IMMERSIONS IN A MANIFOLD WITH A PAIR
OF SYMPLECTIC FORMS

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Let $N$ be a manifold with a pair of symplectic forms $\sigma_1, \sigma_2$, and $M$ a manifold with a pair of closed two-forms $\omega_1$ and $\omega_2$. For certain pairs of symplectic forms on $N$, we prove the existence of smooth immersions $f : M \to N$ such that $f^* \sigma_i = \omega_i$ for $i = 1, 2$.

1. Introduction

Let $(N, \sigma)$ be a symplectic manifold with the symplectic form $\sigma$ and $M$ a manifold with a closed two-form $\omega$. An immersion $f : M \to N$ is said to be a symplectic immersion if $f$ pulls back the form $\sigma$ onto $\omega$. All manifolds and maps in this article are assumed to be smooth. The symplectic immersion theorem of Gromov states that the symplectic immersions $f : M \to N$ satisfy the $C^0$ dense $h$-principle near the continuous maps $f_0 : M \to N$ which pull back the deRham cohomology class of $\sigma$ onto that of $\omega$ [8, 3.4.2(A)]. The aim of this paper is to generalize this theorem when the manifold $N$ comes equipped with a pair of symplectic forms $\sigma_1$ and $\sigma_2$ and $M$ has a pair of closed two-forms $\omega_1$ and $\omega_2$. An immersion $f : M \to N$ will be called a bisymplectic immersion if it satisfies the relations $f^* \sigma_1 = \omega_1$ and $f^* \sigma_2 = \omega_2$. The bisymplectic immersions are solutions to a system of first-order partial differential equations (PDEs) on a manifold. In fact, we can associate a first-order partial differential operator $D$ defined on the space of $C^\infty$ maps from $M$ to $N$ such that the bisymplectic maps are solutions to the equation $D = (\omega_1, \omega_2)$ for a given pair of closed two-forms $\omega_1, \omega_2$ on $M$. This takes us into the theory of $C^\infty$ operators.

Generally, to solve a PDE we need to prove an appropriate implicit function theorem so as to obtain a local inversion of the operator $D$. The implicit function theorem in the present case should ensure the $C^\infty$-smoothness (regularity) of the inversions. Gromov proves in [8, 2.3] that if an $r$th-order $C^\infty$ operator $D$ is infinitesimally invertible on an open subset $U$ in the space
of admissible maps defined by a $d$th-order differential relation for $d \geq r$, then the operator $\mathcal{D}$ restricted to $\mathcal{U}$ is an open map relative to the fine $C^\infty$-topologies on the function spaces and there is a smooth local inversion.

In this case, the associated differential operator takes values in closed forms only, so it cannot admit local inversion. However, we observe that there is a first-order differential operator $\mathcal{D}$ such that the solutions of the associated PDE give rise to the solutions of the original equation (see Section 3). Moreover, the operator $\mathcal{D}$ is infinitesimally invertible on a set of maps which are solutions to some open differential relation. Such maps will be referred as $(\sigma_1, \sigma_2)$-regular maps in this paper (see Definition 2.3).

We observe in Section 2 that a generic map is $(\sigma_1, \sigma_2)$-regular under mild dimension restriction. Applying the implicit function theorem of Gromov we then derive the following result in Section 3.

**Theorem A.** Let $\sigma_1 = \sum_{k=1}^{2q} dx_k \wedge dy_k$ and $\sigma_2 = \sum_{k=1}^{q} (dx_{2k-1} \wedge dy_{2k} - dx_{2k} \wedge dy_{2k-1})$ be two linear symplectic forms on $\mathbb{R}^{4q}$. Let $M$ be a closed manifold with two exact two-forms $\omega_1$ and $\omega_2$. If $2q \geq 3 \dim M$ and $q$ is even, then there exists a $(\sigma_1, \sigma_2)$-regular bisymplectic immersion $f : M \to \mathbb{R}^{4q}$.

We also partially answer a question of Gromov pertaining to inducing square four-forms from a small perturbation of square symplectic forms. Note that a symplectic immersion $f : M \to N$ (i.e., $f^* \sigma = \omega$) satisfies $f^*(\sigma^2) = \omega^2$. Suppose we break the symmetry of $\sigma^2$ by perturbing it a little to $\Omega$. Will it still be possible to induce the form $\omega^2$ by an immersion from the perturbed four-form $\Omega$? We take a linear symplectic form $\sigma_1$ on $\mathbb{R}^{2q}$ and some specific perturbation $\sigma_2$ of $\sigma_1$ which allow $(\sigma_1, \sigma_2)$-regular immersions. If we set the wedge of two such forms as $\Omega$ then it is possible to induce the square forms $\omega^2$ from $\Omega$ by means of immersions $f : M \to \mathbb{R}^{2q}$.

**Theorem B.** Let $\sigma_1 = \sum_{k=1}^{2q} dx_k \wedge dy_k$ and $\sigma_2 = \sum_{k=1}^{q} \lambda_k (dx_{2k-1} \wedge dy_{2k-1} + dx_{2k} \wedge dy_{2k})$ be two linear symplectic forms on $\mathbb{R}^{4q}$, where $\lambda_k$'s are distinct real numbers. Then given any exact two-form $\omega$ on a closed manifold $M$, there exists an immersion $f : M \to \mathbb{R}^{4q}$ such that $f^*(\sigma_1 \wedge \sigma_2) = \omega^2$ for $2q \geq 3 \dim M$.

We also prove the following h-principle in Section 4.

**Theorem C.** Suppose $N$ is a smooth manifold with closed two-forms $\sigma_1$ and $\sigma_2$, and $M$ is an open manifold with a closed two-form $\omega$. Then in the following two cases (a) and (b), the $(\sigma_1, \sigma_2)$-regular immersions $f : M \to N$ which pull back both the forms $\sigma_1$ and $\sigma_2$ onto $\omega$, satisfy the h-principle in the space of continuous maps $f_0 : M \to N$ such that $f_0^*[\sigma_i] = [\omega]$ for $i = 1, 2$.

(a) $\omega$ is the zero form on $M$.

(b) $M$ is a symplectic manifold and $\omega$ is a symplectic form on $M$.

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1The question was posed by Gromov in discussions with the author at the IHES in the year 2005.
The problem of inducing a pair of structures by maps were first considered in [1] and later in [2–4]. In all these articles, one of the two structures was a Riemannian metric, while the other structure was a contact form (in [1]), a Riemannian metric (in [2, 3]) or a symplectic form (in [4]), and except in [3], the authors exhibit the existence of \( C^1 \)-immersions which induce the given pair of structures by adapting Nash’s technique [11]. In [3], on the other hand, the authors prove the existence of Lipschitz solutions to the given problem by the convex integration technique [8, 2.4]. In this paper, we consider a pair of symplectic forms and prove the existence of smooth immersions which induce a given pair of closed two-forms. We employ, in contrast with earlier works, the analytic technique and the sheaf technique in the theory of \( h \)-principle which we discuss in Appendix A.

2. \((\sigma_1, \sigma_2)\)-Regular maps

In this section, we introduce the notion of \((\sigma_1, \sigma_2)\)-regular maps into a manifold \( N \) which comes with a pair of closed two-forms \( \sigma_1 \) and \( \sigma_2 \). The main result of this section gives a sufficient condition for the existence of such regular maps when \( \sigma_1, \sigma_2 \) is a symplectic pair.

**Definition 2.1.** Let \( \sigma_1 \) and \( \sigma_2 \) be a pair of linear two-forms on a vector space \( W \). A subspace \( V \) of \( W \) is said to be \((\sigma_1, \sigma_2)\)-regular (or simply regular) if the linear map

\[
(\bar{\sigma}_1, \bar{\sigma}_2) : W \rightarrow \Lambda^1(V) \times \Lambda^1(V)
\]

defined by

\[
\partial \mapsto (\partial.\sigma_1|_V, \partial.\sigma_2|_V)
\]

is an epimorphism.

A necessary condition for the existence of regular subspaces is that \( \dim W \geq 2 \dim V \). If \( V \) is a regular subspace of \((W, \sigma_1, \sigma_2)\) then any subspace of \( V \) is also regular.

**Proposition 2.1.** \( V \) is \((\sigma_1, \sigma_2)\)-regular if and only if \( W = \ker \bar{\sigma}_1 + \ker \bar{\sigma}_2 \); in other words, \( \ker \bar{\sigma}_1 \) is transversal to \( \ker \bar{\sigma}_2 \).

