Newton Sum Rules and Monotonicity Properties of the Zeros of Scaled Co-recursive Associated Polynomials

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Dedicated to Prof. Lee Lorch for his 80th birthday

ABSTRACT. Let $Q_n(x; \beta, \gamma, c)$ be the polynomials of degree n which satisfy the recurrence relation:

$$\alpha_{n+c} Q_{n+1}(x; \beta, \gamma, c) + \alpha_{n+c-1} Q_{n-1}(x; \beta, \gamma, c) + (\beta_{n+c} + \beta \delta_{n,0}) Q_n(x; \beta, \gamma, c)$$

$$= x(1 + (\gamma - 1)\delta_{n,0}) Q_n(x; \beta, \gamma, c),$$

$$Q_{-1}(x; \beta, \gamma, c) = 0, \qquad Q_0(x; \beta, \gamma, c) = 1.$$

In the above, β is real, $\gamma > 0$, α_{n+c} and β_{n+c} are real sequences with $\alpha_{n+c} > 0$, and $\delta_{n,0}$ is the Kronecker symbol. These polynomials are called scaled co-recursive associated polynomials. The co-recursive associated orthogonal polynomials are obtained from the above for $\gamma = 1$.

In this paper, the Newton sum rules for the k-th power of the zeros of scaled co-recursive associated orthogonal polynomials are determined in terms of the Newton sum rules of associated orthogonal polynomials. Some monotonicity properties of the zeros also are given.

1. Introduction

The orthogonal polynomials $P_n(x)$, $n \geq 0$, of degree n, with respect to a positive Borel measure on the real line with infinite mass points, can be defined by the recurrence relation:

$$\alpha_n P_{n+1}(x) + \alpha_{n-1} P_{n-1}(x) + \beta_n P_n(x) = x P_n(x)$$

$$P_{-1}(x) = 0, \qquad P_0(x) = 1, \tag{1.1}$$

where $\alpha_n > 0$ and β_n are real sequences.

The associated polynomials $P_n(x; c)$ of the above polynomials are obtained when we replace n by $n + c$ in the coefficients α_n and β_n of (1.1), i.e.,

$$\alpha_{n+c} P_{n+1}(x; c) + \alpha_{n+c-1} P_{n-1}(x; c) + \beta_{n+c} P_n(x; c) = x P_n(x; c),$$

$$P_{-1}(x; c) = 0, \qquad P_0(x; c) = 1. \tag{1.2}$$

The associated polynomials are called, associated of order c if c is an integer and numerator polynomials if $c = 1$. For particular cases of associated polynomials, many authors [3, 4, 17, 18, 37, 38, 43] gave explicitly the corresponding 4th-order differential equations. Very recently [46] for the associated polynomials of arbitrary order c which satisfy (1.2), the corresponding unique differential equation was calculated with the help of a program built in Mathematica symbolic language. Results for particular

cases of associated polynomials concerning explicit forms, orthogonality measures, monotonicity properties, and differential inequalities for their zeros were given in [2, 18, 19, 24, 28, 32, 43].

The co-recursive polynomials $Q_n(x)$ of $P_n(x)$, which first were introduced by T. S. Chihara [8], satisfy the same recurrence relation (1.1) with the restriction $Q_1 = \frac{x - \beta_0 - \beta}{\alpha_0}$. They can be defined directly by

$$\alpha_n Q_{n+1}(x) + \alpha_{n-1} Q_{n-1}(x) + (\beta_n + \beta \delta_{n,0}) Q_n(x) = x Q_n(x),$$
$$Q_{-1}(x) = 0, \qquad Q_0(x) = 1 \tag{1.3}$$

where

$$\delta_{n,0} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$

Orthogonal polynomials with perturbed recurrence relations play an important role not only with the Orthogonal Polynomials Theory (see [1, 8, 33, 34, 39]), but also in some other areas such as Quantum Mechanics [42], Birth and Death Processes [27], etc. Recently A. Ronveaux and F. Marcellan [39] gave explicitly the 4th-order differential equation satisfied by the co-recursive polynomials of the classical orthogonal polynomials.

