ON POISSON FUNCTIONS

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In this paper, defining Poisson functions on super manifolds, we show that the graphs of Poisson functions are Dirac structures, and find Poisson functions which include as special cases both quasi-Poisson structures and twisted Poisson structures.

1. Introduction

In this paper, we define Poisson functions on super manifolds as a generalization of Poisson structures on manifolds, and show that quasi-Poisson and twisted Poisson structures are both special cases of Poisson functions on some supermanifolds. Quasi-Poisson structure are introduced by Alekseev, Kosmann-Schwarzbach, and Meinrenken [AK, AKM]. They are defined by an invariant bivector field $\pi$ on a manifold $M$ with a group action such that the Schouten bracket $[\pi, \pi]$ equals the trivector field generated by the Cartan 3-tensor $\Psi$. A twisted Poisson structure is a bivector field $\pi$ on a manifold $M$ such that the Schouten bracket $[\pi, \pi]$ equals the trivector field associated to a closed 3-form $\Phi$ on $M$ [P, KS, SW].

In the work [SW], Ševera and Weinstein interpret twisted Poisson structures in terms of Courant algebroid and Dirac structure, and ask whether there is a general notion which incorporates both quasi-Poisson and twisted Poisson structures. In this paper, first, generalizing Theorem 6.1 of Liu–Weinstein–Xu [LWX], we show that the graphs of Poisson functions are Dirac structures. Second, we show that the notion of Poisson function includes various notions: Poisson structure, twisted Poisson structure, quasi-Poisson structure, Lie algebra action, Lie bialgebra action, Poisson action, etc. In particular, we find Poisson functions which include as special cases both quasi-Poisson structures and twisted Poisson structures. Moreover, a Lie algebroid structure in Theorem 4.1 of Lu [L] associated to a Poisson action of a Poisson Lie group is understood in this more general context.
In the interesting paper [K], Kosmann-Schwarzbach, following Roytenberg [R2], studies weaker versions of Poisson structures by using Poisson functions as “twistings,” and, with many other results, points out a similarity of quasi-Poisson structures and twistings of Lie quasi-bialgebroids. Independently, Bursztyn and Crainic also relate Hamiltonian quasi-Poisson structures and twisted Poisson structures in [BC], and give a geometric way to construct Lie algebroids associated with quasi-Poisson structures in [BCS] with Ševera.

This paper is mainly based on ideas of Vaintrob [V] who interprets Lie algebroid structures as homological vector fields on supermanifolds, and Roytenberg [R1] who gets Courant algebroids from homological functions on supermanifolds.

2. Poisson functions

For a smooth vector bundle $V \to M$ on a smooth manifold $M$, we have a supermanifold $T^*\Pi V$ with canonical Poisson bracket $\{\ ,\ }$. A choice of a local coordinate system $(x^i)$ on $M$ and a local basis $(\xi^a)$ of sections of $V^*$ induces a local coordinate system $(x^i, \xi^a)$ on $\Pi V$ and a local coordinate system $(x^i, \xi^a, p_i, \theta_a)$ on $T^*\Pi V$. The ring of functions on the supermanifold $T^*\Pi V$ is equipped with a bidegree which is compatible with the parity, by assigning bidegree $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ to $(x^i, \xi^a, p_i, \theta_a)$, respectively.

Definition 2.1. A homological function on a supermanifold with an even Poisson bracket $\{\ ,\ }$ is an odd function $S$ satisfying $\{S, S\} = 0$.

