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Fractional integration for Laguerre expansions

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1. Introduction

The aim of this note is to provide a fractional integration theorem in the framework of Laguerre expansions. The method of proof consists of establishing an asymptotic estimate for the kernel involved and then applying a method of Hedberg [6]. We combine this result with the sufficient (p,p) multiplier criteria of Stempak and Trebels [13]. The resulting sufficient (p,q) multiplier criteria are comparable with the necessary ones of Gasper and Trebels [3,4].

Our notation is essentially that in [13]. Thus we consider the Lebesgue spaces

$$L^p_{v(\gamma)} = \Big\{ f\colon \|f\|_{L^p_{v(\gamma)}} = \Big(\int_0^\infty |f(x)|^p x^\gamma dx\Big)^{1/p} < \infty \Big\}, \quad 1 \le p < \infty, \quad \gamma > -1,$$

and define the Laguerre function system $\{l_k^\alpha\}$ by

$$l_k^\alpha(x) = (k!/\Gamma(k+\alpha+1))^{1/2} e^{-x/2} L_k^\alpha(x), \quad \alpha > -1, \quad k \in \mathbf{N}_0.$$

This system is an orthonormal basis in $L^2(\mathbf{R}_+, x^\alpha dx)$ and, for $\gamma < p(\alpha+1)-1$, we can associate to any $f \in L^p_{v(\gamma)}$ the Laguerre series

$$f(x) \sim \sum_{k=0}^\infty a_k l_k^\alpha(x), \qquad a_k = \int_0^\infty f(x) l_k^\alpha(x) x^\alpha dx.$$

It is convenient to introduce the vector space

$$E = \{ f(x) = p(x)e^{-x/2} : 0 \le x < \infty, \quad p(x) \text{ a polynomial} \}$$

which is dense in $L^p_{v(\gamma)}$. We note that $f \in E$ has only finitely many non-zero Fourier-Laguerre coefficients. Analogous to the definition of the Hardy and Littlewood fractional integral operator for Fourier series (see [15, Chap. XII, Sec. 8]), we define a fractional integral operator I_σ, $\sigma > 0$, for Laguerre expansions by

$$I_\sigma f(x) = \sum_{k=0}^\infty (k+1)^{-\sigma} a_k l_k^\alpha(x) \quad f \in E.$$

Observing that the l_k^α are eigenfunctions with eigenvalues λ_k of the differential operator

$$L = -\left(x\frac{d^2}{dx^2} + (\alpha+1)\frac{d}{dx} - \frac{x}{4} \right), \quad \lambda_k = k + (\alpha+1)/2$$

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(see [14, (5.1.2)]), one realizes that I_1 is an integral operator essentially inverse to L (see the following Remark 2). As we will see in Section 2, the fractional integral $I_\sigma f$ can be interpreted as a twisted convolution of f with a function

$$g_\sigma(x) \sim \Gamma(\alpha+1)\sum_{k=0}^{\infty}(k+1)^{-\sigma}L_k^\alpha(x)e^{-x/2}. \tag{1}$$

From Theorem 3.1 in [4], it easily follows that $g_\sigma \in L^1_{v(\gamma)}$ when $\alpha-\gamma<\sigma$, so that, by the convolution theorem of Görlich and Markett [5], I_σ extends to a bounded operator from $L^p_{v(\gamma)}$ to $L^p_{v(\gamma)}$ when $0\le \alpha p/2\le\gamma\le\alpha$, $1\le p\le\infty$. Our main result is

Theorem 1.1. *Let $\alpha\ge 0$, $1<p\le q<\infty$. Assume further that $0<\sigma<\alpha+1$, $a<(\alpha+1)/p'$, $b<(\alpha+1)/q$, $a+b\ge 0$. Then*

$$\|I_\sigma f\|_{L^q_{v(\alpha-bq)}}\le C\|f\|_{L^p_{v(\alpha+ap)}},\quad \frac{1}{q}=\frac{1}{p}-\frac{\sigma-a-b}{\alpha+1}.$$

Remarks. **(1)** Observe that the upper bounds for a and b imply lower bounds for b and a, respectively.

