Methods and Applications of Analysis
4 (3) 1997, pp. 311–325

ISSN 1073–2772

Asymptotics via iterated Mellin-Barnes integrals: application to the generalised Faxén integral

D. Kaminski and R. B. Paris

ABSTRACT. This paper presents a new method for the development of asymptotic expansions for a class of integrals that includes Faxén's integral. The method employed represents the generalised Faxén integral as an iterated Mellin-Barnes integral and proceeds by systematic application of residue theory to obtain the algebraic asymptotic behaviour of the integral under consideration. Order estimates are provided for the errors committed in the expansion process, and the accuracy of the expansions so obtained is illustrated with numerical examples.

1. Introduction

Faxén's integral is defined by

$$\mathrm{Fi}(\alpha,\beta;y)=\int_0^\infty e^{-\tau+y\tau^\alpha}\tau^{\beta-1}d\tau,$$

subject to the restrictions $0\leq \mathrm{Re}\,\alpha<1, \mathrm{Re}\,\beta>0$; see [5, Ch. 9, §4.1]. We have $\mathrm{Fi}(\alpha,\beta;0)=\Gamma(\beta)$ and $\mathrm{Fi}(0,\beta;y)=e^y\Gamma(\beta)$, and to leading order, Fi is known to exhibit the behaviour

$$\mathrm{Fi}(\alpha,\beta;-y)\sim\frac{\Gamma(\beta/\alpha)}{\alpha y^{\beta/\alpha}},$$

and

$$\mathrm{Fi}(\alpha,\beta;y)\sim\left(\frac{2\pi}{1-\alpha}\right)^{1/2}(\alpha y)^{(2\beta-1)/(2-2\alpha)}\exp\left[(1-\alpha)(\alpha^\alpha y)^{1/(1-\alpha)}\right]$$

for $y\to\infty$; see [5, Ch. 3, ex. 7.3]. The latter approximation exhibits exponential growth due to the presence on the path of integration of a saddle point of the phase function of the integrand. By utilising the simple change of variable $\tau^\beta=t$, the factor $\tau^{\beta-1}$ present in the integrand can be removed, yielding, for $y>0$,

$$\begin{aligned}\mathrm{Fi}(\alpha,\beta;\pm y)&=\frac{1}{\beta}\int_0^\infty \exp(-t^{1/\beta}\pm yt^{\alpha/\beta})\,dt\\ &=\frac{y^{\beta/(1-\alpha)}}{\beta}\int_0^\infty\exp\left[-y^{1/(1-\alpha)}(t^{1/\beta}\mp t^{\alpha/\beta})\right]dt.\end{aligned}$$

Received June 7, 1996, revised March 4, 1997.
1991 *Mathematics Subject Classification*: 34E05, 30E15, 41A60.
Key words and phrases: asymptotics, multiple Mellin-Barnes integrals, remainder estimates, Faxen's integral.

Accordingly, we define the generalised Faxén integral by

$$I(\lambda; c_1, c_2, \ldots, c_k) = \int_0^\infty \exp\left[-\lambda\left(x^\mu + \sum_{r=1}^k c_r x^{m_r}\right)\right] dx \tag{1.1}$$

where

$$\mu > m_1 > m_2 > \cdots > m_k > 0. \tag{1.2}$$

We also shall define the quantities δ_r by

$$\delta_r = 1 - \frac{m_r}{\mu}, \quad r = 1, \ldots, k. \tag{1.3}$$

From (1.2), it follows that every δ_r is positive. We note that (1.1) is entire in each of the parameters c_r and holomorphic in $|\arg \lambda| < \pi/2$.

Observe that the choice of parameters $m_1/\mu = \alpha$ and $\mu = 1/\beta$ for real α and β permits us to write $\mathrm{Fi}(\alpha, \beta; \pm y) = (y^{\beta/(1-\alpha)}/\beta) \cdot I(y^{1/(1-\alpha)}; \mp 1)$ for $y > 0$; extensions to complex values of y can be obtained easily. A small-λ expansion of $I(\lambda; c_1)$ is easy to obtain: develop the factor $\exp(-\lambda c_1 x^{m_1})$ in (1.1) with $k = 1$ into its Maclaurin series. Termwise integration of the result produces

$$I(\lambda) = \frac{\lambda^{-1/\mu}}{\mu} \sum_{r=0}^\infty \frac{(-1)^r}{r!} \Gamma\left(\frac{1 + m_1 r}{\mu}\right) c_1^r \lambda^{\delta_1 r} \tag{1.4}$$

for finite, non-zero λ and arbitrary c_1. A small-λ expansion of $I(\lambda; c_1, \ldots, c_k)$ is provided at the end of §3.

The special case $k = 1$ allows us to form other representations of both $I(\lambda; c_1)$ and Fi in terms of the integral function $U_{n,p}$ introduced in [7]. There, in §3.2, the function $U_{n,p}$ is a generalised hypergeometric function defined for $p = 0, 1, 2, \ldots$ by

$$U_{n,p}(z; \vec{\beta}) = \sum_{r=0}^\infty \prod_{j=1}^p \Gamma\left(\frac{r + \beta_j}{n}\right) \frac{(n^{p/n} z)^r}{r!} \qquad (|z| < \infty)$$

where $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)$ are constants, and n is an arbitrary positive number subject only to $n > p$. The particular choices $p = 1$, $m_1 = 1/\beta$, and $\mu = m_1 n$ in (1.4) allow us to write

$$I(\lambda; \pm 1) = \frac{\lambda^{-1/\mu}}{\mu} U_{n,1}(\mp n^{-1/n} \lambda^{\delta_1}; 1/m_1), \qquad n = \mu/m_1,$$

and from the earlier expression of $\mathrm{Fi}(\alpha, \beta; y)$ in terms of $I(\lambda; -1)$, where $n = 1/\alpha$ and $\mu = 1/\beta$, we find

$$\mathrm{Fi}(\alpha, \beta; y) = U_{1/\alpha,1}(\alpha^\alpha y; \beta/\alpha).$$

The known behaviour of the function $U_{n,p}(z; \vec{\beta})$ for $|z| \to \infty$ then provides a means of constructing the dominant algebraic and exponential asymptotic expansions of Fi for large complex values of y, and so for $I(\lambda; c_1)$.

The asymptotic behaviour of the function $I(\lambda; c_1, \ldots, c_k)$ can be obtained through the use of well-known techniques. One technique would be to make the change of variable $u = x^\mu + c_1 x^{m_1} + \cdots + c_k x^{m_k}$ in (1.1), use reversion to construct a differentiable small-u expansion of $x = x(u)$, and proceed by applying Watson's lemma. A less rigorous approach would be to formally develop all but one of the exponential functions in (1.1), $\exp(-\lambda x^\mu)$ and $\exp(-\lambda c_r x^{m_r})$ for $r = 1, \ldots, k-1$, and then termwise integrate the resulting product. If all exponential factors in the integrand, save the last one, i.e., $\exp(-\lambda c_k x^{m_k})$, are expanded into their Maclaurin series, the result of termwise integration will produce the correct asymptotic expansion. One drawback to this approach is that the construction of error bounds, if so desired, is complicated enough that the economy of the method is soon lost. Yet another is that the process can be carried out for any complex c_r and if done with a negative c_r, results in an algebraic asymptotic expansion when a more careful approach would reveal the presence of exponentially dominated contributions. The special case of $x^\mu + c_1 x^{m_1} + c_2 x^{m_2} = x^3 - \frac{9}{2}x^2 + 6x$ illustrates this difficulty.