The scaled co-recursive associated polynomials satisfy the recurrence relation

$$\alpha_{n+c} Q_{n+1}(x; \beta, \gamma, c) + \alpha_{n+c-1} Q_{n-1}(x; \beta, \gamma, c)$$
$$+ (\beta_{n+c} + \beta \delta_{n,0}) Q_n(x; \beta, \gamma, c) = x(1 + (\gamma - 1) \delta_{n,0}) Q_n(x; \beta, \gamma, c), \tag{1.4}$$
$$Q_{-1}(x; \beta, \gamma, c) = 0, \qquad Q_0(x; \beta, \gamma, c) = 1,$$

where $\gamma > 0$. For $\gamma = 1$ these polynomials are the co-recursive associated orthogonal polynomials, which were introduced by J. Letessier in [29, 30, 31] for special cases. The case $\gamma = 0$ is also of some interest in connection with the results in [13, 14, 21, 41].

Given an orthogonal polynomial family $\{P_n(x)\}_{n=0}^{\infty}$, one of the most interesting problems in this subject is to study how the distribution of zeros of $P_n(x)$ is deformed when the coefficients of the recurrence relation satisfied by the orthogonal sequence are modified. It is a well-known fact that when standard modifications of the recurrence (associated, co-recursive, scaled co-recursive, co-modified, or any of its combinations) are considered, the asymptotic zero distribution of both the starting and perturbed families coincide. This means that, in this context, the finite n case is the one of interest. In this situation, one of the most natural ways of dealing with this problem is to compute the moments of the zero distribution of $P_n(x)$ or, equivalently, the Newton sum rules of their zeros, i.e., the sum of the k-th power of the zeros, $k \geq 1$, from which valuable information can be derived (see, e.g., [40, 46]).

Some of the well-known methods for determining Newton sum rules of the zeros of orthogonal polynomials [5–7, 12, 36], semiclassical polynomials [45], and associated polynomials of any order [46 and the references therein] are based on the differential equations which are satisfied by these polynomials. In the case of co-recursive associated polynomials, to the best of our knowledge, only for the Laguerre polynomials [29] and for some special cases of Jacobi polynomials [30, 31] are the differential equations known. Another difficulty is that the known differential equations for the co-recursive associated polynomials are very complicated and difficult to handle. So methods based on the differential equations are not the natural way of dealing with these polynomials. Also some other well-known methods for determining Newton sum

rules are based on the three-term recurrence relation, [9, 15, 16], on the explicit expression of the polynomials [35], and on the weight function [10, 11 and the references therein]. For a summary of all these methods, see [44].

In Section 2, we determine the Newton sum rules $\sum_{n=0}^{N-1} \omega_n^k(\beta, \gamma, c)$, $k = 1, 2, \ldots$, for the zeros $\omega_n(\beta, \gamma, c)$, $n = 0, 1, 2, \ldots, N-1$, of the scaled co-recursive associated polynomials $Q_n(x; \beta, \gamma, c)$ in terms of the Newton sum rules $\sum_{n=0}^{N-1} \lambda_n^k(c)$, $k = 1, 2, \ldots$, of the zeros $\lambda_n(c) \equiv \omega_n(0, 1, c)$ of the corresponding associated polynomials $P_n(x; c)$.

Some examples of Newton sum rules of the classical associated polynomials are given in Section 3.

In Section 4, we give some monotonicity properties of the zeros of the scaled co-recursive associated polynomials.

2. The main result

The method we use is general and is valid also for other cases in addition to the particular case considered in this paper. It has been used successfully by the authors in previous work [20–26]. It is well-known (see for instance [25]) that $\omega_n(\beta, \gamma, c)$, $n = 0, 1, 2, \ldots, N-1$ is a zero of the polynomial $Q_n(x; \beta, \gamma, c)$, defined by (1.4), if and only if it is an eigenvalue of the following generalized eigenvalue problem

$$(T_0 + \beta P_0)x_n = \omega_n(\beta, \gamma, c)(1 + (\gamma - 1)P_0)x_n, x_n \in H_N, \quad n = 0, 1, 2, \ldots, N-1.$$
(2.1)