An impressive result of D. Roytenberg [R1] is that for a homological function $S$ of total degree 3 on $T^*\Pi V$ we have a Courant algebroid structure on $V \oplus V^*$ with

- Loday bracket on $\Gamma(V \oplus V^*)$:
  
  $[a, b]_S := \{\{a, S\}, b\}$

- anchor map on $\Gamma(V \oplus V^*)$:

  $(\tau(a))(f) := \{\{a, S\}, f\}$

- inner product on $\Gamma(V \oplus V^*)$:

  $(a, b) := \{a, b\}$

- map $\epsilon : C^\infty(M) \to \Gamma(V \oplus V^*)$:

  $\epsilon(f) := -\frac{1}{2}\{f, S\}$,

where we identify sections of $V \oplus V^*$ with functions of total degree 1 on $T^*\Pi V$. 
For a function $\sigma$ of degree $(0, 2)$, a canonical transformation

$$e^\sigma(a) := a + \{a, \sigma\} + \frac{1}{2} \{\{a, \sigma\}, \sigma\} + \cdots$$

preserves the total degree and the Poisson bracket $\{\ , \}$:

$$\{e^\sigma(a), e^\sigma(b)\} = e^\sigma\{a, b\}.$$

Therefore, for a homological function $S$ of total degree 3, the function $e^\sigma(S)$ is also of total degree 3 and homological.

**Definition 2.2** (see [K, P, R2]). Let $X$ be a super manifold with an even Poisson bracket and a compatible bidegree, and let $S$ be a homological function on $X$ of total degree 3. A Poisson function with respect to $S$ is a function $\sigma$ of degree $(0, 2)$ such that the $(0, 3)$-component $(e^{-\sigma}(S))^{0, 3}$ vanishes.

**Remark 2.3.** This condition is equivalent to the “Maurer–Cartan” equation:

$$S^{0, 3} - \{S^{1, 2}, \sigma\} + \frac{1}{2!} \{\{S^{2, 1}, \sigma\}, \sigma\} - \frac{1}{3!} \{\{\{S^{3, 0}, \sigma\}, \sigma\}, \sigma\} = 0.$$

**Remark 2.4.** As D. Roytenberg [R2] observes, this condition gives a quasi-Lie bialgebroid structure on $(V, V^*)$ (see also [K, P, HP]).

**Theorem 2.5.** The graph $\Gamma_\sigma = \{\alpha + \{\alpha, \sigma\} : \alpha \in \Gamma(V^*)\}$ is an isotropic and integrable subbundle, i.e., a Dirac subbundle, of the Courant algebroid $V \oplus V^*$ if and only if $\sigma$ is a Poisson function.

**Proof.** First, we note that

$$e^\sigma(\alpha) = \alpha + \{\alpha, \sigma\}$$

for any $(1, 0)$-function $\alpha$ because the bracket $\{\ , \}$ has degree $(-1, -1)$. Then, for any $(1, 0)$-functions $\alpha, \beta$, we have

$$(\alpha + \{\alpha, \sigma\}, \beta + \{\beta, \sigma\}) = (e^\sigma(\alpha), e^\sigma(\beta))$$

$$= \{e^\sigma(\alpha), e^\sigma(\beta)\}$$

$$= e^\sigma\{\alpha, \beta\}$$

$$= 0$$

and

$$[\alpha + \{\alpha, \sigma\}, \beta + \{\beta, \sigma\}] = [e^\sigma(\alpha), e^\sigma(\beta)]$$

$$= e^\sigma[e^\sigma(\alpha), e^\sigma(\beta)S]$$

$$= e^\sigma[e^\sigma(\alpha), e^\sigma(\beta)]e^{-\sigma}(S)$$

$$= e^\sigma[e^\sigma(\alpha), e^\sigma(\beta)](e^{-\sigma}(S))^{0, 3} + (e^{-\sigma}(S))^{1, 2},$$
where we use in the last equation the fact that the bracket $\{ \ , \ \}$ has degree $(-1,-1)$. Therefore, $\sigma$ is a Poisson function if and only if
\[
[\alpha + \{\alpha, \sigma\}, \beta + \{\beta, \sigma\}] S = e^\sigma [\alpha, \beta]_{(e^{-\sigma(S)})^{1.2}}
\]
\[
= [\alpha, \beta]_{(e^{-\sigma(S)})^{1.2}} + \{[\alpha, \beta]_{(e^{-\sigma(S)})^{1.2}}, \sigma\},
\]
which means that the graph $\Gamma_\sigma$ is integrable. This completes the proof of Theorem 2.5. \qed

When a given homological function $S$ has degree $(1,2) + (2,1)$, this proof gives a proof of Theorem 6.1 in Liu–Weinstein–Xu [LWX].

