Let $S$ be a $C^\infty$ compact connected oriented surface whose genus is at least two. Let $\mathcal{P}(S)$ be the moduli space of isotopic classes of projective structures associated to $S$. The natural holomorphic symplectic form on $\mathcal{P}(S)$ will be denoted by $\Omega_P$. The natural holomorphic symplectic form on the holomorphic cotangent bundle $T^*\mathcal{T}(S)$ of the Teichmüller space $\mathcal{T}(S)$ associated to $S$ will be denoted by $\Omega_T$. Let $e : \mathcal{T}(S) \to \mathcal{P}(S)$ be the holomorphic section of the canonical holomorphic projection $\mathcal{P}(S) \to \mathcal{T}(S)$, given by the Earle uniformization. Let $T_e : T^*\mathcal{T}(S) \to \mathcal{P}(S)$ be the biholomorphism constructed using the section $e$. We prove that $T_e^*\Omega_P = \pi \cdot \Omega_T$. This remains true if $e$ is replaced by a large class of sections that include the one given by the Schottky uniformization.

1. Introduction

A projective structure on a smooth compact connected oriented surface $S$ is defined by giving a covering of $S$ by coordinate charts, where the coordinate functions are orientation preserving diffeomorphisms to open subsets of $\mathbb{C}$, such that all the transition functions are Möbius transformations. Two projective structures are called equivalent if they differ by a diffeomorphism of $S$ homotopic to the identity map. Let $\mathcal{P}(S)$ denote the equivalence classes of projective structures on $S$.

The Teichmüller space $\mathcal{T}(S)$ for $S$ parametrizes all the equivalence classes of complex structures on $S$ compatible with its orientation; two complex structures are called equivalent if they differ by a diffeomorphism of $S$ homotopic to the identity map. Both $\mathcal{P}(S)$ and $\mathcal{T}(S)$ are complex manifolds,
and \( \dim \mathcal{P}(S) = 2 \cdot \dim \mathcal{T}(S) \). There is a natural surjective holomorphic submersion

\[
\varphi : \mathcal{P}(S) \longrightarrow \mathcal{T}(S)
\]

that sends a projective structure on \( S \) to the underlying complex structure on \( S \).

The above projection \( \varphi \) makes \( \mathcal{P}(S) \) a torsor over \( \mathcal{T}(S) \) for the holomorphic cotangent bundle \( T^*\mathcal{T}(S) \). This means in particular that the fiber of \( \varphi \) over any point \( X \in \mathcal{T}(S) \) is an affine space for the space of all holomorphic quadratic differentials on the Riemann surface corresponding to \( X \).

Consequently, any smooth section \( f : \mathcal{T}(S) \longrightarrow \mathcal{P}(S) \) of the above projection \( \varphi \) produces a diffeomorphism

\[
T_f : T^*\mathcal{T}(S) \longrightarrow \mathcal{P}(S)
\]

that sends \((X, \omega)\) to the projective structure \( f(X) + \omega \), where \( X \in \mathcal{T}(S) \) and \( \omega \) is a holomorphic quadratic differential on the Riemann surface \( X \). If the section \( f \) is holomorphic, then \( T_f \) is a biholomorphism.

Both \( T^*\mathcal{T}(S) \) and \( \mathcal{P}(S) \) are equipped with natural holomorphic symplectic structures. Let \( \Omega_{\mathcal{T}} \) (respectively, \( \Omega_{\mathcal{P}} \)) denote the canonical symplectic form on \( T^*\mathcal{T}(S) \) (respectively, \( \mathcal{P}(S) \)).

Assume that \( \text{genus}(S) \geq 2 \). We prove the following (Theorem 3.1):

**Theorem 1.1.** Let

\[
e : \mathcal{T}(S) \longrightarrow \mathcal{P}(S)
\]

be the holomorphic section given by the Earle uniformization. Then

\[
T^*_e \Omega_P = \pi \cdot \Omega_{\mathcal{T}},
\]

where \( T_e : T^*\mathcal{T}(S) \longrightarrow \mathcal{P}(S) \) is the biholomorphism given by the section \( e \).

Theorem 1.1 extends to sections \( f : \mathcal{T}(S) \longrightarrow \mathcal{P}(S) \) as above that satisfy certain conditions (see Remark 3.2). Another example of \( f \) with this property is the section given by the Schottky uniformization.

The proof of Theorem 1.1 is based on theorems of S. Kawai and C. T. McMullen.