In (2.1), $T_0 = AV^* + VA + B$ where A, B are the diagonal operators $Ae_n = \alpha_{n+c}e_n$, $Be_n = \beta_{n+c}e_n$ where $\alpha_{n+c} > 0$ and β_{n+c} are real sequences in a finite dimensional Hilbert space H_N with the orthonormal basis e_n, $n = 0, 1, 2, \ldots, N-1$ and scalar product $(\cdot, \cdot)$, V is the truncated shift operator ($Ve_n = e_{n+1}$, $n = 0, 1, \ldots, N-2$, $Ve_{N-1} = 0$), and V^* is its adjoint ($V^*e_n = e_{n-1}$, $n = 1, 2, \ldots, N-1$, $V^*e_0 = 0$). Finally P_0 is the projection operator on the one-dimensional space spanned by e_0, i.e., $P_0x = (x, e_0)e_0$, $x \in H_N$. The operator C is defined by

$$Ce_n = c_ne_n,$$
(2.2)

where $c_0 = \gamma$ and $c_n = 1$, $n = 1, 2, \ldots, N-1$. Since $\gamma > 0$, the operator C is positive-definite, so that both C^{-1} and $C^{-1/2}$ exist.

There is a simple and important result in operator theory which finds here a useful application. This result is known for operators of trace class and Hilbert-Schmidt operators. For a symmetric operator M in a finite-dimensional real Hilbert space, e.g. the space H_N, the result can be stated as follows:

The sum $\sum_{n=0}^{N-1}(Me_n, e_n)$ is independent of the orthonormal basis e_n, $n = 0, 1, 2, \ldots, N-1$, i.e., if e_n and v_n are two orthonormal bases in H_N, then

$$\sum_{n=0}^{N-1}(Me_n, e_n) = \sum_{n=0}^{N-1}(Mv_n, v_n).$$
(2.3)

If x_n, $n = 0, 1, 2, \ldots, N-1$ is the complete orthonormal system of the eigenvectors of T_0 in H_N and $\omega_n(0, 1, c) \equiv \lambda_n(c)$ are the corresponding simple eigenvalues, i.e., the zeros of the polynomial $P_n(x; c)$, then for $M = T_0^k$, we find from (2.1) that

$$\sum_{n=0}^{N-1}(T_0^k e_n, e_n) = \sum_{n=0}^{N-1}(T_0^k x_n, x_n) = \sum_{n=0}^{N-1}\lambda_n^k(c).$$
(2.4)

From the relation,

$$T_0 e_n = \alpha_{n+c-1} e_{n-1} + \alpha_{n+c} e_{n+1} + \beta_{n+c} e_n,$$

$$T_0 e_0 = \alpha_{0+c} e_1 + \beta_{0+c} e_0, \tag{2.5}$$

we get for $1 \le k \le 4$

$$\sum_{n=0}^{N-1} \lambda_n(c) = \sum_{n=0}^{N-1} \beta_{n+c}, \tag{2.6}$$

$$\sum_{n=0}^{N-1} \lambda_n^2(c) = 2 \sum_{n=0}^{N-2} \alpha_{n+c}^2 + \sum_{n=0}^{N-1} \beta_{n+c}^2, \tag{2.7}$$

$$\sum_{n=0}^{N-1} \lambda_n^3(c) = 3 \sum_{n=0}^{N-2} \alpha_{n+c}^2 \beta_{n+c} + 3 \sum_{n=0}^{N-2} \alpha_{n+c}^2 \beta_{n+c+1} + \sum_{n=0}^{N-1} \beta_{n+c}^3, \tag{2.8}$$

$$\sum_{n=0}^{N-1} \lambda_n^4(c) = \sum_{n=0}^{N-1} \beta_{n+c}^4 + 2 \sum_{n=0}^{N-1} \alpha_{n+c}^4 + 4 \sum_{n=0}^{N-2} \alpha_{n+c}^2 \alpha_{n+c+1}^2 + 4 \sum_{n=0}^{N-3} \alpha_{n+c}^2 \beta_{n+c}^2$$

$$+ 4 \sum_{n=0}^{N-2} \alpha_{n+c}^2 \beta_{n+c+1}^2 + 4 \sum_{n=0}^{N-2} \alpha_{n+c}^2 \beta_{n+c} \beta_{n+c+1}. \tag{2.9}$$

The following theorem is useful in the case of scaled co-recursive associated polynomials where the differential equations in general are unknown or very difficult to handle.

