Methods and Applications of Analysis
2 (2) 1995, pp. 212–221

ISSN 1073–2772

Asymptotic solution to a class of singularly perturbed Volterra integral equations

L. A. Skinner

Abstract. A uniformly valid asymptotic expansion for the solution to a class of singularly perturbed Volterra integral equations displaying exponential boundary layer behavior is established. Certain quasilinear ordinary differential equations are a noted special case, and a model for population growth with attrition is briefly discussed.

1. Introduction

Let $f(x,\epsilon) \in C^\infty([0,1]\times[0,1])$ and $k(x,t,\epsilon,y) \in C^\infty([0,1]\times[0,x]\times[0,1]\times[a,b])$, where (a,b) contains the derivative $f_\epsilon(0,0)$, and assume $f(0,0)=0$. We are interested in the asymptotic behavior as $\epsilon \to 0^+$ of the solution to the Volterra integral equation

$$\epsilon y(x,\epsilon) + \int_0^x k(x,t,\epsilon,y(t,\epsilon))dt = f(x,\epsilon). \tag{1.1}$$

The assumption $f(0,0)=0$, which implies $y(0,\epsilon) = f_\epsilon(0,0)+O(\epsilon)$, is nontrivial unless (1.1) is linear. If $k_{yy}(x,t,\epsilon,y)=0$, we can get $f(0,0)=0$ by changing the unknown (if necessary) to $\epsilon y(x,\epsilon)$.

In the special case

$$k(x,t,\epsilon,y) = p(t,\epsilon)y + (x-t)[q(t,\epsilon,y) - p_t(t,\epsilon)]y, \tag{1.2}$$

$$f(x,\epsilon) = \alpha\epsilon + [\beta\epsilon + \alpha p(0,\epsilon)]x, \tag{1.3}$$

equation (1.1) is equivalent to the singularly perturbed initial-value problem

$$\epsilon y'' + p(x,\epsilon)y' + q(x,\epsilon,y) = 0, \tag{1.4}$$

$$y(0)=\alpha, \quad y'(0)=\beta. \tag{1.5}$$

As is well known [5], if $p(x,0)>0$ and the reduced problem

$$p(x,0)u' + q(x,0,u) = 0, \quad y(0)=\alpha, \tag{1.6}$$

has a solution for $0 \le x \le 1$, then the solution to (1.4)–(1.5) exists for $0 \le x \le 1$, and it has a uniformly valid asymptotic expansion of the form

$$y(x,\epsilon) = \sum_{n=0}^{N-1} \epsilon^n [u_n(x) + v_n(x/\epsilon)] + O(\epsilon^N), \tag{1.7}$$

where $u_n(x) \in C^\infty[0,1]$ and $v_n(X) \in C^\infty[0,\infty)$. Furthermore, $v_n(X) = o(X^{-\infty})$. That is, $v_n(X) = o(X^{-r})$ as $X \to \infty$ for any r. It can also be seen from the linear

Received June 29, 1994, revised December 5, 1994.
1991 *Mathematics Subject Classification.* 41A60, 45M05.
Key words and phrases: singular perturbation, Volterra integral equation, asymptotic expansion.

Fredholm equation theory in [4] that the solution to (1.1) is expressible in the form (1.7) if $k(x,t,\epsilon,y) = c(x,t)y$ and $c(x,x) > 0$.

The object of this paper is to establish the validity of (1.7), under appropriate conditions, for the full nonlinear problem (1.1). In so doing, we develop a procedure for successively computing the individual terms of (1.7). This procedure is rigorous and more explicit than the one presented in [1] and [2], which is a formal treatment of (1.1) and a variety of related problems including singular integral equations problems. We also briefly discuss a model for population growth with attrition.

Insisting that $v_n(X) = o(X^{-\infty})$ in (1.7) may at first seem too restrictive. For instance, if

$$k(x,t,\epsilon,y) = y^2 - 2ty, \quad f(x,\epsilon) = \epsilon(1+x) - \frac{1}{3}x^3, \tag{1.8}$$

the solution to (1.1) is $y(x,\epsilon) = x + (1+x/\epsilon)^{-1}$. However, this example is exceptional. Considering just the first term of (1.7), if we are to have $y(x,\epsilon) = u(x) + v(x/\epsilon) + O(\epsilon)$, then (1.1) implies

$$u(0) + v(X) + \int_0^X k(0,0,0,u(0)+v(T))dT = Xf_x(0,0) + f_\epsilon(0,0). \tag{1.9}$$