**2. Symplectic structure on the moduli of projective structures**

Fix a connected compact oriented \( C^\infty \) surface \( S \) of genus \( g \) with \( g \geq 2 \). Let \( \mathcal{T}(S) \) denote the Teichmüller space associated to \( S \). Therefore,

\[
\mathcal{T}(S) = \text{Conf}(S)/\text{Diff}^0(S),
\]

where \( \text{Conf}(S) \) is the space of all conformal structures on \( S \) compatible with the orientation of \( S \), and \( \text{Diff}^0(S) \) is the group of all diffeomorphisms of \( S \) homotopic to the identity map of \( S \). The Teichmüller space \( \mathcal{T}(S) \) is a
complex manifold of complex dimension $3g - 3$, and it is diffeomorphic to $\mathbb{R}^{6g - 6}$.

Similarly, we have the moduli space of projective structures associated to $S$. To explain this with more detail, we first recall the definition of a projective structure on $S$.

A projective structure on $S$ is given by data $\{U_i, \phi_i\}_{i \in I}$, where

- $U_i \subset S$ are open subsets with $\bigcup_{i \in I} U_i = S$, and
- $\phi_i : U_i \rightarrow \mathbb{CP}^1$ are orientation preserving diffeomorphisms from $U_i$ to $\phi_i(U_i)$ satisfying the condition that for each ordered pair $i, k \in I$, there is some element $G_{i,k} \in \text{PGL}(2, \mathbb{C}) = \text{Aut}(\mathbb{CP}^1)$ such that the map
  \begin{equation}
  \phi_k \circ \phi_i^{-1} : \phi_i(U_i \cap U_k) \rightarrow \phi_k(U_i \cap U_k)
  \end{equation}
  coincides with the restriction of the automorphism $G_{i,k}$ of $\mathbb{CP}^1$.

Two data $\{U_i, \phi_i\}_{i \in I}$ and $\{U_j, \phi_j\}_{j \in J}$ of the above type are called equivalent if their union $\{U_k, \phi_k\}_{k \in I \cup J}$ also satisfies the above conditions. A projective structure on $X$ is an equivalence class of data. (See [2] for various alternative descriptions of a projective structure.)

Define
\begin{equation}
P(S) = \text{Proj}(S)/\text{Diff}^0(S),
\end{equation}
where $\text{Proj}(S)$ is the space of all projective structures on $S$, and $\text{Diff}^0(S)$ is the group in (2.1).

It is known that $P(S)$ is a complex manifold of complex dimension $6g - 6$, and it is diffeomorphic to $\mathbb{R}^{12g - 12}$. The complex manifold $P(S)$ has a natural holomorphic symplectic structure. We will briefly recall its description.

A projective structure $P$ on $S$ gives a flat principal $\text{PGL}(2, \mathbb{C})$-bundle over $S$. For any given data $\{U_i, \phi_i\}_{i \in I}$ of above type defining $P$, consider the trivial principal $\text{PGL}(2, \mathbb{C})$-bundle $U_i \times \text{PGL}(2, \mathbb{C})$ on each $U_i$. For any ordered pair $i, k \in I$, these trivial principal $\text{PGL}(2, \mathbb{C})$-bundles on $U_i$ and $U_k$ may be glued together over $U_i \cap U_k$ using $G_{i,k} \in \text{PGL}(2, \mathbb{C})$ as the transition function, where $G_{i,k} \in \text{PGL}(2, \mathbb{C})$ is the element giving the map in (2.2). This way we get a flat principal $\text{PGL}(2, \mathbb{C})$-bundle over $S$ associated to $P$. Consequently, we get a map
\begin{equation}
h : P(S) \rightarrow \text{Hom}(\pi_1(S), \text{PGL}(2, \mathbb{C}))/\text{PGL}(2, \mathbb{C})
\end{equation}
from $P(S)$ in (2.3) that sends any projective structure to the holonomy of the corresponding flat principal $\text{PGL}(2, \mathbb{C})$-bundle on $S$.

We note that for two different base points $s_1$ and $s_2$ of $S$, there is an identification of $\pi_1(S, s_1)$ with $\pi_1(S, s_2)$ unique up to an inner automorphism (by fixing a path connecting $s_1$ to $s_2$). Therefore, the quotient space...
Hom(\(\pi_1(S)\), \(\text{PGL}(2, \mathbb{C})/\text{PGL}(2, \mathbb{C})\)) in (2.4) does not depend on the choice of the base point needed to define the fundamental group.