Instead of computing the algebraic asymptotic expansion of (1.1) using these methods, we will develop a different technique entirely, based on the computation of residues of a multidimensional contour integral representation of $I(\lambda; c_1, \ldots, c_k)$ along with the bounding of error terms. The principal advantage of the technique is that it can be readily extended to deal with multidimensional analogues of (1.1), a claim that cannot be made for the approaches briefly described above.

We begin by representing the function $I(\lambda) = I(\lambda; c_1, \ldots, c_k)$ as an iterated Mellin-Barnes integral after the fashion employed in [3] and [4]. For each factor $\exp(-\lambda c_r x^{m_r})$ present in the integrand of (1.1), we apply the representation

$$e^{-z} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(t) z^{-t} dt, \qquad |\arg z| < \frac{\pi}{2},\ z \neq 0,$$

where the integration contour is indented to the right of the origin. After interchanging the order of integration, (1.1) becomes

$$I(\lambda; \vec{\mathbf{c}}) = \frac{\lambda^{-1/\mu}}{\mu}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^k \Gamma(\vec{\mathbf{t}})\,\Gamma\!\left(\frac{1-\vec{\mathbf{m}}\cdot\vec{\mathbf{t}}}{\mu}\right) \vec{\mathbf{c}}^{-\vec{\mathbf{t}}}\lambda^{-\vec{\delta}\cdot\vec{\mathbf{t}}} d\vec{\mathbf{t}} \tag{1.5}$$

where

$$\vec{\mathbf{m}} = (m_1, m_2, \ldots, m_k), \qquad \vec{\mathbf{t}} = (t_1, t_2, \ldots, t_k),$$
$$\vec{\mathbf{c}} = (c_1, c_2, \ldots, c_k), \qquad \vec{\delta} = (\delta_1, \delta_2, \ldots, \delta_k),$$

and where we have written

$$\Gamma(\vec{\mathbf{t}}) = \Gamma(t_1)\Gamma(t_2)\cdots\Gamma(t_k),$$
$$\vec{\mathbf{c}}^{-\vec{\mathbf{t}}} = c_1^{-t_1} c_2^{-t_2}\cdots c_k^{-t_k},$$
$$d\vec{\mathbf{t}} = dt_1\, dt_2 \cdots dt_k.$$

The 'dot' appearing between two vector quantities is just the usual Euclidean inner product. The integration contours in (1.5) are indented to the right, away from the origin, to avoid the pole of the integrand present there.

By considering each integral in (1.5) separately, for example

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(t_r)\,\Gamma\!\left(\frac{1-\vec{\mathbf{m}}\cdot\vec{\mathbf{t}}}{\mu}\right) c_r^{-t_r}\lambda^{-\delta_r t_r} dt_r,$$

and considering estimates for the Γ function of large argument, we can show each integral converges absolutely in the sector

$$\left|\arg\lambda + \frac{1}{\delta_r}\arg c_r\right| < \frac{\pi}{2}\frac{1 + m_r/\mu}{\delta_r}. \tag{1.6}$$

The inequality (1.6) is fully developed in [3, (3.7)]. For positive values of $c_1, c_2, \ldots, c_k$, for example, we have $\arg c_r = 0$, so the representation (1.5) in this case provides an analytic continuation of $I(\lambda)$ to a sector wider than the right half-plane for λ. Consequently, a restricted class of Fourier-type integrals also may be analysed using the method which follows. On the other hand, if $\arg\lambda = 0$, (1.6) implies that $|\arg c_r| < \frac{\pi}{2}(1 + m_r/\mu) < \pi$, thereby excluding the case of negative c_r.

In the sections that follow, we assume the constants c_r and the parameter λ satisfy (1.6). Some mention of the case of negative c_r is made in the closing section of the paper.

2. Large-λ expansions for trinomial phases

We shall proceed to determine the large-λ behaviour of $I(\lambda)$ from the representation (1.5), after the fashion exploited in [3]. The method proceeds by repeatedly displacing the integration contours for an integration variable and applying the residue theorem as the contour is shifted past a pole of the integrand. In general, these residues will depend on unevaluated integration variables, so the derivation will not be complete until all k integration contours have undergone translations parallel to the imaginary axis. For simplicity of presentation, we first shall assume $k = 2$, so the integral (1.5) under examination is

$$I(\lambda; c_1, c_2) = \frac{\lambda^{-1/\mu}}{\mu}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^2 \Gamma(t_1)\,\Gamma(t_2)\,\Gamma\left(\frac{1 - m_1t_1 - m_2t_2}{\mu}\right) \times c_1^{-t_1}c_2^{-t_2}\lambda^{-\delta_1t_1-\delta_2t_2}dt_1\,dt_2.$$

Let us displace the t_1 contour first. If we set $t_1 = \rho e^{i\theta}$, with ρ and θ real and $|\theta| < \pi/2$, then we find that the logarithm of the modulus of the integrand has the large-ρ behaviour[1]

$$\rho\cos\theta\log\rho\cdot\delta_1 + \mathcal{O}(\rho)$$

which, in view of (1.2) and (1.3), must tend to ∞. Accordingly, we displace the t_1 contour to the right to obtain large-λ asymptotic behaviour, and, in so doing, encounter poles of the integrand at the sequence of t_1 values given by

$$t_1' = t_1'(t_2) = \frac{1}{m_1}(1 + \mu r_1 - m_2t_2) \tag{2.1}$$

where r_1 is a nonnegative integer. Since t_2 everywhere along its integration contour has $\mathrm{Re}\,t_2 = 0$, except for an indentation to the right near the origin where $\mathrm{Re}\,t_2$ is positive but arbitrarily small, we find that every choice of nonnegative r_1 gives rise to a pole of the t_1 integrand.

Fix a positive integer N_1 and a positive number ϵ_1 satisfying $0 < \epsilon_1 < 1$. We shift the t_1 contour to the vertical line $\mathrm{Re}\,t_1 = \{1 + \mu(N_1 + \epsilon_1)\}/m_1$ in conventional fashion by considering the integral (in t_1) taken over the rectangular contour with vertices $\pm iM$ and $\{1 + \mu(N_1 + \epsilon_1)\}/m_1 \pm iM$, $M > 0$. It is routine to show that the

---

[1] We note that as $\theta \to \pm\frac{1}{2}\pi$, the $\mathcal{O}(\rho)$ term yields the contribution $\rho\{\pm\arg(c_1\lambda^{\delta_1}) - \frac{1}{2}\pi(1 + m_1/\mu)\}$ which generates the convergence condition in (1.6). Subsequent displacement of the t_1 contour therefore implicitly implies this latter condition. In the subsequent analysis, we shall omit details of the $\mathcal{O}(\rho)$ estimate of the logarithm of the modulus of the integrand on the understanding that the analogue of (1.6) applies on each occasion.