**Proof.** Let

\[
\bar{\sigma}_1 = \bar{\sigma}_1|_{\ker \bar{\sigma}_2} : \ker \bar{\sigma}_2 \rightarrow \Lambda^1(V) \quad \text{and} \quad \bar{\sigma}_2 = \bar{\sigma}_2|_{\ker \bar{\sigma}_1} : \ker \bar{\sigma}_1 \rightarrow \Lambda^1(V).
\]

Observe that \( \ker \bar{\sigma}_1 = \ker \bar{\sigma}_2 = \ker(\bar{\sigma}_1, \bar{\sigma}_2) = \ker \bar{\sigma}_1 \cap \ker \bar{\sigma}_2 \). If \( V \) is regular then both \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \) are surjective. The converse is also true. To see this let \((\alpha_1, \alpha_2) \in \Lambda^1(V) \times \Lambda^1(V)\). Then there exist vectors \( \partial_1 \in \ker \bar{\sigma}_1 \) and \( \partial_2 \in \ker \bar{\sigma}_2 \) such that

\[
\bar{\sigma}_2(\partial_1) = \alpha_2 \quad \text{and} \quad \bar{\sigma}_1(\partial_2) = \alpha_1.
\]
Hence, \( \tilde{\sigma}_1(\partial_1 + \partial_2) = \alpha_1 \) and \( \tilde{\sigma}_2(\partial_1 + \partial_2) = \alpha_2 \) which proves \((\sigma_1, \sigma_2)\)-regularity of \( V \). Consequently, \( V \) is regular if and only if the following equalities hold:

\[
\dim \ker \tilde{\sigma}_1 = \dim V + \dim[\ker \tilde{\sigma}_1 \cap \ker \tilde{\sigma}_2] = \dim \ker \tilde{\sigma}_2,
\]

which implies that

\[
\dim(\ker \tilde{\sigma}_1 + \ker \tilde{\sigma}_2) = \dim \ker \tilde{\sigma}_1 + \dim \ker \tilde{\sigma}_2 - \dim[\ker \tilde{\sigma}_1 \cap \ker \tilde{\sigma}_2] = 2 \dim V + \dim \ker(\tilde{\sigma}_1, \tilde{\sigma}_2) = \dim W.
\]

If \( \sigma_1 \) and \( \sigma_2 \) are two linear symplectic forms on \( W \) then we can characterize the regularity condition as follows:

**Corollary 2.1.** Suppose \( \sigma_1 \) and \( \sigma_2 \) are two linear symplectic forms on \( W \). A subspace \( V \) is \((\sigma_1, \sigma_2)\)-regular if and only if

\[
V^{\sigma_1} \text{ is transversal to } V^{\sigma_2},
\]

where \( V^{\sigma_i} = \{ w \in W : \sigma_i(v, w) = 0 \text{ for all } v \in V \} \) is the symplectic complement of \( V \) relative to \( \sigma_i \), \( i = 1, 2 \).

**Proof.** We may identify \( \ker \tilde{\sigma}_1 \) as the symplectic complement of \( V \) relative to \( \sigma_1 \). Similarly \( \ker \tilde{\sigma}_2 \) is the symplectic complement of \( V \) relative to \( \sigma_2 \). \( \square \)

**Proposition 2.2.** Let \( \sigma_1 \) and \( \sigma_2 \) be two linear symplectic forms on \( W \). Let \( \Lambda : W \to W \) denote the unique vector space isomorphism determined by the relation \( \sigma_2(v, w) = \sigma_1(v, Aw) \) for all \( v, w \in W \). A subspace \( V \) of \( W \) is \((\sigma_1, \sigma_2)\)-regular if and only if \( V + A(V) \) has the maximum possible dimension.

**Proof.** We observe that \( V^{\sigma_1} \) is transversal to \( V^{\sigma_2} \) if and only if \( V^{\sigma_1} \) is transversal to \( A(V)^{\sigma_1} \). Since \( (V + A(V))^{\sigma_1} = V^{\sigma_1} \cap A(V)^{\sigma_1} \), we obtain \( \text{codim}(V + A(V))^{\sigma_1} = 2n \). Consequently, \( V + A(V) \) has the maximum possible dimension, as \( \sigma_1 \) is a symplectic form. \( \square \)

**Definition 2.2.** Let \( W \) be a vector space with two linear two-forms \( \sigma_1, \sigma_2 \). A linear map \( \ell : V \to W \) from a vector space \( V \) to \( W \) will be called \((\sigma_1, \sigma_2)\)-regular if \( \ell(V) \) is a regular subspace of \( W \). Equivalently, the linear map

\[
W \to \Lambda^1(V) \times \Lambda^1(V),
\]

\[
\partial \mapsto (\ell^*(\partial, \sigma_1), \ell^*(\partial, \sigma_2))
\]

is an epimorphism.

**Observation.** Let \( p_1 \) and \( p_2 \) denote the projection maps of the product vector space \( V \times W \) onto \( V \) and \( W \), respectively. Suppose that \( \sigma_1, \sigma_2 \) are two linear two-forms on \( W \) and \( \omega_1, \omega_2 \) are two linear two-forms on \( V \). Let \( \tilde{\sigma}_1 = p_1^*(\omega_1) - p_2^*(\sigma_1) \) and \( \tilde{\sigma}_2 = p_1^*(\omega_2) - p_2^*(\sigma_2) \). If \( \ell : V \to W \) is \((\sigma_1, \sigma_2)\)-regular then the graph map of \( \ell \), \( \ell : V \to V \times W \), is \((\tilde{\sigma}_1, \tilde{\sigma}_2)\)-regular. This follows from the relation that \( \ell^*(\partial, \tilde{\sigma}_i) = \ell^*(\partial, \sigma_i) \) for all \( \partial \in W \), \( i = 1, 2 \).
**Definition 2.3.** Let \( \sigma_1, \sigma_2 \) be two closed two-forms on a manifold \( N \). A smooth immersion \( f : M \rightarrow N \) is \((\sigma_1, \sigma_2)\)-regular if the derivative map \( df_x : T_x M \rightarrow T_{f(x)} N \) is \((\sigma_1(f(x)), \sigma_2(f(x)))\)-regular for each \( x \in M \). We shall often refer to the \((\sigma_1, \sigma_2)\)-regular maps as regular maps.

**Theorem 2.1.** Consider two linear symplectic forms \( \sigma_1 \) and \( \sigma_2 \) on \( \mathbb{R}^{2q} \) which are related by \( \sigma_2(v, w) = \sigma_1(v, Aw) \) for some linear isomorphism \( A \) of \( \mathbb{R}^{2q} \). Suppose that \( k \) is the maximum of the geometric multiplicities of real eigenvalues of \( A \). Then generic maps \( f : M \rightarrow \mathbb{R}^{2q} \) are \((\sigma_1, \sigma_2)\)-regular immersions for \( 2q \geq \max\{3 \dim M, 2 \dim M + k\} \).

In particular, if \( A \) has no real eigenvalues then generic maps \( f : M \rightarrow \mathbb{R}^{2q} \) are \((\sigma_1, \sigma_2)\)-regular immersions for \( 2q \geq 3 \dim M \).

**Proof.** Let \( \dim M = n \). Let \( \Sigma \) be the subset of the Grassmannian manifold \( Gr_n(\mathbb{R}^{2q}) \) which consists of all \( n \)-planes \( T \) in \( \mathbb{R}^q \) such that \( T \cap A(T) \neq \emptyset \). Then \( \Sigma \) is the union of two sets \( \Sigma' \) and \( \Sigma'' \) where

1. \( \Sigma' \) consists of all \( n \)-planes \( T \) in \( \mathbb{R}^{2q} \) which contains an eigenvector of \( A \), where the eigenspaces of \( A \) are at most \( k \)-dimensional,
2. \( \Sigma'' \) consists of all \( n \)-planes \( T \) in \( \mathbb{R}^{2q} \) which contains a two-dimensional subspace spanned by a pair \( \{v, Av\} \) for some \( v \in T \).

Since the geometric multiplicities of the eigenvalues of \( A \) are at the most \( k \), the dimension of \( \Sigma' \) is less than or equal to \( k - 1 + (n - 1)(2q - n) \). On the other hand, the dimension of \( \Sigma'' \) is less than or equal to \( (2q - 1) + (n - 2)(2q - n) \). Therefore, \( \dim \Sigma = \max(\dim \Sigma', \dim \Sigma'') \).

Let \( \mathcal{R} \) denote the open subset of \( J^1(M, \mathbb{R}^{2q}) \) consisting of one-jet of germs of immersions from \( M \) to \( \mathbb{R}^{2q} \) and let \( p : \mathcal{R} \rightarrow Gr_n(\mathbb{R}^{2q}) \) be the canonical projection which maps an one-jet \( j^1_x f(x) \), \( x \in M \), onto the \( n \)-dimensional subspace \( \text{Im} df_x \) in \( \mathbb{R}^{2q} \). A smooth map \( f : M \rightarrow \mathbb{R}^{2q} \) is a regular immersion if and only if \( p \circ j^1_x f \) misses the set \( \Sigma \), if and only if \( j^1_x f \) misses the set \( p^{-1}(\Sigma) \).

Now, observe that if \( 2q \geq \max\{3n, 2n + k\} \) then \( \text{codim} \Sigma > n \), and hence the same is true for the codimension of \( p^{-1}(\Sigma) \), since \( p \) is a submersion. Therefore, by an application of the Thom Transversality Theorem \( [7] \) \( j^1 f \) misses \( p^{-1}(\Sigma) \) for a generic \( f \). Thus, a generic map \( f : M \rightarrow \mathbb{R}^{2q} \) is regular if \( 2q \geq \max\{3n, 2n + k\} \). \( \square \)

**Remark 2.1.** If \( \sigma_1 \) and \( \sigma_2 \) are two symplectic forms on a manifold \( N \), then there is a bundle isomorphism \( \Lambda : TN \rightarrow TN \) satisfying the following relation: \( \sigma_2(v, w) = \sigma_1(v, Aw) \) for all \( v, w \in T_x N \) and \( x \in N \). Suppose that

\[
 k = \max_{x \in M} \{\text{geometric multiplicities of the real eigenvalues of } A_x\}.
\]

Then, we can obtain an exact analogue of Theorem 2.1 for maps \( f : M \rightarrow (N, \sigma_1, \sigma_2) \) with this \( k \).
Let $\sigma_1, \sigma_2$ be two linear symplectic forms on $W$ and let $A$ satisfy $\sigma_2(v, w) = \sigma_1(v, Aw)$ for all $v, w \in W$. If $\{u_1, \ldots, u_q, v_1, \ldots, v_q\}$ is a canonical symplectic basis for $\sigma_1$ then relative to this basis $A$ can be represented by a matrix of the following form:

$$M(A) = \begin{pmatrix} B & C \\ D & B^t \end{pmatrix},$$

where $B, C, D$ are $q \times q$ square matrices of which $C$ and $D$ are skew-symmetric. Indeed, writing $\sigma_1 = \sum_{k=1}^q u_k^* \wedge v_k^*$ we have $B = (\sigma_2(u_j, v_i))_{i,j}$, $C = (\sigma_2(v_i, v_j))_{i,j}$ and $D = (\sigma_2(u_i, u_j))_{i,j}$.