Thus, in order to have $v(X) = O(1)$ for $0 \le X < \infty$, it must be that $v(X)$ approaches an equilibrium point of

$$v' + k(0,0,0,u(0)+v) = f_x(0,0) \tag{1.10}$$

as $X \to \infty$. Without loss of generality, $v(\infty) = 0$, by choice of $u(0)$; therefore $k_y(0,0,0,u(0)) \ge 0$. If $k_y(0,0,0,u(0)) > 0$, then $k_y(0,0,0,y) > 0$ in a neighborhood of $y = u(0)$; therefore $v(X) \to 0$ exponentially as $X \to \infty$. That is, $v(X) = o(X^{-\infty})$ if $k_y(0,0,0,u(0)) > 0$. On the other hand, if $k_y(0,0,0,u(0)) = 0$, as in (1.8), then $f_x(0,0)$ must just happen to be an extreme value of $k(0,0,0,y)$. Indeed, (1.10) also implies $k(0,0,0,u(0)) = f_x(0,0)$. We shall therefore limit our investigation of (1.1) to solutions having asymptotic expansions of the form (1.7) with $v_n(X) = o(X^{-\infty})$.

2. Integral expansion

Let

$$s_N(t,T,\epsilon) = \sum_{n=0}^{N-1} \epsilon^n [u_n(t) + v_n(T)], \tag{2.1}$$

where $u_n(t) \in C^\infty[0,1]$, $v_n(T) \in C^\infty[0,\infty)$, and $v_n(T) = o(T^{-\infty})$, but $u_n(t)$, $v_n(T)$ are not yet tied to the solution of (1.1). We begin by determining a uniformly valid expansion for

$$Y_N(x,\epsilon) = \int_0^x k(x,t,\epsilon,y_N(t,\epsilon))dt, \tag{2.2}$$

where $y_N(t,\epsilon) = s_N(t,t/\epsilon,\epsilon)$. Note that we are presuming $a < u_0(t) + v_0(T) < b$ for all $(t,T) \in [0,1] \times [0,\infty)$, so that $y_N(t,\epsilon) \in (a,b)$ for $0 \le t \le 1$ and all sufficiently small $\epsilon > 0$. We will use the following two theorems.

Theorem 2.1. *If $h(t,T,\epsilon) \in C^\infty([0,1]\times[0,\infty)\times[0,1])$ and $h(t,T,\epsilon) = o(T^{-\infty})$ as $T\to\infty$, then*

$$h(t,t/\epsilon,\epsilon) = \sum_{n=0}^{N-1} \epsilon^n h_n(t/\epsilon) + O(\epsilon^N), \tag{2.3}$$

where $h_n(T) \in C^\infty[0,\infty)$ is the coefficient of ϵ^n in the Taylor expansion of $h(\epsilon T, T, \epsilon)$, and $h_n(T) = o(T^{-\infty})$.

Theorem 2.2. *If $\phi(T,\epsilon) \in C^\infty([0,\infty)\times[0,1])$ and $g(t,\epsilon,y) \in C^\infty([0,1]\times[0,1]\times[a,b])$, where $a < \phi(T,\epsilon) < b$ for all $(T,\epsilon) \in [0,\infty)\times[0,1]$, and if $\phi(T,\epsilon) = o(T^{-\infty})$, then*

$$g(t,\epsilon,\phi(t/\epsilon,\epsilon)) = \sum_{n=0}^{N-1} \epsilon^n [g_n(t) + h_n(t/\epsilon)] + O(\epsilon^N), \tag{2.4}$$

where $g_n(t) \in C^\infty[0,1]$ is the coefficient of ϵ^n in the Taylor expansion of $g(t,\epsilon,0)$, $h_n(T) \in C^\infty[0,\infty)$ is the coefficient of ϵ^n in the expansion of $g(\epsilon T,\epsilon,\phi(T,\epsilon)) - g(\epsilon T,\epsilon,0)$, and $h_n(T) = o(T^{-\infty})$.

Proofs. Let $p_n(t,T)$ be the coefficient of ϵ^n in the Taylor expansion of $h(t,T,\epsilon)$, and let $p_{nk}(T)$ be the coefficient of t^k in the expansion of $p_n(t,T)$. Then $p_{nk}(T) \in C^\infty[0,\infty)$, $p_{nk}(T) = o(T^{-\infty})$, and

$$p_n(t,T) = \sum_{k=0}^{N-n-1} t^k p_{nk}(T) + t^{N-n} r_{N-n}(t,T), \tag{2.5}$$

