SQUARE ROOTS OF HAMILTONIAN DIFFEOMORPHISMS

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In this paper, we prove that on any closed symplectic manifold there exists an arbitrarily $C^\infty$-small Hamiltonian diffeomorphism not admitting a square root.

1. Introduction

Let $(M, \omega)$ be a closed symplectic manifold, i.e., $\omega \in \Omega^2(M)$ is a non-degenerate, closed 2-form. To a function $L : S^1 \times M \to \mathbb{R}$ we associate the Hamiltonian vector field $X_L$ by setting

$$\omega(X_L, \cdot) = -dL_t(\cdot),$$

where $L_t(x) := L(t, x)$. The flow $\phi^t_L : M \to M$ of the vector field $X_L$ is called a Hamiltonian flow. For simplicity we abbreviate

$$\phi_L = \phi^1_L.$$

The Hamiltonian diffeomorphisms form the Lie group $\text{Ham}(M, \omega)$ with Lie algebra being the smooth functions modulo constants. We refer the reader to the book [MS98] for the basics in symplectic geometry.

In this paper, we prove the following Theorem.

**Theorem 1.** In any $C^\infty$-neighborhood of the identity in $\text{Ham}(M, \omega)$ there exists a Hamiltonian diffeomorphism $\phi$ which has no square root, i.e., for all Hamiltonian diffeomorphism $\psi$ (not necessarily close to the identity)

$$\psi^2 \neq \phi$$

holds.

An immediate corollary of Theorem 1 is the following.
Corollary 2. The exponential map
\[ \text{Exp} : C^\infty(M, \mathbb{R})/\mathbb{R} \to \text{Ham}(M, \omega) \]
(1.4)
\[ [L] \mapsto \phi_L \]

is not a local diffeomorphism.

In the proof of the Theorem, we use the following beautiful observation by Milnor [Mil84, Warning 1.6]. Milnor observed that an obstruction to the existence of a square root is an odd number of $2^k$-cycles, see the next section for details. The main work in this paper is to construct an example in the symplectic category.

2. Milnor’s observation

We define
(2.1)
\[ CM^k := M^k / (\mathbb{Z}/k), \]
where $\mathbb{Z}/k$ acts by cyclic shifts on $M^k$. We write elements of $CM^k$ as
(2.2)
\[ [x_1, \ldots, x_k] \in CM^k. \]
The space of $k$-cycles of a diffeomorphism $\phi : M \to M$ is
(2.3)
\[ \mathcal{C}^k(\phi) := \left\{ [x_1, \ldots, x_k] \in CM^k \mid \phi^j(x_i) \neq x_i \forall j = 1, \ldots, k-1, \phi(x_i) = x_{i+1} \right\}. \]

We point out that if $[x_1, \ldots, x_k] \in \mathcal{C}^k(\phi)$ then $\phi^k(x_i) = x_i$ for $i = 1, \ldots, k$.

Proposition 3 (Milnor [Mil84]). If $\phi = \psi^2$ then $\mathcal{C}^{2k}(\phi)$ admits a free $\mathbb{Z}/2$-action. In particular, $\#\mathcal{C}^{2k}(\phi)$ is even if $\mathcal{C}^{2k}(\phi)$ is a finite set.

For the convenience of the reader we include a proof of Milnor’s ingenious observation.

Proof. We define
(2.4)
\[ I : \mathcal{C}^{2k}(\phi) \to \mathcal{C}^{2k}(\phi), \]
\[ [x_1, \ldots, x_{2k}] \mapsto [\psi(x_1), \ldots, \psi(x_{2k})]. \]

Since $\psi \circ \phi = \phi \circ \psi$ and $\psi^2 = \phi$ the map $I$ is well-defined and an involution. We assume by contradiction that $[x_1, \ldots, x_{2k}]$ is a fixed point of $I$, i.e., there exists $0 \leq r \leq 2k-1$
(2.5)
\[ \psi(x_i) = x_{i+r}, \]
where we read indices $\mathbb{Z}/2k$-cyclically. Using $x_{i+r} = \phi^r(x_i)$ we get
(2.6)
\[ \psi(x_i) = \phi^r(x_i) = \psi^{2r}(x_i) \]
and thus
\[(2.7)\quad \psi^{2r-1}(x_i) = x_i.\]
In particular,
\[(2.8)\quad x_i = \psi^{2r-1}(x_i) = \psi^{2r-1}(\psi^{2r-1}(x_i)) = \psi^{4r-2}(x_i) = \phi^{2r-1}(x_i).\]
In summary, we have
\[(2.9)\quad x_i = \phi^{2r-1}(x_i) \quad \text{and} \quad x_i = \phi^{2k}(x_i).\]
In general, if
\[(2.10)\quad z = \phi^a(z) \quad \text{and} \quad z = \phi^b(z)\]
for \(a, b \in \mathbb{Z}\) then
\[(2.11)\quad z = \phi^{\text{lcd}(a,b)}(z)\]
since by the Euclidean algorithm there exists \(n_1, n_2 \in \mathbb{Z}\) with
\[(2.12)\quad \text{lcd}(a, b) = n_1a + n_2b.\]
In our specific situation, \(2r - 1\) is odd and \(2k\) is even and thus
\[(2.13)\quad 1 \leq \text{lcd}(2r - 1, 2k) < 2k\]
contradicting the assumption \(\phi^j(x_i) \neq x_i \forall j = 1, \ldots, 2k - 1\). This proves the Proposition. \(\square\)