Theorem 2.1. *Let $\lambda_n(c)$ and $\omega_n(\beta, \gamma, c)$ be the zeros of the polynomials $P_n(x; c)$ and $Q_n(x; \beta, \gamma, c)$, respectively, and let k be a positive integer. Then*

$$\sum_{n=0}^{N-1} \omega_n^k(\beta, \gamma, c) = \sum_{n=0}^{N-1} \lambda_n^k(c) + (T^k e_0, e_0) + (T^k e_1, e_1) - (T_0^k e_0, e_0) - (T_0^k e_1, e_1) \tag{2.10}$$

where

$$T = C^{-1/2}(T_0 + \beta P_0) C^{-1/2} \tag{2.11}$$

and

$$T_0 = AV^* + VA + B.$$

Proof. By setting $x_n = C^{-1/2} y_n$ where $C = 1 + (\gamma - 1) P_0$, the problem (2.1) is transformed to

$$C^{-1/2}(T_0 + \beta P_0) C^{-1/2} y_n = \omega_n(\beta, \gamma, c) y_n, \quad \|y_n\| = 1.$$

From the above, we find

$$T y_n = \omega_n(\beta, \gamma, c) y_n$$

or

$$(T y_n, y_n) = \omega_n(\beta, \gamma, c) \tag{2.12}$$

and

$$\sum_{n=0}^{N-1} (T y_n, y_n) = \sum_{n=0}^{N-1} (T e_n, e_n). \tag{2.13}$$

So, from (2.12) and (2.13), we get

$$\sum_{n=0}^{N-1} \omega_n(\beta, \gamma, c) = \sum_{n=0}^{N-1} (Te_n, e_n)$$

and

$$\sum_{n=0}^{N-1} \omega_n^k(\beta, \gamma, c) = \sum_{n=0}^{N-1} (T^k e_n, e_n), \qquad k \geq 1,$$

or

$$\sum_{n=0}^{N-1} \omega_n^k(\beta, \gamma, c) = (T^k e_0, e_0) + (T^k e_1, e_1) + \sum_{n=2}^{N-1} (T^k e_n, e_n). \qquad (2.14)$$

Since

$$T = C^{-1/2} T_0 C^{-1/2} + \beta C^{-1/2} P_0 C^{-1/2},$$

we obtain

$$Te_0 = C^{-1/2} T_0 C^{-1/2} e_0 + \frac{\beta}{\gamma} e_0,$$

$$Te_1 = C^{-1/2} T_0 C^{-1/2} e_1,$$

$$Te_n = T_0 e_n, \qquad 2 \leq n \leq N - 1,$$

and from (2.14), we get (2.10) because

$$\sum_{n=0}^{N-1} \lambda_n^k(c) = \sum_{n=0}^{N-1} (T_0^k e_n, e_n). \qquad (2.15)$$

$\square$

Remark 2.1. For $1 \leq k \leq 4$, equation (2.10) gives
for $k = 1$:

$$\sum_{n=0}^{N-1} \omega_n(\beta, \gamma, c) = \sum_{n=0}^{N-1} \lambda_n(c) + \frac{\beta + \beta_{0+c}}{\gamma} - \beta_{0+c}, \qquad (2.16)$$

for $k = 2$:

$$\sum_{n=0}^{N-1} \omega_n^2(\beta, \gamma, c) = \sum_{n=0}^{N-1} \lambda_n^2(c) + \frac{(\beta + \beta_{0+c})^2}{\gamma^2} + 2\alpha_{0+c}^2 \left(\frac{1}{\gamma} - 1 \right) - \beta_{0+c}^2, \qquad (2.17)$$

for $k = 3$:

$$\sum_{n=0}^{N-1} \omega_n^3(\beta, \gamma, c) = \sum_{n=0}^{N-1} \lambda_n^3(c) + \frac{(\beta + \beta_{0+c})^3}{\gamma^3} + \frac{3\alpha_{0+c}^2}{\gamma^2}(\beta + \beta_{0+c})$$