(2) By the above, it is clear that the sequence $\{(k+1)^{-\sigma}\}$ which generates the fractional integral can be replaced by any sequence $\{\Omega_\sigma(k)\}$ satisfying

$$\Omega_\sigma(k)=\sum_{j=0}^{J}c_j(k+1)^{-\sigma-j}+O((k+1)^{-\sigma-J})$$

for sufficiently large J, say $J\ge\alpha+2$. Thus, in particular, the same result is obtained for the sequence $\{\Gamma(k+1)/\Gamma(\sigma+k+1)\}$, see [9].

(3) A weaker version of Theorem 1.1 (e.g., in the case $a=b=0$) can easily be deduced by the following argument. By a slight modification of the proof of Theorem 3.1 in [4], we have:

Let $\alpha>-1$ and $N\in\mathbf{N}_0$, $N>(2\alpha+2)(1/r-1/2)-1/3$. If $\{f_k\}$ is a bounded sequence with $\lim_{k\to\infty} f_k=0$ and

$$\sum_{k=0}^{\infty}(k+1)^{N+(\alpha+1)/r'}|\Delta^{N+1}f_k|\le K_r(f)<\infty,\quad 1\le r<\infty,$$

then there exists a function $f\in L^r_{v(\alpha)}$ with

$$\|f\|_{L^r_{v(\alpha)}}\le CK_r(f),\quad f(x)\sim\sum_{k=0}^{\infty}f_kL_k^\alpha(x)e^{-x/2}.$$

This applied to the sequence $\{(k+1)^{-\sigma}\}$ gives, by Young's inequality [5],

$$\|I_\sigma f\|_{L^q_{v(\alpha)}}\le C\|f\|_{L^p_{v(\alpha)}},\quad \frac{1}{q}>\frac{1}{p}-\frac{\sigma}{\alpha+1},$$

where $\alpha\ge 0$, $\sigma>0$ and $1\le p,\, q\le\infty$.

Next we indicate how Theorem 1.1 can be used to gain some insight into the structure of $M^{p,q}$-Laguerre multipliers. For the sake of simplicity, let us restrict ourselves to the case $\gamma = \alpha$. Consider a sequence $m = \{m_k\}$ of numbers and associate to m the operator

$$T_m f(x) = \sum_{k=0}^{\infty} m_k a_k l_k^\alpha(x), \quad f \in E.$$

The sequence m is called a bounded (p,q)-multiplier, notation $m \in M_{\alpha,\alpha}^{p,q}$, if

$$\|m\|_{M_{\alpha,\alpha}^{p,q}} := \inf\{\, C : \|T_m f\|_{L^q_{v(\alpha)}} \le C \|f\|_{L^p_{v(\alpha)}} \text{ for all } f \in E \,\}$$

is finite. If $p \le 2 \le q$, then sufficient conditions follow at once in the following way: observe that $M_{\alpha,\alpha}^{2,2} = l^\infty$, choose σ_0, $\sigma_1 \ge 0$ such that $I_{\sigma_0} \colon L^p \to L^2$, $I_{\sigma_1} \colon L^2 \to L^q$, $\sigma_0 + \sigma_1 = \sigma$, and hence

$$\|m\|_{M_{\alpha,\alpha}^{p,q}} \le \|k^{-\sigma_0}\|_{M_{\alpha,\alpha}^{p,2}} \; \|\{k^\sigma m_k\}\|_{M_{\alpha,\alpha}^{2,2}} \; \|k^{-\sigma_1}\|_{M_{\alpha,\alpha}^{2,q}},$$

in particular, $m \in M_{\alpha,\alpha}^{p,q}$ when $\{k^\sigma m_k\} \in l^\infty$. Thus, when $1 < p \le 2 \le q < \infty$ and σ is as in Theorem 1.1, $\{(-1)^k (k+1)^{-\sigma}\} \in M_{\alpha,\alpha}^{p,q}$ generates a bounded operator which does not fall under the scope of the above introduced fractional integral operators. To formulate a corollary based on a combination of Theorem 1.1 and the multiplier result in [13], we need the notion of a difference operator Δ^s of fractional order s given by

$$\Delta^s m_k = \sum_{j=0}^{\infty} A_j^{-s-1} m_{k+j}, \quad A_j^t = \frac{\Gamma(j+t+1)}{j!\,\Gamma(t+1)}, \quad t \in \mathbf{R},$$

whenever the sum converges. In view of the remark concerning the case $p < 2 < q$ and on account of duality, we may restrict ourselves to the case $1 < p < q < 2$.