3. Proof of Theorem 1

Let \((M, \omega)\) be a closed symplectic manifold. We fix a Darboux chart \(B^{2N}(R) \cong B \subset M\) where \(B^{2N}(R)\) is the open ball of radius \(R\) in \(\mathbb{R}^{2N}\). For an integer \(k \geq 1\) and a positive number \(\delta > 0\), we choose a smooth function \(\rho : [0, R^2] \to \mathbb{R}\) satisfying the following:

\[
\begin{cases}
\frac{\pi}{2k} \geq \rho'(r) > 0, \\
\rho'(r) = \frac{\pi}{2k} \iff r = \frac{1}{2}R^2, \\
\rho'|_{\frac{8}{9}R^2, R^2} = \delta > 0.
\end{cases}
\]

We set for \(1 \leq \nu \leq N\)

\[
(3.2)\quad \zeta(\nu) := \begin{cases} 
1 & \nu = N, \\
\frac{9}{10} & \text{else}
\end{cases}
\]
and define

$$H : B^{2N}(R) \to \mathbb{R}$$

(3.3)

$$z \mapsto \rho \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_{\nu}|^2 \right).$$

We denote by $\phi^t_H : B^{2N}(R) \to B^{2N}(R)$ the induced Hamiltonian flow. We recall that the Hamiltonian flow of $z \mapsto |z|^2$ is given by $z \mapsto \exp(2it)z$ thus

$$\left( \phi^t_H(z) \right)_\nu = \exp \left[ \rho' \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_{\nu}|^2 \right) 2i\zeta(\nu)t \right] z_{\nu}. \quad (3.4)$$

We point out that $\phi^t_H$ preserves the quantities $|z_{\nu}|, \nu = 1, \ldots, N$.

**Lemma 4.** The fixed points of $\phi^{2k}_H$ are precisely $z = 0$ and the circle

$$C := \{(z_1, \ldots, z_N) \in B^{2N}(R) \mid |z_N|^2 = \frac{1}{2}R^2 \text{ and } z_1 = \cdots = z_{N-1} = 0\}. \quad (3.5)$$

Moreover, $\phi_H$ acts on $C$ by rotation of the last coordinate by an angle of $\frac{\pi}{k}$.

**Proof.** Assume that $\phi^{2k}_H(z) = z$, which is equivalent to

$$\exp \left[ \rho' \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_{\nu}|^2 \right) 2i\zeta(\nu)2k \right] z_{\nu} = z_{\nu}, \quad \nu = 1, \ldots, N, \quad (3.6)$$

thus, either $z_{\nu} = 0$ or

$$\rho' \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_{\nu}|^2 \right) 4k\zeta(\nu) \in 2\pi\mathbb{Z}. \quad (3.7)$$

From $\rho'(r) \leq \frac{\pi}{2k}$ we conclude that $z_1 = \cdots = z_{N-1} = 0$. Moreover, $z_N = 0$ or

$$\rho' \left( \sum_{\nu=1}^{N} \zeta(\nu)|z_{\nu}|^2 \right) = \rho'(|z_N|^2) = \frac{\pi}{2k}, \quad (3.8)$$

holds. In summary, either $z = 0$ or $z \in C$. This together with (3.4) proves the Lemma. \hfill \Box

We now perturb $H$. For this we fix a smooth cut-off function $\beta : [0, R^2] \to [0, 1]$ satisfying

$$\beta|_{[\frac{1}{3}R^2, \frac{2}{3}R^2]} = 1 \quad \text{and} \quad \beta|_{[0, \frac{1}{3}R^2] \cup [\frac{2}{3}R^2, R^2]} = 0 \quad (3.9)$$

and set

$$F(z) := \beta(|z_N|^2) \cdot \text{Re} \left( \frac{z_N^k}{|z_N|^k} \right) : B^{2N}(R) \to \mathbb{R}, \quad (3.10)$$
where \( \text{Re} \) is the real part. If we introduce new coordinates \( (z_1, \ldots, z_{N-1}, r, \vartheta) \), where \( z_N = r \exp(i \vartheta) \), the function \( F \) equals
\[
F(z) = \beta(r^2) \cos(k \vartheta).
\]
We point out that the Hamiltonian diffeomorphism \( \phi_H \circ \phi_{\epsilon F} \) maps \( B^{2N}(R) \) into itself.

