COHOMOLOGICALLY SYMPLECTIC SOLVMANIFOLDS ARE SYMPLECTIC

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We consider aspherical manifolds with torsion-free virtually polycyclic fundamental groups, constructed by Baues. We prove that if those manifolds are cohomologically symplectic then they are symplectic. As a corollary we show that cohomologically symplectic solvmanifolds are symplectic.

1. Introduction

A $2n$-dimensional compact manifold $M$ is called cohomologically symplectic (c-symplectic) if we have $\omega \in H^2(M, \mathbb{R})$ such that $\omega^n \neq 0$. A compact symplectic manifold is c-symplectic but the converse is not true in general. For example $\mathbb{C}P^2 \# \mathbb{C}P^2$ is c-symplectic but not symplectic. But for some class of manifolds these two conditions are equivalent. For examples, nilmanifolds, i.e., compact homogeneous spaces of nilpotent simply connected Lie groups. In [7], for a nilpotent simply connected Lie group $G$ with a cocompact discrete subgroup $\Gamma$ (such subgroup is called a lattice), Nomizu showed that the de Rham cohomology $H^*(G/\Gamma, \mathbb{R})$ of $G/\Gamma$ is isomorphic to the cohomology $H^*(\mathfrak{g})$ of the Lie algebra of $G$. By the application of Nomizu’s theorem, if $G/\Gamma$ is c-symplectic then $G/\Gamma$ is symplectic (see [3, p. 191]). Every nilmanifold can be represented by such $G/\Gamma$ (see [6]).

Consider solvmanifolds, i.e., compact homogeneous spaces of solvable simply connected Lie groups. Let $G$ be a solvable simply connected Lie group with a lattice $\Gamma$. We assume that for any $g \in G$ the all eigenvalues of the adjoint operator $\text{Ad}_g$ are real. With this assumption, in [5] Hattori extended Nomizu’s theorem. By Hattori’s theorem, for such case, without difficulty, we can similarly show that if $G/\Gamma$ is c-symplectic, then $G/\Gamma$ is symplectic. But the isomorphism $H^*(G/\Gamma, \mathbb{R}) \cong H^*(\mathfrak{g})$ fails to hold for general solvable Lie groups, and not all solvmanifolds can be represented by $G/\Gamma$. Thus it is a considerable problem whether every c-symplectic solvmanifold is symplectic.
Let \( \Gamma \) be a torsion-free virtually polycyclic group. In [1] Baues constructed the compact aspherical manifold \( M_\Gamma \) with \( \pi_1(M_\Gamma) = \Gamma \). Baues proved that every infra-solvmanifold (see [1] for the definition) is diffeomorphic to \( M_\Gamma \). In particular, the class of such aspherical manifolds contains the class of solvmanifolds. We prove that if \( M_\Gamma \) is c-symplectic then \( M_\Gamma \) is symplectic. In other words, for a torsion-free virtually polycyclic group \( \Gamma \) with \( 2n = \text{rank} \Gamma \), if there exists \( \omega \in H^2(\Gamma, \mathbb{R}) \) such that \( \omega^n \neq 0 \) then we have a symplectic aspherical manifold with the fundamental group \( \Gamma \).

2. Notation and conventions

A general reference here is [2]. Let \( k \) be a subfield of \( \mathbb{C} \). A group \( G \) is called a \( k \)-algebraic group if \( G \) is a Zariski-closed subgroup of \( GL_n(\mathbb{C}) \) which is defined by polynomials with coefficients in \( k \). Let \( G(k) \) denote the set of \( k \)-points of \( G \) and \( U(G) \) the maximal Zariski-closed unipotent normal \( k \)-subgroup of \( G \) called the unipotent radical of \( G \). Let \( U_n(k) \) denote the \( n \times n \) \( k \)-valued upper triangular unipotent matrix group.

3. Aspherical manifolds with torsion-free virtually polycyclic fundamental groups

**Definition 3.1.** A group \( \Gamma \) is polycyclic if it admits a sequence

\[
\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}
\]

of subgroups such that each \( \Gamma_i \) is normal in \( \Gamma_{i-1} \) and \( \Gamma_{i-1}/\Gamma_i \) is cyclic. We denote \( \text{rank} \Gamma = \sum_{i=1}^{k} \text{rank} \Gamma_{i-1}/\Gamma_i \).

**Proposition 3.1 [8, Proposition 3.10].** The fundamental group of a solvmanifold is torsion-free polycyclic.

Let \( k \) be a subfield of \( \mathbb{C} \). Let \( \Gamma \) be a torsion-free virtually polycyclic group. For a finite index polycyclic subgroup \( \Delta \subset \Gamma \), we denote \( \text{rank} \Gamma = \text{rank} \Delta \).