contributions to the value of the integral from the segments parallel to the real axis tend to zero as $M \to \infty$. The residue theorem then yields

$$I(\lambda; c_1, c_2) = \frac{\lambda^{-1/\mu}}{\mu}\left\{\frac{\mu}{m_1}\sum_{r_1=0}^{N_1}\frac{(-1)^{r_1}}{r_1!}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(t_1')\,\Gamma(t_2)\,\vec{\mathbf{c}}^{-\vec{\mathbf{t}}}\lambda^{-\vec{\delta}\cdot\vec{\mathbf{t}}}dt_2 + R_1\right\} \tag{2.2}$$

where the integrands appearing in the finite sum have

$$\vec{\mathbf{c}}^{-\vec{\mathbf{t}}} = c_1^{-t_1'}c_2^{-t_2}, \quad \vec{\delta}\cdot\vec{\mathbf{t}} = \delta_1 t_1' + \delta_2 t_2,$$

with t_1' given in (2.1) and remainder term

$$R_1 = \left(\frac{1}{2\pi i}\right)^2\int_{C_1}\int_{-i\infty}^{i\infty}\Gamma(t_1)\,\Gamma(t_2)\,\Gamma\left(\frac{1-m_1t_1-m_2t_2}{\mu}\right)c_1^{-t_1}\,c_2^{-t_2}\,\lambda^{-\delta_1t_1-\delta_2t_2}dt_2\,dt_1. \tag{2.3}$$

In (2.3), the contour C_1 is the vertical line $\operatorname{Re} t_1 = \{1+\mu(N_1+\epsilon_1)\}/m_1$.

An order estimate of R_1 is easily obtained. Observe that the t_2 integral in (2.3) has $\operatorname{Re} t_2 = 0$ everywhere, except for an indentation near the origin where $\operatorname{Re} t_2$ is positive but arbitrarily small. The t_1 variable everywhere along C_1 has $\operatorname{Re} t_1 = \{1+\mu(N_1+\epsilon_1)\}/m_1$, so that

$$R_1 = \mathcal{O}\big(\lambda^{-\delta_1(1+\mu(N_1+\epsilon_1))/m_1}\big)$$

for $\lambda \to \infty$. Indeed, in view of the Riemann-Lebesgue lemma and the oscillatory nature of the integrand in (2.3), this can be strengthed to an o-estimate, if so desired.

Let us now turn to the integrals present in the finite series in (2.2),

$$I_1(\lambda; r_1) \equiv I_1(\lambda; c_1, c_2, r_1) \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(t_1')\,\Gamma(t_2)\,c_1^{-t_1'}\,c_2^{-t_2}\,\lambda^{-\delta_1t_1'-\delta_2t_2}dt_2 \tag{2.4}$$

where t_1' is given in (2.1). To determine which direction we displace the t_2 contour, we proceed as before and set $t_2 = \rho e^{i\theta}$, $|\theta| < \pi/2$, from which we see that the logarithm of the modulus of the integrand has the large-ρ behaviour

$$\rho\cos\theta\log\rho\cdot\left(\frac{m_1-m_2}{m_1}\right) + \mathcal{O}(\rho).$$

From $m_1 > m_2 > 0$ (recall (1.2)), we see that this estimate tends to ∞ as $\rho \to \infty$ which indicates that the t_2 contour also must be displaced to the right to recover the large-λ behaviour of each $I_1(\lambda; r_1)$. In displacing the t_2 contour, we encounter poles at solutions to $t_1' = -r_2$ for r_2 a nonnegative integer; i.e., at the points

$$t_2' = \frac{1}{m_2}(1+\mu r_1+m_1r_2). \tag{2.5}$$

Thus, we find

$$I_1(\lambda; r_1) = \frac{m_1}{m_2}\sum_{r_2=0}^{N_2(r_1)}\frac{(-1)^{r_2}}{r_2!}\Gamma(t_2')\,\vec{\mathbf{c}}^{-\vec{\mathbf{t}}}\lambda^{-\vec{\delta}\cdot\vec{\mathbf{t}}} + R_2(r_1) \tag{2.6}$$

where now, since $t'_1(t'_2) = -r_2$,

$$\vec{\mathbf{c}}^{-\vec{\mathbf{t}}} = c_1^{-t'_1(t'_2)} c_2^{-t'_2} = c_1^{r_2} c_2^{-(1+\mu r_1+m_1 r_2)/m_2}, \tag{2.7}$$

$$-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}} = -\delta_1 t'_1(t'_2) - \delta_2 t'_2 = -\frac{(m_1-m_2)}{m_2} r_2 - \frac{\delta_2}{m_2}(1+\mu r_1) \tag{2.8}$$

in view of the identity

$$m_i\delta_j - m_j\delta_i = m_i - m_j, \quad i \neq j. \tag{2.9}$$

The remainder term $R_2(r_1)$ is given by

$$R_2(r_1) = \frac{1}{2\pi i}\int_{C_2} \Gamma(t'_1)\,\Gamma(t_2)\, c_1^{-t'_1}\, c_2^{-t_2} \lambda^{-\delta_1 t'_1 - \delta_2 t_2}\, dt_2 \tag{2.10}$$

where C_2 is the vertical line $\operatorname{Re} t_2 = \{1+\mu r_1 + m_1(N_2+\epsilon_2)\}/m_2$ for some $0<\epsilon_2<1$. The precise nature of the dependence of N_2 on r_1 will be detailed below.

Collecting together the approximations (2.2) and (2.6), we obtain

$$I(\lambda) = \frac{\lambda^{-1/\mu}}{\mu}\left\{\frac{\mu}{m_2}\sum_{r_1=0}^{N_1}\left\{\sum_{r_2=0}^{N_2(r_1)} \frac{(-1)^{r_1+r_2}}{r_1!\,r_2!}\Gamma(t'_2)\, c_1^{r_2} c_2^{-t'_2} \lambda^{-\vec{\delta}\cdot\vec{t}} + R_2(r_1)\right\} + R_1\right\}$$

with t'_2 given in (2.5) and $-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}$ in (2.8). From (2.10), we find that as $\lambda\to\infty$,

$$R_2(r_1) = \mathcal{O}\big(\lambda^{-\operatorname{Re}\{\delta_1 t'_1(t_2)+\delta_2 t_2\}}\big) = \mathcal{O}\big(\lambda^{-\delta_1(1+\mu r_1)/m_1 - (m_1-m_2)\operatorname{Re}(t_2)/m_2}\big)$$
$$= \mathcal{O}\big(\lambda^{-\delta_2(1+\mu r_1)/m_2 - (m_1-m_2)(N_2+\epsilon_2)/m_2}\big)$$

using (2.1), (2.9), and the fact that on C_2, we have $\operatorname{Re}(t_2) = \{1+\mu r_1 + m_1(N_1+\epsilon_1)\}/m_2$. If the combined errors

$$\sum_{r_1=0}^{N_1} R_2(r_1)$$

are to be of the same order as the error R_1, then it suffices to have, for each r_1, the order estimate

$$R_2(r_1) = \mathcal{O}\big(\lambda^{-\delta_1(1+\mu(N_1+\epsilon_1))/m_1}\big).$$