where $r_{N-n}(t,T) \in C^\infty([0,1]\times[0,\infty))$ and $r_{N-n}(t,T) = o(T^{-\infty})$. Therefore $t^{N-n} r_{N-n}(t,t/\epsilon) = \epsilon^{N-n}[(t/\epsilon)^{N-n} r_{N-n}(t,t/\epsilon)] = O(\epsilon^{N-n})$. Similarly, $t^k p_{nk}(t/\epsilon) = \epsilon^k P_{nk}(t/\epsilon)$, where $P_{nk}(T) = T^k p_{nk}(T) \in C^\infty[0,\infty)$ and $P_{nk}(T) = o(T^{-\infty})$. Thus, we have (2.3) with

$$h_n(T) = \sum_{k=0}^{n} P_{k,n-k}(T). \tag{2.6}$$

Theorem 2.2 is obtained by applying Theorem 2.1 to

$$h(t,T,\epsilon) = g(t,\epsilon,\phi(T,\epsilon)) - g(t,\epsilon,0). \tag{2.7}$$

By applying Theorem 2.2 to

$$g(t,\epsilon,y,x) = k\big(x,t,\epsilon,\sum_{n=0}^{N-1} \epsilon^n u_n(t) + y\big) \tag{2.8}$$

with

$$\phi(T,\epsilon) = \sum_{n=0}^{N-1} \epsilon^n v_n(T) \tag{2.9}$$

(and x as an uninvolved parameter), it is apparent that

$$k(x,t,\epsilon,y_N(t,\epsilon)) = \sum_{n=0}^{N-1} \epsilon^n [\phi_n(x,t) + \psi_n(x,t/\epsilon)] + O(\epsilon^N), \tag{2.10}$$

where $\phi_n(x,t) \in C^\infty([0,1]\times[0,x])$, $\psi_n(x,T) \in C^\infty([0,1]\times[0,\infty))$, and $\psi_n(x,T) = o(T^{-\infty})$. A little computation also reveals

$$\phi_0(x,t) = k(x,t,0,u_0(t)), \tag{2.11}$$
$$\phi_1(x,t) = k_y(x,t,0,u_0(t))u_1(t) + k_\epsilon(x,t,0,u_0(t)), \tag{2.12}$$

and, if we let

$$h(x,t,\epsilon,y) = k(x,t,\epsilon,u_0(0)+y) - k(x,t,\epsilon,u_0(0)), \tag{2.13}$$

then

$$\psi_0(x,T) = h(x,0,0,v_0(T)), \tag{2.14}$$
$$\psi_1(x,T) = k_y(x,0,0,u_0(0)+v_0(T))v_1(T) + \widehat{\psi}_1(x,T), \tag{2.15}$$

where $\widehat{\psi}_1(x,T)$ is the combination

$$\widehat{\psi}_1(x,T) = (h_\epsilon + Th_t + u_0'(0)Th_y + u_1(0)h_y)(x,0,0,v_0(T)). \tag{2.16}$$

In general, for $1 \le n \le N-1$,

$$\phi_n(x,T) = k_y(x,t,0,u_0(t))u_n(t) + \widehat{\phi}_n(x,t), \tag{2.17}$$
$$\psi_n(x,T) = k_y(x,0,0,u_0(0)+v_0(T))v_n(T) + \widehat{\psi}_n(x,T), \tag{2.18}$$

where $\widehat{\phi}_n(x,t)$ and $\widehat{\psi}_n(x,T) - u_n(0)h_y(x,0,0,v_0(T))$ are determined by $u_k(t), v_k(T)$ for $0 \le k \le n-1$.

We need to substitute (2.10) into (2.2). Upon applying Theorem 2.1 to

$$\Psi_n(x,X) = \int_X^\infty \psi_n(x,T)dT, \tag{2.19}$$

we get the desired result. Namely,

$$Y_N(x,\epsilon) = \sum_{n=0}^{N-1} \epsilon^n[U_n(x) + \epsilon V_n(x/\epsilon)] + \epsilon^N \theta_N(x,\epsilon), \tag{2.20}$$

where $U_n(x) \in C^\infty[0,1]$, $V_n(X) \in C^\infty[0,\infty)$, $V_n(X) = o(X^{-\infty})$ and $\theta_N(x,\epsilon) = O(1)$ for $0 \le x \le 1$ as $\epsilon \to 0^+$. In particular,

$$U_0(x) = \int_0^x k(x,t,0,u_0(t))dt, \quad V_0(X) = -\int_X^\infty h(0,0,0,v_0(T))dT. \tag{2.21}$$