A homomorphism \(\rho_0 : \pi_1(S) \to \text{PGL}(2, \mathbb{C})\) is called **irreducible** if the subgroup \(\text{image}(\rho_0) \subseteq \text{PGL}(2, \mathbb{C})\) does not fix any point of \(\mathbb{CP}^1\). Let

\[
\mathcal{R} \subset \text{Hom}(\pi_1(S), \text{PGL}(2, \mathbb{C}))/\text{PGL}(2, \mathbb{C})
\]

be the space of all irreducible representations. This irreducible representation space \(\mathcal{R}\) is a complex manifold of complex dimension \(6g - 6\) equipped with a holomorphic symplectic structure \([4]\). The image of the map \(h\) in (2.4) lies in \(\mathcal{R}\).

The map \(h\) is locally a biholomorphism, which means that \(h\) is holomorphic, and for each point \(P \in \mathcal{P}(S)\), the differential of \(h\) at \(P\) is an isomorphism of tangent spaces \([6, 7]\). Therefore, the holomorphic symplectic form on \(\mathcal{R}\) pulls back, by \(h\), to a holomorphic symplectic form on \(\mathcal{P}(S)\). Let

\[
(2.5) \quad \Omega_P \in H^0(\mathcal{P}(S), \Omega^2_{\mathcal{P}(S)})
\]

be the holomorphic symplectic form on \(\mathcal{P}(S)\) obtained this way.

There is a natural map from \(\mathcal{P}(S)\) to the Teichmüller space

\[
(2.6) \quad \varphi : \mathcal{P}(S) \to \mathcal{T}(S)
\]

that sends any projective structure on \(S\) to the underlying complex structure on \(S\). It is known that \(\varphi\) is a holomorphic surjective submersion. For any \(X \in \mathcal{T}(S)\), the fiber \(\varphi^{-1}(X)\) is an affine space for the vector space \(H^0(X, K_X^{\otimes 2})\) of all holomorphic quadratic differentials on the Riemann surface \(X\) (see \([2, 5, 7]\) for details).

Let \(T^*\mathcal{T}(S)\) be the total space of the holomorphic cotangent bundle of \(\mathcal{T}(S)\). Therefore, the fiber of \(T^*\mathcal{T}(S)\) over any \(X \in \mathcal{T}(S)\) is \(H^0(X, K_X^{\otimes 2})\). Take any smooth section

\[
(2.7) \quad f : \mathcal{T}(S) \to \mathcal{P}(S)
\]

of the projection \(\varphi\) in (2.6); so \(\varphi \circ f = \text{Id}_{\mathcal{T}(S)}\). Using \(f\) we have a diffeomorphism

\[
(2.8) \quad T_f : T^*\mathcal{T}(S) \to \mathcal{P}(S)
\]

that sends any \((X, \omega) \in T^*\mathcal{T}(S)\), where \(\omega\) is a holomorphic quadratic differential on the Riemann surface \(X\), to the projective structure \(f(X) + \omega\) on \(X\). If \(f\) is a holomorphic section, then the diffeomorphism \(T_f\) is a biholomorphism.

The complex manifold \(T^*\mathcal{T}(S)\) being the total space of the cotangent bundle of a complex manifold has a canonical holomorphic symplectic structure. To describe this symplectic form, let \(\sigma\) be the tautological Liouville one-form on \(T^*\mathcal{T}(S)\) that sends any tangent vector \(v\) at a point \((z, w) \in T^*\mathcal{T}(S)\), where \(z \in \mathcal{T}(S)\) and \(w \in T^*_z\mathcal{T}(S)\), to \(w(dp(v)) \in \mathbb{C}\); here

\[
dp : TT^*\mathcal{T}(S) \to p^*T\mathcal{T}(S)
\]
is the differential of the natural projection
\[ p : T^* \mathcal{T}(S) \rightarrow \mathcal{T}(S). \]

The two-form \( d\sigma \) defines a holomorphic symplectic structure on \( T^* \mathcal{T}(S) \).
This symplectic form \( d\sigma \) on \( T^* \mathcal{T}(S) \) will also be denoted by \( \Omega_T \).

For particular choices of the section \( f \) in (2.7) we may ask whether the
diffeomorphism \( T_f \) in (2.8) takes the symplectic form \( \Omega_T \) on \( T^* \mathcal{T}(S) \) defined
above to a constant multiple of the symplectic form \( \Omega_P \) constructed in (2.5).

