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RATIONAL INTERPOLATION OF FUNCTIONS ON THE UNIT CIRCLE

Xin Li

Dedicated to Dick Askey on the occasion of his sixty-fifth birthday

ABSTRACT. We establish various results on the convergence of a sequence of rational functions that interpolate a function on the unit circle. In particular, we extend Walsh's equi-convergence theorem and the Walsh-Sharma theorem on L_2 convergence (a special case of a theorem of Lozinski) for the interpolating rational functions.

1. Introduction

Let $\mathbf{T}$ be the unit circle in the complex plane $\mathbf{C}$, and let $\mathbf{D}$ and $\overline{\mathbf{D}}$ denote the open and closed unit disks, respectively. Also, let $z_{n,k} = e^{i\theta_{n,k}}$, $0 \le \theta_{n,0} < \theta_{n,1} < \cdots < \theta_{n,n} < 2\pi$, and for a function f defined on $\mathbf{T}$, let $L_n(f;\cdot) \in \mathcal{P}_n$, the set of polynomials of degree at most n, such that $L_n(f; z_{n,k}) = f(z_{n,k})$ for $k = 0, 1, \ldots, n$. As early as in 1884, Méray showed that if $\{z_{n,k} : k = 0, 1, \ldots, n\}$ are the $n+1$st roots of unity, i.e., $\theta_{n,k} = 2k\pi/(n+1)$, $k = 0, 1, \ldots, n$, then the polynomials $L_n(f;\cdot)$ do not necessarily converge to f. Indeed, Méray took the function $f_M(z) = 1/z$. So, $L_n(f_M; z) = z^n$, $n = 1, 2, 3, \ldots$. These polynomials do not approach the function $f_M(z)$ for all $z \in \mathbf{D}$, as $n \to \infty$, but approach the limit zero. Méray's example was examined later by Walsh [24] who pointed out that the singularity of $f_M(z) = 1/z$ at $z = 0$ caused the sequence $\{L_n(f_M;\cdot)\}$ to fail to approach $f_M(z)$ for $z \in \mathbf{D}$. Walsh also furnished the following lovely companion to Méray's example.

Walsh's Theorem (1932, [24, Theorem 1]). *Let f be continuous (or more generally Riemann integrable) on $\mathbf{T}$, and let $\{z_{n,k} : k = 0, 1, \ldots, n\}$ be the $n+1$st roots of unity. Then the sequence $\{L_n(f;\cdot)\}$ converges to the limit*

$$f_{\mathbf{D}}(z) = \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{f(t)dt}{t-z}$$

for $z \in \mathbf{D}$, uniformly on any compact subset of $\mathbf{D}$.

It is worthwhile to recall some steps in the historical development prior to Walsh's Theorem. Runge (1904, [19]) proved that if $\{z_{n,k} : k = 0, 1, \ldots, n\}$ are the $n+1$st roots of unity and if f is analytic on $\overline{\mathbf{D}}$, then the sequence $\{L_n(f;z)\}$ converges to $f(z)$ on $\overline{\mathbf{D}}$. After that, Fejér (1918, [9]) showed that $\{L_n(f;z)\}$ converges to $f(z)$ for $z \in \mathbf{D}$ if $f(z)$ is assumed only to be continuous on $\overline{\mathbf{D}}$ and analytic in $\mathbf{D}$. See also [11].

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There were further developments after Walsh's Theorem. When the points of interpolation are the roots of unity, Lozinski (1940, [15]) proved, under Fejér's assumption, the L_p $(p > 0)$ convergence of $\{L_n(f;\cdot)\}$ on the unit circle, i.e.,

$$\lim_{n\to\infty} \int_{\mathbf{T}} |f(z) - L_n(f;z)|^p \, |dz| = 0, \qquad p > 0. \tag{1}$$

This result, when $p = 2$, was rediscovered later by Walsh and Sharma (1964, [26]). Some more recent results on interpolation at the roots of unity can be found in the work of Saff and Walsh [21], Cavaretta, et al. [6, 7], and Sharma and Vertesi [22].

