On cohomological obstructions for the existence of log-symplectic structures

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We prove that a compact log-symplectic manifold has a class in the second cohomology group whose powers, except maybe for the top, are nontrivial. This result gives cohomological obstructions for the existence of log-symplectic structures similar to those in symplectic geometry.

A Poisson structure $\pi$ on a smooth manifold $M$ of dimension $2n$ is called a log-symplectic structure if the map

$$\wedge^n \pi : M \rightarrow \bigwedge^{2n} TM, \quad x \mapsto \wedge^n \pi(x)$$

is transverse to the zero section.

These structures were initially studied in the framework of deformation quantization in [4], where they are called $b$-symplectic structures. Later, their complex analogue was considered in [1], where they were first given the name log symplectic. In the context of Poisson geometry, this class of Poisson structures was introduced on two-dimensional surfaces in [5] (under the name of topologically stable Poisson structures) where a complete classification was obtained. In higher dimensions, a systematic investigation of the geometric properties of log-symplectic structures appeared in [3]. Their integrations by symplectic groupoids were studied in [2].

Our interest in log-symplectic structures comes from the fact that these can be used to construct regular corank-one Poisson structures. First, the singular locus of a log-symplectic structure $Z := (\wedge^n \pi)^{-1}(0)$, if nonempty, carries a regular corank-one Poisson structure with a very special property: it has a transverse Poisson vector field [3]. Secondly, a log-symplectic structure can be used to construct a regular corank-one Poisson on $M \times S^1$, simply given by

$$\pi + X \wedge \frac{\partial}{\partial \theta},$$

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where $X$ is a representative of the modular vector field of $(M, \pi)$. However, our result excludes the possibility of using this procedure to construct corank-one Poisson structures in some interesting examples, e.g., on $S^4 \times S^1$.

Our result is the following:

**Theorem.** Let $(M^{2n}, \pi)$ be a compact log-symplectic manifold. Then there exists a class $c \in H^2(M)$ such that $c^{n-1} \in H^{2n-2}(M)$ is nonzero.

**Proof.** Denote by $Z := (\wedge^n \pi)^{-1}(0)$ the singular locus of $\pi$. If $Z = \emptyset$, we can apply the usual argument from symplectic geometry. Assume that $Z \neq \emptyset$.

We first assume that $M$ is orientable. Let $\mu$ be a volume form on $M$ and denote by $t := \langle \pi^n, \mu \rangle$. The singular locus becomes $Z = \{ t = 0 \}$. The log-symplectic condition implies that $t$ is a submersion along $Z$, so we can find a retraction $r : U \to Z$, where $U$ is an open around $Z$, such that $(r, t) : U \to Z \times (-\delta, \delta)$ is a diffeomorphism. Since $Z$ is a Poisson submanifold (it is fixed by all Poisson automorphisms, hence all Hamiltonians are tangent to $Z$), in this open set $U$ we can write

$$\pi|_U = t \partial/\partial t \wedge X_t + w_t$$

for a vector field $X_t$ and a bivector $w_t$ on $Z$, both depending smoothly on $t$. Since $(1/t)\pi^n = n \partial/\partial t \wedge X_t \wedge w_t^{n-1}$ is nowhere vanishing, we have that the bivector $\partial/\partial t \wedge X_t + w_t$ is invertible. Denote its inverse by $\alpha_t \wedge dt + \beta_t$, with $\alpha_t$ and $\beta_t$ forms on $Z$ depending smoothly on $t \in (-\delta, \delta)$. Then $\omega := \pi|_{M \setminus Z}$ can be written as

$$\omega|_{U \setminus Z} = \alpha_t \wedge dt/t + \beta_t.$$

Since $\omega$ is closed we get that $\alpha_0$ and $\beta_0$ are closed, and since $dt \wedge \alpha_0 + \beta_0$ is invertible, it follows that $\alpha_0 \wedge \beta_0^{n-1}$ is a volume form on $Z$. Since $Z$ is compact, this implies that $\beta_0^{n-1}$ cannot be exact. We will construct a closed 2-form $\omega'$ on $M$ whose pullback to $Z$ is $\beta_0$; hence $c := [\omega']$ will satisfy the conclusion of the theorem.

Let $\chi : (-\delta, \delta) \to \mathbb{R}$ be a bump function that takes the value 1 for $|t| \leq \delta/4$, and 0 for $|t| \geq \delta/2$. Consider the 2-form $\omega'$ on $M \setminus Z$ that coincides with $\omega$ outside of $U$ and on $U \setminus Z$ it is given by

$$\omega'|_{U \setminus Z} = (\alpha_t - \chi(t)\alpha_0) \wedge dt/t + \beta_t.$$

The form $\omega'$ extends smoothly to $Z$, since for $|t| \leq \delta/4$ it can be written as $\omega' = \lambda_t \wedge dt + \beta_t$, where $\lambda_t = \int_0^1 \alpha_{ts} ds$, or equivalently $\alpha_t = \alpha_0 + t \lambda_t$. So $\omega'$ is a closed 2-form on $M$ whose pullback to $Z$ is $\beta_0$; thus $[\omega']^{n-1} \neq 0$. 
If $M$ is not orientable, consider $p: \tilde{M} \to M$ the orientable double cover, and let $\gamma: M \to \tilde{M}$ be the corresponding deck transformation. We first construct a tubular neighborhood $(\tilde{\tau}, t): \tilde{U} \to \tilde{Z} \times (-\delta, \delta)$ of the singular locus $\tilde{Z} := p^{-1}(Z)$ of $\tilde{\pi} := p^*(\pi)$, with $\tilde{U} = p^{-1}(U)$, and such that the action of $\gamma$ corresponds to $\gamma(z, t) = (\gamma(z), -t)$, for $(z, t) \in \tilde{Z} \times (-\delta, \delta)$. The map $\tilde{\tau}: \tilde{U} \to \tilde{Z}$ can be constructed by lifting a retraction $r: U \to Z$. Consider a volume form $\mu_0$, and denote by $f$ the smooth function satisfying $\gamma^* \mu_0 = -e^f \mu_0$. Then the volume form $\mu := e^{f/2} \mu_0$ satisfies $\gamma^* \mu = -\mu$. Thus, by shrinking $U$, we can use $t := (\tilde{\tau}^{n}, \mu)$ to construct the desired tubular neighborhood. As before, on $\tilde{Z} \times (-\delta, \delta)$ we can write $p^* (\omega|_{U\backslash Z}) = \alpha_t \wedge dt/t + \beta_t$. Invariance under $\gamma$ implies that $(\gamma|_{\tilde{Z}})^* (\alpha_t) = \alpha_{-t}$ and $(\gamma|_{\tilde{Z}})^* (\beta_t) = \beta_{-t}$. In particular $\alpha_0$ and $\beta_0$ are invariant. Thus, choosing the function $\chi(t)$ from the construction from the orientable case to satisfy $\chi(t) = \chi(-t)$, we obtain an invariant closed 2-form $\omega'$ on $\tilde{M}$ that satisfies $[\omega']^{n-1} \neq 0$. Invariance implies that $\omega' = p^* (\omega'')$ for a closed 2-form $\omega''$ on $M$; hence $c := [\omega'']$ satisfies the conclusion.

**Remark.** Observe that for $Z \neq \emptyset$ the proof of the theorem uses only the compactness of $Z$ and not that of $M$.

**References**


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