$$+ \frac{3\alpha_{0+c}^2 \beta_{1+c}}{\gamma} - \beta_{0+c}^3 - 3\alpha_{0+c}^2(\beta_{0+c} + \beta_{1+c}), \qquad (2.18)$$

and for $k = 4$:

$$\sum_{n=0}^{N-1} \omega_n^4(\beta, \gamma, c) = \sum_{n=0}^{N-1} \lambda_n^4(c) + \frac{(\beta + \beta_{0+c})^4}{\gamma^4} + 2\alpha_{0+c}^4 \left(\frac{1}{\gamma^2} - 1\right)$$

$$+ \frac{4\alpha_{0+c}^2}{\gamma^3}(\beta_{0+c} + \beta)^2 + \frac{2\alpha_{0+c}^2\beta^2}{\gamma^3} + \frac{4\alpha_{0+c}^2\beta_{1+c}^2}{\gamma} + \frac{4\alpha_{0+c}^2\beta_{1+c}}{\gamma^2}(\beta + \beta_{0+c})$$

$$+ 3\left(\frac{1}{\gamma} - 1\right)\alpha_{0+c}^2\alpha_{1+c}^2 - \beta_{0+c}^4 - 4\alpha_{0+c}^2(\beta_{0+c}^2 + \beta_{1+c}^2 + \beta_{0+c}\beta_{1+c}). \qquad (2.19)$$

3. Newton sum rules for the associated classical polynomials

(1) Consider the associated Laguerre polynomials $L_n^{(a)}(x; c)$, $a > -1$, $c \geq 0$, which satisfy the recurrence relation

$$(n + c + 1)L_{n+1}^{(a)}(x; c) + (n + c + a)L_{n-1}^{(a)}(x; c)$$
$$- (2n + 2c + a + 1)L_n^{(a)}(x; c) = -xL_n^{(a)}(x; c), \qquad n = 0, 1, 2, \ldots, \qquad (3.1)$$
$$L_{-1}^{(a)}(x; c) = 0, \qquad L_0^{(a)}(x; c) = 1.$$

Setting $L_n^{(a)}(x; c) = (-1)^n \sqrt{U_n} Q_n^{(a)}(x; c)$ where U_n satisfies

$$U_{-1} = 0, \quad U_0 = 1, \quad U_n = \frac{n + c + a}{n + c} U_{n-1}, \quad n \geq 1,$$

we obtain from (3.1)

$$\sqrt{(n + c + 1)(n + c + a + 1)} Q_{n+1}^{(a)}(x; c) + \sqrt{(n + c + a)(n + c)} Q_{n-1}^{(a)}(x; c)$$
$$+ (2n + 2c + a + 1)Q_n^{(a)}(x; c) = xQ_n^{(a)}(x; c), \qquad n \geq 1, \qquad (3.2)$$
$$Q_{-1}^{(a)}(x; c) = 0, \qquad Q_0^{(a)}(x; c) = 1.$$

The polynomials $Q_n^{(a)}(x; c)$ and the associated Laguerre polynomials $L_n^{(a)}(x; c)$ have the same zeros $\lambda_n(c)$. In this case, we have: $\alpha_{n+c}(a) = \sqrt{(n + c + 1)(n + c + a + 1)}$, $\beta_{n+c}(a) = (2n + 2c + a + 1)$, $n = 0, 1, 2, \ldots, N - 1$, so, from the relations (2.6)–(2.9), we get:

$$\sum_{n=0}^{N-1} \lambda_n(c) = N(N + 2c + a), \qquad (3.3)$$

$$\sum_{n=0}^{N-1} \lambda_n^2(c) = 2N^3 + N^2(6c + 3a - 1) + N(6c^2 + 6ca - 2c + a^2 - a) - 2c^2 - 2ca,$$
$$(3.4)$$

$$\sum_{n=0}^{N-1} \lambda_n^3(c) = 5N^4 + N^3(20c + 10a - 6) + N^2(30c^2 + 30ac - 18c + 6a^2 - 9a + 2)$$