Walsh gave an example to show that when the points of interpolation $\{z_{n,k}\}$ are not the roots of unity, $\{L_n(f;z)\}$ may fail to converge in $\mathbf{D}$ even though $\{z_{n,k}\}$ are sufficiently dense on $\mathbf{T}$.

Walsh's Example ([25, pp. 293–294]). Let $z_{n,k}$ be the roots of

$$\left(\frac{1-\alpha z}{\alpha - z}\right)^{n+1} = 1, \qquad \alpha > 1.$$

Let $f_W(z) = 1/(\zeta + \beta)$, $0 < \beta < 1$, where $\zeta = (1-\alpha z)/(\alpha - z)$. Then

$$L_n(f_W; z) = \frac{1}{\zeta + \beta} + \frac{(\alpha+\beta)^n(\zeta^{n+1} - 1)}{[(-1)^n + \beta^{n+1}](\zeta+\beta)(\zeta-\alpha)^n},$$

which converges for z in only a part of the unit disk $\overline{\mathbf{D}}$.

Walsh's example shows that the distribution of the points of interpolation $\{z_{n,k}\}$ on $\mathbf{T}$ should be taken into account when we study the convergence of interpolating polynomials $\{L_n(f;\cdot)\}$. Closely related to this observation, the following result is now well known (cf. [25, Chapter 7]).

The Uniform Distribution Theorem. *A necessary and sufficient condition for*

$$\lim_{n\to\infty} L_n(f;z) = f(z),$$

uniformly for $z \in \overline{\mathbf{D}}$, for every f analytic on $\overline{\mathbf{D}}$, is that the sequence of points $\{z_{n,k} : k = 0, 1, \ldots, n\}$ is uniformly distributed on $\mathbf{T}$, i.e.,

$$\lim_{n\to\infty} \left| \prod_{k=0}^{n} (z - z_{n,k}) \right|^{1/n} = |z|, \qquad |z| > 1.$$

Now, let us take a closer look at the points of interpolation $\{z_{n,k}\}$ in Walsh's example, which satisfy, for $|z| > 1$,

$$\lim_{n\to\infty} \left| \prod_{k=0}^{n} (z - z_{n,k}) \right|^{1/n} = \lim_{n\to\infty} \left| \frac{(1-\alpha z)^{n+1} - (\alpha - z)^{n+1}}{(-\alpha)^{n+1} - (-1)^{n+1}} \right|^{1/n} = \left| z - \frac{1}{\alpha} \right|.$$

They are not uniformly distributed unless $\alpha = \infty$. How do we recover a function f analytic on $\overline{\mathbf{D}}$ if we are given the values of f at the points $\{z_{n,k}\}$ as in Walsh's example? According to the Uniform Distribution Theorem, the interpolating polynomials do not necessarily converge to f on $\overline{\mathbf{D}}$; we must use interpolating functions that are more general than polynomials. Indeed, in Walsh's example, we just need to use rational functions of the form

$$\frac{p_n(z)}{(\alpha - z)^{n+1}}, \qquad p_n \in \mathcal{P}_n$$

that interpolate f at $\{z_{n,k} : k = 0, 1, \ldots, n\}$. Then, by a simple transformation $z = (1-\alpha\zeta)/(\alpha-\zeta)$, Runge's result mentioned earlier would imply that this sequence of interpolating rational functions converges to f on $\overline{\mathbf{D}}$. It is the purpose of this paper to explore this direction further and hence extend Walsh's Theorem to interpolation on the unit circle $\mathbf{T}$ by rational functions.