3. Quasi-Poisson and twisted Poisson structures

For a smooth manifold $M$ and a Lie algebra $\mathfrak{g}$ with structure constants $f_{ab}^c$ for a basis $(\tau_a)$, we consider the supermanifold $X = T^*\Pi TM \times \Pi \mathfrak{g}^*$ with local coordinates $(x^i, \xi^i, \tau_a)$ on $\Pi TM \times \Pi \mathfrak{g}^*$ and conjugate local coordinates $(p_i, \theta_i, \eta_a)$. Each 3-form
\[
\Phi = \frac{1}{3!} \Phi_{ijk} \xi^i \xi^j \xi^k
\]
and each skew-symmetric 3-tensor
\[
\Psi = \frac{1}{3!} \Psi_{abc} \tau_a \tau_b \tau_c
\]
give a homological function
\[
S = S_\mathfrak{g} + S_M + \Psi + \Phi
\]
\[
= \frac{1}{2!} f_{abc} \eta^a \eta^b \tau_c + \xi^i p_i + \frac{1}{3!} \Psi_{abc} \tau_a \tau_b \tau_c + \frac{1}{3!} \Phi_{ijk} \xi^i \xi^j \xi^k
\]
of degree 3 on $X$ when $\Phi$ is closed.

\[
\{S_M, \Phi\} = 0
\]
with respect to $S_M$ and $\Psi$ is closed.

\[
\{S_\mathfrak{g}, \Psi\} = 0
\]
with respect to $S_\mathfrak{g}$. A $(0,2)$-function
\[
\sigma = \pi + \rho
\]
is a Poisson function with respect to $S$ if and only if

- $-\{S_\mathfrak{g}, \rho\} + \frac{1}{2!}\{\{S_M, \rho\}, \rho\} = 0$
- $\{\{S_M, \pi\}, \rho\} + \{\{S_M, \rho\}, \pi\} = 0$
- $\frac{1}{2!}\{\{S_M, \pi\}, \pi\} - \frac{1}{3!}\{\{\Psi, \rho\}, \rho\} - \frac{1}{4!}\{\{\Phi, \pi\}, \pi\} = 0$.

In the special case when $\Phi = 0$, we have

- $-\{S_\mathfrak{g}, \rho\} + \frac{1}{2!}\{\{S_M, \rho\}, \rho\} = 0$
• \{\{S_M, \pi\}, \rho\} + \{\{S_M, \rho\}, \pi\} = 0
• \frac{1}{2!}\{\{S_M, \pi\}, \pi\} - \frac{1}{3!}\{\{\Psi, \rho\}, \rho\} = 0.

These conditions correspond to the following.
• \(\rho\) is a representation of: \(\rho : g \rightarrow \Gamma(TM)\)
• \(\pi\) is invariant for the action \(\rho\)
• \(\pi\) is a quasi-Poisson structure with respect to \(\Psi\) and \(\rho\), when there exists an invariant inner product on \(g\) and \(\Psi\) is the associated Cartan 3-tensor. In the special case when \(\rho = 0\), we have
\[
\frac{1}{2!}\{\{S_M, \pi\}, \pi\} - \frac{1}{3!}\{\{\Phi, \pi\}, \pi\} = 0
\]
which means that \(\pi\) is a twisted Poisson structure with respect to \(\Phi\).