**Lemma 5.** There exists \( \epsilon_0 > 0 \) such that for all \( 0 < \epsilon < \epsilon_0 \)
\[
\# C^{2k}(\phi_H \circ \phi_{\epsilon F}) = 1.
\]

**Proof.** We set
\[
D := \left\{ (z_1, \ldots, z_{N-1}, r, \vartheta) \in C \mid \vartheta = \frac{j\pi}{k}, j = 0, \ldots, 2k - 1 \right\},
\]
where \( C \) is defined in Lemma 4. The same lemma implies that \( \phi_H \) acts on \( D \) as a cyclic permutation sending \( \frac{j\pi}{k} \) to \( \frac{(j+1)\pi}{k} \). Moreover, we have
\[
\phi_{\epsilon F} z = z
\]
for \( z \in D \) since \( D \subset \text{Crit} F \). In particular, \( D \) corresponds precisely to a single element in \( C^{2k}(\phi_H \circ \phi_{\epsilon F}) \). It remains to show that there are no other \( 2k \)-cycles. We prove something stronger, namely that for sufficiently small \( \epsilon > 0 \) the only other fixed point of \( (\phi_H \circ \phi_{\epsilon F})^{2k} \) is \( z = 0 \).

For \( 0 < a < b \), we set
\[
A(a, b) := \left\{ (z_1, \ldots, z_{N-1}, r, \vartheta) \in B^{2N}(R) \mid r \in [aR^2, bR^2] \right\}.
\]
We observe that on \( A(\frac{1}{3}, \frac{2}{3}) \) we have \( \beta = 1 \) and thus the flow of \( \epsilon F \) is given by
\[
(z_1, \ldots, z_{N-1}, r, \vartheta) \mapsto (z_1, \ldots, z_{N-1}, \sqrt{-2ck \sin(k \vartheta)t + r^2}, \vartheta).
\]

In particular, if we set
\[
\bar{\epsilon} := \frac{7R^4}{324k^2}
\]
then for \( 0 < \epsilon < \bar{\epsilon} \) we conclude that
\[
(\phi_H \circ \phi_{\epsilon F})^{2k} \left( A(\frac{4}{9}, \frac{5}{9}) \right) \subset A(\frac{1}{3}, \frac{2}{3}),
\]
since \( \phi_H^j \) preserves the \( r \) coordinate. Fix \( w \in A(\frac{4}{9}, \frac{5}{9}) \) with \( (\phi_H \circ \phi_{\epsilon F})^{2k}(w) = w \) and set for \( j = 0, \ldots, 2k \)
\[
\begin{align*}
z^{(j)}_v &:= P_{z_w} \left( (\phi_H \circ \phi_{\epsilon F})^{j}(w) \right), \quad \nu = 1, \ldots, N - 1, \\
r^{(j)} &:= P_r \left( (\phi_H \circ \phi_{\epsilon F})^{j}(w) \right), \\
\vartheta^{(j)} &:= P_{\vartheta} \left( (\phi_H \circ \phi_{\epsilon F})^{j}(w) \right),
\end{align*}
\]
where $P_{z}$, $P_{r}$, and $P_{\theta}$ are the projections on the respective coordinates. It follows from equation (3.16) that

\[(3.20)\quad P_{z} \left( (\phi_{H} \circ \phi_{eF})^{j}(w) \right) = P_{z} \left( \phi_{H}^{j}(w) \right), \quad \nu = 1, \ldots, N - 1 .\]

By the same argument as in the proof of Lemma 4, we conclude

\[(3.21)\quad z_{\nu}^{j} = 0, \quad \forall \nu = 1, \ldots, N - 1 \text{ and } \forall j = 0, \ldots, 2k .\]

Next, it follows from the flow equations (3.4) and (3.16)

\[(3.22)\quad 0 < \vartheta_{j+1} - \vartheta_{j} \leq \frac{\pi}{k} \text{ mod } 2\pi .\]

By (3.1) equality holds if and only if $r_{j+1} = \frac{1}{2}R^{2}$. Using again $(\phi_{H} \circ \phi_{eF})^{2k}(w) = w$ we deduce

\[(3.23)\quad \vartheta_{2k} - \vartheta_{0} = 0 \text{ mod } 2\pi\]

and therefore

\[(3.24)\quad r_{0} = r_{1} = \cdots = r_{2k} = \frac{1}{2}R^{2} .\]