2. More Notation

Let α_n, $n = 0, 1, \ldots,$ be given (not necessarily distinct) points in $\mathbf{D}$, and let

$$w_n(z) := \prod_{k=0}^{n} (z - \alpha_k) \quad \text{and} \quad w_n^*(z) := \prod_{k=0}^{n} (1 - \overline{\alpha_k} z).$$

We now define the space of rational functions for each n. Let

$$\mathcal{R}_n := \left\{ \frac{p(z)}{w_n^*(z)} : p \in \mathcal{P}_n \right\},$$

and let

$$\mathcal{R}_n^* := \left\{ \overline{r(1/\bar{z})} : r \in \mathcal{R}_n \right\}.$$

Then we see that, when we restrict z to $\mathbf{T}$, we have $\mathcal{R}_n^* = \{\bar{r} : r \in \mathcal{R}_n\}$. In general, an element in $\mathcal{R}_n^*$ has the form

$$\frac{zp(z)}{w_n(z)} \qquad \text{for some } p \in \mathcal{P}_n.$$

Define

$$\mathcal{R} = \bigcup_{n=0}^{\infty} \mathcal{R}_n \qquad \text{and} \qquad \mathcal{R}^* = \bigcup_{n=0}^{\infty} \mathcal{R}_n^* .$$

We will write $B_n(z) = w_n(z)/w_n^*(z)$. From now on, we will let the points of interpolation $\{z_{n,k} : k = 0, 1, \ldots, n\}$ be given as the roots of

$$w_n(z) = w_n^*(z) \qquad \text{or, equivalently,} \qquad B_n(z) = 1.$$

It is easy to see that $z_{n,k} \in \mathbf{T}$ for all $k = 0, 1, \ldots, n$ and $n = 1, 2, \ldots$. Moreover, the $z_{n,k}$s are distinct as implied by formula (9) to be given later. For a function f defined on $\mathbf{T}$, let $R_n(f; \cdot)$ denote the unique rational function from $\mathcal{R}_n$ that interpolates f at $\{z_{n,k} : k = 0, 1, \ldots, n\}$. Then, it is easy to verify that

$$R_n(f; z) = \sum_{k=0}^{n} \frac{f(z_{n,k})(B_n(z) - 1)}{B_n'(z_{n,k})(z - z_{n,k})}.$$

We will use $C(\mathbf{T})$ to denote the space of functions continuous on $\mathbf{T}$ equipped with the sup-norm on $\mathbf{T}$. Let $A(\mathbf{D})$ denote the disk algebra, that is, the set of functions continuous on $\overline{\mathbf{D}}$ and analytic in $\mathbf{D}$.

Finally, let

$$\sigma_n := \sum_{k=0}^{n} (1 - |\alpha_k|) \qquad \text{and} \qquad \delta_n := \min_{0 \le k \le n} (1 - |\alpha_k|).$$

3. Main results

We will establish various convergence results of the sequence $\{R_n(f;\cdot)\}$ according to different assumptions on f. First, we give a simple and natural extension of Walsh's Theorem to rational interpolation.

Theorem 1. *Assume that* $\lim_{n\to\infty} \sigma_n = \infty$. *If* $f \in C(\mathbf{T})$, *then*

$$\lim_{n\to\infty} R_n(f;z) = f_{\mathbf{D}}(z)$$

for $z \in \mathbf{D}$, *and the convergence is uniform on any compact subset of* $\mathbf{D}$.

We remark that the requirement $\lim_{n\to\infty} \sigma_n = \infty$ is equivalent to saying $\lim_{n\to\infty} B_n(z) = 0$ for $z \in \mathbf{D}$ (see, e.g., [25, §10.1]). It is also a necessary and sufficient condition for the denseness of certain rational functions in $C(\mathbf{T})$ (cf.[1, 12]); see Lemma 3 for a more precise statement.

Our proof of Theorem 1 depends on the following result of weak-star convergence of a sequence of discrete measures supported at the points of interpolation. Let $d\delta_z$ denote the unit measure supported at the single point z. For $n = 0, 1, 2, \ldots$, define

$$d\nu_n = \sum_{k=0}^{n} \frac{d\delta_{z_{n,k}}}{|B_n'(z_{n,k})|}.$$

Theorem 2. *If* $\lim_{n\to\infty} \sigma_n = \infty$, *then* $\{d\nu_n\}$ *converges in the weak-star topology to the uniform measure on* $\mathbf{T}$, $d\theta/(2\pi)$. *More precisely,*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f(z_{n,k})}{|B_n'(z_{n,k})|} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta, \tag{2}$$

for every $f \in C(\mathbf{T})$.