If $M(A)$ is symmetric then all eigenvalues are real and if $M(A)$ is skew-symmetric then all eigenvalues are purely imaginary. We consider two special cases under the above criteria. The first, when $B$ is symmetric and $C = D = 0$, and the second, when $B$ is skew-symmetric and $C = D = 0$.

**Example 2.1.** Let $\sigma_1 = u_k^* \wedge v_k^*$. If $\sigma_2 = \sum_{k=1}^q \lambda_k u_k^* \wedge v_k^*$, then $M(A)$ is symmetric. If $q = 2n$ is even and $\sigma_2 = \sum_{k=1}^n (u_{2k-1}^* \wedge v_{2k}^* - u_{2k}^* \wedge v_{2k-1}^*)$, then $M(A)$ is skew-symmetric.

We can now easily deduce the following two corollaries from Theorem 2.1.

**Corollary 2.2.** Let $\sigma_1 = \sum_{i=1}^q dx_i \wedge dy_i$ and $\sigma_2 = \sum_{i=1}^q \lambda_i dx_i \wedge dy_i$ be two symplectic forms on $\mathbb{R}^{2q}$, where the multiplicities of $\lambda_i$’s are less than equal to $k$. Then a generic map $f : M \to \mathbb{R}^{2q}$ is a $(\sigma_1, \sigma_2)$-regular immersion for $2q \geq \max\{3 \dim M, 2 \dim M + k\}$.

**Corollary 2.3.** If $q$ is even, say $q = 2n$ and $\sigma_1 = \sum_{i=1}^{2n} dx_i \wedge dy_i$ and $\sigma_2 = \sum_{k=1}^n (dx_{2k-1} \wedge dy_{2k} - dx_{2k} \wedge dy_{2k-1})$, then generic maps $f : M \to \mathbb{R}^{2q}$ are $(\sigma_1, \sigma_2)$-regular immersions for $2q \geq 3 \dim M$.

We end this section by formulating equivalent criteria for the symmetry and the skew-symmetry conditions on $M(A)$. The following is a standard fact from symplectic geometry [9].

**Lemma 2.1.** Let $W$ be a vector space with a linear symplectic form $\sigma$ which is invariant under an almost complex structure $J$. Define a bilinear form $g$ on $W$ by

$$g(u, v) = \sigma(u, Jv)$$

for all $u, v \in W$.

Then

1. $\sigma(u, v) = g(Ju, v)$;
2. $g$ is a non-degenerate symmetric form;
3. $g$ is $J$-invariant.

The triple $(g, J, \sigma)$ is such that given any two of these structures the third structure is obtained uniquely by the relation $g(u, v) = \sigma(u, Jv)$.
Lemma 2.2. Let \( \sigma_1 \) and \( \sigma_2 \) be two linear symplectic forms on \( W \). Let \( A \) denote the unique vector space isomorphism \( W \to W \) determined by the relation \( \sigma_2(v, w) = \sigma_1(v, Aw) \). Let \( J \) be an almost complex structure on \( W \) such that \( \sigma_1 \) is invariant under \( J \). Define \( g_1 \) by \( g_1(v, w) = \sigma_1(v, Jw) \) for \( v, w \in W \). Then the following are equivalent:

1. \( \sigma_2 \) is invariant under \( J \).
2. \( A \) commutes with \( J \).
3. \( A \) is symmetric relative to the symmetric form \( g_1 \).

Therefore, under any of the above conditions, eigenvalues of \( A \) are all real and they have even multiplicities.

Proof. (1) \( \implies \) (2): Since both \( \sigma_1 \) and \( \sigma_2 \) are \( J \) invariant, \( \sigma_1(Ju, JAv) = \sigma_1(u, Av) = \sigma_2(u, v) = \sigma_2(Ju, Jv) = \sigma_1(Ju, AJv) \) for all \( v, w \). Now, the non-degeneracy of \( \sigma_1 \) implies that \( AJ = JA \). Consequently the eigenvalues occur in pairs.

(1) \( \implies \) (3): \( g_1(Au, v) = \sigma_1(Au, Jv) = -\sigma_1(Jv, Au) = -\sigma_2(Jv, u) = \sigma_2(u, Jv) \). Since \( \sigma_2 \) is \( J \)-invariant, \( \sigma_2(u, Jv) = \sigma_2(v, Jw) \) and hence we have \( g_1(Au, v) = g_1(u, Av) \).

(2) \( \implies \) (1): Since \( A \) commutes with \( J \) we obtain \( \sigma_2(Jv, Jw) = \sigma_1(Jv, AJw) = \sigma_1(Jv, JAw) \). Further, since \( \sigma_1 \) is \( J \)-invariant, \( \sigma_1(Jv, JAw) = \sigma_1(v, Av) = \sigma_2(v, w) \). Thus \( \sigma_2 \) is \( J \)-invariant.

(3) \( \implies \) (1): For any \( v, w \in W \), \( \sigma_2(Jv, Jw) = \sigma_1(Jv, AJw) = -g_1(v, AJw) = -g_1(Av, Jw) \) (since \( A \) is symmetric with respect to \( g_1 \)).

If \( A \) is symmetric with respect to \( g_1 \) then there is a \( g_1 \)-orthonormal basis \( u_1, u_2, \ldots, u_{2n} \) consisting of eigenvectors of \( A \). Suppose \( Au_i = \lambda_i u_i \) for \( i = 1, 2, \ldots, 2n \), where \( \lambda_i \) are real numbers. Consider \( u_1 \in W \). There exists at least one \( u_{n_1} \), \( n_1 \neq 1 \), such that \( \sigma_2(u_1, u_{n_1}) \neq 0 \). Using the relation between \( \sigma_1 \) and \( \sigma_2 \) we obtain that \( \lambda_1 = \lambda_{n_1} \). Now, the \( g_1 \)-orthogonal complement \( W_1 \) of the span of \( u_1 \) and \( u_{n_1} \) is spanned by the set \( \{ u_i | i \neq 1, i \neq n_1 \} \). Hence, we can repeat the above argument for the pair \( (W_1, A|_{W_1}) \), where \( \dim W_1 < \dim W \). Consequently, an induction on \( \dim W \) proves that the eigenvalues are real and repeated even number of times.

Analogously, we can prove:

Lemma 2.3. Let \( \sigma_1 \) and \( \sigma_2 \) be two linear symplectic forms on \( W \) and \( A, J \) and \( g_1 \) be defined as in Lemma 2.2. Then the following are equivalent:

1. \( \sigma_2(v, Jw) = \sigma_2(Jv, w) \) for all \( v, w \in W \); in other words, \( J^* \sigma_2 = -\sigma_2 \).
2. \( A \) anticommutes with \( J \), that is, \( AJ = -JA \).
3. \( A \) is skew-symmetric relative to the symmetric bilinear form \( g_1 \).

Therefore, under any of the above conditions, eigenvalues of \( A \) are purely imaginary.
3. Existence of immersions inducing a given pair of forms

In this section we study the existence of bisymplectic immersions in a manifold \((N, \sigma_1, \sigma_2)\), where \(\sigma_1 = d\tau_1\) and \(\sigma_2 = d\tau_2\) are two exact two-forms on \(N\). Let \(M\) be a manifold with a pair of exact two-forms \(\omega_1 = d\alpha_1\) and \(\omega_2 = d\alpha_2\). Consider the differential operator

\[
D : C^\infty(M, N) \to \Omega^2(M) \times \Omega^2(M)
\]

which takes an \(f \in C^\infty(M, N)\) onto the pair \((f^*\sigma_1, f^*\sigma_2)\), where \(C^\infty(M, N)\) denotes the space of smooth maps from \(M\) to \(N\) and \(\Omega^2(M)\) denotes the space of two-forms on \(M\). The bisymplectic immersions \(f : M \to N\) are solutions to the differential equation \(Df = (\omega_1, \omega_2)\) for the given pair of two-forms on \(M\). If \(f : M \to N\) is a bisymplectic immersion, then \(f^*\tau_1 - \alpha_1\) and \(f^*\tau_2 - \alpha_2\) are closed one-forms, and conversely.

Let us now consider the following first-order differential operator:

\[
\tilde{D} : C^\infty(M, N) \times C^\infty(M) \times C^\infty(M) \to \Omega^1(M) \times \Omega^1(M)
\]

defined by

\[
(f, \phi_1, \phi_2) \mapsto (f^*\tau_1 + d\phi_1, f^*\tau_2 + d\phi_2),
\]

where \(C^\infty(M)\) is the space of smooth real valued functions on \(M\) and \(\Omega^1(M)\) is the space of one-forms on \(M\). This operator is closely associated with the operator \(D\); indeed, if \((f, \phi_1, \phi_2)\) is a solution of the equation \(\tilde{D} = (\alpha_1, \alpha_2)\), then clearly \(f\) satisfies the equations \(f^*\sigma_1 = \omega_1\) and \(f^*\sigma_2 = \omega_2\).

