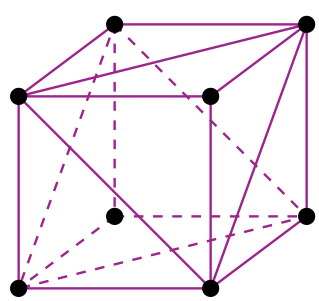
# Triangulations of Manifolds

## by Ciprian Manolescu[*,†]

In topology, a basic building block for spaces is the $n$-simplex. A 0-simplex is a point, a 1-simplex is a closed interval, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. In general, an $n$-simplex is the convex hull of $n+1$ vertices in $n$-dimensional space. One constructs more complicated spaces by gluing together several simplices along their faces, and a space constructed in this fashion is called a *simplicial complex*. For example, the surface of a cube can be built out of twelve triangles—two for each face, as in the following picture:



Apart from simplicial complexes, manifolds form another fundamental class of spaces studied in topology. An $n$-dimensional topological manifold is a space that looks locally like the $n$-dimensional Euclidean space; i.e., such that it can be covered by open sets (charts) homeomorphic to $\mathbb{R}^n$. One can also consider topological manifolds with additional structure:

(i) A *smooth manifold* is a topological manifold equipped with a (maximal) open cover by charts such that the transition maps between charts are smooth ($C^\infty$);

(ii) A $C^k$ *manifold* is similar to the above, but requiring that the transition maps are only $C^k$, for $0 \leq k < \infty$. In particular, $C^0$ manifolds are the same as topological manifolds. For $k \geq 1$, it can be shown that every $C^k$ manifold has a unique compatible $C^\infty$ structure. Thus, for $k \geq 1$ the study of $C^k$ manifolds reduces to that of smooth manifolds.

(iii) A *piecewise linear manifold* is a topological manifold equipped with a (maximal) open cover by charts such that the transition maps are piecewise linear.

Manifolds are ubiquitous in many parts of mathematics; for instance, they can appear as spaces of solutions to systems of polynomial equations, or to systems of differential equations. However, knowing that a space is a manifold does not tell us much about its global structure. To study the properties of a manifold, it is helpful to triangulate it, that is, to construct a homeomorphism to a simplicial complex. For example, the surface of a sphere is a two-dimensional manifold, homeomorphic to the cube. Hence, the sphere admits a triangulation with twelve triangles. (Of course, it also admits many other triangulations.) A triangulation yields a combinatorial description for the manifold. Furthermore, if we have two manifolds and we try to tell them apart, the first thing to do is to check if their topological invariants (such as their homology groups) are the same. If we are able to triangulate the manifolds, it is straightforward to compute their homology groups in terms of the two triangulations.

The first question about triangulating manifolds was formulated by Poincaré [Poi99] in 1899. In modern language, it reads:

**Question 1.** *Does every smooth manifold admit a triangulation?*

In 1924, Kneser [Kne26] asked a more general question:

[*] Department of Mathematics, University of California at Los Angeles
E-mail: cm@math.ucla.edu
[†] The author was supported by NSF grant DMS-1104406.

**Question 2.** *Does every topological manifold admit a triangulation?*

One class of manifolds that are easy to triangulate are the piecewise linear ones. In fact, every piecewise linear manifold admits a *combinatorial triangulation*, that is, one in which the manifold structure is evident. Technically, in a simplicial complex $K$, one defines the link of a simplex $\sigma$ to be the union of the simplices $\tau \in K$ such that $\sigma \cap \tau = \emptyset$ and $\sigma$ and $\tau$ are both faces of a simplex in $K$. Then, a triangulation is called combinatorial if the link of every simplex is a sphere.

Notice that most of the triangulations of a manifold that one can think of are combinatorial. The simplest way to construct a non-combinatorial triangulation is to first triangulate a non-trivial homology sphere (a manifold with the same homology groups as the sphere, but not a sphere), and then to take its double suspension. One then needs to appeal to the Double Suspension Theorem, proved in the 1970's by Edwards and Cannon, to see that the resulting space is a manifold (in fact, a sphere).

One can ask the following strengthened version of Question 2:

**Question 3.** *Does every topological manifold admit a combinatorial triangulation (or, equivalently, a piecewise linear structure)?*

Answering the three questions above has inspired much research in topology. The first attempt to give an affirmative answer to Question 1 was made by Poincaré himself, but his proof lacked rigor. A complete proof was found in the 1940's:

**Theorem 1** (Cairns [Cai35]; Whitehead [Whi40]). *Every smooth manifold admits an (essentially unique) compatible piecewise linear structure.*

The answers to Questions 2 and 3 are affirmative in low dimensions ($\leq 3$). Indeed, in Kneser's time, it was already known that every two-dimensional surface has a piecewise linear structure, due to the work of Radó [Rad25]. In 1952, Moise showed that any three-dimensional manifold is smooth, and thus piecewise linear [Moi52].

In 1969, Kirby and Siebenmann showed that there exist manifolds without piecewise linear structures in any dimension greater than 4. They also answered in the negative the related Hauptvermutung for manifolds—the question of uniqueness for piecewise linear structures:

**Theorem 2** (Kirby-Siebenmann [KS69]). *A topological manifold of dimension $d \geq 5$ admits a piecewise linear structure if and only if a certain obstruction class $\Delta(M) \in H^4(M;\mathbb{Z}/2)$ vanishes. For every $d \geq 5$, there exist manifolds $M$ such that $\Delta(M) \neq 0$. Further, if $\Delta(M) = 0$, then piecewise linear structures on $M$ are classified by elements in $H^3(M;\mathbb{Z}/2)$.*

This shows that the answer to Question 3 is "no" in dimensions $\geq 5$.