Corollary 1.2. *If $\alpha \ge 0$, $1 < p < q < 2$, and $s > \max\{(2\alpha+2)(1/q - 1/2),\ 1\}$, then for some constant C, independent of the sequence $\{m_k\}$, there holds*

$$\|\{m_k\}\|^2_{M_{\alpha,\alpha}^{p,q}} \le C\Big(\|\{k^\sigma m_k\}\|_\infty^2 + \sup_N \sum_{k=N}^{2N} |(k+1)^{s+\sigma} \Delta^s m_k|^2 (k+1)^{-1} \Big).$$

The proof follows as in [2]. The result itself should be compared with the corresponding necessary condition in [3] which also shows that $\{(-1)^k (k+1)^{-\sigma}\} \notin M_{\alpha,\alpha}^{p,q}$ provided $1 < p < q < 2$ and α is sufficiently large.

The plan of the paper is as follows. In Section 2, we derive an asymptotic estimate of the function g_σ defined above by (1). Then the twisted generalized convolution is used to dominate $I_\sigma f$ by a generalized Euclidean convolution of g_σ with f. The latter's mapping behavior is discussed by a method of Hedberg [6] which uses maximal functions, thus giving Theorem 1.1 in the standard weight case $a = b = 0$. In Section 3, we extend this result to some power weights, modifying an argument in Stein and Weiss [10].

2. Proof of the standard weight case

We start by deriving the required asymptotic estimate and showing

Lemma 2.1. *Let $\alpha > -1$. Then, for $x > 0$ and $0 < \sigma < \alpha + 1$, there holds*

$$|g_\sigma(x)| \le C x^{\sigma - \alpha - 1}. \tag{2}$$

Proof. First note that by subordination [3, p. 1234] there holds for $N > \alpha+2$, $N \in \mathbf{N}$,

$$g_\sigma(x) = C\sum_{k=0}^{\infty}\left(\Delta^N(k+1)^{-\sigma}\right)L_k^{\alpha+N}(x)e^{-x/2}.$$

Then the assertion (2) follows after it is proved that

$$\sum_{k=0}^{\infty}(k+1)^{-\sigma-N}|x^{\alpha+1-\sigma}L_k^{\alpha+N}(x)e^{-x/2}| \le C \tag{3}$$

uniformly in $x > 0$. With the notation

$$\mathcal{L}_k^\alpha(x) = (k!/\Gamma(k+\alpha+1))^{1/2}x^{\alpha/2}e^{-x/2}L_k^\alpha(x), \quad k \in \mathbf{N}_0,$$

this is equivalent to

$$\sum_{k=0}^{\infty}(k+1)^{(\alpha-N)/2-\sigma}|\mathcal{L}_k^{\alpha+N}(x)| \le Cx^{\sigma+(N-\alpha-2)/2}, \tag{4}$$

uniformly in $0 < x < \infty$. We will make use of the pointwise estimates for the Laguerre functions in [1, Sec. 2]:

$$|\mathcal{L}_k^{\alpha+N}(x)| \le C\begin{cases}(x(k+1))^{(\alpha+N)/2}, & \text{if } \ 0 \le x \le c/(k+1),\\ (x(k+1))^{-1/4}, & \text{if } \ c/(k+1) \le x \le d(k+1),\end{cases} \tag{5}$$

for fixed positive constants c and d. These and two further estimates in [8, (2.5)] imply that

$$\sup_{x>0}|\mathcal{L}_k^{\alpha+N}(x)| \le C, \quad k \in \mathbf{N}_0, \tag{6}$$

and so it is obvious that (6) implies (4) for $x \ge 1$.