The linearization of \(\tilde{D}\) at \((f, \phi_1, \phi_2)\) is an operator

\[
L : \Gamma^\infty(f^*TN) \times C^\infty(M) \times C^\infty(M) \to \Omega^1(M) \times \Omega^1(M),
\]

which is given by

\[
(\partial, \psi_1, \psi_2) \mapsto (f^*(\partial, \sigma_1 + d(\partial, \tau_1)) + d\psi_1, f^*(\partial, \sigma_2 + d(\partial, \tau_2)) + d\psi_2),
\]

where \(\partial\) is a vector field on \(N\) along \(f\), and \(\psi_1\) and \(\psi_2\) are smooth functions on \(M\). \(L\) is right invertible if we can solve the following system of equations in \(\partial, \psi_1\) and \(\psi_2\) for arbitrary one-forms \(g_1\) and \(g_2\) on \(M:\)

\[
\begin{align*}
&f^*(\partial, \sigma_1) = g_1, & f^*(\partial, \sigma_2) = g_2, \\
&f^*(\partial, \tau_1) + \psi_1 = 0, & f^*(\partial, \tau_2) + \psi_2 = 0.
\end{align*}
\]

If \(f\) is \((\sigma_1, \sigma_2)\)-regular (see Definition 2.3), then the first two equations can be solved for \(\partial\), the value of which is then inserted in the second set of equations to obtain \(\psi_1\) and \(\psi_2\). Thus, \(L\) is right invertible by a zeroth-order operator \(L^{-1} : (g_1, g_2) \to (\partial, \psi_1, \psi_2)\) when \(f\) is \((\sigma_1, \sigma_2)\)-regular. Hence, \(\tilde{D}\) is infinitesimally invertible on \((\sigma_1, \sigma_2)\)-regular immersions. Now, the \((\sigma_1, \sigma_2)\)-regular immersions are solutions to an open differential relation \(A \subset J^1(M, N)\). Consequently, the set of regular \(C^\infty\) immersions, \(A\), form an open subspace...
of $C^\infty(M,N)$ in the fine $C^\infty$-topology. Hence, we obtain the following result by an application of Theorem A.2 (see Section 5).

**Proposition 3.1.** The restriction of $\tilde{\mathcal{D}}$ to the space of regular immersions is an open map relative to the fine $C^\infty$-topologies on the function spaces.

We are now in a position to prove Theorem A.

**Proof of Theorem A.** Denote the coordinates on $\mathbb{R}^{2q}$ by $x_1, y_1, \ldots, x_q, y_q$. Suppose that $q = 2n$ and let

$$\bar{\sigma}_1 = \sum_{k=1}^{2n} dx_k \wedge dy_k \quad \text{and} \quad \bar{\sigma}_2 = \sum_{k=1}^{n} (dx_{2k-1} \wedge dy_{2k} - dx_{2k} \wedge dy_{2k-1}),$$

so that $\sigma_1 = \bar{\sigma}_1 \oplus \bar{\sigma}_2$ and $\sigma_2 = \bar{\sigma}_2 \oplus \bar{\sigma}_1$. Take a $(\bar{\sigma}_1, \bar{\sigma}_2)$-regular immersion $h : M \to \mathbb{R}^{2q}$; such an $h$ is guaranteed by Corollary 2.3 since $2q \geq 3 \dim M$.

Define $h' : M \to \mathbb{R}^{2q}$ by $h'(x) = (h_1(x), -h_2(x), \ldots, h_{2q-1}(x), -h_{2q}(x))$, and set $\bar{h} = (h, h') : M \to \mathbb{R}^{4q}$. Clearly, $\bar{h}$ is $(\sigma_1, \sigma_2)$-regular and it pulls back both $\sigma_1$ and $\sigma_2$ onto the zero form on $M$, that is, $\bar{h}^*\sigma_i = 0$ for $i = 1, 2$. Since both $\sigma_1$ and $\sigma_2$ are exact, we can write $\sigma_1 = d\tau_1$ and $\sigma_2 = d\tau_2$ for some one-forms $\tau_1$ and $\tau_2$ on $\mathbb{R}^{2q}$, and this implies that $\bar{h}^*\tau_1$ and $\bar{h}^*\tau_2$ are closed one-forms on $M$. Therefore, if we define $\mathcal{D}$ as above then its image contains an ordered pair $(c_1, c_2)$ of closed one-forms on $M$, where $c_i = \bar{h}^*\tau_i$ for $i = 1, 2$. Since $M$ is a closed manifold and $\mathcal{D}$ is an open map (by Proposition 3.1), for every pair of one-forms $(\alpha_1, \alpha_2)$ on $M$ there exists a scalar $\lambda > 0$ such that $(c_1 + \lambda \alpha_1, c_2 + \lambda \alpha_2)$ also belongs to the image of $\mathcal{D}$. This implies that, we have a triple $(\bar{f}, \phi_1, \phi_2)$ such that $\bar{f}^*\tau_1 + d\phi_1 = c_1 + \lambda \alpha_1$ and $\bar{f}^*\tau_2 + d\phi_2 = c_2 + \lambda \alpha_2$, where $\bar{f} : M \to \mathbb{R}^{4q}$ is a regular immersion and $\phi_1$ and $\phi_2$ are smooth functions on $M$. Consequently, $\bar{f}^*(\sigma_1) = \lambda d\alpha_1$ and $\bar{f}^*(\sigma_2) = \lambda d\alpha_2$. The required map $\bar{f}$ is then defined as $\bar{f} = \lambda^{-\frac{1}{2}}\bar{f}$.

**Remark 3.1.** Let $(N, \sigma_1, \sigma_2)$ and $(M, \omega_1, \omega_2)$ be as described in the beginning of this section. Now, suppose that

- the pair $(\sigma_1, \sigma_2)$ admits regular immersions $M \to N$ and
- there exists a diffeomorphism $\phi$ of $N$ such that $\phi^*\sigma_i = -\sigma_i$ for $i = 1, 2$.

If $M$ is a closed manifold, then by setting $h' = \phi \circ h$ it may be seen as in the proof of Theorem A that there exists a regular immersion $f : M \to N \times N$ which satisfies $f^*(\sigma_1 \oplus \sigma_1) = \omega_1$ and $f^*(\sigma_2 \oplus \sigma_2) = \omega_2$.

Also, if we have

- a pair of closed two-forms $\sigma_1$ and $\sigma_2$ on $N$ and
- a $(\sigma_1, \sigma_2)$-regular embedding $f : M \to N$ such that $f^*\sigma_1 = 0 = f^*\sigma_2$ then both $\sigma_1$ and $\sigma_2$ are exact on a tubular neighbourhood of image $f$ in $N$. Therefore, as in the above theorem, an arbitrary pair of exact forms on $M$ can be induced from the pair $(\sigma_1, \sigma_2)$ by a regular immersion $f$. Moreover, we can choose $\bar{f}$ sufficiently $C^0$ close to $f$. 

Corollary 3.1. Let

\[ \sigma_1 = \sum_{k=1}^{2q} dx_k \wedge dy_k \quad \text{and} \quad \sigma_2 = \sum_{k=1}^{q} \lambda_k (dx_{2k-1} \wedge dy_{2k-1} + dx_{2k} \wedge dy_{2k}) \]

be two symplectic forms on \( \mathbb{R}^{4q} \), where the multiplicity of each \( \lambda_k \) is less than or equal to \( k \). If \( M \) is a closed manifold, then for \( 2q \geq \max\{3\dim M, 2\dim M + k\} \), there exists a smooth regular immersion \( f : M \to \mathbb{R}^{4q} \) such that \( f^*(\sigma_1) = \omega_1 \) and \( f^*(\sigma_2) = \omega_2 \), where \( \omega_1 \) and \( \omega_2 \) are given exact two-forms on \( M \).

Theorem B is now immediate from the above corollary if we take \( \omega_1 = \omega_2 = \omega \).

We end this section with the following result in \( h \)-principle.

Theorem 3.1. Let \( \sigma_1, \sigma_2 \) be two exact two-forms on a manifold \( N \), and \( \omega_1 \) and \( \omega_2 \) two exact two-forms on \( M \). Then \( (\sigma_1, \sigma_2) \)-regular maps \( f : M \times \mathbb{R} \to N \) which pull back the forms \( \sigma_1 \) and \( \sigma_2 \) onto \( p^*\omega_1 \) and \( p^*\omega_2 \), respectively, satisfy the \( h \)-principle.

We postpone the proof of this theorem as of now.

4. \( h \)-Principle of immersions inducing given pair of forms

In this section, we assume that \( \sigma_1 \) and \( \sigma_2 \) are arbitrary closed two-forms on \( N \) and \( \omega_1, \omega_2 \) are two closed two-forms on \( M \). We aim to see if the regular maps \( f : M \to N \) satisfying \( f^*\sigma_i = \omega_i \), \( i = 1, 2 \) follow the \( h \)-principle. We first note that such an \( f \) pulls back the deRham cohomology classes of \( \sigma_1 \) and \( \sigma_2 \), respectively, onto those of \( \omega_1 \) and \( \omega_2 \). Therefore, the \( h \)-principle can at most be \( C^0 \)-dense (Definition A.2) in the space of continuous maps \( f_0 : M \to N \) such that \( f_0^*\sigma_i = [\omega_i] \) for \( i = 1, 2 \), in which case the solution space, if non-empty, is dense in the space of such continuous maps \( f_0 \). In view of this we start with a smooth map \( f_0 : M \to N \) which satisfies these cohomology conditions. Let \( p_1 \) and \( p_2 \) denote the projection maps of the product manifold \( M \times N \) onto \( M \) and \( N \), respectively. Consider the two product forms on \( M \times N \):

\[ \bar{\sigma}_1 = p_1^*\omega_1 - p_2^*\sigma_1 \quad \text{and} \quad \bar{\sigma}_2 = p_1^*\omega_2 - p_2^*\sigma_2. \]

Since the graph of \( f_0 \) is an embedded submanifold of the product manifold \( M \times N \), the cohomology condition on \( f_0 \) implies that both \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \) are exact in a tubular neighbourhood \( Y \) of graph \( f_0 \). Suppose, \( \bar{\sigma}_i = d\tau_i \) for some one-forms \( \tau_i \) on \( Y \), \( i = 1, 2 \).
If $\tilde{f} : M \to M \times N$ is a section of $p_1$ then we will denote the underlying map of $\tilde{f}$, namely $p_2 \circ \tilde{f}$, by $f$. If $f$ is $(\sigma_1, \sigma_2)$-regular then $\tilde{f}$ is $(\sigma_1, \sigma_2)$-regular (see Observation in Section 2).