If we fix a base point \( X_0 \in \mathcal{T}(S) \), there a section
(2.9) \[ B := B_{X_0} : \mathcal{T}(S) \rightarrow \mathcal{P}(S) \]
constructed by Bers using the notion of simultaneous uniformization. More
precisely, for any \( X \in \mathcal{T}(S) \), the projective structure \( B_{X_0}(X) \) is given by the
quasifuchsian group that uniformizes \( X \) and \( \overline{X}_0 \), where \( \overline{X}_0 \) is the quotient
of the lower half plane by the Fuchsian group for \( X_0 \) (see [1]). In [8], Kawai
showed that when \( f \) in (2.7) is the section \( B \) in (2.9), then
(2.10) \[ T_B^* \Omega_P = \pi \cdot \Omega_T, \]
where \( T_B \) is constructed as in (2.8) (see [8, p. 165, Theorem]).

3. Earle uniformization

In [3], Earle constructed a canonical holomorphic section
(3.1) \[ e : \mathcal{T}(S) \rightarrow \mathcal{P}(S). \]
The section \( e \) depends on the marked surface defined by the fixed surface
\( S \) as well as on the choice of an involution of the fundamental group of \( S \)
induced by some orientation reversing diffeomorphism of \( S \). In particular,
unlike the section \( B \) in (2.9) constructed by Bers, the section \( e \) does not
require fixing a base point of \( \mathcal{T}(S) \) for its definition. In this sense, this
section \( e \) is intrinsic (see the first paragraph of [3, p. 527]). It should be
clarified that this section \( e \) is not equivariant for the natural actions of the
mapping class group \( \text{Diff}_+(S)/\text{Diff}^0(S) \) on \( \mathcal{T}(S) \) and \( \mathcal{P}(S) \) (here \( \text{Diff}_+(S) \)
is the group of orientation preserving diffeomorphisms of \( S \)). Let
(3.2) \[ T_e : T^* \mathcal{T}(S) \rightarrow \mathcal{P}(S) \]
be the biholomorphism constructed as in (2.8) from the section \( e \) in (3.1).

**Theorem 3.1.** For the biholomorphism \( T_e \) in (3.2),
\[ T_e^* \Omega_P = \pi \cdot \Omega_T, \]
where \( \Omega_P \) and \( \Omega_T \) are the natural holomorphic symplectic forms on \( \mathcal{P}(S) \)
and \( T^* \mathcal{T}(S) \), respectively.
Proof. Let

\begin{equation}
\theta := e - B \in C^\infty(\mathcal{T}(S), T^*\mathcal{T}(S))
\end{equation}

be the smooth \((1,0)\)-form on \(\mathcal{T}(S)\), where \(e\) and \(B\) are the sections in (3.1) and (2.9) respectively. Recall that the space of projective structures on a given Riemann surface compatible with its complex structure is an affine space for the space of all holomorphic quadratic differentials on it, which implies that \(e - B\) is a \((1,0)\)-form on \(\mathcal{T}(S)\). We will first show the following.

For the biholomorphism \(T_e\) in (3.2),

\begin{equation}
T_e^*\Omega_P = \pi \cdot \Omega_T
\end{equation}

if and only if

\begin{equation}
d\theta = 0,
\end{equation}

where \(\theta\) is constructed in (3.3).

To prove this, let

\begin{equation}
F_\theta : T^*\mathcal{T}(S) \rightarrow T^*\mathcal{T}(S)
\end{equation}

be the diffeomorphism defined by \((X, \eta) \mapsto (X, \eta + \theta(X))\), where \(\theta\) is the \((1,0)\)-form in (3.3). It is easy to see that

\begin{equation}
T_B \circ F_\theta = T_e,
\end{equation}

where \(T_B\) (respectively, \(T_e\)) is the diffeomorphism in (2.10) (respectively, (3.2)), and \(F_\theta\) is constructed in (3.6).