In summary

\[(3.25)\quad w = (0, \ldots, 0, \frac{1}{2}R^{2}, \vartheta_{0})\]

with $\vartheta_{0} \in \frac{\pi}{k}\mathbb{Z}$, i.e., $w \in D$. Thus, we proved that the only 2k-cycle of $\phi_{H} \circ \phi_{eF}$ in the region $A(\frac{4}{7}, \frac{5}{9})$ is the one corresponding to the set $D$. Therefore it remains to prove that after possibly shrinking $\bar{\epsilon}$ there are no other fixed points of $(\phi_{H} \circ \phi_{eF})^{2k}$ outside $A(\frac{4}{7}, \frac{5}{9})$ except for $z = 0$. We argue by contradiction.

We assume that there exists a sequence $\epsilon_{m} \to 0$ and a sequence $(z^{m})_{m \in \mathbb{N}}$ of points in $B^{2N}(R) \setminus A(\frac{4}{7}, \frac{5}{9})$ with

\[(3.26)\quad (\phi_{H} \circ \phi_{eF})^{2k}(z^{m}) = z^{m} \quad \forall m \in \mathbb{N} .\]

By compactness we may assume that $z^{m} \to z^{*} \in B^{2N}(R) \setminus \text{int}A(\frac{4}{7}, \frac{5}{9})$ with

\[(3.27)\quad \phi_{H}^{2k}(z^{*}) = z^{*} .\]

It follows from Lemma 4 that $z^{*} = 0$ and thus for $M$ sufficiently large

\[(3.28)\quad z^{m} \in B^{2N}(\frac{1}{3}R) \quad \forall m \geq M .\]

Then by definition of $\beta$ the restriction of $\phi_{eF}$ to the ball $B^{2N}(\frac{1}{3}R)$ equals the identity. Moreover, since $\phi_{H}$ fixes all balls centered at zero we have

\[(3.29)\quad z^{m} = (\phi_{H} \circ \phi_{eF})^{2k}(z^{m}) = \phi_{H}^{2k}(z^{m}) \quad \forall m \geq M .\]

Applying again Lemma 4 we conclude that $z^{m} = 0$ for all $m \geq M$. This proves the Lemma. \qed
Remark 6. Proposition 3 together with Lemma 5 implies that for all $0 < \epsilon < \epsilon_0$ the Hamiltonian diffeomorphism $\phi_H \circ \phi_{\epsilon F} : B^{2N}(R) \to B^{2N}(R)$ has no square root.

We are now in the position to prove Theorem 1.

Proof of Theorem 1. We choose $k \in \mathbb{Z}$, $\delta > 0$ and $0 < \epsilon < \epsilon_0$ (cp. Lemma 5) so that the Hamiltonian diffeomorphism

$$\phi_H \circ \phi_{\epsilon F} : B^{2N}(R) \to B^{2N}(R)$$

has precisely one $2k$-cycle. By construction $\phi_H \circ \phi_{\epsilon F}$ equals the map

$$\left(z_1, \ldots, z_N\right) \mapsto \left(e^{\frac{9\delta}{N}} z_1, \ldots, e^{\frac{9\delta}{N}} z_{N-1}, e^{2i\delta} z_N\right)$$

near the boundary of $B^{2N}(R)$. Indeed, if $z \in \partial B^{2N}(R)$, then we conclude

$$\sum_{\nu=1}^{N} \zeta(\nu) |z_{\nu}|^2 \geq \frac{9}{10} \sum_{\nu=1}^{N} |z_{\nu}|^2 = \frac{9}{10} R^2 > \frac{8}{9} R^2$$

and therefore $\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu) |z_{\nu}|^2\right) = \delta$. Next, we extend the Hamiltonian function of $\phi_H \circ \phi_{\epsilon F}$ to $\tilde{H} : S^1 \times M \to \mathbb{R}$ which we can choose to be autonomous outside the Darboux ball $B$. If we choose $\delta > 0$ sufficiently small, then we can guarantee that outside $B$ the only periodic orbits of $\tilde{H}$ of period less or equal to $2k$ are critical points of $\tilde{H}$, see [HZ94], in particular line 4 & 5 on page 185. In particular, $\phi_{\tilde{H}}$ has still precisely one $2k$-cycle. Finally, by choosing $k$ sufficiently large and $\delta$ and $\epsilon$ sufficiently small, $\phi_H \circ \phi_{\epsilon F}$ and thus $\phi_{\tilde{H}}$ can be chosen to lie in an arbitrary $C^\infty$-neighborhood of the identity on $B^{2N}(R)$ resp. $M$. Therefore, with Proposition 3 the Theorem follows. \qed

References


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