Let $\Gamma^\infty(Y)$ denote the sheaf of $C^\infty$ sections of the product bundle $M \times N \to M$ whose images lie in $Y$. Define a differential operator $\mathcal{D}$ as follows:

$$\Gamma^\infty(Y) \times C^\infty(M) \times C^\infty(M) \xrightarrow{\mathcal{D}} \Omega^1(M) \times \Omega^1(M),$$

$$(f, \phi_1, \phi_2) \mapsto (f^*\tau_1 + d\phi_1, f^*\tau_2 + d\phi_2).$$

If $\mathcal{D}(\tilde{f}, \phi_1, \phi_2) = 0$ then $f^*\tau_1 = \omega_1$ and $f^*\tau_2 = \omega_2$.

Now, as we observed in the previous section, $\mathcal{D}$ is infinitesimally invertible at all triples $(\tilde{f}, \phi_1, \phi_2)$ for which $p_2 \circ f$ is $(\sigma_1, \sigma_2)$-regular. Consequently, $\mathcal{D}$ is a first-order differential operator which admits a zeroth-order inversion of defect 1 (see Section 5).

Let $E$ denote the fibre bundle over $M$ whose total space is $Y \times \mathbb{R} \times \mathbb{R}$ and the projection map $\pi : E \to M$ is defined by $\pi(y, t, s) = p_1(y)$, where $(y, t, s) \in Y \times \mathbb{R} \times \mathbb{R}$. The sections of $E$ are in one-to-one correspondence with triples $(\tilde{f}, \phi_1, \phi_2)$, where $\tilde{f} \in \Gamma^\infty(Y)$ and $\phi_1, \phi_2 \in C^\infty(M)$. Let $G$ denote the vector bundle $\Lambda^1(M) \oplus \Lambda^1(M)$, where $\Lambda^1(M)$ is the cotangent bundle of $M$.

The operator $\mathcal{D}$ induces a sequence of bundle maps $\tilde{\Delta}_\alpha : E^{(\alpha + 1)} \to G^{\alpha}$, $\alpha \geq 0$, satisfying the relations $\tilde{\Delta}_\alpha \circ \tilde{j}^{(\alpha + 1)}_{(f, \phi_1, \phi_2)} = \tilde{j}^{\alpha}_{\mathcal{D}(f, \phi_1, \phi_2)}$ (see Section 5).

For each non-negative integer $\alpha$, we now define a differential relation $R^\alpha = \tilde{\Delta}_\alpha^{-1}(0)$. All these relations have the same $C^\infty$ solutions as the equation $\mathcal{D} = 0$.

Let $R_2$ be the subset of $\tilde{R}^2$ consisting of all three-jets at $x, x \in M$, which can be represented by a local section $(\tilde{f}, \phi_1, \phi_2)$ of $E$ such that $p_2 \circ \tilde{f}$ is regular at $x$. Therefore, $(\tilde{f}, \phi_1, \phi_2)$ is a solution of $R_2$ if

1. $f = p_2 \circ \tilde{f}$ is $(\sigma_1, \sigma_2)$-regular, and
2. $f^*\tau_1 + d\phi_1 = 0$ and $f^*\tau_2 + d\phi_2 = 0$, that is $\mathcal{D}(\tilde{f}, \phi_1, \phi_2) = 0$.

Let $\Phi$ denote the solution sheaf of $R_2$ and $\Psi_2$ the sheaf of sections of $R_2$.

The next result follows from Theorem A.2 and Proposition A.1 in Section 5.

**Proposition 4.1.** $\Phi$ is a microflexible sheaf. Moreover, the three-jet map $j^3 : \Phi \to \Psi_2$ is a local weak homotopy equivalence; in other words, $\mathcal{R}_2$ satisfies the local parametric $h$-principle.

**Remark 4.1.** Let $\Phi$ denote the sheaf of regular solutions of the original differential equation, namely, $\mathcal{D}f \equiv (f^*\tau_1, f^*\tau_2) = (\omega_1, \omega_2)$. Let $R_1$ be the subset of $J^2(M, N)$ consisting of two-jets of infinitesimal regular solutions of order 1 of the differential equation $\mathcal{D} = (\omega_1, \omega_2)$ and $\Psi_1$ the sheaf of sections of $R_1$. There is a canonical map $p' : \Phi \to \Phi$ which takes the triple $(\tilde{f}, \phi_1, \phi_2)$ onto $p_2 \circ \tilde{f}$. Then $p'$ induces a map $p : R_2 \to R_1$ defined by $(j^3_{f, \phi_1, \phi_2})_*(x) \mapsto j^3_{p_2 \circ f}(x)$. To see this, we note that the exterior differential
operator \( d \) determines, for each \( k \geq 1 \), a sequence of bundle maps \( d_\alpha : (\Lambda^k(M))^{(\alpha+1)} \rightarrow (\Lambda^{k+1}(M))^{(\alpha)} \), \( \alpha = 0, 1, 2, \ldots \) such that \( d_\alpha \circ j^{\alpha+1}_\tau = j^{\alpha}_d \tau \), where \( \tau \) is a \( k \)-form. Therefore, if \( j^2_\tau (\bar{f}^* \tau_1 + d\phi_1) = 0 \) and \( j^2_\tau (\bar{f}^* \tau_2 + d\phi_2) = 0 \) at \( x \), then applying \( d_1 \) on both sides we get \( j^1_\tau (\bar{f}^* \omega_1) = j^1_\omega_1(x) \) and \( j^1_\tau (\bar{f}^* \omega_2) = j^1_\omega_2(x) \), where \( f = p_2 \circ \bar{f} \). Thus, \( f \) is an infinitesimal solution of order 1 of the equation \( D = (\omega_1, \omega_2) \).

Further, we have the following commutative diagram which relates the solution sheaves of the two differential equations:

\[
\begin{array}{ccc}
\xi & \xrightarrow{\alpha} & \Psi_2 \\
\downarrow{\rho} & & \downarrow{\rho} \\
\phi & \xrightarrow{\alpha} & \Psi_1
\end{array}
\]

where \( p_* \) is the map induced by \( p \). It can be proved following [5] that \( p : R_2 \rightarrow R_1 \) is a surjective submersion and the fibres of \( p \) are affine subspaces; hence \( p \) has a section. These are, in fact, consequences of the following sequence of vector bundles and maps which is exact by the formal Poincaré Lemma:

\[
\cdots \rightarrow (\Lambda^{k-2}(M))^{(3)} \xrightarrow{d_2} (\Lambda^{k-1}(M))^{(2)} \xrightarrow{d_1} (\Lambda^k(M))^{(1)} \rightarrow \cdots
\]

Since \( p \) has a section, \( p_* \) is onto. It is now easy to see from the above commutative square, that if \( \bar{R}_2 \) satisfies the \( h \)-principle, then \( R_1 \) also satisfies the \( h \)-principle.

We recall the following definitions from [8, 3.4.1(B)].

**Definition 4.1.** Let \( M \) be a smooth manifold with a closed two-form \( \omega \). A vector field \( \partial \) on \( M \) is said to be \( \omega \)-isometric if the Lie derivative \( L_\partial \omega = 0 \), in other words, \( \partial.\omega \) is a closed form. The vector field \( \partial \) is said to be \( \omega \)-exact if there exists a zero-form \( \alpha \) (i.e., a function on \( M \)) such that \( \partial.\omega = d\alpha \).

A (local) isotopy \( \delta_t : U \rightarrow M \) is called exact if \( \delta'_t = \frac{d\delta_t}{dt} \) is an exact vector field on \( \delta_t(U) \) for all \( t \in [0, 1] \) and there exists a homotopy of zero-forms \( \alpha_t \) defined on \( \delta_t(U) \) such that \( \delta'_t.\omega = d\alpha_t \).

**Observation.** The isotopy defined by a \( \omega \)-isometric vector field consists of diffeomorphisms which preserve the form \( \omega \). If \( \delta_t \) is a \( \omega \) exact diffeotopy which fixes an open set \( U_0 \) pointwise then the exact one-forms \( \delta'_t.\omega \) vanish on \( U_0 \). This means that any primitive of \( \delta'_t.\omega \) takes a constant real value on \( U_0 \). Hence, we can choose a primitive \( \phi_t \) which also vanishes on the set \( U_0 \). We would like to remark here that diffeotopies with this property are referred as strictly exact diffeotopy in [8, 3.4.1].

**Lemma 4.1.** Suppose that \( \omega_1 = \omega_2 = \omega \). Then the \( \omega \)-exact diffeotopies of \( M \) act on the sheaf \( \bar{\Phi} \).
Proof. We follow \[8, 3.4.1(B)\] to define an action of strictly \(\omega\)-exact diffeotopies on \(\Phi\). First note that each diffeotopy \(\delta_t: U \to V\) of open subsets of \(M\) lifts to a diffeotopy \(\bar{\delta}_t: \bar{U} \times N \to V \times N\) by \(\bar{\delta}_t(u, x) = (\delta_t(u), x)\).

Suppose that, \(Y' \subset Y \cap (U \times N)\) and \(\bar{\delta}_t(Y') \subset Y\) for all \(t\). Then \(\delta_t\) has a natural action on sections \(\bar{f}: V \to V \times N\) whose images lie in \(Y\). The action is given by \(\delta_t \bar{f} = \bar{\delta}_t^{-1} \bar{f} \delta_t\) (see Example A.1 in Section 5).