Therefore, decomposing the interval $(0,1)$ dyadically, it suffices to check that

$$\sum_{k=0}^{\infty}(k+1)^{(\alpha-N)/2-\sigma}|\mathcal{L}_k^{\alpha+N}(x)| \le C(2^j)^{\sigma+(N-\alpha-2)/2}$$

provided $2^j \le x \le 2^{j+1}$, $j < 0$. Using the first line of (5), we get

$$\sum_{k=0}^{2^{-j}}(k+1)^{(\alpha-N)/2-\sigma}|\mathcal{L}_k^{\alpha+N}(x)| \le C(2^j)^{(\alpha+N)/2}\sum_{k=0}^{2^{-j}}(k+1)^{\alpha-\sigma} \le C(2^j)^{\sigma+(N-\alpha-2)/2},$$

while the second line of (5) gives

$$\sum_{k=2^{-j}}^{\infty} k^{(\alpha-N)/2-\sigma}|\mathcal{L}_k^{\alpha+N}(x)| \le C2^{-j/4}\sum_{k=2^{-j}}^{\infty} k^{(\alpha-N)/2-\sigma-1/4} \le C(2^j)^{\sigma+(N-\alpha-2)/2}.$$

This completes the proof of Lemma 2.1.

As already mentioned, we will apply Hedberg's method, which involves maximal functions. To make use of the corresponding results in [11], we switch to the system

$$\psi_k^\alpha(x) = (2k!/\Gamma(k+\alpha+1))^{1/2} e^{-x^2/2} L_k^\alpha(x^2), \quad k \in \mathbf{N}_0,$$

which is obviously orthonormal on $L^2(\mathbf{R}_+, d\mu_\alpha)$, $d\mu_\alpha(x) = x^{2\alpha+1}dx$, $\alpha > -1$. For the sake of simplicity, we write the norm on $L^p(\mathbf{R}_+, d\mu_\alpha)$ as

$$\|F\|_p = \left(\int_0^\infty |F(x)|^p d\mu_\alpha(x)\right)^{1/p}.$$

We adopt the notion of the twisted generalized convolution on $L^1(\mathbf{R}_+, d\mu_\alpha)$ from [11,12],

$$F \times G(x) = \int_0^\infty \tau_x F(y) G(y) d\mu_\alpha(y),$$

where the twisted generalized translation operator τ_x is given by

$$\tau_x F(y) = \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)} \int_0^\pi F((x,y)_\theta) \mathcal{J}_{\alpha-1/2}(xy\sin\theta)(\sin\theta)^{2\alpha} d\theta,$$

$\mathcal{J}_\beta(x) = \Gamma(\beta+1) J_\beta(x)/(x/2)^\beta$, J_β denoting the Bessel function of order β, and

$$(x,y)_\theta = (x^2+y^2-2xy\cos\theta)^{1/2}.$$

With respect to the system $\{\psi_k^\alpha\}$, this convolution has the following transform property: if $F \sim \sum c_k \psi_k^\alpha$ and $F \times G \sim \sum c_k d_k \psi_k^\alpha$, then $G(x) \sim \Gamma(\alpha+1) \sum d_k L_k^\alpha(x^2) e^{-x^2/2}$.

If we set $f(y^2) = F(y)$, $g_\sigma(y^2) = G_\sigma(y)$, we see that

$$|I_\sigma f(x^2)| = |F \times G_\sigma(x)|.$$

Apart from Lemma 2.1, the proof of Theorem 1.1 will be based on the fact that for $\alpha \geq 0$ and suitable F and G,

$$|F \times G| \leq |F| * |G|,$$

which follows at once from the definition of the generalized Euclidean $*$-convolution

$$F * G(x) = \int_0^\infty \tau_x^E F(y) G(y) d\mu_\alpha(y)$$

with associated generalized Euclidean translation

$$\tau_x^E F(y) = \int_0^\pi F((x,y)_\theta) d\nu_\alpha(\theta), \quad d\nu_\alpha(\theta) = \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)} (\sin\theta)^{2\alpha} d\theta.$$

Therefore we restrict ourselves to fractional integrals defined via the generalized Euclidean convolution.

