On the Generalized Wazewski inequality
for retarded systems

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1. Introduction and statement of the main result

Let us consider in a Euclidean space $\mathbf{C}^n$ with the Euclidean norm $\| \cdot \|_{C^n}$ the equation

$$\dot{x} = A(t)x, \quad \dot{u} = \frac{du}{dt}, \quad t \geq 0, \tag{1.1}$$

where $A(t)$ is a variable matrix. Denote by $(\cdot, \cdot)$ the scalar product in $\mathbf{C}^n$. As is well known, Wazewski [6] established the very useful inequality

$$\|U_A(t, s)\|_{C^n} \leq e^{\int_s^t \alpha(A(\tau))d\tau}, \quad t, s \geq 0, \tag{1.2}$$

where $U_A(t, s)$ is the evolution operator of the equation (1.1) and

$$\alpha(A(t)) = \sup_{x \in \mathbf{C}^n} \text{Re}\,(A(t)x, x)(x, x)^{-1}$$

(see also Winter [7], Daleckii and Krein [1, Section 3.4.6]).

The present paper is devoted to a generalization of inequality (1.2) to the equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(t - h), \quad t \geq 0, \ h = \text{const.} > 0 \tag{1.3}$$

where $B(t)$ is a variable matrix. Note that an estimate for the Lyapunov exponents of equation (1.3) is presented in Kolmanovskii [3], among other interesting results. Under some restrictions below, we make the Kolmanovskii result more precise. In the scalar case, our results supplement the well-known stability conditions from Yoneyama [8].

Take the initial condition

$$x(t) = \Phi(t) \quad \text{for } -h \leq t \leq 0 \tag{1.4}$$

where $\Phi(t)$ is a given continuous vector-valued function. A solution of (1.3) is an absolutely continuous function $x(t) : [-h, \infty) \to \mathbf{C}^n$ which satisfies that equation on $[0, \infty)$ almost everywhere with a given initial condition (cf. Hale and Verduyn Lunel [2], Kolmanovskii and Myshkis [4]). The existence of solutions is assumed.

Let us suppose that the functions $\alpha(A(t))$ and

$$\alpha(B(t)) \equiv \sup_{x \in \mathbf{C}^n} \text{Re}\,(B(t)x, x)(x, x)^{-1}$$

are Riemann integrable. Put

$$\alpha_W(t) = \alpha(A(t)) + \alpha(B(t + h)).$$

The aim of the present paper is to prove the following:

Theorem 1.1. *Let the inequality*

$$J_B \equiv \sup_{t \geq 0} \int_t^{t+h} \|B(s)\|_{C^n} ds < 1 \tag{1.5}$$

be fulfilled. Then for any solution $x(t)$ of (1.3) with a continuous initial function $\Phi(t)$, the estimate

$$\|x(t)\|_{C^n} \leq (1 - J_B)^{-1} c_W(\Phi) \exp\left[\int_0^t (\alpha_W(s) + \eta_W(s)) ds\right], \qquad t \geq h, \tag{1.6}$$

is valid with the notation

$$\eta_W(s) = (1 - J_B)^{-1} \|B(s+h)\|_{C^n} \int_s^{s+h} \|A(\tau) + B(\tau+h)\|_{C^n} e^{\int_\tau^s \alpha_W(\tau_1) d\tau_1} d\tau$$

and

$$c_W(\Phi) = \|\Phi(0)\|_{C^n} + \int_0^h e^{-\int_0^s \alpha_W(\tau) d\tau} \|B(s)\Phi(s-h)\|_{C^n} ds.$$

If $B(t) \equiv 0$, then (1.6) coincides with the Wazewski inequality (1.2).

2. Proof of Theorem 1.1

Put $\tilde{A}(t) = A(t) + B(t+h)$ and denote by $U(t,s)$ the evolution operator of the equation

$$\dot{x}(t) = \tilde{A}(t)x(t).$$

Let us suppose that there is a Riemann integrable function $\alpha(t)$ and a positive constant K such that

$$\|U(t,s)\|_{C^n} \leq K e^{\int_s^t \alpha(\tau) d\tau}, \qquad t, s \geq 0. \tag{2.1}$$

Lemma 2.1. *Let the inequalities (2.1) and*

$$\theta \equiv K \sup_{t \geq 0} \int_t^{t+h} \|B(s)\|_{C^n} ds < 1$$

be fulfilled. Then for any solution $x(t)$ of (1.3) with a continuous initial function $\Phi(t)$, the estimate

$$\|x(t)\|_{C^n} \leq (1-\theta)^{-1} c(\Phi) \exp\left[\int_0^t (\alpha(s) + \eta(s)) ds\right], \quad t \geq h,$$

is valid with the notation

$$\eta(s) = K(1-\theta)^{-1} \|B(s+h)\|_{C^n} \int_s^{s+h} \|A(\tau) + B(\tau+h)\|_{C^n} e^{\int_\tau^s \alpha(\tau_1) d\tau_1} d\tau$$

and

$$c(\Phi) = K\left[\|\Phi(0)\|_{C^n} + \int_0^h e^{-\int_0^s \alpha(\tau) d\tau} \|B(s)\Phi(s-h)\|_{C^n} ds\right].$$