For $1 \le n \le N-1$,

$$U_n(x) = \int_0^x k_y(x,t,0,u_n(t))dt + \widehat{U}_n(x), \tag{2.22}$$
$$V_n(X) = -\int_X^\infty k_y(0,0,0,u_0(0)+v_0(T))v_n(T)dT + \widehat{V}_n(X), \tag{2.23}$$

where, letting $\lambda_{mn}(X)$ denote the coefficient of x^m in the Taylor expansion of $\Psi_{n-m}(x,X)$,

$$\widehat{U}_n(x) = \int_0^x \widehat{\phi}_n(x,t)dt + \Psi_{n-1}(x,0), \tag{2.24}$$

$$\widehat{V}_n(X) = -\int_X^\infty \widehat{\psi}_n(0,T)dT - \sum_{m=1}^{n} X^m \lambda_{mn}(X). \tag{2.25}$$

In particular,

$$\widehat{U}_1(x) = \int_0^x k_\epsilon(x,t,0,u_0(t))dt + \int_0^\infty h(x,0,0,v_0(T))dT, \tag{2.26}$$

$$\widehat{V}_1(X) = -\int_X^\infty \left[\widehat{\psi}_1(0,T) + Xh_x(0,0,0,v_0(T))\right]dT. \tag{2.27}$$

We could, of course, have absorbed $V_{N-1}(x/\epsilon)$ into $\theta_N(x,\epsilon)$ in (2.20). However, in its present form, since the expansion need not stop at N terms, $\theta_N(x,\epsilon) = p_N(x) + \epsilon\rho_N(x,\epsilon)$, where $p_N(x) \in C^\infty[0,1]$ and $\rho_N(x,\epsilon) = O(1)$ as $\epsilon \to 0^+$. Thus, we have the following key result.

Theorem 2.3. *The derivative of the error term in* (2.20), $\theta_{Nx}(x,\epsilon) = O(1)$ *for* $0 \le x \le 1$ *as* $\epsilon \to 0^+$.

Proof. Just as we established (2.20) for $Y_N(x,\epsilon)$, we also have

$$Y_{Nx}(x,\epsilon) = \sum_{n=0}^{N} \epsilon^n \left[\widetilde{U}_n(x) + \widetilde{V}_n(x/\epsilon)\right] + \epsilon^{N+1}\widetilde{\theta}_{N+1}(x,\epsilon), \tag{2.28}$$

where $\widetilde{U}_n(x) \in C^\infty[0,1]$, $\widetilde{V}_n(X) \in C^\infty[0,\infty)$, $\widetilde{V}_n(X) = o(X^{-\infty})$, and $\widetilde{\theta}_{N+1}(x,\epsilon) = O(1)$ for $0 \le x \le 1$ as $\epsilon \to 0^+$, since

$$Y_{Nx}(x,\epsilon) = k(x,x,\epsilon,y_N(x,\epsilon)) + \int_0^x k_x(x,t,\epsilon,y_N(t,\epsilon))dt. \tag{2.29}$$

Integrating (2.28) and comparing with (2.20) shows that

$$\rho_N(x,\epsilon) = \int_0^{x/\epsilon} \widetilde{V}_N(T)dT + \int_0^x \widetilde{\theta}_{N+1}(t,\epsilon)dt. \tag{2.30}$$

Therefore $\rho_{Nx}(x,\epsilon) = \epsilon^{-1}\widetilde{V}_N(x/\epsilon) + \widetilde{\theta}_{N+1}(x,\epsilon)$, so $\theta_{Nx}(x,\epsilon) = p'_N(x) + \epsilon\rho_{Nx}(x,\epsilon) = O(1)$.

3. Asymptotic solution

In this section, we establish the preliminary result that, under appropriate conditions, $y_N(x,\epsilon) = s_N(x,x/\epsilon,\epsilon)$ satisfies (1.1) asymptotically, in the sense that

$$\epsilon y_N(x,\epsilon) + Y_N(x,\epsilon) = f(x,\epsilon) - \epsilon^N \phi_N(x,\epsilon), \tag{3.1}$$

where $\phi_N(x,\epsilon) = O(1)$ uniformly in x for $0 \le x \le 1$ as $\epsilon \to 0^+$. From (2.1) and (2.20) it is clear that for (3.1) to hold, we must have

$$u_{n-1}(x) + U_n(x) = f_n(x), \quad v_n(X) + V_n(X) = 0, \tag{3.2}$$

for $0 \le n \le N-1$, where $f_n(x)$ is the coefficient of ϵ^n in the Taylor expansion of $f(x,\epsilon)$ and $u_{-1}(x) = 0$. Furthermore, (3.1) implies $u_n(0) + v_n(0) = f_{n+1}(0)$ and