Theorem 2.2. *Let* $1 < p < q < \infty$, $\alpha > -1/2$, $K_\sigma(x) = x^{2(\sigma-\alpha-1)}$. *Then*

$$\|F * K_\sigma\|_q \leq C\|F\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{\sigma}{\alpha+1}.$$

By the above, it is clear that Theorem 1.1 for the case $a = b = 0$ follows from Theorem 2.2.

Proof. Following Hedberg [6], we want to estimate $F * K_\sigma(x)$ pointwise by a suitable maximal function which in this setting turns out to be (see Stempak [11, p. 138])

$$F^*(x) = \sup_{\varepsilon>0} \varepsilon^{-(2\alpha+2)} \int_0^\varepsilon \tau_x^E(|F|)(y)d\mu_\alpha(y)$$

with the usual boundedness property $\|F^*\|_r \le C\|F\|_r$, $1 < r \le \infty$. Now there holds

$$|F * K_\sigma(x)| \le C\left(\int_0^\delta + \int_\delta^\infty\right) \tau_x^E(|F|)(y)y^{2(\sigma-\alpha-1)}d\mu_\alpha(y) = J_1 + J_2,$$

where $\delta > 0$ will be chosen later appropriately. Clearly,

$$\begin{aligned} J_1 = \sum_{k=0}^\infty \int_{2^{-k-1}\delta}^{2^{-k}\delta} \cdots &\le C\sum_{k=0}^\infty (2^{-k}\delta)^{2\sigma}(2^{-k}\delta)^{-2\alpha-2}\int_{2^{-k-1}\delta}^{2^{-k}\delta} \tau_x^E(|F|)(s)s^{2\alpha+1}ds \\ &\le C\delta^{2\sigma}\sum_{k=0}^\infty 2^{-k2\sigma}(2^{-k}\delta)^{-2\alpha-2}\int_0^{\delta 2^{-k}} \tau_x^E(|F|)(s)s^{2\alpha+1}ds \le C\delta^{2\sigma}F^*(x). \end{aligned}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} J_2 = \int_\delta^\infty \tau_x^E(|F|)(s)s^{2(\sigma-\alpha-1)}s^{2\alpha+1}ds &\le \|\tau_x^E(|F|)\|_p\left(\int_\delta^\infty s^{2(\sigma-\alpha-1)p'}s^{2\alpha+1}ds\right)^{1/p'} \\ &\le C\delta^{2\sigma-(2\alpha+2)/p}\|F\|_p, \end{aligned}$$

since τ_x^E are contractions on $L^p(\mathbf{R}_+, d\mu_\alpha)$. Hence

$$|F * K_\sigma(x)| \le C\big(\delta^{2\sigma}F^*(x) + \delta^{2\sigma-(2\alpha+2)/p}\|F\|_p\big).$$

Choosing $\delta = \big(F^*(x)/\|F\|_p\big)^{-p/(2\alpha+2)}$ (where $\|F\|_p \ne 0$ is assumed), we obtain

$$|F * K_\sigma(x)| \le CF^*(x)^{1-\sigma p/(\alpha+1)}\|F\|_p^{\sigma p/(\alpha+1)}$$

and then $\|F * K_\sigma\|_q \le C\|F\|_p$ due to the inequality for the maximal function with $r = q\,(1 - \sigma p/(\alpha+1)) = p > 1$. This finishes the proof of Theorem 2.2.

3. Extension to power weights

The proof of Theorem 1.1 in the general case follows along the lines in Section 2 from

Theorem 3.1. *Let $\alpha > -1/2$, $0 < \sigma < \alpha+1$ and $a < (\alpha+1)/p'$, $b < (\alpha+1)/q$, with $a+b \ge 0$. If $1 < p \le q < \infty$, then*

$$\|K_\sigma * F(x)x^{-2b}\|_q \le C\|F(x)x^{2a}\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{\sigma-a-b}{\alpha+1}.$$