$$+ N(20c^3 + 30ac^2 - 18c^2 + 12a^2c - 18ac + 4c + a^3 - 3a^2 + 2a)$$

$$- (12c^3 + 18ac^2 + 6a^2c), \qquad (3.5)$$

and

$$\sum_{n=0}^{N-1} \lambda_n^4(c) = 14N^5 + N^4(70c + 35a - 29)$$

$$+ N^3(140c^2 + 140ac - 111c + 30a^2 - 58a + 22)$$

$$+ N^2(140c^3 + 210ac^2 - 174c^2 + 90a^2c - 174ac + 66c + 10a^3 - 35a^2 + 33a - 6)$$

$$+ N(70c^4 + 140ac^3 - 116c^3 + 90a^2c^2 + 174ac^2 - 66c^2 + 20a^3c - 70a^2c - 66ac$$

$$- 12c + a^4 - 6a^3 + 11a^2 - 6a)$$

$$- (58c^4 + 116ac^3 + 70a^2c^2 + 12a^3c + 12ac). \tag{3.6}$$

(2) Consider the associated Hermite polynomials defined by

$$H_{n+1}(x; c) + 2(n + c)H_{n-1}(x; c) = 2xH_n(x; c), \qquad n = 0, 1, 2, \ldots, \tag{3.7}$$

$$H_{-1}(x; c) = 0, \qquad H_0(x; c) = 1.$$

Setting $H_n(x; c) = \sqrt{U_n}P_n(x; c)$ where U_n satisfies

$$U_{-1} = 0, \qquad U_0 = 1, \qquad U_n = 2(n + c)U_{n-1}, \qquad n \geq 1,$$

we obtain from (3.7)

$$\sqrt{\frac{n + c + 1}{2}}P_{n+1}(x; c) + \sqrt{\frac{n + c}{2}}P_{n-1}(x; c) = xP_n(x; c), \tag{3.8}$$

$$P_{-1}(x; c) = 0, \qquad P_0(x; c) = 1.$$

The $P_n(x; c)$ and the associated Hermite polynomials $H_n(x; c)$ have the same zeros $\lambda_n(c)$. In this case, we have $\alpha_{n+c} = \sqrt{\frac{n+c+1}{2}}$, $\beta_{n+c} = 0$, $n = 0, 1, 2, \ldots, N - 1$, so, from the relations (2.6)–(2.9), we get

$$\sum_{n=0}^{N-1} \lambda_n(c) = \sum_{n=0}^{N-1} \lambda_n^3(c) = 0 \tag{3.9}$$

because $\beta_{n+c} = 0$,

$$\sum_{n=0}^{N-1} \lambda_n^2(c) = \frac{(N - 1)(N + 2c)}{2}, \tag{3.10}$$

and

$$\sum_{n=0}^{N-1} \lambda_n^4(c) = \frac{1}{4}(N - 1)N(2N - 3) + \frac{c}{2}(3N - 5)(N + c). \tag{3.11}$$

(3) Consider the associated Jacobi polynomials $P_n^{(a,b)}(x; c)$, which are defined by

$$2(n + c + 1)(n + c + a + b + 1)(2n + 2c + a + b)P_{n+1}^{(a,b)}(x; c)$$

$$+ 2(n + c + a)(n + c + b)(2n + 2c + a + b + 2)P_{n-1}^{(a,b)}(x; c)$$

$$+ (b^2 - a^2)(2n + 2c + a + b + 1)P_n^{(a,b)}(x; c)$$

$$= (2n + 2c + a + b + 1)(2n + 2c + a + b + 2)(2n + 2c + a + b)xP_n^{(a,b)}(x; c), \tag{3.12}$$

$$P_{-1}^{(a,b)}(x; c) = 0, \qquad P_0^{(a,b)}(x; c) = 1.$$

Setting $P_n^{(a,b)}(x;c) = U_n R_n^{(a,b)}(x;c)$ where

$$U_{-1} = 0, \qquad U_0 = 1,$$

$$U_{n+1} = \sqrt{\frac{(n+c+a+1)(n+c+b+1)(2n+2c+a+b+1)}{(n+c+1)(n+c+a+b+1)(2n+2c+a+b+3)}} U_n,$$

we find

$$\sqrt{\frac{4(n+c+1)(n+c+a+1)(n+c+b+1)(n+c+a+b+1)}{(2n+2c+a+b+2)^2(2n+2c+a+b+1)(2n+2c+a+b+3)}} R_{n+1}^{(a,b)}(x;c)$$