$$\phi_N(x,\epsilon) = \theta_N(x,\epsilon) + u_{N-1}(x) - \epsilon^{-N}\Big[f(x,\epsilon) - \sum_{n=0}^{N-1} \epsilon^n f_n(x)\Big], \tag{3.3}$$

where $\theta_N(x,\epsilon)$ is the error term in (2.20). Thus, in light of Theorem 2.3, $\phi_{Nx}(x,\epsilon) = O(1)$ as $\epsilon \to 0^+$. Also, $\theta_N(0,\epsilon) = -V_{N-1}(0) = v_{N-1}(0)$ by (2.20) and (3.2), and thus $\phi_N(0,\epsilon) = v_{N-1}(0) + u_{N-1}(0) - [f_N(0) + 0(\epsilon)] = O(\epsilon)$.

Of course, we need to show that (3.2) is consistent with the conditions imposed on $u_n(x)$ and $v_n(X)$ in deriving expansion (2.20). This requires some additional hypotheses, which we list below as H2 and H3. We have been assuming H1 all along.

H1. The function $f(x,\epsilon) \in C^\infty([0,1]\times[0,1])$, and $k(x,t,\epsilon,y) \in C^\infty([0,1]\times[0,x]\times[0,1]\times[a,b])$. Also, $f_0(0) = 0$ and $f_1(0) \in (a,b)$.

H2. The integral equation $U_0(x) = f_0(x)$ has a solution $u_0(x) \in C^\infty[0,1]$ with $a + |f_1(0) - u_0(0)| < u_0(x) < b - |f_1(0) - u_0(0)|$.

H3. There exists $\kappa > 0$ such that $k_y(x,x,0,u_0(x)) \ge \kappa$ for all $x \in [0,1]$, and $y = 0$ is the only root of $h(0,0,0,y)$ in an interval containing $f_1(0) - u_0(0)$.

Condition H3 ensures $v_0(X) = o(X^{-\infty})$. Indeed, $k_y(0,0,0,u_0(0)) \ge \kappa$ means $v_0 = 0$ is an attractor for $v_0' + h(0,0,0,v_0) = 0$, as noted in Section 1, and the $h(0,0,0,y) \ne 0$ assumption means $v_0(0) = f_1(0) - u_0(0)$ is in the domain of attraction. Furthermore, H3 implies $v_0(X) \to 0$ monotonically, hence $a < u_0(x) + v_0(X) < b$.

For $1 \le n \le N-1$, the equations for $u_n(x)$ and $v_n(X)$ are linear. Indeed, (3.2), (2.22), and (2.23) imply

$$\int_0^x k_y(x,t,0,u_0(t))u_n(t)dt = f_n(x) - u_{n-1}(x) - \widehat{U}_n(x), \tag{3.4}$$

$$v_n' + k_y(0,0,0,u_0(0) + v_0(X))v_n = -\widehat{V}_n'(X), \tag{3.5}$$

for $1 \le n \le N-1$. Thus, by induction, $\widehat{U}_n(x) \in C^\infty[0,1]$, so, since $k_y(x,x,0,u_0(x)) > 0$, $u_n(x) \in C^\infty[0,1]$. Also, $\widehat{V}_n(X) \in C^\infty[0,\infty)$ with $\widehat{V}_n(X) = o(X^{-\infty})$, so $v_n(X) \in C^\infty[0,\infty)$ and $v_n(X) = o(X^{-\infty})$, since $k_y(0,0,0,u_0(0)+v_0(X)) \ge \kappa$ for X sufficiently large.

Of course, regarding H2, $U_0(x) = f_0(x)$ may have no solution. For example, there is no solution if $k(x,t,0,y) = y^2$ and $f(x,0) = -x$. On the other hand, there can be at most one solution satisfying $k_y(x,x,0,u_0(x)) \ge \kappa$. Indeed, there can be only one solution to (1.1) of the form (1.7).

If $k(x,t,\epsilon,y) = y^3 - y$ and $f(x,\epsilon) = \epsilon(c+\epsilon)$, for example, then $U_0(x) = 0$, since $f_0(x) = 0$. Hence, referring to (2.21), there are three possibilities, namely, $u_0(x) = \pm 1$ and $u_0(x) = 0$. However, if $u_0(x) = 0$, then $v_0' + v_0^3 - v_0 = 0$ and $v_0(0) = c$. Thus, $v_0(X) \to \pm 1$ as $X \to \infty$, unless $c = 0$. But if $c = 0$, condition H3 fails to hold; indeed, the solution to (1.1) in this case, $y(x,\epsilon) = \epsilon/[(1-\epsilon^2)e^{-2x/\epsilon} + \epsilon^2]$, is not expressible in the form (1.7). If $u_0(x) = 1$, then $h(0,0,0,v_0) = v_0^3 + 3v_0^2 + 2v_0$ and $v_0(0) = c - 1$, so $v_0(X) = o(X^{-\infty})$ only if $c > 0$. The remaining possibility, $u_0(x) = -1$, is the proper solution if $c < 0$.