**Remark 3.1.** This interpretation of quasi-Poisson structures gives a clear view to the quasi-Poisson cohomology defined by [AKM]. In fact, the differential of the quasi-Poisson cohomology is the restriction to the subspace of \(G\)-invariant multivectors \(C^\infty(M, \wedge TM)^G\) of the differential
\[
d = \{e^{-\sigma}(S))^{1,2}, \cdot\}
\]
\[
= \{-\{S_M, \pi\} - \{S_M, \rho\} + \frac{1}{2}\{\{\Psi, \rho\}, \rho\}, \cdot\}
\]
on the space of \((0, \ast)\)-functions \(C^\infty(M, \wedge TM) \otimes \wedge g^*\).

4. Lu’s Lie algebroid

For a smooth manifold \(M\) and a Lie bialgebra \((g, g^*)\) with structure constants \(f^a_{bc}, \gamma^a_{bc}\) for a basis \((\tau_a, \eta^a)\) we consider the supermanifold \(X = T^*(\Pi TM \times \Pi g^*)\) with local coordinates \((x^i, \xi^i, \tau_a)\) on \(\Pi TM \times \Pi g^*\) and conjugate local coordinates \((p_i, \theta_i, \eta^a)\). Each 3-form
\[
\Phi = \frac{1}{3!}\Phi_{ijk}\xi^i\xi^j\xi^k
\]
and each skew-symmetric 3-tensor
\[
\Psi = \frac{1}{3!}\Psi^{abc}\tau_a\tau_b\tau_c
\]
give a homological function
\[
S = S_g + S_{g^*} + S_M + \Psi + \Phi
\]
\[
= \frac{1}{2!} f^c_{ab}\eta^b\eta^c + \frac{1}{2!} \gamma^a_{bc}\eta^b\tau_c + \xi^i p_i + \frac{1}{3!}\Psi^{abc}\tau_a\tau_b\tau_c + \frac{1}{3!}\Phi_{ijk}\xi^i\xi^j\xi^k
\]
of degree 3 on \(X\) when \(\Phi\) is closed
\[
\{S_M, \Phi\} = 0
\]
with respect to \(S_M\) and \(\Psi\) is closed
\[
\{S_g + S_{g^*}, \Psi\} = 0
\]
with respect to $S_g + S_{g^*}$. A $(0, 2)$-function
\[ \sigma = \pi + \rho \]
\[ = \frac{1}{2!} \pi^{ij} \theta_i \theta_j + \rho^a \eta^a \theta_j \]
is a Poisson function with respect to $S$ if and only if
\[ -\{ S_g, \rho \} + \frac{1}{2!} \{ \{ S_M, \pi \}, \rho \} = 0 \]
\[ \{ \{ S_M, \pi \}, \rho \} \]
\[ + \frac{1}{2!} \{ \{ S_M, \pi \}, \pi \} - \frac{1}{3!} \{ \{ \Psi, \rho \}, \rho \} - \frac{1}{3!} \{ \{ \Phi, \pi \}, \pi \} = 0. \]

In the special case when $\Phi = 0$ and $\Psi = 0$, we have
\[ -\{ S_g, \rho \} + \frac{1}{2!} \{ \{ S_M, \rho \}, \rho \} = 0 \]
\[ \{ \{ S_M, \pi \}, \rho \} \]
\[ + \frac{1}{2!} \{ \{ S_M, \pi \}, \pi \} = 0. \]

These conditions correspond to the following.
\[ \rho \] is a representation of: $\rho : g \rightarrow \Gamma(TM)$
\[ \rho \] is an infinitesimal Poisson action of the Lie bialgebra $(g, g^*)$
\[ \pi \] is a Poisson structure.

For each Poisson function $\sigma$, Theorem 2.5 gives a Lie algebroid structure on the graph $\Gamma_\sigma$ which is equivalent to the Lie algebroid structure on $T^*M \times g$ in Theorem 4.1 of J.-H. Lu [L] associated to a Poisson action of a Poisson Lie group.

References


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