. They proved that the Willmore energy for any torus is not less than $2\pi^2$ and is equal to it only for the Clifford torus in the three-sphere. We shall explain briefly how it was achieved.

It is already clear from the conformal area argument of Li-Yau that minimal surfaces of the sphere should play an important role and also that certain canonical deformations of surfaces in three-sphere are important: the effective action of the Möbius group of conformal deformations of the three-sphere on a surface can be parametrized by the four-dimensional ball bounded by the three-sphere.

In fact, for each vector v in this four-dimensional ball, one can associate a conformal diffeomorphism of the three-sphere and hence a surface which is the image of the conformal diffeomorphism. Then one can create an one-parameter family of embedded surfaces by taking signed distance from this new embedded surface. It was observed by Ros [43] that the area of each surface is less than the Willmore energy of the original surface (due to conformal invariance).

All together, we create a five-dimensional family of surfaces in the three-sphere. And associate to such family of surfaces, we can perform a Min-Max argument to obtain an embedded surface in the three-sphere whose area is less than the Willmore energy of the original surface.

The crucial problem is to prove that we can keep the topology to be nontrivial and that the minimal surface has area greater than 4π. Marques and Neves do this through a topological argument. They show that if the minimal surface had area 4π then the map from the boundary of the five-parameter family into the space of round spheres would be homotopically trivial. On the other hand we can compute the degree of this map and show it is the genus of the initial surface, and hence non-zero by hypothesis. Also we need to prove the index of this new minimal surface in the three-sphere is not greater than five. Then a theorem of Urbano [46] proved that the minimal surface must be the Clifford torus which is the product of two circles with radius $\frac{1}{\sqrt{2}}$ siting in the first coordinate plane and the last coordinate plane respectively. The area of this Clifford torus is exactly $2\pi^2$. And hence the proof of the Willmore conjecture. There is tremendous technical problems to be overcome in this proposed approach. The Min-Max argument is building on the theory of Almgren-Pitts [3, 39] where the convergence is based on mass norm.

Ian Agol, Marques and Neves [1] made use the argument of the proof of the Willmore conjecture to solve a conjecture made by Freedman, He and Wang [17] in 1994 that the Möbius energy of a nontrivial link in $\mathbb{R}^3$ is always not less than $2\pi^2$ and that it is equal to $2\pi^2$ if and only if the link is given by the stereographic projection of the standard Hopf link in three-sphere: two separate circles in each coordinate two-planes.

The Möbius energy of two space curves is defined by

$$E(\gamma_1,\gamma_2) = \int_{S^1\times S^1} \frac{|\gamma_1'(s)||\gamma_2'(t)|}{|\gamma_1(s)-\gamma_2(t)|^2} dsdt.$$

and it is invariant under Möbius transformations of $\mathbb{R}^3$.

After the settlement of the Willmore conjecture, Simon Brendle [11] used the "non-collapsing" argument which was first introduced by Ben Andrews [4] in 2011, and exploited the additional special structure coming from the minimal surface condition to solve the Lawson conjecture ([26], 1970): the only embedded minimal torus Σ in the three-sphere is the Clifford torus. By modifying Andrews' function $Z(x,y)$, Brendle can show that the largest interior ball touching Σ at any point $x\in\Sigma$ has curvature equal to the maximum principal curvature of Σ at x. Since Σ is minimal, there is no difference between the inside and outside. So the same thing holds for exterior touching ball. In fact, Brendle showed that the principal curvatures have to be constant. Therefore the surface Σ is congruent to the Clifford torus.

After Simon Brendle announced his paper on Lawson conjecture in the Internet, Ben Andrews and Haizhong Li [5] extended the same "non-collapsing" argument to give a complete classification of embedded CMC tori in the three-sphere. When the constant mean curvature is equal to zero or $\pm\frac{1}{\sqrt{3}}$, the only embedded torus is the Clifford torus or $S^1(\frac{1}{2})\times S^1(\frac{\sqrt{3}}{2})$. For other values of the mean curvature, there exist embedded tori which are not $S^1(r)\times S^1(\sqrt{1-r^2})$. In fact, by studying the monotonicity of the period of a family of immersion rotational surface with constant mean curvature H, they give a complete classification of embedded CMC tori in S^3. We remark that the uniqueness of the embedded torus with constant mean curvature $H=\pm\frac{1}{\sqrt{3}}$ is unexpected, which is an analogue to the Lawson conjecture in the minimal case. As a Corollary, Andrews and Li's Theorem solved the famous Pinkall-Sterling conjecture ([38], 1989): the embedded tori with CMC in the three-sphere are surfaces of revolution.

There are many questions remained to be answered, after the Willmore conjecture is proved. Naturally one likes to know the higher genus generalization. We also like to know the answer to the following question: after fixing the conformal structure on the Riemann surface, one can find an embedded surface in three-space that minimizes the Willmore energy, how unique such surfaces are? Would it be unique for a generic conformal structure? When we deform the conformal structures, will the representative in three-space move smoothly?

The Willmore energy also defines a function on the Teichmüller space, what kind of function is it? For the Riemann surface which is the quotient by an arithmetic group, there are special points of arithmetic interest on the surface, can we find special properties of these points when we realize the surface in three-space? Can one relate the spectrum of the Laplacian on the Riemann surface with the spectrum of the Laplacian of the embedded surface?

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