We conclude this section with a brief discussion of the population growth problem modeled by (1.1) when $f(x,\epsilon) = \epsilon s(x)$ and $k(x,t,\epsilon,y) = -s(x-t)y(1-y/c)$. In this model, the survival function $s(x)$ gives the fraction of the initial population which is still alive at time x (so s(0) = 1), $y(x,\epsilon)$ is the (relative) total population size at time x, and $\epsilon^{-1}y(1-y/c)$ is the (rapid) rate of reproduction. Here we again have $U_0(x) = 0$ and, thus, $u_0(x) = 0$ or $u_0(x) = c$. To satisfy H3, we must take $u_0(x) = c$ and $c > 0$. It follows that $v_0' = -v_0(1 + v_0/c)$, $v_0(0) = 1 - c$. Therefore

$$u_0(x) + v_0(X) = \frac{c}{1 + (c-1)e^{-X}}, \tag{3.6}$$

which is notably independent of $s(x)$. In fact, $y_0(x,\epsilon) = u_0(x) + v_0(x/\epsilon)$ is the well-known s-shaped exact solution to the differential equation form of this model which exists when $s(x) = 1$.

With $y_0(x,\epsilon)$ determined, it follows from (3.4) and (2.26) that

$$\int_0^x s(x-t)u_1(t)dt = -c[1 - s(x)]. \tag{3.7}$$

Hence $u_1(0) = cs'(0)$; therefore, from (3.5) and (2.27),

$$v_1' + [1 + (2/c)v_0(x)]v_1 = -s'(0)v_0(x), \quad v_1(0) = -cs'(0). \tag{3.8}$$

It is apparent from (3.7) that $u_1(x) < 0$ for $x > 0$. Thus, we see that the effect of attrition is that the population size never reaches the saturation level $y = c$. For instance, if $s(x) = (1 + \alpha x)e^{-\alpha x}$, then

$$u_1(x) = -(\alpha c/2)(1 - e^{-2\alpha x}), \quad v_1(X) = 0. \tag{3.9}$$

In addition to fertility being high, lifetimes are short in this example if α is large. Such problems are also studied in [3], but with the two rates connected. Obviously, here we would at least need $\alpha = o(1/\epsilon)$.

To calculate $u_2(x)$ and $v_2(X)$, we would need to determine $\phi_2(x,t)$ and $\psi_2(x,T)$ in (2.10) and then use the fact that

$$U_2(x) = \int_0^x \phi_2(x,t)dt + \int_0^\infty \psi_1(x,T)dT, \tag{3.10}$$

$$V_2(X) = -\int_X^\infty \left[\psi_2(0,T) + X\psi_{1x}(0,T) + \frac{1}{2}X^2\psi_{0xx}(0,T)\right] dT. \tag{3.11}$$

4. Confirmation

We are in position now to prove that, under assumptions H1–H3, for all $\epsilon > 0$ sufficiently small, (1.1) has a solution for $0 \le x \le 1$ with an asymptotic expansion given by (1.7), where $u_n(x)$ and $v_n(X)$ are the functions determined in Sections 2 and 3. We shall follow a procedure patterned after the proof of comparable results for singularly perturbed ordinary differential equations. For example, see [6, pp. 197-205].

First, we consider the linear Volterra equation

$$\epsilon z(x,\epsilon) + \int_0^x k_y(x,t,\epsilon,y_N(t,\epsilon))z(t,\epsilon)dt = g(x,\epsilon). \tag{4.1}$$

We expect that if $y(x,\epsilon)$ is the solution to (1.1), then the difference $\epsilon^{-N}[y(x,\epsilon) - y_N(x,\epsilon)]$ nearly satisfies (4.1) when $g(x,\epsilon) = \phi_N(x,\epsilon)$, where $\phi_N(x,\epsilon)$ is the function in (3.1).

Theorem 4.1. *Choose* $\epsilon_0 > 0$ *so that* $a < y_N(x,\epsilon) < b$ *on* $D = \{(x,\epsilon) : 0 \le x \le 1,\ 0 < \epsilon \le \epsilon_0\}$. *Assume* $g(x,\epsilon)$ *is continuously differentiable with respect to* x *on* D, *and assume* $g(0,\epsilon) = O(\epsilon)$, $g_x(x,\epsilon) = O(1)$. *Then the solution to* (4.1) *is uniformly bounded on* D.