Differentiating the homotopy of one-forms \(\bar{\delta}_t^* \tau_1\) with respect to \(t\) we obtain

\[L_{\bar{\delta}_t} \tau_1 = \bar{\delta}_t^* \bar{\sigma}_1 + d(\bar{\delta}_t^* \tau_1) = p_1^* (\alpha_t, \omega) + d(\bar{\delta}_t^* \tau_1).\]

If \(\delta_t\) is strictly \(\omega\)-exact, then there exists a homotopy of \(C^\infty\) functions \(\alpha_t\) along \(\delta_t(U)\) such that \(\delta_t^* \omega = d\alpha_t\). Hence

\[(4.1) \quad \bar{\delta}_t^* \tau_1 = \tau_1 + d\phi_t, \quad \text{where } \phi_t = \int_0^t (p_1^* \alpha_t + \bar{\delta}_t^* \tau_1) dt.
\]

Further, if \(\delta_t\) is constant on \(U_0\) for \(t \leq t_0\), we can and we do choose \(\phi_t = 0\) for \(t \leq t_0\).

Consider a triple \((\bar{f}, \phi, \psi)\) in \(\Phi\) so that \(\bar{f}: V \to V \times N\) has its image in \(Y\) and \(\bar{f}^* \tau_1 + d\phi = 0\), \(\bar{f}^* \tau_2 + d\psi = 0\). If we define \(\delta_t \bar{f} = \bar{\delta}_t^{-1} \bar{f} \delta_t\), then using equation (4.1) we obtain

\[
(\delta_t \bar{f})^* \tau_1 = \bar{\delta}_t^* \bar{f}^* (\bar{\delta}_t^{-1})^* \tau_1 = \delta_t^* \bar{f}^* [\tau_1 - d(\bar{\delta}_t^{-1})^* \phi_t] = -\delta_t^* d\phi + d(\delta_t \bar{f})^* \phi_t = -d(\delta_t^* \phi - (\delta_t \bar{f})^* \phi_t).
\]

Finally, since \(\delta_t\) is strictly \(\omega\)-exact diffeotopy, \(\phi \mapsto \delta_t^* \phi - (\delta_t \bar{f})^* \phi_t\) defines an action on the space of \(C^\infty\) functions on \(M\) (for a fixed \(\bar{f}\)). Indeed, if \(\delta_t\) is constant in \(t\) on a maximal open set \(U_0\) then we can choose \(\phi_t = 0\) on \(U_0\) and then \(\delta_t^* \phi - (\delta_t \bar{f})^* \phi_t\) is constant on \(U_0\).

Therefore, we can define the action of a strictly exact diffeotopy \(\delta_t\) satisfying \(\delta_t(Y') \subset Y\) on \(\Phi\) by

\[
(\delta_t \bar{f}, \phi, \psi) = (\delta_t \bar{f}, \delta_t^* \phi - (\delta_t \bar{f})^* \phi_t, \delta_t^* \psi - (\delta_t \bar{f})^* \psi_t),
\]

where \(\delta_t \bar{f} = \bar{\delta}_t^{-1} \bar{f} \delta_t\) and \(\phi_t, \psi_t\) satisfy the relations \(\bar{\delta}_t^* \tau_1 = \tau_1 + d\phi_t, \bar{\delta}_t^* \tau_2 = \tau_2 + d\psi_t\).

\[\square\]

Proposition 4.2. Suppose that \(\omega_1 = \omega_2 = \omega\), where \(\omega\) is the zero-form or a symplectic form on \(M\). If \(M_0\) is a submanifold of \(M\) of positive codimension then \(j^3: \Phi|_{M_0} \to \bar{\Phi}_2|_{M_0}\) is a weak homotopy equivalence.

Proof. In view of Theorem A.3 in Section 5 and Proposition 4.1 we need to show that there is an appropriate class of (local) diffeotopies which act on the sheaf \(\Phi\) and have the desired sharply moving property. We first consider
the case when $\omega$ is a symplectic form. Recall that $\omega$ defines a canonical isomorphism $I_\omega : TM \to T^*M$ which in turn defines a one-to-one correspondence between vector fields and one-forms on $M$. Indeed, if $\partial$ is a vector field on $M$ then $\partial \cdot \omega$ is a global one-form on $M$. The $\omega$-exact (local) diffeotopies are obtained by integrating the vector fields which correspond to the exact one-forms under this correspondence. As we observed in Lemma 4.1, these diffeotopies act on the sheaf $\Phi$. It is also known that these diffeotopies sharply move any submanifold of $M$ of positive codimension (see [8, 3.4.2]). This proves the proposition for the case when $\omega$ is symplectic.

If $\omega$ is the zero form, it is enough to observe that any local diffeotopy is $\omega$-exact. □

We are now in a position to prove Theorem C.

**Proof of Theorem C.** Since $M$ is an open manifold, it admits a Morse function without any critical point of index equal to the dimension of $M$ [10]. It then follows from Morse theory that, there is a simplicial complex $K$ in $M$ of positive codimension which is a strong deformation retract of $M$. In fact, $M$ is isotopic to an arbitrarily small open neighbourhood of $K$. Now, by the above proposition, the $h$-principle for $\Phi$ localizes near $K$, and hence the $h$-principle for the sheaf $\Phi$ also localizes near $K$ (see Remark 4.1). This means that a section of $R_1$ can be homotoped to a solution $f_1$ of the differential equation $D = (\omega, \omega)$ near $K$, where $\omega$ is either the zero-form or a symplectic form on $M$.

To prove (a), take an isotopy $\phi_t$ such that $\phi_1$ brings $M$ into the domain of $f_1$. Then the composition map $f_1 \circ \phi_1$ is a global solution of the equation $D = (0, 0)$.

To obtain global $h$-principle stated in (b) we observe that there is a homotopy of symplectic immersions $\phi_t : (M, \omega) \to (M, \omega)$ such that $\phi_0 = \text{id}$ and $\phi_1$ maps $M$ into $K$ ([5, 6]). Composing $f_1$ with $\phi_1$ we obtain a global solution of the equation $D = (\omega, \omega)$. □

**Remark 4.2.** If $(M_0, \omega_0)$ is a symplectic manifold then as a direct consequence of the above theorem we obtain the $h$-principle with $M = M_0 \times \mathbb{R}^2$ and $\omega = \omega_0 \oplus dx \wedge dy$.

**Proof of Theorem 3.1.** Let $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$ for some one-forms $\alpha_1, \alpha_2$ on $M$. Let $\hat{\Phi}$ denote the sheaf of solutions of the differential equation $D = (p^*\alpha_1, p^*\alpha_2)$, where $D$ is defined as in Section 3. A diffeomorphism $\lambda : M \times \mathbb{R} \to M \times \mathbb{R}$ is said to be fibre-preserving if $p \circ \lambda = \lambda$, where $p : M \times \mathbb{R} \to M$ is the projection onto the first factor. Hence $\lambda^* p^* \alpha_i = p^* \alpha_i, i = 1, 2$, for such a $\lambda$. This allows us to define an action of fibre-preserving diffeomorphisms on the sheaf $\hat{\Phi}$. Indeed, if $D(f, \phi_1, \phi_2) = (p^*\alpha_1, p^*\alpha_2)$ and $f$ is regular, then $f^* \tau_i + d\phi_i = p^* \alpha_i$, where $\sigma_i = d\tau_i$ for $i = 1, 2$. If $\lambda$ is a fibre-preserving diffeomorphism then we define an action
by the following simple rule:

\[ \lambda(f, \phi_1, \phi_2) = (f \circ \lambda, \phi_1 \circ \lambda, \phi_2 \circ \lambda). \]

(Note that if \( \lambda_t \) is a fibre-preserving diffeotopy, then the vector fields \( \lambda'_t \) has no component along \( M_0 \). Hence, \( \lambda'_t \) is \( \omega \)-exact for any two-form \( \omega \) of the form \( p^*_0 \omega_0 \), where \( \omega_0 \) is a two-form on \( M \).) Since we observed in Proposition 3.1 that \( \tilde{D} \) is infinitesimally invertible on regular maps, the hypothesis of Theorem A.4 is satisfied (by Theorem A.2 and Proposition A.1 in Section 5). This proves that \( \hat{\Phi} \) satisfies the \( h \)-principle. This \( h \)-principle then descends to the desired \( h \)-principle for \( \Phi \) by an argument similar to that in Remark 4.1. □

5. Appendix A. Preliminaries of \( h \)-principle

Here we briefly discuss the sheaf technique and the analytic technique in the theory of \( h \)-principle following [8].

Let \( p : E \rightarrow M \) be a \( C^\infty \)-fibration, and let \( E^{(r)} \) denote the \( r \)-jet space of \( C^\infty \)-sections of \( E \) for \( r \geq 1 \). Then the canonical projection \( p^{(r)} : E^{(r)} \rightarrow M \) is also a fibration. We endow the space of sections of \( p \) and \( p^{(r)} \) with the \( C^\infty \) and \( C^0 \)-compact open topologies, respectively. The canonical projection maps \( E^{(r)} \rightarrow E^{(i)} \) are denoted by \( p^r_i \).

A partial differential relation of order \( r \) for sections of \( E \) is a subset \( \mathcal{R} \) of \( E^{(r)} \). A section \( f : M \rightarrow E \) is said to be a solution of \( \mathcal{R} \) if the \( r \)-jet map \( j^r_f \) (which is a section of \( p^{(r)} \)) maps \( M \) into \( \mathcal{R} \). A section of \( p^{(r)} \) is called holonomic if it is the \( r \)-jet map of a solution of \( \mathcal{R} \).