$$+ \sqrt{\frac{4(n+c)(n+c+a)(n+c+b)(n+c+a+b)}{(2n+2c+a+b)^2(2n+2c+a+b-1)(2n+2c+a+b+1)}} R_{n-1}^{(a,b)}(x;c)$$

$$+ \frac{(b^2 - a^2)}{(2n+2c+a+b+2)(2n+2c+a+b)} R_n^{(a,b)}(x,c) = x R_n^{(a,b)}(x;c),$$

$$R_{-1}^{(a,b)}(x;c) = 0, \qquad R_0^{(a,b)}(x;c) = 1.$$

$$(3.13)$$

The polynomials $R_n^{(a,b)}(x;c)$ and the associated Jacobi polynomials $P_n^{(a,b)}(x;c)$ have the same zeros $\lambda_n(c)$. In this case, we have

$$\alpha_{n+c}(a,b) = \sqrt{\frac{4(n+c+1)(n+c+a+1)(n+c+b+1)(n+c+a+b+1)}{(2n+2c+a+b+2)^2(2n+2c+a+b+1)(2n+2c+a+b+3)}},$$

$$\beta_{n+c}(a,b) = \frac{(b^2 - a^2)}{(2n+2c+a+b+2)(2n+2c+a+b)}, \qquad n = 0, 1, 2, \ldots, N-1,$$

and the relations (2.6), (2.7) give, respectively,

$$\sum_{n=0}^{N-1} \lambda_n(c) = \frac{N(b^2 - a^2)}{(2c+a+b)(2N+2c+a+b)}, \qquad (3.14)$$

$$\sum_{n=0}^{N-1} \lambda_n^2(c) = \frac{(N-1)[2c(c+a+b+N)+N(a+b+1)]}{(2c+a+b+1)(2N+2c+a+b-1)}$$

$$+ \frac{(b-a)^2[(3N-1)(b+a)^2(2c+a+b)(2N+2c+a+b)]}{2(2c+a+b+1)(2N+2c+a+b-1)(2c+a+b)^2(2N+2c+a+b)^2}$$

$$- \frac{(b-a)^2[(N-1)(2c+a+b)^2(2N+2c+a+b)^2 + 2N^2(2N-1)(b+a)^2]}{2(2c+a+b+1)(2N+2c+a+b-1)(2c+a+b)^2(2N+2c+a+b)^2}.$$

$$(3.15)$$

Remark 3.1. For $c = 0$, in (3.3)–(3.6), (3.10), (3.11), (3.14), and (3.15), we get the known [5, 7] sum rules for the classical Laguerre, Hermite, and Jacobi polynomials, respectively, which were found using differential equations methods.

Remark 3.2. The sum rules (3.9), (3.10), (3.11) for the zeros of the associated Hermite polynomials have been found in [46], as an application of a program built in the Mathematica symbolic language.

4. Monotonicity properties of the zeros $\omega_n(\beta, \gamma, c)$

The operator (2.11) can be written

$$T = C^{-1/2}T_0C^{-1/2} + \beta C^{-1/2}P_0C^{-1/2}.$$

The eigenvalues $\omega_n(\beta, \gamma, c)$ of T, i.e., the zeros of the polynomial $Q_n(x; \beta, \gamma, c)$, are differentiable functions of β and the derivatives are given [20] by

$$\frac{\partial \omega_n(\beta, \gamma, c)}{\partial \beta} = (C^{-1/2}P_0C^{-1/2}y_n, y_n) = \left|(C^{-1/2}y_n, e_0)\right|^2 = |(y_n, C^{-1/2}e_0)|^2 \quad (4.1)$$

where y_n, $n = 0, 1, 2, \ldots, N-1$, are the normalized eigenvectors of T which correspond to the eigenvalues $\omega_n(\beta, \gamma, c)$: $Ty_n = \omega_n(\beta, \gamma, c)y_n$.