Proof. In terms of

$$w(x,\epsilon) = g_x(x,\epsilon) - \int_0^x k_{xy}(x,t,\epsilon,y_N(t,\epsilon))z(t,\epsilon)dt, \tag{4.2}$$

(4.1) is equivalent to

$$\epsilon z' + k_y(x,x,\epsilon,y_N(x,\epsilon))z = w(x,\epsilon), \quad z(0,\epsilon) = \epsilon^{-1}g(0,\epsilon). \tag{4.3}$$

Thus, in terms of

$$A(x,\epsilon) = \int_0^x k_y(s,s,\epsilon,y_N(s,\epsilon))ds, \tag{4.4}$$

if we let

$$G(x,\epsilon) = \epsilon^{-1}g(0,\epsilon)e^{-A(x,\epsilon)/\epsilon} + \epsilon^{-1}\int_0^x e^{-[A(x,\epsilon)-A(t,\epsilon)]/\epsilon}g_x(t,\epsilon)dt, \tag{4.5}$$

$$K(x,t,\epsilon) = \epsilon^{-1}\int_t^x e^{-[A(x,\epsilon)-A(t,\epsilon)]/\epsilon}k_{xy}(s,t,\epsilon,y_N(t,\epsilon))ds, \tag{4.6}$$

it follows that (4.1) also is equivalent to

$$z(x,\epsilon) + \int_0^x K(x,t,\epsilon)z(t,\epsilon)dt = G(x,\epsilon). \tag{4.7}$$

By an application of Theorem 2.2,

$$k_y(s,s,\epsilon,y_N(s,\epsilon)) = k_y(s,s,0,u_0(s)) + h_y(0,0,0,v_0(s/\epsilon)) + O(\epsilon). \tag{4.8}$$

Therefore

$$A(x,\epsilon) - A(t,\epsilon) = \int_t^x k_y(s,s,0,u_0(s))ds + O(\epsilon), \tag{4.9}$$

hence $A(x,\epsilon) - A(t,\epsilon) \ge \kappa(x-t) + O(\epsilon)$. Thus, we have

$$\int_0^x e^{-[A(x,\epsilon)-A(t,\epsilon)]/\epsilon}g_x(t,\epsilon)dt = O(\epsilon), \tag{4.10}$$

hence $G(x,\epsilon) = O(1)$ on D. Similarly, $K(x,t,\epsilon) = O(1)$ for $0 \le x \le 1$, $0 \le t \le x$, $0 < \epsilon \le \epsilon_0$. Hence, from (4.7), by an application of Gronwall's inequality, $z(x,\epsilon) = O(1)$ on D.

If we denote the solution to (4.1) by $z(x,\epsilon) = \mathcal{K}g(x,\epsilon)$, then $\mathcal{K}$ is a linear operator whose domain is all $g(x,\epsilon)$ satisfying the conditions of Theorem 4.1. Let $\mathbb{B}$ denote the Banach space of all $f(x) \in C^0[0,1]$ with $\|f\| = \max|f(x)|$, and let $S = \{f(x) \in \mathbb{B} : \|f\| \le 2d\}$, where d is chosen so that $d > \|\mathcal{K}\phi_N(x,\epsilon)\|$ for $0 < \epsilon \le \epsilon_0$. For example, $f(x) = \epsilon x$ is in S for any $\epsilon \le 2d$. Finally, let

$$E(x,t,\epsilon,y,\rho) = \epsilon^{-2N}\left[k(x,t,\epsilon,y+\epsilon^N\rho) - k(x,t,\epsilon,y) - \epsilon^N\rho k_y(x,t,\epsilon,y)\right], \tag{4.11}$$

which is continuously differentiable. In particular, for ϵ_0 sufficiently small, there exists $M \geq 0$ such that both

$$|E(x,t,\epsilon,y_N(t,\epsilon),\rho)| \leq M \quad \text{and} \quad |E_\rho(x,t,\epsilon,y_N(t,\epsilon),\rho)| \leq M, \tag{4.12}$$

for all $(x,t,\epsilon,\rho) \in [0,1] \times [0,x] \times (0,\epsilon_0] \times [-2d,2d]$.

Theorem 4.2. *There exists $\epsilon_0 > 0$ such that the nonlinear problem*

$$\epsilon z(x,\epsilon) + \int_0^x F(x,t,\epsilon,z(t,\epsilon))dt = \phi_N(x,\epsilon), \tag{4.13}$$

where

$$F(x,t,\epsilon,z) = k_y(x,t,\epsilon,y_N(t,\epsilon))z + \epsilon^N E(x,t,\epsilon,y_N(t,\epsilon),z), \tag{4.14}$$

has a solution in S, provided $0 < \epsilon \leq \epsilon_0$.