Then $N_2 = N_2(r_1)$ can be determined from the requirement

$$\delta_1(1+\mu(N_1+\epsilon_1))/m_1 = \delta_2(1+\mu r_1)/m_2 + (m_1-m_2)(N_2+\epsilon_2)/m_2,$$

which implies that $N_2(r_1)$ is the largest nonnegative integer strictly less than

$$N_2^*(r_1) = \frac{1}{m_1-m_2}\left\{\frac{\delta_1 m_2}{m_1}(1+\mu(N_1+\epsilon_1)) - \delta_2(1+\mu r_1)\right\}. \tag{2.11}$$

This determines the upper range of the summation in (2.6). One final simplification follows from the identity $1/\mu + \delta_r/m_r = 1/m_r$, whence we obtain as $\lambda \to \infty$

$$I(\lambda) = \frac{\lambda^{-1/m_2}}{m_2} \sum_{r_1=0}^{N_1} \sum_{r_2=0}^{N_2(r_1)} \frac{(-1)^{r_1+r_2}}{r_1!\, r_2!} \Gamma\left(\frac{1+\mu r_1 + m_1 r_2}{m_2}\right) \vec{\mathbf{c}}^{-\vec{\mathbf{t}}} \lambda^{-\varphi(r_1,r_2)} + \mathcal{O}\left(\lambda^{-1/m_1 - \delta_1\mu(N_1+\epsilon_1)/m_1}\right) \tag{2.12}$$

with $\vec{\mathbf{c}}^{-\vec{\mathbf{t}}}$ as in (2.7) and

$$\varphi(r_1, r_2) = \frac{\mu\delta_2}{m_2} r_1 + \frac{(m_1 - m_2)}{m_2} r_2.$$

3. Large-λ expansions for general phases

The problem of determining the asymptotic behaviour of $I(\lambda)$ for general k can be handled in similar fashion. As before, we displace the t_1 contour in (1.5) to the right and encounter poles of the integrand at points

$$t_1' = t_1'(\vec{\mathbf{t}}^{(1)}) = \frac{1}{m_1}(1+\mu r_1) - \frac{1}{m_1}(\vec{\mathbf{m}}^{(1)} \cdot \vec{\mathbf{t}}^{(1)}) \tag{3.1}$$

for every choice of nonnegative integer r_1. Here, we have written $\vec{\mathbf{m}}^{(i)} \equiv (m_{i+1}, m_{i+2}, \ldots, m_k)$ and $\vec{\mathbf{t}}^{(i)} \equiv (t_{i+1}, t_{i+2}, \ldots, t_k)$, for notational convenience. As was the case for trinomial phases (i.e., $k = 2$), we displace the t_1 contour to the vertical line C_1 given by $\operatorname{Re} t_1 = \{1 + \mu(N_1 + \epsilon_1)\}/m_1$ for some $0 < \epsilon_1 < 1$, and so arrive at

$$I(\lambda; \vec{\mathbf{c}}) = \frac{\lambda^{-1/\mu}}{\mu} \left\{ \frac{\mu}{m_1} \sum_{r_1=0}^{N_1} \frac{(-1)^{r_1}}{r_1!} I_1(\lambda; \vec{\mathbf{c}}, r_1) + R_1 \right\} \tag{3.2}$$

where now

$$I_1(\lambda; \vec{\mathbf{c}}, r_1) \equiv \left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^{k-1} \Gamma(t_1')\,\Gamma(t_2)\cdots\Gamma(t_k)\, \vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t_1')} \lambda^{-\vec{\delta}\cdot\vec{\mathbf{t}}(t_1')} dt_2 \cdots dt_k \tag{3.3}$$

and

$$R_1 \equiv \frac{1}{2\pi i}\int_{C_1} \left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^{k-1} \Gamma(\vec{\mathbf{t}})\,\Gamma\left(\frac{1-\vec{\mathbf{m}}\cdot\vec{\mathbf{t}}}{\mu}\right) \vec{\mathbf{c}}^{-\vec{\mathbf{t}}} \lambda^{-\vec{\delta}\cdot\vec{\mathbf{t}}} dt_2 \cdots dt_k\, dt_1. \tag{3.4}$$

In (3.3), the quantity $\vec{\mathbf{t}}(t_1')$ denotes $\vec{\mathbf{t}}$ with $t_1 = t_1'$; compare (2.3) and (2.4).

In each I_1 present in the sum (3.2), we displace the t_2 contour. Setting $t_2 = \rho e^{i\theta}$, $|\theta| < \pi/2$, we find that the logarithm of the modulus of each integrand has the large-ρ behaviour

$$\rho\cos\theta\log\rho \cdot \left(\frac{m_1 - m_2}{m_1}\right) + \mathcal{O}(\rho),$$

which in view of (1.2) must tend to ∞ as $\rho \to \infty$. Consequently, the t_2 contour must be shifted to the right, and as the contour is displaced, we encounter poles at

$$t_2' = t_2'(\vec{\mathbf{t}}^{(2)}) = \frac{1}{m_2}(1+\mu r_1 + m_1 r_2) - \frac{1}{m_2}\vec{\mathbf{m}}^{(2)} \cdot \vec{\mathbf{t}}^{(2)} \tag{3.5}$$

resulting from solutions to $t_1' = -r_2$, for nonnegative integral r_2. The result of displacing the t_2 contour to the vertical line C_2, given by $\operatorname{Re} t_2 = \{1+\mu r_1 + m_1(N_2+\epsilon_2)\}/m_2$

for some $0 < \epsilon_2 < 1$ and some as yet undetermined $N_2 = N_2(r_1)$, is the finite series with remainder

$$I_1(\lambda; \vec{\mathbf{c}}, r_1) = \frac{m_1}{m_2} \sum_{r_2=0}^{N_2(r_1)} \frac{(-1)^{r_2}}{r_2!} I_2(\lambda; \vec{\mathbf{c}}, r_1, r_2) + R_2(r_1) \tag{3.6}$$

where

$$I_2(\lambda; \vec{\mathbf{c}}, r_1, r_2) \equiv \left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^{k-2} \Gamma(t_2')\,\Gamma(t_3)\cdots\Gamma(t_k)\vec{\mathbf{c}}^{\,-\vec{\mathbf{t}}(t_1',t_2')}\lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t_1',t_2')}dt_3\cdots dt_k \tag{3.7}$$

and

$$R_2 \equiv \frac{1}{2\pi i}\int_{C_2}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^{k-2} \Gamma(t_1')\,\Gamma(t_2)\cdots\Gamma(t_k)\vec{\mathbf{c}}^{\vec{\mathbf{t}}}\lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}}dt_3\cdots dt_k\,dt_2. \tag{3.8}$$

In (3.7), the quantity $\vec{\mathbf{t}}(t_1', t_2')$ denotes $\vec{\mathbf{t}}$ with $t_2 = t_2'$ and $t_1 = t_1'(t_2')$.