We denote the space of solutions of \( \mathcal{R} \) by \( \text{Sol} \mathcal{R} \), while \( \Gamma(\mathcal{R}) \) denotes the space of sections of \( E^{(r)} \rightarrow M \) whose images lie in \( \mathcal{R} \).

**Definition A.1.** A relation \( \mathcal{R} \) is said to satisfy the \( h \)-principle if a section of \( \mathcal{R} \) can be homotoped within \( \Gamma(\mathcal{R}) \) to a holonomic section.

\( \mathcal{R} \) satisfies the parametric \( h \)-principle if the \( r \)-jet map \( j^r : \text{Sol} \mathcal{R} \rightarrow \Gamma(\mathcal{R}) \) is a weak homotopy equivalence.

**Definition A.2.** Let \( S \) be a subspace of the space of continuous sections of \( E \). A relation \( \mathcal{R} \subset E^{(r)} \) is said to satisfy the \( h \)-principle \( C^0 \)-dense in \( S \) if for every \( f_0 \in S \), for every neighbourhood \( U \) of graph \( f_0 \) and for every section \( \phi_0 : M \rightarrow \mathcal{R} \) satisfying \( p^0_0 \circ \phi_0 = j^0_f \), there exists a homotopy of sections \( \phi_t : M \rightarrow \mathcal{R} \) such that the image of \( p^0_0 \circ \phi_t \) is contained in \( U \) and \( \phi_1 \) is holonomic.

Let \( \Phi \) denote the sheaf of solutions of a given relation \( \mathcal{R} \), and \( \Psi \) the sheaf of sections of \( \mathcal{R} \). The topologies on \( \Phi(U) \) and \( \Psi(U) \) are, respectively, the \( C^\infty \) and \( C^0 \)-compact open topologies. The \( r \)-jet map \( j^r \) defines a sheaf homomorphism from \( \Phi \) to \( \Psi \). This takes us into the realm of topological sheaves.
Sometimes, by an abuse of language, we say that the sheaf \( \Phi \) satisfies the \( h \)-principle without giving any reference to a partial differential relation. But one should be careful at this point, since two subsets \( \mathcal{R} \) and \( \mathcal{R}' \) possibly in different jet spaces \( E^{(r)} \) and \( E^{(s)} \) may have the same set of solutions, but one of the relations may satisfy the \( h \)-principle while the other may not. In fact, given an \( r \)th-order relation \( \mathcal{R} \), we can form an \( s \)th-order relation \( \mathcal{R}' \) (for any \( s > r \)) by taking \( s \)-jets of \( C^s \) solutions of \( \mathcal{R} \).

We recall some general definitions and terminology from [8].

**Definition A.3.** Let \( \mathcal{F} \) be a topological sheaf over \( M \) and \( A \) a compact set in \( M \). Then \( \mathcal{F}(A) \) will denote the direct limit of the sets \( \mathcal{F}(U) \), where \( U \) runs over all open sets containing \( A \).

However, these sets, \( \mathcal{F}(A) \), will have only quasi-topological structures [8, 1.4.1]. A map \( f : P \rightarrow \mathcal{F}(A) \) on a polyhedron \( P \) is called continuous (in the quasi-topological sense) if there exists an open set \( U \supset A \) such that each \( f_p \) is defined over \( U \) and the resulting map \( P \rightarrow \mathcal{F}(U) \) is continuous with respect to the given topology on \( \mathcal{F}(U) \).

**Definition A.4.** \( \mathcal{R} \) satisfies the local parametric \( h \)-principle if for each \( x \in M \), \( j^r : \Phi(x) \rightarrow \Psi(x) \) is a weak homotopy equivalence.

**Definition A.5.** A topological sheaf \( \mathcal{F} \) over \( M \) is flexible if the restriction maps \( \mathcal{F}(A) \rightarrow \mathcal{F}(B) \) are Serre fibrations for every pair of compact sets \( (A,B) \), \( A \supset B \). The restriction map \( \mathcal{F}(A) \rightarrow \mathcal{F}(B) \) is called a microfibration if given a continuous map \( f_0' : P \rightarrow \mathcal{F}(A) \) on a polyhedron \( P \) and a homotopy \( f_t, 0 \leq t \leq 1 \), of \( f_0' | B \) there exists an \( \varepsilon > 0 \) and a homotopy \( f_t' \) of \( f_0' \) such that \( f_t' | \text{Op} B = f_t \) for \( 0 \leq t \leq \varepsilon \). If for every pair of compact sets the restriction morphism is a microfibration, then the sheaf \( \mathcal{F} \) is called microflexible.

The following topological result provides a sufficient condition for a sheaf homomorphism to be a weak homotopy equivalence.

**Theorem A.1 ([8, 2.2.1(B)]).** Let \( \mathcal{F} \) and \( \mathcal{G} \) be two flexible sheaves over \( M \), and let \( \alpha : \mathcal{F} \rightarrow \mathcal{G} \) be a continuous sheaf homomorphism such that \( \alpha(x) : \mathcal{F}(x) \rightarrow \mathcal{G}(x) \) is a weak homotopy equivalence for each \( x \in M \). Then \( \alpha \) is a weak homotopy equivalence.

Thus, if the solution sheaf \( \Phi \) is flexible and if \( \mathcal{R} \) satisfies the local parametric \( h \)-principle, then \( \mathcal{R} \) satisfies the parametric \( h \)-principle (because \( \Psi \) is always flexible [8, 1.4.2(A')]).

The solution sheaf turns out to be non-flexible in many important problems, though microflexibility is a much more common property. For example, when \( \mathcal{R} \) is open the solution sheaf is easily seen to be microflexible; however, many of the relations that are of special interest are fibrewise closed in the
jet space. This is the case, when the solutions of $\mathcal{R}$ also arise as solutions to some PDEs $\mathcal{D}(f) = g$, where $\mathcal{D}$ is defined on sections of the fibre bundle $E$ taking values in the space of sections of a vector bundle. Gromov proves that $\Phi$ (or possibly a subsheaf of $\Phi$) is microflexible when the operator $\mathcal{D}$ is infinitesimally invertible over an open subset of $\Gamma^\infty(E)$ in the fine $C^\infty$ topology. On the other hand, he observes that there are higher order relations $\mathcal{R}^\alpha \subset J^{(r+\alpha)}$, $\alpha = 0, 1, 2, \ldots$, that have the same solution space as $\mathcal{R}$ and which satisfy the local parametric $h$-principle for $\alpha > \alpha_0$, where $\alpha_0$ is some positive integer.

To elaborate this let $E \longrightarrow M$ be a $C^\infty$-fibration and $G \longrightarrow M$ be a $C^\infty$ vector bundle over a manifold $M$. We denote by $\mathcal{E}^\alpha$ and $\mathcal{G}^\alpha$, respectively, the spaces of $C^\alpha$ sections of $E$ and $G$ with the fine $C^\alpha$ topologies, for $\alpha = 1, 2, \ldots, \infty$. Let $\mathcal{D} : \mathcal{E}^r \longrightarrow \mathcal{G}^0$ be a $C^\infty$ differential operator of order $r$, which means that $\mathcal{D}$ is given by a $C^\infty$ bundle map $\Delta : E^{(r)} \to G$ such that $\mathcal{D}(f) = \Delta \circ j_f^r$. As a consequence, we obtain a sequence of bundle maps $\Delta_\alpha : E^{(r+\alpha)} \to G^{(\alpha)}$ such that $j^{\alpha}_{\mathcal{D}(f)} = \Delta_\alpha \circ j_f^{r+\alpha}$, where $\alpha$ is any non-negative integer.

Let $V$ denote the subbundle of $TE$ consisting of all vectors which are tangent to the fibres of $E$ over points of $M$. We shall refer $V$ as the vertical tangent bundle of $E$. For any section $f$ of $E$, the vector space of $C^\beta$ sections of the pullback bundle $f^*V$ will be denoted by $\mathcal{E}^\beta_f$. The space $\mathcal{E}^\beta_f$ is defined as the infinite-dimensional tangent space of $\mathcal{E}$ at $f$. It is not difficult to see that when $E$ is a vector bundle, $f^*V$ is canonically isomorphic to $E$ and therefore $\mathcal{E}^\beta_f$ is isomorphic to $\mathcal{E}^\beta$.

The linearization $L_f$ of $\mathcal{D}$ at $f$ is a map $L_f : \mathcal{E}^r_f \longrightarrow \mathcal{G}^0$ which is defined as follows:

$$L_f(y) = \lim_{t \to -\infty} \frac{\partial}{\partial t} \mathcal{D}(f(t))|_{t=0},$$

where $f_t$ is a differentiable curve in $\mathcal{E}^r$ such that $f_0 = f$ and the tangent to $f_t$ at $t = 0$ is $y \in \mathcal{E}^r_f$. Clearly, $L_f$ is a linear differential operator of order $r$ in $y$ and $L(f, y) = L_f(y)$ is a differential operator of order $r$ in both $f$ and $y$.

Let $A \subset E^{(d)}$ be an open relation of order $d$ for some $d \geq r$, and $\mathcal{A}$ denote the space of solutions of the relation $A$. Clearly, $\mathcal{A}$ is contained in $\mathcal{E}^d$, and $\mathcal{A}^{\alpha+d} = \mathcal{A} \cap \mathcal{E}^{\alpha+d}$ is an open subset of $\mathcal{E}^{\alpha+d}$ in the fine $C^{\alpha+d}$ topology. A solution of $A$ will be referred as an $A$-regular section of $E$.