Proof. An equivalent version of the inequality above is

$$\left(\int_0^\infty |Sf(x)|^q d\mu_\alpha(x)\right)^{1/q} \le C\left(\int_0^\infty |f(x)|^p d\mu_\alpha(x)\right)^{1/p}$$

where

$$f(x) = x^{2a}F(x), \quad Sf(x) = \int_0^\infty K(x,y)f(y)d\mu_\alpha(y),$$

and
$$K(x,y) = x^{-2b}\left(\int_0^\pi (x,y)_\theta^{2(\sigma-\alpha-1)}d\nu_\alpha(\theta)\right)y^{-2a}.$$
We first consider the case $p=q$. Then $\sigma = a+b$ and, therefore, the kernel $K(x,y)$ is homogeneous of degree $-(2\alpha+2)$: $K(rx,ry) = r^{-(2\alpha+2)}K(x,y)$, $r>0$. It now suffices to check that (cf. Section 2 of [13])
$$\int_0^\infty K(1,y)y^{-(2\alpha+2)/p}d\mu_\alpha(y) < \infty.$$
We first note that the function $K(1,y)$ has at most an integrable singularity at $y=1$, since for $1/2 \le y \le 2$ we have
$$\int_0^\pi \frac{(\sin\theta)^{2\alpha}d\theta}{\left((1-y)^2+4y\sin^2(\theta/2)\right)^{\alpha+1-\sigma}} \le C(1+|1-y|^{\sigma-1}).$$
To deal with the singularity at 0, we note that for $y<1/2$ we have $(1,y)_\theta \approx 1$, and therefore $\int_0^\pi (1,y)_\theta^{2(\sigma-\alpha-1)}d\nu_\alpha(\theta) \approx 1$ as well. Hence $K(1,y) \le Cy^{-2a}$ and thus
$$\int_0^{1/2} K(1,y)y^{-(2\alpha+2)/p}d\mu_\alpha(y) < \infty, \quad a < (\alpha+1)/p'.$$
For $y>2$, we have $(1,y)_\theta \approx y$, $\int_0^\pi (1,y)_\theta^{2(\sigma-\alpha-1)}d\nu_\alpha(\theta) \approx y^{2(\sigma-\alpha-1)}$, and hence
$$\int_2^\infty K(1,y)y^{-(2\alpha+2)/p}d\mu_\alpha(y) < \infty, \quad b < (\alpha+1)/q,$$
since $\sigma < (\alpha+1)/p$.

Now, consider the case $1<p<q<\infty$. We use still another equivalent version of the inequality to be proved, namely
$$\int_0^\infty \int_0^\infty K(x,y)f(y)g(x)d\mu_\alpha(y)d\mu_\alpha(x) \le C\|f\|_p\,\|g\|_{q'},$$
assuming for simplicity that f and g are nonnegative. Writing $\mathbf{R}_+ \times \mathbf{R}_+ = D_1 \cup D_2 \cup D_3$ where
$$D_1 = \{y/2 \le x \le 2y\}, \quad D_2 = \{2y < x\}, \quad D_3 = \{x < y/2\},$$
it suffices to check for $i=1,2,3$ that
$$I_i = \iint_{D_i} K(x,y)f(y)g(x)d\mu_\alpha(y)d\mu_\alpha(x) \le C\|f\|_p\,\|g\|_{q'}.$$
Consider I_1 first. Since $a+b \ge 0$, for $x,y \in D_1$ and $\theta \in (0,\pi)$ we have
$$((x,y)_\theta)^{2(a+b)} \le Cx^{2(a+b)} \le Cx^{2b}y^{2a},$$
so
$$I_1 \le \iint_{D_1} \widetilde{K}(x,y)f(y)g(x)d\mu_\alpha(y)d\mu_\alpha(x)$$
with $\widetilde{K}(x,y) = \tau_x^E K_{\sigma-a-b}(y)$. Hence we are reduced to showing that
$$\int_0^\infty K_{\sigma-a-b} * f(x)g(x)d\mu_\alpha(x) \le \|f\|_p\,\|g\|_{q'},$$
which is implied by Theorem 2.2.

To estimate I_2 and I_3, we need the following lemma.