From (3.7), we have

\begin{equation}
T_e^*\Omega_P = F_\theta^*(T_B^*\Omega_P).
\end{equation}

Therefore, in view of (2.10), we now conclude that (3.4) holds if and only if

\begin{equation}
F_\theta^*\Omega_T = \Omega_T.
\end{equation}

On the other hand, from the definition of \(F_\theta\) it follows that

\begin{equation}
F_\theta^*\Omega_T - \Omega_T = p^*d\theta,
\end{equation}

where \(p\) is the natural projection from \(T^*\mathcal{T}(S)\) to \(\mathcal{T}(S)\). To prove (3.8), we recall that the canonical symplectic form on the total space \(T^*M\) of the cotangent bundle of a \(C^\infty\) manifold \(M\) is the exterior derivative of a tautological one-form \(\alpha_M\) on \(T^*M\). For any smooth one-form \(\mu\) on \(M\), the diffeomorphism

\begin{equation}
D_\mu : T^*M \rightarrow T^*M
\end{equation}

defined by \((x, \omega) \mapsto (x, \omega + \mu(x))\) has the property that

\begin{equation}
D_\mu^*\alpha_M = \alpha_M + q^*\mu,
\end{equation}

where \(q : T^*M \rightarrow M\) is the natural projection. The identity (3.8) follows immediately from this fact.
Since $p$ in (3.8) is a submersion, the two-form $p^*d\theta$ vanishes if and only if $d\theta$ vanishes. Consequently, using (3.8) we now conclude that (3.4) holds if and only if (3.5) holds.

To prove that (3.5) holds, we first note that since both $B$ and $e$ are holomorphic sections of the projection $\varphi$ in (2.6), the $(1, 0)$-form $\theta$ in (3.3) is holomorphic. Hence $d\theta$ is a $(2, 0)$-form, or in other words,

\begin{equation}
(3.9) \quad d\theta \in C^\infty(\mathcal{T}(S), \Omega^{2,0}_{\mathcal{T}(S)}).
\end{equation}

Let

\begin{equation}
(3.10) \quad \phi : \mathcal{T}(S) \longrightarrow \mathcal{P}(S)
\end{equation}

be the smooth section given by the Fuchsian uniformization. Let

\begin{equation}
(3.11) \quad \alpha := e - \phi \in C^\infty(\mathcal{T}(S), T^*\mathcal{T}(S))
\end{equation}

and

\begin{equation}
(3.12) \quad \beta := B - \phi \in C^\infty(\mathcal{T}(S), T^*\mathcal{T}(S))
\end{equation}

be the smooth $(1, 0)$-forms on $\mathcal{T}(S)$, where $\phi$, $e$ and $B$ are the sections in (3.10), (3.1) and (2.9), respectively.

From a theorem due to McMullen, [9, p. 350, Theorem 7.1], we have that

\begin{equation}
(3.13) \quad d\beta \in C^\infty(\mathcal{T}(S), \Omega^{1,1}_{\mathcal{T}(S)})
\end{equation}

(in fact, $d\beta = \sqrt{-1} \cdot \omega_{WP}$, where $\omega_{WP}$ is the Weil–Petersson symplectic form on $\mathcal{T}(S)$). Moreover, Theorem 9.2 in [9, p. 355] states that

\begin{equation}
(3.14) \quad d\alpha = d\beta.
\end{equation}

We note that in [9, Theorem 9.2], this statement is proved for the Schottky uniformization. However, the proof remains unchanged for any smooth section $f$ (as in (2.7)) as long as $f$ is holomorphic and Theorem 9.1 of [9, p. 355] applies to it\(^1\). Both the sections $e$ and $B$ clearly satisfy these two conditions. (We also note that [9, Theorem 9.2] gives an alternative proof of a theorem of Takhtazan and Zograf in [10].)

We note that $\theta$ in (3.3) satisfies the identity

$$\theta = \alpha - \beta,$$

where $\alpha$ and $\beta$ are constructed in (3.11) and (3.12), respectively. Therefore, from (3.13) and (3.14), we have

$$d\theta \in C^\infty(\mathcal{T}(S), \Omega^{1,1}_{\mathcal{T}(S)}).$$

Comparing this with (3.9) we now conclude that

$$d\theta = 0.$$

\(^1\)We thank Curtis T. McMullen for clarifying this.
As we observed earlier, this implies that (3.4) holds. This completes the proof of the theorem. □

Remark 3.2. Take any section $f: \mathcal{T}(S) \to \mathcal{P}(S)$ as in (2.7) such that

1) $f$ is holomorphic, and

Consider the biholomorphism $T_f$ constructed in (2.8). The proof of Theorem 3.1 gives

$$T_f^*\Omega_P = \pi \cdot \Omega_T.$$

Apart from the section $\mathcal{T}(S) \to \mathcal{P}(S)$ given by the Earle uniformization, the section given by the Schottky uniformization also satisfies the two conditions stated above.

References