Note that when $\alpha_n = 0$ for all $n = 0, 1, 2, \ldots$, we have $B_n(z) = z^{n+1}$ and $z_{n,k}$ are the $n+1$st roots of unity, and $|B_n'(z)| = n+1$ for $z \in \mathbf{T}$. So, equation (2) becomes the well-known equality

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f(e^{2i\pi/(n+1)})}{n+1} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

In general, one should compare Theorem 2 with the quadrature formulas for the Poisson integrals as obtained recently by Bultheel et al. [3]. It is essentially (though not exactly) equivalent to the convergence result in [3] (via suitable transformations and modifications). We will give a proof of Theorem 2 that is independent of [3] to indicate a different and direct approach.

Walsh's Theorem holds for all f that are Riemann integrable on $\mathbf{T}$, while in Theorem 1 we only managed to obtain the case when f is assumed continuous on $\mathbf{T}$. With more restrictive conditions on the rate in which the poles are allowed to approach the unit circle $\mathbf{T}$, we do have an extension of Walsh's theorem for all Riemann integrable functions as follows.

Theorem 3. *Assume that*

$$\lim_{n\to\infty} \sigma_n \delta_n = \infty. \tag{3}$$

If f is Riemann integrable on $\mathbf{T}$, *then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f(z_{n,k})}{|B_n'(z_{n,k})|} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta \tag{4}$$

and

$$\lim_{n\to\infty} R_n(f;z) = f_{\mathbf{D}}(z) \tag{5}$$

for $z \in \mathbf{D}$, and the convergence is uniform on any compact subset of $\mathbf{D}$.

Condition (3) still allows α_n to approach $\mathbf{T}$ but is more restrictive on its rate of convergence. For example, $\alpha_n = 1 - 1/n^\delta$, $n = 1, 2, \ldots$, with $\delta \in [0, 1/2)$ satisfy (3). Since $\sigma_n > \sigma_n \delta_n$, (3) implies $\lim_{n\to\infty} \sigma_n = \infty$, which is what we assumed in Theorem 1. It is not clear whether (3) can be replaced by the weaker condition $\lim_{n\to\infty} \sigma_n = \infty$ in the assumption of Theorem 3.

Next, we present the L_2 convergence of $\{R_n(f;\cdot)\}$ for $f \in A(\mathbf{D})$, which generalizes the result of Walsh and Sharma [26] and that of Lozinski [15] for $p = 2$ in (1).

Theorem 4. *If* $\lim_{n\to\infty} \sigma_n = \infty$, *then*

$$\lim_{n\to\infty} \int_{\mathbf{T}} |R_n(f;z) - f(z)|^2\, |dz| = 0,$$

for every $f \in A(\mathbf{D})$.

Similar to the polynomial case, this implies a rational version of Fejer's result mentioned in the Introduction.

The truths of Theorems 1 and 4 not only reveal a clear analogy between polynomial and rational interpolations on the unit circle $\mathbf{T}$, but also implicitly indicate how the extension from polynomial cases to rational cases can be carried out. For example, the well-known result of Walsh, the equi-convergence theorem [25, Theorem 1, p. 153], is extended for rational interpolation in the following result.

Theorem 5. *Assume that the set $\{\alpha_0, \alpha_1, \ldots\}$ has no limit point on $\mathbf{T}$ and its closure* $\mathrm{cls}(\{\alpha_0, \alpha_1, \ldots\})$ *does not separate the plane* $\mathbf{C}$. *If*

$$\lim_{n\to\infty} |B_n(z)|^{1/n} = \psi(z) \not\equiv \textit{constant} \tag{6}$$

holds locally uniformly on $\mathbf{C} \setminus (\mathbf{D} \cup \overline{\{1/\overline{\alpha_0}, 1/\overline{\alpha_1}, \ldots\}})$, *then*

$$\lim_{n\to\infty} [R_n(f;z) - r_n(f;z)] = 0 \qquad \textit{for} \quad \psi(z) < \rho^2$$

where f is assumed analytic for $\psi(z) < \rho$ ($\rho > 1$) and $r_n(f;z)$ is the least-square approximation to f out of $\mathcal{R}_n$, i.e.,

$$\int_{\mathbf{T}} |f(z) - r_n(f;z)|^2 |dz| = \min_{r \in \mathcal{R}_n} \int_{\mathbf{T}} |f(z) - r(z)|^2 |dz|.$$

This result is analogous to one proved by Saff and Sharma in [20] for a different rational system in which poles are the roots of $z^n - r^n = 0$ for some $r > 1$ (and so the poles are not from a single sequence as we have assumed in this paper).