We can continually reapply this process until all integrals have been evaluated. Let us set

$$\begin{aligned} I_l &\equiv I_l(\lambda; \vec{\mathbf{c}}, r_1, \ldots, r_l) \\ &= \left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^{k-l} \Gamma(t_l')\,\Gamma(t_{l+1})\cdots\Gamma(t_k)\,\vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t_1',\ldots,t_l')}\lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t_1',\ldots,t_l')}\,dt_{l+1}\cdots dt_k \end{aligned} \tag{3.9}$$

where

$$t_l' \equiv t_l'(\vec{\mathbf{t}}^{(l)}) = \frac{1}{m_l}(1 + \mu r_1 + m_1 r_2 + \cdots + m_{l-1} r_l) - \frac{1}{m_l}\vec{\mathbf{m}}^{(l)}\cdot\vec{\mathbf{t}}^{(l)} \tag{3.10}$$

and with $\vec{\mathbf{t}}(t_1', \ldots, t_l')$ in (3.9) denoting $\vec{\mathbf{t}}$ with $t_l = t_l', t_{l-1} = t_{l-1}'(t_l'), \ldots, t_1 = t_1'(t_2', \ldots, t_l')$. The t_{l+1} contour in I_l must be displaced to the right since if $t_{l+1} = \rho e^{i\theta}$ for $|\theta| < \pi/2$, then the logarithm of the modulus of I_l has the large-ρ behaviour

$$\rho\cos\theta\log\rho\cdot\left(\frac{m_l - m_{l+1}}{m_l}\right) + \mathcal{O}(\rho).$$

In view of (1.2), this tends to ∞ as $\rho \to \infty$. As the t_{l+1} contour is displaced, poles are encountered at $t_l' = -r_{l+1}$, or

$$t_{l+1}' = t_{l+1}'(\vec{\mathbf{t}}^{(l+1)}) = \frac{1}{m_{l+1}}(1 + \mu r_1 + \cdots + m_l r_{l+1}) - \frac{1}{m_{l+1}}\vec{\mathbf{m}}^{(l+1)}\cdot\vec{\mathbf{t}}^{(l+1)}$$

for nonnegative integral r_{l+1}. If the t_{l+1} contour is shifted so as to pass over the poles corresponding to $r_{l+1} = 0, 1, \ldots, N_{l+1}$, but not over the pole corresponding to $r_{l+1} = 1 + N_{l+1}$, then we find

$$I_l = \frac{m_l}{m_{l+1}} \sum_{r_{l+1}=0}^{N_{l+1}} \frac{(-1)^{r_{l+1}}}{r_{l+1}!} I_{l+1}(\lambda) + R_{l+1} \tag{3.11}$$

where

$$R_{l+1} \equiv \frac{1}{2\pi i}\int_{C_{l+1}}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^{k-(l+1)} \Gamma(t_l')\,\Gamma(t_{l+1})\cdots\Gamma(t_k)\,\vec{\mathbf{c}}^{-\vec{\mathbf{t}}}\lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}}dt_{l+2}\cdots\,dt_k\,dt_{l+1}. \tag{3.12}$$

In this expression for R_{l+1}, the contour C_{l+1} is taken over a vertical line along which $\operatorname{Re} t_{l+1} = \{1+\mu r_1+m_1r_2+\cdots+m_{l-1}r_l+m_l(N_{l+1}+\epsilon_{l+1})\}/m_{l+1}$ for some $0<\epsilon_{l+1}<1$. The precise character of $N_{l+1} = N_{l+1}(r_1, r_2, \ldots, r_l)$ is described below.

The recurrence in (3.11) terminates when $l = k-1$, at which point (3.11) becomes

$$I_{k-1} = \frac{m_{k-1}}{m_k} \sum_{r_k=0}^{N_k} \frac{(-1)^{r_k}}{r_k!} \Gamma(t'_k)\, \vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t'_1,\ldots,t'_k)} \lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t'_1,\ldots,t'_k)} + R_k \tag{3.13}$$

with

$$R_k = \frac{1}{2\pi i} \int_{C_k} \Gamma(t'_{k-1})\, \Gamma(t_k)\, \vec{\mathbf{c}}^{-\vec{\mathbf{t}}} \lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}}\, dt_k \tag{3.14}$$

where C_k is a vertical line along which $\operatorname{Re} t_k = \{1+\mu r_1+m_1r_2+\cdots+m_{k-2}r_{k-1}+m_{k-1}(N_k+\epsilon_k)\}/m_k$ for some $0<\epsilon_k<1$ and

$$t'_k = \frac{1}{m_k}(1+\mu r_1+m_1r_2+\cdots+m_{k-1}r_k). \tag{3.15}$$

At this point, we can assemble the results of the recursion (3.11) together with the starting values (3.2) and (3.6) and the final step (3.13) to write

$$\begin{aligned} I(\lambda) = {} & \frac{\lambda^{-1/\mu}}{m_k} \sum_{\substack{0\le r_i\le N_i\\ 1\le i\le k}} \frac{(-1)^{r_1+r_2+\cdots+r_k}}{r_1!\,r_2!\cdots r_k!} \Gamma(t'_k)\, \vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t'_1,\ldots,t'_k)} \lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t'_1,\ldots,t'_k)} \\ & + \frac{\lambda^{-1/\mu}}{\mu} \Bigg\{ \sum_{\substack{0\le r_i\le N_i\\ 1\le i\le k-1}} R_k(r_1,\ldots,r_{k-1}) + \sum_{\substack{0\le r_i\le N_i\\ 1\le i\le k-2}} R_{k-1}(r_1,\ldots,r_{k-2}) + \cdots + R_1 \Bigg\}. \end{aligned} \tag{3.16}$$

Before estimating the remainder terms, let us evaluate the factors $\vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t'_1,\ldots,t'_k)}$ and $\lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t'_1,\ldots,t'_k)}$ appearing in the first sum of (3.16). These values will depend on the result of the chain of successive evaluations $t_k = t'_k$, $t_{k-1} = t'_{k-1}(t'_k)$ and so on. Beginning with t'_k in (3.15) and repeatedly applying (3.10) with $l = k-1, k-2, \ldots, 1$, we find

$$t'_{k-1} = -r_k, \quad t'_{k-2} = -r_{k-1}, \quad \ldots \quad , \quad t_1 = -r_2,$$

from which, with the aid of (2.9), it follows that

$$\begin{aligned} -\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t'_1,\ldots,t'_k) &= (\delta_1,\delta_2,\ldots,\delta_{k-1},\delta_k) \\ &\qquad \cdot \Big(r_2, r_3, \ldots, r_k, -\frac{1}{m_k}(1+\mu r_1+m_1r_2+\cdots+m_{k-1}r_k)\Big) \\ &= -\frac{1}{m_k}\sum_{j=1}^{k-1}(m_j-m_k)r_{j+1} - \frac{\delta_k}{m_k}(1+\mu r_1). \end{aligned}$$

Thus, the first term in (3.16), the finite asymptotic approximation of $I(\lambda)$, can be rendered in the form

$$\frac{\lambda^{-1/\mu}}{m_k} \sum_{\substack{0\le r_i\le N_i\\ 1\le i\le k}} \frac{(-1)^{r_1+r_2+\cdots+r_k}}{r_1!\,r_2!\cdots r_k!} \Gamma(t'_k) \vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t'_1,\ldots,t'_k)} \lambda^{-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t'_1,\ldots,t'_k)}$$

where the quantities $\vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t_1',\ldots,t_k')}$ and $\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t_1',\ldots,t_k')$ are given by

$$\vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t_1',\ldots,t_k')} = c_1^{r_2}\cdots c_{k-1}^{r_k}\,c_k^{-(1+\mu r_1+m_1r_2+\cdots+m_{k-1}r_k)/m_k} \tag{3.17}$$

and

$$\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}(t_1',\ldots,t_k') = \frac{1}{m_k}\sum_{j=1}^{k-1}(m_j-m_k)r_{j+1} + \frac{\delta_k}{m_k}(1+\mu r_1). \tag{3.18}$$