Dimension four is very special in topology, and new techniques were needed in that case. In the early 1980's, Freedman revolutionized four-dimensional topology, and in particular gave an example of a four-manifold (the $E_8$ manifold) that has no piecewise linear structures [Fre82]. Thus, Question 3 has the answer "no" in dimension four.

The first non-triangulable manifolds were also found in dimension four: In the mid 1980's, Casson introduced a new invariant of homology 3-spheres [AM90]. This can be used it to show that, for example, Freedman's $E_8$ manifold is not triangulable. Hence, Question 2 has a negative answer in dimension four, too.

This left open Question 2 in dimensions greater than 4. In the 1970's, this problem had been shown to be equivalent to a different problem, about 3-manifolds and homology cobordism. The three-dimensional homology cobordism group $\Theta_3^H$ is generated by oriented homology 3-spheres, modulo the following equivalence relation: $Y_0 \sim Y_1$ if there exists a smooth (or, equivalently, piecewise linear), compact, oriented four-manifold $W$ with $\partial W = (-Y_0) \cup Y_1$ and such that $W$ has the homology of $[0,1] \times S^3$. The group $\Theta_3^H$ makes an appearance in questions about triangulations because the links of codimension four simplices are homology 3-spheres; and modifying the triangulation in a certain way produces cobordisms $W$ of the form above. In principle, one could also consider links of simplices of arbitrary codimension. However, the analogous homology cobordism group is trivial in all dimensions $\neq 3$, by the work of Kervaire [Ker69].

In dimension 3, the group $\Theta_3^H$ is non-trivial. This can be shown by the existence of a surjective homomorphism $\mu : \Theta_3^H \to \mathbb{Z}/2$, called the *Rokhlin homomorphism*.

Consider the following short exact sequence:

$$0 \longrightarrow \ker(\mu) \longrightarrow \Theta_3^H \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \tag{1}$$

and the associated long exact sequence in cohomology

$$\begin{aligned} \dots \longrightarrow H^4(M;\Theta_3^H) \xrightarrow{\mu_*} H^4(M;\mathbb{Z}/2) \\ \xrightarrow{\delta} H^5(M;\ker(\mu)) \longrightarrow \dots, \end{aligned} \tag{2}$$

**Theorem 3** (Galewski-Stern [GS80]; Matumoto [Mat78]). *A topological manifold $M$ of dimension $\geq 5$ is triangulable if and only if $\delta(\Delta(M)) = 0 \in H^5(M;\ker(\mu))$, where $\Delta(M)$ is the Kirby-Siebenmann obstruction. If*

$\delta(\Delta(M)) = 0$, *then triangulations are classified (up to a relation called concordance) by elements in* $H^4(M;\ker(\mu))$. *Further, in any dimension* $\geq 5$, *there exists a* $d$*-dimensional manifold* $M$ *with* $\delta(\Delta(M)) \neq 0$ *if and only if the short exact sequence* (1) *does not split.*

In view of this theorem, Question 3 in higher dimensions is equivalent to the question of whether (1) splits. A splitting would consist of an element $[Y] \in \Theta_3^H$ such that $2[Y] = 0$ and $\mu([Y]) = 1$. One way of showing that such a $[Y]$ does not exist is to construct a lift of the Rokhlin homomorphism to $\mathbb{Z}$. This was done by the author in [Man13]:

**Theorem 4** (Manolescu [Man13]). *There exists a map* $\beta : \Theta_3^H \to \mathbb{Z}$ *such that* $\beta(\bmod 2) = \mu$ *and* $\beta(-[Y]) = -\beta([Y])$, *for all* $Y$. *Hence, the exact sequence* (1) *does not split, and the answer to Question 3 is "no" in all dimensions* $\geq 5$.

The construction of $\beta$ involves techniques from gauge theory: namely, a new version of Floer homology called Pin(2)-equivariant Seiberg-Witten Floer homology. Gauge theory is the study of certain elliptic partial differential equations that first appeared in physics—they govern the weak and strong interactions between particles. In the 1980's, Donaldson pioneered the use of gauge theory in low-dimensional topology [Don83]. Out of gauge theory came Floer homology, an invariant associated to three-manifolds that is particularly useful in studying cobordisms. (A cobordism between two three-manifolds $Y$ and $Y'$ is a four-manifold with initial boundary $Y$ and final boundary $Y'$.) Floer homology is what Atiyah called a topological quantum field theory (TQFT) [Ati88]. The main property of a TQFT is that a cobordism from $Y$ to $Y'$ induces a map between the respective invariants (in this case, their Floer homologies). This should be contrasted with what happens in ordinary homology, where we need an actual map (not a cobordism!) between $Y$ and $Y'$ to get a map between their homologies. The various kinds of Floer homologies (instanton, Seiberg-Witten, Heegaard Floer) are the main tool for studying cobordisms between 3-manifolds, and the answer to the Galewski-Stern-Matumoto problem is only one of their many applications.

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