Proof. Problem (4.13) is equivalent to $z(x,\epsilon) = \mathcal{N}z(x,\epsilon)$, where, for any $z(x,\epsilon) \in S$, the operator $\mathcal{N}$ is defined by

$$\mathcal{N}z(x,\epsilon) = \mathcal{K}\big[\phi_N(x,\epsilon) - \epsilon^N \int_0^x E(x,t,\epsilon,y_N(t,\epsilon),z(t,\epsilon))dt\big]. \tag{4.15}$$

There exists $m > 0$ such that $\|\mathcal{K}g(x,\epsilon)\| \leq m\|g(x,\epsilon)\|$ for any $g(x,\epsilon)$ in the domain of $\mathcal{K}$, hence $\|\mathcal{N}z(x,\epsilon)\| \leq d + \epsilon^N mM$ for any $z(x,\epsilon) \in S$. Thus, there exists $\epsilon_0 > 0$ such that, if $z(x,\epsilon) \in S$, then $\mathcal{N}z(x,\epsilon) \in S$ for $0 < \epsilon \leq \epsilon_0$. Furthermore, since (4.12) also implies

$$\begin{aligned} \big|E(x,t,\epsilon,y_N(t,\epsilon),z(t,\epsilon)) - E(x,t,\epsilon,y_N(t,\epsilon),w(t,\epsilon))\big| \\ \leq M|z(t,\epsilon) - w(t,\epsilon)| \end{aligned} \tag{4.16}$$

whenever $z(t,\epsilon), w(t,\epsilon) \in S$, we know

$$\|\mathcal{N}z(x,\epsilon) - \mathcal{N}w(x,\epsilon)\| \leq mM\epsilon^N \|z(x,\epsilon) - w(x,\epsilon)\|. \tag{4.17}$$

Thus, $\mathcal{N}$ is contracting on S for $0 < \epsilon \leq \epsilon_0 < (mM)^{-1/N}$.

Finally, we can state our main result, and its proof is now a simple computation.

Theorem 4.3. *Let $R_N(x,\epsilon)$ be the solution to (4.13) determined by Theorem 4.2. Then $y(x,\epsilon) = y_N(x,\epsilon) + \epsilon^N R_N(x,\epsilon)$ satisfies (1.1) for $0 \leq x \leq 1$, $0 < \epsilon \leq \epsilon_0$.*

Proof. For $0 \leq x \leq 1$, $0 \leq t \leq x$, $0 < \epsilon \leq \epsilon_0$, we have

$$k\big(x,t,\epsilon,y_N(t,\epsilon) + \epsilon^N R_N(t,\epsilon)\big) = k(x,t,\epsilon,y_N(t,\epsilon)) + \epsilon^N F(x,t,\epsilon,R_N(t,\epsilon)), \tag{4.18}$$

and $y_N(x,\epsilon)$ satisfies (3.1). Therefore

$$\begin{aligned} &\int_0^x k(x,t,\epsilon,y_N(t,\epsilon) + \epsilon^N R_N(t,\epsilon))dt \\ &\qquad = f(x,\epsilon) - \epsilon^N[\phi_N(x,\epsilon) - \epsilon y_N(x,\epsilon)] + \epsilon^N[\phi_N(x,\epsilon) - \epsilon R_N(x,\epsilon)]. \end{aligned} \tag{4.19}$$

References

1. J. S. Angell and W. E. Olmstead, *Singularly perturbed Volterra integral equations*, SIAM J. Appl. Math. **47** (1987), 1–14.
2. ———, *Singularly perturbed Volterra integral equations II*, SIAM J. Appl. Math. **47** (1987), 1150–1162.
3. F. C. Hoppensteadt, *An algorithm for approximate solutions to weakly filtered synchronous control systems and nonlinear renewal processes*, SIAM J. Appl. Math. **43** (1983), 834–843.
4. Charles G. Lange and Donald R. Smith, *Singular perturbation analysis of integral equations*, Stud. Appl. Math. **79** (1988), 1–63.
5. R. E. O'Malley, Jr., *Singular Perburbation Methods for Ordinary Differential Equations*, Springer-Verlag, New York, 1991.
6. Donald R. Smith, *Singular Perturbation Theory*, Camb. Univ. Press, Cambridge, 1985.

Department of Mathematical Sciences, University of Wisconsin, Milwaukee, WI 53201, U.S.A.