Proof. For the sake of simplicity, throughout this proof, we set $\|\cdot\|_{C^n} = |\cdot|$. Rewrite (1.3) in the form

$$\dot{x}(t) = \tilde{A}(t)x(t) + B(t)x(t-h) - B(t+h)x(t), \quad t \geq 0.$$

This equation is equivalent to the following:

$$x(t) = U(t,0)x(0) + \int_0^t U(t,s)\big[B(s)x(s-h) - B(s+h)x(s)\big]ds, \quad t \geq 0.$$

Hence,

$$x(t) = U(t,0)x(0) + \int_0^{t-h} \big[U(t,s+h) - U(t,s)\big]B(s+h)x(s)ds$$

$$- \int_{t-h}^t U(t,s)B(s+h)x(s)ds + f_1(t), \qquad t \geq h, \tag{2.2}$$

where

$$f_1(t) = \int_0^h U(t,s)B(s)\Phi(s-h)ds.$$

By (2.1), we get

$$\left|\int_{t-h}^t U(t,s)B(s+h)x(s)ds\right| \leq K \int_{t-h}^t e^{\int_s^t \alpha(\tau)d\tau}\big|B(s+h)\big|\,\big|x(s)\big|\,ds. \tag{2.3}$$

Further, it can be written,

$$\tilde{J}(t) \equiv \left|\int_0^{t-h}\big[U(t,s+h) - U(t,s)\big]B(s+h)x(s)\,ds\right|$$

$$\leq \int_0^{t-h}\big|U(t,s+h) - U(t,s)\big|\,\big|B(s+h)\big|\,\big|x(s)\big|\,ds.$$

Take into account that

$$\frac{\partial}{\partial s}U(t,s) = -U(t,s)\tilde{A}(s)$$

(cf. Tanabe [5, p. 89]). Then, by (2.1),

$$|U(t,s+h) - U(t,s)| \leq \int_s^{s+h} |U(t,\tau)\tilde{A}(\tau)|d\tau$$

$$\leq e^{\int_s^t \alpha(\tau_1)d\tau_1} K \int_s^{s+h} e^{\int_\tau^s \alpha(\tau_1)d\tau_1}|\tilde{A}(\tau)|d\tau. \tag{2.4}$$

Therefore,

$$\tilde{J}(t) \leq \int_0^{t-h} e^{\int_s^t \alpha(\tau)d\tau}w(s)|x(s)|\,ds \tag{2.5}$$

where

$$w(s) = K|B(s+h)| \int_s^{s+h} e^{\int_\tau^s \alpha(\tau_1)d\tau_1}|\tilde{A}(\tau)|d\tau.$$

In addition, thanks to (2.1),

$$|U(t,0)x(0) + f_1(t)| \leq K\Big[e^{\int_0^t \alpha(\tau)d\tau}|x(0)| + \int_0^h e^{\int_s^t \alpha(\tau)d\tau}|B(s)\Phi(s-h)|ds\Big]$$

$$= e^{\int_0^t \alpha(\tau)d\tau}c(\Phi).$$

Now (2.2)–(2.4) imply

$$|x(t)| \leq e^{\int_0^t \alpha(\tau)d\tau} c(\Phi) + \int_0^{t-h} e^{\int_s^t \alpha(\tau)d\tau} w(s)|x(s)|ds$$

$$+ K \int_{t-h}^t e^{\int_s^t \alpha(\tau)d\tau} |B(s+h)| \, |x(s)| \, ds.$$

Put

$$y(t) = |x(t)| e^{-\int_0^t \alpha(\tau)d\tau}. \tag{2.6}$$

Clearly,

$$y(t) \leq c(\Phi) + K \int_{t-h}^t |B(s+h)| \, y(s) \, ds + \int_0^{t-h} w(s) \, y(s) \, ds.$$

Denote

$$\zeta(t) = \max_{0 \leq s \leq t} y(s).$$

Then, for a fixed positive $T < \infty$,

$$\zeta(T) \leq c(\Phi) + \theta\zeta(T) + \int_0^{T-h} w(s)\zeta(s) \, ds.$$

Under the condition $\theta < 1$, this gives

$$\zeta(T) \leq (1-\theta)^{-1}\left[c(\Phi) + \int_0^T w(s)\zeta(s) \, ds\right].$$