To place the remainder terms in (3.16) in simpler form, let us first observe that in (3.12), we have $\operatorname{Re} t_{l+2} = \operatorname{Re} t_{l+3} = \cdots = \operatorname{Re} t_k = 0$ everywhere along the contours in the inner integrals constituting R_{l+1}, except for an indentation near the origin for each variable where the real part is positive but arbitrarily small. Notice also that $\operatorname{Re} t_{l+1} = \{1+\mu r_1+m_1r_2+\cdots+m_{l-1}r_l+m_l(N_{l+1}+\epsilon_{l+1})\}/m_{l+1}$, and that the remaining t_is present in the integrand must be evaluated through repeated substitution using

$$t_i = \frac{1}{m_i}(1+\mu r_1+m_1r_2+\cdots+m_{i-1}r_i) - \frac{1}{m_i}\vec{\mathbf{m}}^{(i)}\cdot\vec{\mathbf{t}}^{(i)}.$$

As was the case above when examining the first term in (3.16), we find that

$$t_1=-r_2,\quad t_2=-r_3,\quad \ldots,\quad t_{l-1}=-r_l,\quad t_l=-(N_{l+1}+\epsilon_{l+1}),$$

whence along the integration contours in (3.12) away from the origins in the $\{t_{l+2},t_{l+3},\ldots,t_k\}$–planes,

$$\begin{aligned}\operatorname{Re}(-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}) &= (\delta_1,\delta_2,\ldots,\delta_k)\cdot\Big(r_2,r_3,\ldots,r_l,N_{l+1}+\epsilon_{l+1},\\ &\quad -\frac{1}{m_{l+1}}(1+\mu r_1+m_1r_2+\cdots+m_{l-1}r_l+m_l(N_{l+1}+\epsilon_{l+1})),0,\ldots,0\Big)\\ &= -\frac{1}{m_{l+1}}\sum_{j=1}^{l-1}(m_j-m_{l+1})r_{j+1} - \frac{m_l-m_{l+1}}{m_{l+1}}(N_{l+1}+\epsilon_{l+1}) - \frac{\delta_{l+1}}{m_{l+1}}(1+\mu r_1).\end{aligned}$$

Any indentations to the right of the origins in the $\{t_{l+2},t_{l+3},\ldots,t_k\}$–planes will produce estimates for the real part of $-\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{t}}$ that are even more negative, so we have for positive l

$$R_{l+1} = \mathcal{O}\Big(\lambda^{-\frac{1}{m_{l+1}}\{\sum_{j=1}^{l-1}(m_j-m_{l+1})r_{j+1}+(m_l-m_{l+1})(N_{l+1}+\epsilon_{l+1})+\delta_{l+1}(1+\mu r_1)\}}\Big) \tag{3.19}$$

as $\lambda\to\infty$. As we remarked in the previous section, this $\mathcal{O}$-estimate can be strengthened to an o-estimate by virtue of the Riemann-Lebesgue lemma. The estimate for R_1 that we had for the case of trinomial phases applies to (3.4) without change, so R_1 in the case of general k is

$$R_1 = \mathcal{O}\big(\lambda^{-\delta_1(1+\mu(N_1+\epsilon_1))/m_1}\big). \tag{3.20}$$

The order estimates (3.19) and (3.20) can be used to determine the upper summation indices $N_2 = N_2(r_1), N_3 = N_3(r_1,r_2),\ldots,N_k = N_k(r_1,\ldots,r_{k-1})$ present in (3.16) and so provide a bound on the error (the bracketed quantity in (3.16)) in our approximation of $I(\lambda)$.

To bound the error in (3.16), fix N_1 and ϵ_1 and set

$$\Delta = \frac{\delta_1}{m_1}(1+\mu(N_1+\epsilon_1)), \tag{3.21}$$

so that the estimate (3.20) has the compact form $R_1 = \mathcal{O}(\lambda^{-\Delta})$. For each $l = 1, 2, \ldots, k-1$, let us determine the maximum admissible value for $N_{l+1} + \epsilon_{l+1}$ for which the estimate (3.19) is $\mathcal{O}(\lambda^{-\Delta})$. This happens when the exponent of λ in (3.19) is equal to $-\Delta$, or

$$\frac{1}{m_{l+1}}\left\{\sum_{j=1}^{l-1}(m_j - m_{l+1})r_{j+1} + (m_l - m_{l+1})(N_{l+1} + \epsilon_{l+1}) + \delta_{l+1}(1 + \mu r_1)\right\} = \Delta.$$

Thus, the maximum admissible value for $N_{l+1} + \epsilon_{l+1}$, which we denote by N^*_{l+1}, is

$$N^*_{l+1} = N^*_{l+1}(r_1, \ldots, r_l) = \frac{m_k}{m_l - m_k}\left\{\Delta - \frac{\delta_k}{m_k}(1 + \mu r_1) - \sum_{j=1}^{l-1}\frac{m_j - m_k}{m_k}r_{j+1}\right\} \tag{3.22}$$

for $l = 1, 2, \ldots, k-1$. In the particular case where $l = 1$, the sum is empty; compare (2.11).

With N^*_{l+1} available, we define N_{l+1} to be the largest integer satisfying

$$N_{l+1} < N^*_{l+1}. \tag{3.23}$$

The approximation in (3.16) then can be rewritten as

$$I(\lambda) = \frac{\lambda^{-1/\mu}}{m_k}\left\{\sum_{r_1=0}^{N_1}\sum_{r_2=0}^{N_2(r_1)}\cdots\sum_{r_k=0}^{N_k(r_1,\ldots,r_{k-1})}\frac{(-1)^{r_1+r_2+\cdots+r_k}}{r_1!r_2!\cdots r_k!}\Gamma(t'_k)\right.$$
$$\left.\times\, \vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t'_1,\ldots,t'_k)}\lambda^{-\vec{\delta}\cdot\vec{\mathbf{t}}(t'_1,\ldots,t'_k)} + \mathcal{O}(\lambda^{-\Delta})\right\}$$

where $\vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t'_1,\ldots,t'_k)}$ and $\vec{\delta}\cdot\vec{\mathbf{t}}(t'_1,\ldots,t'_k)$ are given in (3.17) and (3.18), respectively, Δ is given in (3.21), and t'_k is displayed as (3.15). From the identity $1/\mu + \delta_r/m_r = 1/m_r$, we can simplify this further to arrive at the final form