$\mathcal{D}$ is said to be infinitesimally invertible over the subset $A \subset \mathcal{E}^d$ if for every $f \in \mathcal{A}$ there is a linear differential operator $M_f : \mathcal{G}^s \longrightarrow \mathcal{E}^0_f$ of a certain order $s$ (independent of $f$) such that the following properties are satisfied:
(1) The global operator
\[ M : \mathcal{A}^d \times \mathcal{G}^s \rightarrow T(\mathcal{E}^0) \]

is a differential operator that is given by a \( C^\infty \) map \( \mathcal{A} \oplus G(s) \rightarrow V \).

(2) \( L(f, M(f,g)) = g \) for all \( f \in \mathcal{A}^{d+r} \) and \( g \in \mathcal{G}^{r+s} \), where \( M(f,g) = M_f(g) \). In other words, \( M_f \) is a right inverse of \( L_f \).

The integer \( d \) is called the defect of the infinitesimal inversion \( M \) [8, 2.3.1].

We now quote two results from [8] which are consequences of an Implicit Function Theorem (due to Gromov) in the context of differential operators.

**Theorem A.2** ([8, 2.3.2(B),(D′′)]. Suppose that \( \mathcal{D} \) is a \( C^\infty \) differential operator of order \( r \) and it admits an infinitesimal inversion of defect \( d \) on \( \mathcal{A} \).

(i) The operator \( \mathcal{D} : \mathcal{A}^\infty \rightarrow \mathcal{G}^\infty \) is an open map in the respective fine \( C^\infty \) topologies.

(ii) The sheaf of \( \mathcal{A} \)-regular solutions of the differential equation \( \mathcal{D} = g \) is microflexible, where \( g \) is a smooth section of \( \mathcal{G} \).

**Definition A.6.** A local section \( f \) of \( E \), defined on a neighbourhood of some \( x \in M \), is said to be an infinitesimal solution of \( \mathcal{D} = g \) of order \( \alpha \) if the \( \alpha \)-jet of \( \mathcal{D}(f) - g \) is zero at \( x \).

Let \( \mathcal{R}^\alpha \subset E^{(\alpha+r)} \) consist of \( (\alpha+r) \)-jets of infinitesimal solutions of \( \mathcal{D} = g \) of order \( \alpha \) and let \( \mathcal{R}^0 \) be denoted as \( \mathcal{R} \). Since \( j_{\mathcal{D}(f)}^\alpha = \Delta_\alpha \circ j_f^{r+\alpha} \), therefore, \( \mathcal{R}^\alpha = (\Delta_\alpha)^{-1}(j_g^0) \).

Define
\[ \mathcal{R}_\alpha = \mathcal{R}^\alpha \cap (p^\alpha_d)^{-1}(A), \]
where \( p^\alpha_d : E^{(\alpha+r)} \rightarrow E^{(d)} \) is the canonical projection map for \( \alpha \geq d - r \). The relations \( \mathcal{R}_\alpha \) have the same \( C^\infty \) solutions for all \( \alpha \geq d - r \), namely the \( C^\infty \) solutions of the equation \( \mathcal{D}(x) = g \) in \( \mathcal{A} \).

Let \( \Phi_{\text{reg}} \) denote the sheaf of \( \mathcal{A} \)-regular solutions of the equation \( \mathcal{D} = g \) with the \( C^\infty \) compact open topology and let \( \Psi_\alpha \) be the sheaf of sections of \( \mathcal{R}_\alpha \) with \( C^0 \) compact open topology.

**Proposition A.1** ([8, 2.3.2(D′),(D′′)]. If \( \mathcal{D} \) admits an infinitesimal inversion of order \( s \) and defect \( d \) on \( \mathcal{A} \) then the map \( J : \Phi_{\text{reg}} \rightarrow \Psi_\alpha \), defined by \( J(\phi) = j^{r+\alpha}_\phi \), is a local weak homotopy equivalence for each \( \alpha \geq \max(d + s, 2r + 2s) \). In other words, \( \mathcal{R}_\alpha \) satisfies the local parametrical \( h \)-principle.

Thus we see that there is a large class of relations \( \mathcal{R} \) which satisfy the local \( h \)-principle and for which the solution sheaves are microflexible. The following result of Gromov in [8, 2.2.3(C′)] is the central result in the theory of \( h \)-principle as far as the sheaf technique is concerned:
**Theorem A.3.** If $\Phi$ is a microflexible sheaf on a manifold $M$ and $N$ is an embedded submanifold of positive codimension, then $\Phi|N$ is flexible, provided there is a class of ‘acting diffeotopies’ $D_0$ which ‘sharply moves $N$’.

We now explain the notion of ‘acting diffeotopies’ and ‘sharply moving diffeotopies’.

**Action of diffeotopies:** Let $\Phi$ be a topological sheaf over a manifold $M$ and $U'$ an open subset of $M$. Consider a diffeotopy $\delta_t : U \rightarrow U'$ that moves an open subset $U \subset U'$ inside $U'$, where $\delta_0 = \text{id}$. Let $\Phi'$ be a subset of $\Phi(U')$, and the diffeotopy $\delta_t$ act on $\Phi'$ by assigning a homotopy of sections $\delta_t^* \phi$ in $\Phi(U)$ to every $\phi \in \Phi'$ such that $\delta_0^* \phi = \phi|_U$, and the following conditions are satisfied:

1. If two sections $\phi_1$ and $\phi_2$ in $\Phi'$ are such that $\phi_1(u') = \phi_2(u')$ for some $u' \in U'$ and if $\delta_{t_0}(u) = u'$ for some $u \in U$, then $\delta_{t_0}^* \phi_1(u) = \delta_{t_0}^* \phi_2(u)$.
   In particular, if the two sections $\phi_1$ and $\phi_2$ restrict to the same section on $U$, then $(\delta_t|_U)^* \phi_1 = (\delta_t|_U)^* \phi_2$.

2. If $U_0$ is a maximal open subset where $\delta_t$ is constant, (that is, $\delta_t(x) = x$ for all $x \in U_0$), then $\delta_t^* \phi$ is also constant on $U_0$ (that is $\delta_t^* \phi = \phi$ on $U_0$).

3. If the diffeotopy $\delta_t$ is constant for $t \geq t_0$, then $\delta_t^* \phi$ is also constant for $t \geq t_0$ for some $t_0 \in [0, 1]$.

4. If $\phi_p \in \Phi'$, $p \in P$, is a continuous family of sections then the family $\delta_t^* \phi_p$ is jointly continuous in $p$ and $t$.

Conditions (2) and (3) are natural in the sense that they make the action compatible with the presheaf structure.

One must note that this is a partial action as $\delta_t$ need not in general act on $\Phi(U)$. Further, there is no condition on the subset $\Phi'$.

**Example A.1.** Let $\Phi$ denote the sheaf of sections of the product bundle $M \times N$ over $M$. Then $\text{Diff}(M)$, the pseudogroup of local diffeomorphisms of $M$, has a natural action on $\Phi$ given by $\delta \cdot \phi = \delta^{-1} f \delta$, where $f \in \Phi$ and $\delta : U \times N \rightarrow V \times N$ is given by $\delta(x, y) = (\delta(x), y)$. We can extend this action to an action by diffeotopies of $M$. However, if we consider the subsheaf $\Phi_Y$ of sections of $M \times N$ whose images lie in an open subset $Y$ then $\Phi_Y$ is not invariant under this action. In this case we get only a partial action by diffeotopies: indeed, if $\delta_t$ is sufficiently $C^0$-close to the identity map then it acts on $\Phi_Y(U)$. More generally, if $\delta_t$ is a diffeotopy that moves $U$ in $M$ and if there is an open subset $Y' \subset (U \times N)$ such that $\delta_t(Y') \subset Y$ for all $t \in [0, 1]$, then $\delta_t$ acts on the sheaf $\Phi_Y$.

**Definition A.7.** We fix a metric $d$ on $M$. Let $M_0$ be a submanifold of $M$ of positive codimension which lies in an open subset $U'$ of $M$. A class of
diffeotopies $\mathcal{D}$ on $M$ is said to sharply move $M_0$ in $M$ if given any hypersurface $S$ in $M_0$ and any positive numbers $\varepsilon$, we can obtain a diffeotopy $\delta_t : \text{Op} M_0 \rightarrow U'$ in $\mathcal{D}$ which satisfies the following conditions:

1. $\delta_0$ is the identity map;
2. $\delta_t|_{\text{Op} v}$ is identity for all $v \in M_0$ for which $d(v, S) \geq \varepsilon$;
3. $d(\delta_1(S), M_0) > r$ for some positive number $r$.

We end this section with the following result of $h$-principle.

**Theorem A.4.** Let $M = M_0 \times \mathbb{R}$. Suppose that $\mathcal{R}$ satisfies the local parametric $h$-principle and the solution sheaf $\Phi$ of $\mathcal{R}$ is microflexible. If the fibre-preserving diffeotopies of $M$ act on $\Phi$ then $\mathcal{R}$ satisfies the $h$-principle.

**Proof.** Since the fibre-preserving diffeotopies of $M$ sharply move the submanifold $M_0$, it follows from Theorem A.3 that a section of $\mathcal{R}$ can be homotoped to a holonomic section $j_f$ over an open neighbourhood $U$ of $M_0 \times \{0\}$ in $M$. Since $M$ is split as $M_0 \times \mathbb{R}$, we can deform $M$ into $U$ by a smooth one-parameter family of embeddings $F_t : M_0 \times \mathbb{R} \longrightarrow M_0 \times \mathbb{R}, 0 \leq t \leq 1$, such that $F_t$ is fibre-preserving and $F_1$ takes $M$ into $U$. Since $F_t$ acts on $\Phi$, $F_t^* f$ is a global solution of $\mathcal{R}$. This proves the theorem.  

**References**


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