Replacing T by t and applying the Gronwall inequality, we get

$$\zeta(t) \leq (1-\theta)^{-1}c(\Phi) \, \exp\left[(1-\theta)^{-1}\int_0^t w(s)ds\right].$$

Since $y(t) \leq \zeta(t)$, due to (2.5), we arrive at the required estimate.   □

Proof of Theorem 1.1. The result is due to (1.2) and Lemma 2.1.   □

3. Stability conditions

Theorem 1.1 yields

Corollary 3.1. *Let the inequalities* (1.5) *and*

$$\varlimsup_{t\to\infty} \int_0^t \left[\alpha_W(s) + \eta_W(s)\right] ds < \infty \tag{3.1}$$

be fulfilled. Then equation (1.3) *is stable.*

Let us assume that under the condition $\alpha_W(t) \leq 0$, $t \geq 0$, the inequalities

$$J_{A+B} \equiv \sup_{t\geq 0} \int_t^{t+h} \|A(s) + B(s+h)\|_{C^n} ds < \infty,$$

$$\Lambda_W \equiv \sup_{t\geq 0} \int_t^{t+h} |\alpha_W(s)|ds < \infty \tag{3.2}$$

are fulfilled. Then clearly,

$$\int_0^t \eta_W(s)ds \leq J_{A+B}(1 - J_B)^{-1}e^{\Lambda_W} \int_0^t \|B(s+h)\|_{C^n} \, ds.$$

Further, take into account that

$$\varlimsup_{t\to\infty} t^{-1}\int_0^t \|B(s+h)\|_{C^n}ds \le \varlimsup_{j\to\infty}(hj)^{-1}\int_0^{(j+1)h}\|B(s+h)\|_{C^n}ds$$

$$\le \varlimsup_{j\to\infty}(hj)^{-1}\sum_{k=0}^{j}\int_{kh}^{(k+1)h}\|B(s+h)\|_{C^n}ds \le h^{-1}J_B.$$

Now Corollary 3.1 implies

Corollary 3.2. *Under conditions* (1.5) *and* (3.2), *let the inequality*

$$\varlimsup_{t\to\infty} t^{-1}\int_0^t \alpha_W(s)ds + \frac{e^{\Lambda w}J_B J_{A+B}}{h(1-J_B)} \le 0$$

be fulfilled. Then equation (1.3) *is stable.*

If the inequality

$$\int_{t_0}^t \left[\alpha(B(s+h))+\eta_W(s)\right]ds \le \int_{t_0}^t \|B(s)\|_{C^n}\,ds \qquad t\ge t_0$$

holds, then Corollary 3.1 under (1.5) makes the following stability conditions more precise:

$$\int_{t_0}^t \left[\alpha(A(s))+\|B(s)\|_{C^n}\right]ds \le 0, \qquad t\ge t_0,$$

from Kolmanovski [3]. Consider now the scalar equation

$$\dot{x}(t) = -b(t)x(t-h) \tag{3.3}$$

with a continuous positive function $b(t)$. Denote by z_0 the unique real root of the equation $z(1+e^z)=1$. Clearly, $0<z_0<1$. Assume that

$$J_B = \sup_{t\ge 0}\int_t^{t+h} b(s)ds < z_0. \tag{3.4}$$

Then $J_B < 1$, and $J_B(1+e^{J_B}) < 1$, since $z(1+e^z)$ increases as $z>0$ increases. Therefore,

$$(1-J_B)^{-1}J_B e^{J_B} < 1. \tag{3.5}$$

On the other hand,

$$\eta_W(s) = (1-J_B)^{-1}b(s+h)\int_s^{s+h} b(\tau+h)e^{-\int_\tau^s b(\tau_1+h)d\tau_1}d\tau$$

$$\le (1-J_B)^{-1}b(s+h)J_B e^{J_B}.$$

Relation (3.5) implies

$$\int_0^t \left[\eta_W(s)+\alpha_W(s)\right]ds \le \int_0^t \left[-b(s+h)+J_B e^{J_B}(1-J_B)^{-1}b(s+h)\right]ds$$

$$\le \left(-1+J_B e^{J_B}(1-J_B)^{-1}\right)\int_0^t b(s+h)\,ds < 0, \qquad t\ge 0.$$

So condition (3.1) holds, and the following result is true:

Corollary 3.3. *Let inequality* (3.4) *be fulfilled. Then equation* (3.3) *is stable.*

This result supplements the "3/2-stability theorem" (Yoneyama [8]), which gives for equation (3.3) the following stability condition:

$$h \sup_t b(t) \leq \frac{3}{2}.$$

Finally, note that Theorem 1.1 can be easily extended to the equation

$$\dot{x}(t) = \sum_{k=0}^{m} A_k(t)x(t - h_k) \qquad t \geq 0$$

where $A_k(t)$, $k = 0, \ldots, m$, are variable matrices, and $h_0, \ldots, h_m$ are positive numbers.

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