$$I(\lambda) = \frac{\lambda^{-1/m_k}}{m_k}\sum_{r_1=0}^{N_1}\sum_{r_2=0}^{N_2(r_1)}\cdots\sum_{r_k=0}^{N_k(r_1,\ldots,r_{k-1})}\frac{(-1)^{r_1+r_2+\cdots+r_k}}{r_1!r_2!\cdots r_k!}\Gamma(t'_k)$$
$$\times\, \vec{\mathbf{c}}^{-\vec{\mathbf{t}}'}\lambda^{-\varphi(r_1,\ldots,r_k)} + \mathcal{O}(\lambda^{-\Delta-1/\mu}) \tag{3.24}$$

where

$$\varphi(r_1, \ldots, r_k) = \frac{\mu\delta_k}{m_k}r_1 + \frac{1}{m_k}\sum_{j=1}^{k-1}(m_j - m_k)r_{j+1} \tag{3.25}$$

and

$$\vec{\mathbf{c}}^{-\vec{\mathbf{t}}'} \equiv \vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t'_1,\ldots,t'_k)} = c_1^{r_2}\cdots c_{k-1}^{r_k}c_k^{-t'_k},$$
$$t'_k = \frac{1}{m_k}(1 + \mu r_1 + m_1 r_2 + \cdots + m_{k-1}r_k),$$
$$\Delta = \frac{\delta_1}{m_1}(1 + \mu(N_1 + \epsilon_1)).$$

Here, N_1 is a positive integer, $0 < \epsilon_1 < 1$ and each value $N_l(r_1, \ldots, r_{l-1})$ for $l > 1$ is an integer strictly less than the N^*_l given in (3.22). The geometric interpretation of the k-fold summation in (3.24) is that of a sum taken over the integer lattice in the non-negative orthant in $(r_1, \ldots, r_k)$-space bounded by the hyperplane

$$\frac{\delta_k}{m_k}(1+\mu r_1) + \frac{1}{m_k}\sum_{j=1}^{k-1}(m_j - m_k)r_{j+1} = \frac{\delta_k}{m_k} + \varphi(r_1, \ldots, r_k) = \Delta. \tag{3.26}$$

We close this section by remarking that the technique detailed above can be successfully applied to the problem of constructing small-λ expansions of $I(\lambda; c_1, \ldots, c_k)$. Treating $t_1, t_2, \ldots, t_k$ in turn, we find that, with $t_l = \rho e^{i\theta}$ as before, the logarithm of the modulus of the integrand at each step has the behaviour

$$\rho \cos\theta \log\rho \cdot \delta_l + \mathcal{O}(\rho)$$

for large ρ and so tends to ∞ as $\rho \to \infty$. Small-λ asymptotics result when each t_l contour is displaced to the *left*, over the poles located at $t_l = -r_l, r_l$ a nonnegative integer. The resulting convergent (if (1.2) is satisfied) expansion, for finite λ, emerges as

$$I(\lambda) = \frac{\lambda^{-1/\mu}}{\mu}\sum_{r_1,\ldots,r_k=0}^{\infty}\frac{(-1)^{r_1+r_2+\cdots+r_k}}{r_! \, r_2! \cdots r_k!}\Gamma\left(\frac{1+\vec{\mathbf{m}}\cdot\vec{\mathbf{r}}}{\mu}\right)c_1^{r_1}\cdots c_k^{r_k}\lambda^{\vec{\boldsymbol{\delta}}\cdot\vec{\mathbf{r}}} \tag{3.27}$$

where $\vec{\mathbf{r}} = (r_1, r_2, \ldots, r_k)$; compare (1.4). The analysis for error terms used above for the large-λ expansion can be modified for the small-λ expansion (3.27).

4. Numerical examples

In this section, we present a number of numerical examples to illustrate the accuracy of the approximation (3.24) developed in §3. Because of the presence in the asymptotic scale in (3.24) of a number of indices $r_1, \ldots, r_k$, it perhaps is not clear whether the Poincaré asymptotics we have obtained suffer from difficulties associated with multiple asymptotic scales, as described by Olver in [6, §3].

The correct way to interpret the expansion (3.24) is suggested by the error term, which restricts the multiple sum to those indices bounded by the hyperplane (3.26). We group together those terms in (3.24) which are of the same order (corresponding to those $(r_1, \ldots, r_k)$ lying on a common level set of $\varphi(r_1, \ldots, r_k)$) so that

$$\begin{aligned} I(\lambda) &\sim \frac{\lambda^{-1/m_k}}{m_k}\sum_{\varphi\geq 0}\left\{\sum_{\varphi(r_1,\ldots,r_k)=\varphi}\frac{(-1)^{r_1+r_2+\cdots+r_k}}{r_1!\,r_2!\cdots r_k!}\Gamma(t_k')\,\vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t_1',\ldots,t_k')}\right\}\lambda^{-\varphi} \\ &\equiv \frac{\lambda^{-1/m_k}}{m_k}\sum_{\varphi\geq 0} a_\varphi \lambda^{-\varphi} \end{aligned} \tag{4.1}$$

where $\varphi(r_1, \ldots, r_k)$ is given in (3.25) and the quantities t_k' and $\vec{\mathbf{c}}^{-\vec{\mathbf{t}}(t_1',\ldots,t_k')}$ are to be found immediately following (3.25). The latter representation of the expansion is taken over the sequence of positive values of φ for which nonnegative solutions $(r_1, \ldots, r_k)$ of $\varphi = \varphi(r_1, \ldots, r_k)$ exist. Viewed this way, the expansion (3.24) is a conventional Poincaré-type expansion and so is immune to the weaknesses exposed by Olver.

This reformulation of (3.24) is the basis of the computations we present in Tables 1 and 2. In each table, values of $I(\lambda; c_1, c_2)$ (in the case of Table 1) and $I(\lambda; c_1, c_2, c_3)$ (in the case of Table 2) for various choices of c_r and m_r, computed by numerical integration, are compared with the approximate values furnished by the expansions obtained in §§2 and 3. The asymptotic values reported in the Tables were calculated

$\mu = 4 \quad m_1 = 2,\ m_2 = 1$			
$c_1 = c_2 = 1$			
λ	$I(\lambda)$	Asymptotic value	N_{opt}
20	0.04598 53887	0.04560 00000	4
50	0.01927 84645	0.01927 87009	11
60	0.01615 78731	0.01615 78962	13
80	0.01220 80072	0.01220 80074	17
100	0.00981 07596	0.00981 07596	21
$c_1 = 3,\ c_2 = 2$			
λ	$I(\lambda)$	Asymptotic value	N_{opt}
10	0.04453 45530	0.04587 50000	3
20	0.02343 78460	0.02341 07141	6
40	0.01207 58772	0.01207 58992	13
50	0.00972 35443	0.00972 35436	16
60	0.00813 89111	0.00813 89112	19
$c_1 = 5,\ c_2 = 3$			
λ	$I(\lambda)$	Asymptotic value	N_{opt}
10	0.03045 56901	0.03016 84499	4
20	0.01586 32390	0.01586 48470	9
30	0.01073 84365	0.01073 84496	13
40	0.00811 88561	0.00811 88560	18
50	0.00652 74290	0.00652 74290	22

TABLE 1. Comparison of the asymptotic values of $I(\lambda)$ with a trinomial phase.

using optimal truncation (that is, truncation just before the smallest term in absolute value), in the sense of the re-arranged series (4.1); the number of terms used in optimal truncation, N_{opt}, also is indicated in the Tables.