The inequality

$$\frac{\partial \omega_n(\beta, \gamma, c)}{\partial \beta} \leq \frac{1}{\gamma} \quad (4.2)$$

follows immediately from (4.1) because $\|y_n\| = 1$ and $\|C^{-1/2}e_0\|^2 = \frac{1}{\gamma}$. We also observe that

$$\frac{\partial}{\partial \beta}\left(\frac{\omega_n(\beta, \gamma, c)}{\beta}\right) = \frac{\beta\dfrac{\partial \omega_n(\beta, \gamma, c)}{\partial \beta} - \omega_n(\beta, \gamma, c)}{\beta^2} = -\frac{1}{\beta^2}(C^{-1/2}T_0C^{-1/2}y_n, y_n)$$

$$= -\frac{1}{\beta^2}(T_0x_n, x_n) \quad (4.3)$$

where $x_n = C^{-1/2}y_n$.

If $\lambda_0(c)$ is the smallest zero of $P_n(x; c)$, i.e., the first eigenvalue of T_0, then the operator $T_0 + \lambda_0(c)$ is positive, i.e.,

$$\big((T_0 + \lambda_0(c))f, f\big) \geq 0$$

for every $f \in H_N$. Thus from (4.3), we find

$$\frac{\partial}{\partial \beta}\left(\frac{\omega_n(\beta, \gamma, c)}{\beta}\right) \leq \frac{\lambda_0(c)}{\beta^2}\|x_n\|^2 \leq \frac{\lambda_0(c)}{\beta^2}\|C^{-1/2}\|^2. \quad (4.4)$$

From the above, we obtain the following results:

(i) The function $\omega_n(\beta, \gamma, c)$ increases with β. This follows from (4.1).

(ii) The function $\omega_n(\beta, \gamma, c) - \frac{\beta}{\gamma}$ decreases with $\beta > 0$. This follows from (4.2) because

$$\frac{\partial}{\partial \beta}(\omega_n(\beta, \gamma, c) - \frac{\beta}{\gamma}) \leq 0. \quad (4.5)$$

(iii) If $\lambda_0(c) \leq 0$, then the function $\frac{\omega_n(\beta, \gamma, c)}{\beta}$ decreases with $\beta > 0$. This follows immediately from (4.4).

(iv) If $\lambda_0(c) > 0$, then the functions $\frac{\gamma\omega_n(\beta, \gamma, c) + \lambda_0(c)}{\beta}$ and $\frac{\omega_n(\beta, \gamma, c) + \lambda_0(c)}{\beta}$ decrease with $\beta > 0$ for $0 < \gamma \leq 1$ and for $\gamma > 1$, respectively. This follows from (4.4) because $\|C^{-1/2}\|^2 = \frac{1}{\gamma}$ for $0 < \gamma \leq 1$ and $\|C^{-1/2}\|^2 = 1$ for $\gamma > 1$.

Remark 4.1. From (4.5) we have for $\beta > 0$

$$\omega_n(\beta, \gamma, c) < \omega_n(0, \gamma, c) + \frac{\beta}{\gamma}. \quad (4.6)$$

This inequality gives an upper bound for all the zeros $\omega_n(\beta, \gamma, c)$, $n = 0, 1, 2, \ldots, N-1$ of the scaled co-recursive associated polynomials, and the bound is sharp for small β because for $\beta = 0$, (4.6) becomes an equality.

Remark 4.2. From (4.1) we obtain

$$\frac{\partial \omega_n(\beta, \gamma, c)}{\partial \beta} = \frac{1}{\gamma} |(y_n, e_0)|^2$$

and

$$\sum_{n=0}^{N-1} \frac{\partial \omega_n(\beta, \gamma, c)}{\partial \beta} = \frac{1}{\gamma} \sum_{n=0}^{N-1} |(y_n, e_0)|^2 = \frac{1}{\gamma} \|e_0\|^2 = \frac{1}{\gamma}.$$

This means that

$$\frac{\partial}{\partial \beta} \Big(\sum_{n=0}^{N-1} \omega_n(\beta, \gamma, c) - \frac{\beta}{\gamma} \Big) = 0.$$

Thus we obtain the relation

$$\sum_{n=0}^{N-1} \omega_n(\beta, \gamma, c) - \frac{\beta}{\gamma} = \sum_{n=0}^{N-1} \omega_n(0, \gamma, c), \tag{4.7}$$

which follows also from the relation (2.16).

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