Lemma 3.2. *Let* $\alpha > -1$ *and* $V_\delta f(x) = x^{2(\delta-\alpha-1)} \int_0^x f(y) y^{-2\delta} d\mu_\alpha(y)$, $\delta < (\alpha+1)/p'$. *Then*

$$\|V_\delta f\|_p \le C\|f\|_p, \quad |V_\delta f(x)| \le Cx^{-(2\alpha+2)/p}\|f\|_p.$$

Proof. We have that $V_\delta f(x) = \int_0^\infty L(x,y)f(y)d\mu_\alpha(y)$, where $L(x,y) = x^{2(\delta-\alpha-1)}y^{-2\delta}$ for $y < x$ and equals 0 otherwise. Clearly L is homogeneous of degree $-(2\alpha+2)$. Therefore the norm inequality is implied by

$$\int_0^\infty L(1,y)y^{-(2\alpha+2)/p}d\mu_\alpha(y) = \int_0^1 y^{-2\delta-(2\alpha+2)/p+2\alpha+1}dy < \infty,$$

which is finite since $\delta < (\alpha+1)/p'$. For the pointwise estimate, we simply write

$$\begin{aligned}
|V_\delta f(x)| &\le x^{2(\delta-\alpha-1)} \int_0^x |f(y)|y^{-2\delta}d\mu_\alpha(y) \\
&\le Cx^{2(\delta-\alpha-1)} \left(\int_0^x y^{-2\delta p'}d\mu_\alpha(y)\right)^{1/p'} \|f\|_p \\
&\le Cx^{-(2\alpha+2)/p}\|f\|_p,
\end{aligned}$$

which finishes the proof of the lemma.

Estimating I_2, we note that for $x, y \in D_2$ and all θ, $0 < \theta < \pi$, there holds $x < 2(x-y) \le 2((x-y)^2 + 2xy(1-\cos\theta))^{1/2} = 2(x,y)_\theta$. Hence

$$\begin{aligned}
I_2 &\le C\int_0^\infty x^{2(\sigma-\alpha-1-b)}g(x)\left(\int_0^x f(y)y^{-2a}d\mu_\alpha(y)\right)d\mu_\alpha(x) \\
&= C\int_0^\infty g(x)x^{2(\sigma-a-b)}V_a f(x)d\mu_\alpha(x) \le C\|g\|_{q'}\,\|x^{2(\sigma-a-b)}V_a f\|_q.
\end{aligned}$$

It now suffices to estimate the latter L^q norm by $\|f\|_p$. We have

$$\int_0^\infty |x^{2(\sigma-a-b)}V_a f(x)|^q d\mu_\alpha(x) = \int_0^\infty |V_a f(x)|^p |V_a f(x)|^{q-p} x^{2(\sigma-a-b)q}d\mu_\alpha(x).$$

Since $a < (\alpha+1)/p'$, by the pointwise estimate in the above lemma

$$|V_a f(x)| \le Cx^{-(2\alpha+2)/p}\|f\|_p,$$

hence

$$|V_a f(x)|^{q-p}x^{2(\sigma-a-b)q} \le C\|f\|_p^{q-p}$$

due to the identity from assumptions. Hence

$$\int_0^\infty |x^{2(\sigma-a-b)}V_a f(x)|^q d\mu_\alpha(x) \le C\|f\|_p^{q-p}\int_0^\infty |V_a f(x)|^p d\mu_\alpha(x) \le C\|f\|_p^q,$$

by the norm inequality from Lemma 3.2. The estimate of I_3 is similar and we omit it. The proof of the theorem is complete.

After submitting this paper the authors became acquainted with the following result of Kanjin and Sato [7]:

$$\|I_\sigma f\|_{L^q_{v(\alpha q/2)}} \le C\|f\|_{L^p_{v(\alpha p/2)}}, \quad 0 < \frac{1}{q} = \frac{1}{p} - \sigma, \quad p > 1, \quad \alpha \ge 0,$$

whereas for $-1 < \alpha < 0$, there occurs $(1 + \frac{\alpha}{2})^{-1} < p,\ q < -\frac{2}{\alpha}$ as an additional restriction. Combining the above Theorem 1.1 in the case $a = b = \alpha = 0$ with Kanjin's transplantation theorem (cf. [13]) one at once recovers the above stated result of Kanjin and Sato.

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