**Definition 3.2.** We call a \( k \)-algebraic group \( H_\Gamma \) a \( k \)-algebraic hull of \( \Gamma \) if there exists an injective group homomorphism \( \psi : \Gamma \to H_\Gamma(k) \) and \( H_\Gamma \) satisfies the following conditions:

1. \( \psi(\Gamma) \) is Zariski-dense in \( H_\Gamma \).
2. \( Z_{H_\Gamma}(U(H_\Gamma)) \subset U(H_\Gamma) \) where \( Z_{H_\Gamma}(U(H_\Gamma)) \) is the centralizer of \( U(H_\Gamma) \).
3. \( \dim U(H_\Gamma) = \text{rank} \Gamma \).

**Theorem 3.1 [1, Theorem A.1].** There exists a \( k \)-algebraic hull of \( \Gamma \) and a \( k \)-algebraic hull of \( \Gamma \) is unique up to \( k \)-algebraic group isomorphism.

Let \( \Gamma \) be a torsion-free virtually polycyclic group and \( H_\Gamma \) the \( \mathbb{Q} \)-algebraic hull of \( \Gamma \). Denote \( H_\Gamma = H_\Gamma(\mathbb{R}) \). Let \( U_\Gamma \) be the unipotent radical of \( H_\Gamma \).
and $T$ a maximal reductive subgroup. Then $H_\Gamma$ decomposes as a semi-direct product $H_\Gamma = T \times U_\Gamma$. Let $u$ be the Lie algebra of $U_\Gamma$. Since the exponential map $\exp : u \rightarrow U_\Gamma$ is a diffeomorphism, $U_\Gamma$ is diffeomorphic to $\mathbb{R}^n$ such that $n = \text{rank} \Gamma$. For the semi-direct product $H_\Gamma = T \times U_\Gamma$, we denote $\phi : T \rightarrow \text{Aut}(U_\Gamma)$ the action of $T$ on $U_\Gamma$. Then we have the homomorphism $\alpha : H_\Gamma \rightarrow \text{Aut}(U_\Gamma) \times U_\Gamma$ such that $\alpha(t, u) = (\phi(t), u)$ for $(t, u) \in T \times U_\Gamma$. By the property (2) in Definition 3.2, $\phi$ is injective and hence $\alpha$ is injective.

In [1] Baues constructed a compact aspherical manifold $M_\Gamma = \alpha(\Gamma) \backslash U_\Gamma$ with $\pi_1(M_\Gamma) = \Gamma$. We call $M_\Gamma$ a standard $\Gamma$-manifold.

**Theorem 3.2** [1, Theorem 1.2, 1.4]. A standard $\Gamma$-manifold is unique up to diffeomorphism. A solvmanifold with the fundamental group $\Gamma$ is diffeomorphic to the standard $\Gamma$-manifold $M_\Gamma$.

Let $A^*(M_\Gamma)$ be the de Rham complex of $M_\Gamma$. Then $A^*(M_\Gamma)$ is the set of the $\Gamma$-invariant differential forms $A^*(U_\Gamma)^T$ on $U_\Gamma$. Let $(\bigwedge u^*)^T$ be the left-invariant forms on $U_\Gamma$ which are fixed by $T$. Since $\Gamma \subset H_\Gamma = T \times U_\Gamma$, we have the inclusion

$$(\bigwedge u^*)^T = A^*(U_\Gamma)^H_\Gamma \subset A^*(U_\Gamma)^\Gamma = A^*(M_\Gamma).$$

**Theorem 3.3** [1, Theorem 1.8]. This inclusion induces an isomorphism on cohomology.

By the application of the above facts, we prove the main theorem of this paper.

**Theorem 3.4.** Suppose $M_\Gamma$ is c-symplectic. Then $M_\Gamma$ admits a symplectic structure. In particular, cohomologically symplectic solvmanifolds are symplectic.

**Proof.** Since we have the isomorphism $H^*(M_\Gamma, \mathbb{R}) \cong H^*((\bigwedge u^*)^T)$, we have $\omega \in (\bigwedge^2 u^*)^T$ such that $0 \neq [\omega]^n \in H^{2n}((\bigwedge u^*)^T)$. This gives $0 \neq \omega^n \in (\bigwedge u^*)^T$ and hence $0 \neq \omega^n \in \bigwedge u^*$. Since $\omega^n$ is a non-zero invariant 2n-form on $U_\Gamma$, we have $(\omega^n)_p \neq 0$ for any $p \in U_\Gamma$. Hence by the inclusion $(\bigwedge u^*)^T \subset A^*(U_\Gamma)^T = A^*(M_\Gamma)$, we have $(\omega^n)_{\Gamma p} \neq 0$ for any $\Gamma p \in \Gamma \backslash U_\Gamma = M_\Gamma$. This implies that $\omega$ is a symplectic form on $M_\Gamma$. Hence, we have the theorem.  