For Table 1, the particular choice of parameters $\mu = 4$, $m_1 = 2$, and $m_2 = 1$ is made, so the series derived from (2.12) becomes

$$I(\lambda; c_1, c_2) \sim \frac{1}{\lambda c_2} \sum_{r_1, r_2 = 0}^{\infty} \frac{(-1)^{r_1 + r_2}}{r_1!\, r_2!} (4r_1 + 2r_2)!\, c_1^{r_2} c_2^{-4r_1 - 2r_2} \lambda^{-3r_1 - r_2}$$

for $\lambda \to \infty$. If we undertake the rearrangement (4.1), then this asymptotic expansion becomes

$$I(\lambda; c_1, c_2) \sim \frac{1}{\lambda c_2} \sum_{k=0}^{\infty} (-1)^k c_1^k c_2^{-2k} \left\{ \sum_{r=0}^{\lfloor k/3 \rfloor} \frac{(2k - 2r)!}{r!\,(k - 3r)!} c_1^{-3r} c_2^{2r} \right\} \lambda^{-k},$$

a form suitable for pursuing optimal truncation. Here, $\lfloor x \rfloor$ indicates, for $x > 0$, the integer part of x.

$\mu = 5 \qquad m_1 = 3,\ m_2 = 2,\ m_3 = 1$			
$c_1 = c_2 = c_3 = 1$			
λ	$I(\lambda)$	Asymptotic value	N_{opt}
20	0.04567 71062	0.04570 50375	9
40	0.02384 03273	0.02384 01794	12
50	0.01924 67670	0.01924 67708	13
60	0.01613 83594	0.01613 83597	13
80	0.01219 90817	0.01219 90817	18
$c_1 = 3,\ c_2 = 0.5,\ c_3 = 2$			
λ	$I(\lambda)$	Asymptotic value	N_{opt}
5	0.09109 63605	0.08675 00000	3
10	0.04803 22068	0.04800 52222	6
20	0.02457 82979	0.02457 84270	8
30	0.01649 33771	0.01649 33796	8
40	0.01240 69369	0.01240 69369	15
$c_1 = 2,\ c_2 = 1,\ c_3 = 0.5$			
λ	$I(\lambda)$	Asymptotic value	N_{opt}
50	0.03489 53705	0.03513 60000	4
100	0.01857 95472	0.01858 03573	8
150	0.01267 70947	0.01267 70562	12
200	0.00962 34641	0.00962 34640	13
250	0.00775 60993	0.00775 60993	17

TABLE 2. Comparison of the asymptotic values of $I(\lambda; c_1, c_2, c_3)$.

The entries for Table 2 were computed with the particular parameters $\mu = 5$, $m_1 = 3$, $m_2 = 2$, and $m_3 = 1$, whence the asymptotic expansion from (3.24) becomes

$$I(\lambda; c_1, c_2, c_3) \sim \frac{1}{\lambda} \sum_{r_1, r_2, r_3 = 0}^{\infty} \frac{(-1)^{r_1+r_2+r_3}}{r_1!\, r_2!\, r_3!} (5r_1 + 3r_2 + 2r_3)! \\ \times c_1^{r_2} c_2^{r_3} c_3^{-(1+5r_1+3r_2+2r_3)} \lambda^{-4r_1-2r_2-r_3}$$

for $\lambda \to \infty$. The rearrangement in (4.1) in this case leads to the Poincaré expansion

$$I(\lambda; c_1, c_2, c_3) \sim \frac{1}{\lambda c_3} \sum_{k=0}^{\infty} (-1)^k c_2^k c_3^{-2k} a_k \lambda^{-k}$$

where the coefficients a_k are given by

$$a_k = \sum_{l=0}^{\lfloor k/2 \rfloor} \sum_{r=0}^{\lfloor l/2 \rfloor} \frac{(-1)^{l+r}(2k-l-r)!}{r!\,(l-2r)!\,(k-2l)!} c_1^{l-2r} c_2^{-2l} c_3^{l+r}.$$

As above, $\lfloor x \rfloor$ indicates the floor function.

5. Closing remarks

The principal result of the present work is the expansion of $I(\lambda; c_1, \ldots, c_k)$ presented in (3.24), valid (in the sense of Poincaré) for $\lambda \to \infty$ in the intersection of sectors given by (1.6). As we pointed out in the Introduction, the expansion (3.24) could be obtained through the use of other techniques, albeit with either some computational difficulty or by purely formal methods.

In any event, it is the representation of (1.1) as the iterated Mellin-Barnes integral (1.5), and its subsequent evaluation detailed in §§2 and 3, which are of greatest interest, as this technique can be applied to multidimensional Laplace-type integrals with little modification, as done in [3] and [4]. What is provided in the present work is a justification of the validity of the method in a setting where the geometric content explored in [3] and [4] does not obscure the careful analysis of errors made in the approximation process.

If some c_r in (1.1) is negative, then the integral $I(\lambda)$ may have saddle points through which the integration contour, initially the positive real axis, may be deformed. In such an event, the saddle point or steepest descent method will generate an asymptotic expansion which may dominate the algebraic expansions obtained in §§2 and 3. These expansions have been obtained by others under a variety of restrictions, and we direct the interested reader to [1], [2], and [8] for an account of the exponential expansions arising from $I(\lambda)$ when one or more c_r is negative. None of these papers, incidentally, concerns itself with the task of obtaining complete asymptotic expansions. As we mentioned in §1, the construction of dominant algebraic and exponential expansions for large complex values of λ for the case $k = 1$ of (1.1) can be done via the setting of $U_{n,1}$ in [7].

Acknowledgements. The first named author (DK) would like to thank the Division of Mathematical Sciences at the University of Abertay Dundee for its hospitality during the course of this investigation, and to the University of Lethbridge, its Faculty Association, and NSERC Canada for financial support.

References

1. N. G. Bakhoom, *Asymptotic expansions of the function* $F_k(x) = \int_0^\infty e^{-u^k + xu} du$, Proc. Lond. Math. Soc. **35** (1933), 83–100.
2. W. R. Burwell, *Asymptotic expansions of generalised hypergeometric functions*, Proc. Lond. Math. Soc. **22** (1924), 57–72.
3. D. Kaminski and R. B. Paris, *Asymptotics of a class of multidimensional Laplace-type integrals I. Double integrals*, to appear in Phil. Trans. R. Soc. London. (1997).
4. ———, *Asymptotics of a class of multidimensional Laplace-type integrals II. Treble integrals*, to appear in Phil. Trans. R. Soc. London. (1997).
5. F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, London, 1974.
6. ———, *Asymptotic approximations and error bounds*, SIAM Review **22** (1980), 188–203.
7. R. B. Paris, and A. D. Wood, *Asymptotics of high order differential equations*, Pitman Res. Notes in Math. Series 129, Longman Scientific & Technical, Harlow, 1986.
8. E. M. Wright, *The generalised Bessel function of order greater than one*, Q. J. Math. **46** (1940), 36–48.

Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive, Lethbridge AB, Canada T1K 3M4

Division of Mathematical Sciences, University of Abertay Dundee, Dundee DD1 1HG, UK