4. Remarks

Let $G = \mathbb{R} \ltimes \phi U_3(\mathbb{C})$ such that

$$\phi(t) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{\text{i}zt} \cdot x \\ 0 & 1 & e^{-\text{i}zt} \cdot y \\ 0 & 0 & 1 \end{pmatrix},$$
and $D = \mathbb{Z} \ltimes_{\varphi} D'$ with
\[
D' = \left\{ \begin{pmatrix} 1 & x_1 + iy_2 & z_1 + iz_2 \\ 0 & 1 & y_1 + iy_2 \\ 0 & 0 & 1 \end{pmatrix} : x_1, y_2, z_2 \in \mathbb{Z}, x_2, y_1, z_1 \in \mathbb{R} \right\}.
\]
Then $D$ is not discrete and $G/D$ is compact. We have $D/D_0 \sim \mathbb{Z} \ltimes_{\varphi} U_3(\mathbb{Z})$ such that
\[
\varphi(t) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (-1)^tx & z \\ 0 & 1 & (-1)^ty \\ 0 & 0 & 1 \end{pmatrix},
\]
where $D_0$ is the identity component of $D$. Denote $\Gamma = D/D_0$. We have the algebraic hull $H_\Gamma = \{\pm 1\} \ltimes_{\psi} (U_3(\mathbb{R}) \times \mathbb{R})$ such that
\[
\psi(-1) \cdot \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, t \right) = \left( \begin{pmatrix} 1 & -x & z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}, t \right).
\]
The dual of the Lie algebra $u$ of $U_3(\mathbb{R}) \times \mathbb{R}$ is given by $u^* = \langle \alpha, \beta, \gamma, \delta \rangle$ such that the differential is given by
\[
da \alpha = da = d\delta = 0,
\]
and the action of $\{\pm 1\}$ is given by
\[
(-1) \cdot \alpha = -\alpha, (-1) \cdot \beta = -\beta,
\]
\[
(-1) \cdot \gamma = \gamma, (-1) \cdot \delta = \delta.
\]
Then we have a diffeomorphism $M_\Gamma \cong G/D$ and an isomorphism $H^*(M_\Gamma, \mathbb{R}) \cong H^*((\bigwedge u^*)^{\{\pm 1\}})$. By simple computations, $H^2((\bigwedge u^*)^{\{\pm 1\}}) = 0$ and hence the solvmanifold $G/D$ is not symplectic.

**Remark 1.** The proof of the Theorem 3.4 contains a proof of the following proposition.

**Proposition 4.1.** If $M_\Gamma$ admits a symplectic structure, then $U_\Gamma$ has an invariant symplectic form.

Otherwise for the above example, $U_\Gamma = U_3(\mathbb{R}) \times \mathbb{R}$ has an invariant symplectic form but $M_\Gamma$ is not symplectic. Thus the converse of this proposition is not true. If $\Gamma$ is not nilpotent, then $T$ is trivial and any invariant symplectic form on $U_\Gamma$ induces the symplectic form on $M_\Gamma$. Hence for nilmanifolds, the converse of Proposition 4.1 is true.

**Remark 2.** $\Gamma$ is a finite extension of a lattice of $U_\Gamma = U_3(\mathbb{R}) \times \mathbb{R}$. Hence $M_\Gamma$ is finitely covered by a Kodaira–Thurston manifold (see [9], [3, p. 192]). $M_\Gamma$ is an example of a non-symplectic manifold finitely covered by a symplectic manifold.
Let $H = G \times \mathbb{R}$. Then the dual of the Lie algebra $\mathfrak{h}$ of $H$ is given by $\mathfrak{h}^* = \{\sigma, \tau, \zeta_1, \zeta_2, \eta_1, \eta_2, \theta_1, \theta_2\}$ such that the differential is given by

\[d\sigma = d\tau = 0,\]
\[d\zeta_1 = \tau \wedge \zeta_2, \quad d\zeta_2 = -\tau \wedge \zeta_1,\]
\[d\eta_1 = \tau \wedge \eta_2, \quad d\eta_2 = -\tau \wedge \eta_1,\]
\[d\theta_1 = -\zeta_1 \wedge \eta_1 + \zeta_2 \wedge \eta_2, \quad d\theta_2 = -\zeta_1 \wedge \eta_2 - \zeta_2 \wedge \eta_1.\]

By simple computations, any closed invariant 2-form $\omega \in \bigwedge^2 \mathfrak{h}^*$ satisfies $\omega^4 = 0$. Hence $H$ has no invariant symplectic form. Otherwise we have a lattice $\Delta = 2\mathbb{Z} \times U_3(\mathbb{Z} + i\mathbb{Z}) \times \mathbb{Z}$ which is also a lattice of $\mathbb{R}^2 \times U_3(\mathbb{C})$. Thus $H/\Delta$ is diffeomorphic to a direct product of a two-dimensional torus and an Iwasawa manifold (see [4]). Since an Iwasawa manifold is symplectic (see [4]), $H/\Delta$ is also symplectic. By this example we can say:

**Remark 3.** For a simply connected nilpotent Lie group $G$ with a lattice $\Gamma$, if the nilmanifold $G/\Gamma$ is symplectic then $G$ has an invariant symplectic form. But suppose $G$ is solvable we have an example of a symplectic solvmanifold $G/\Gamma$ such that $G$ has no invariant symplectic form.

**References**


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