In the assumption of Theorem 5, the required range of z implied by (6) can be made smaller. We have used the stronger assumption for simplicity. When $\{\alpha_n\}_{n=0}^{\infty}$ is cyclic, i.e.,

$$\alpha_{km+r} = \alpha_r, \qquad r = 0, 1, \ldots, m-1, \quad k = 1, 2, \ldots,$$

for some $m \geq 1$, then (6) is satisfied as

$$\lim_{n\to\infty} |B_n(z)|^{1/n} = \left| \prod_{k=0}^{m-1} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z} \right|^{1/m}.$$

We refer to Walsh's discussion in [25, §§9.4, 9.5, in particular, p. 236] for more properties of $\psi(z)$. Here we only mention the following fact that will be needed in our proof: The locus $\gamma(\rho_1) := \{z : \psi(z) = \rho_1\}$ is rectifiable and, for $1 < \rho_1 < \rho$, $\gamma(\rho_1)$, lies in $\{z : \psi(z) < \rho\}$ and is exterior to $\mathbf{T}$.

Note that for f analytic on $\overline{\mathbf{D}}$, the least square approximation to f out of $\mathcal{P}_n$ is the same as the nth Taylor polynomial of f about $z = 0$. As all our theorems imply the corresponding known results for polynomial interpolation, Theorem 5 will be given the following form when we let $\alpha_n = 0$ for all $n = 0, 1, 2, \ldots$.

Corollary 6. *If f is analytic for $|z| < \rho$ ($\rho > 1$), then*

$$\lim_{n\to\infty} [L_n(f; z) - s_n(f; z)] = 0 \qquad \text{for} \quad |z| < \rho^2$$

where $L_n(f; z)$ is the polynomial interpolating f at the $n+1$st roots of unity, and $s_n(f; z)$ is the Taylor polynomial of degree n of f about $z = 0$.

This is the celebrated Walsh's equi-convergence theorem mentioned above. This theorem has been much extended in recent years; for the latest results, see [4, 5, 17, 18] and the bibliographies therein.

4. Auxiliary results

We collect some known results and prove some new ones that are needed in the proofs of the theorems.

Lemma 1. *For $k = 0, 1, \ldots, n$, we have $z_{n,k} B_n'(z_{n,k}) = |B_n'(z_{n,k})|$.*

Proof. A straightforward calculation (cf. (i) of [14, Lemma 1]) shows that

$$\frac{z B_n'(z)}{B_n(z)} = |B_n'(z)| \qquad \text{for all} \quad z \in \mathbf{T}.$$

Letting $z = z_{n,k}$ yields the desired equality. □

Lemma 2. *For $n = 0, 1, 2, \ldots$, we have*

$$\sum_{k=0}^{n} \frac{1}{|B_n'(z_{n,k})|} = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2}.$$

Proof. Let $f_I(z) := 1 - \overline{B_n(0)}B_n(z)$. Then $f_I \in \mathcal{R}_n$ and

$$f(z_{n,k}) = 1 - \overline{B_n(0)}B_n(z_{n,k}) = 1 - \overline{B_n(0)}, \qquad k = 0, 1, \ldots, n\ .$$

So, by the uniqueness of the interpolating rational function,

$$f_I(z) = R_n(f_I; z) = \sum_{k=0}^{n} \frac{(1 - \overline{B_n(0)}\,)(B_n(z) - 1)}{B_n'(z_{n,k})(z - z_{n,k})}.$$

Hence, we have the identity

$$\sum_{k=0}^{n} \frac{1}{B_n'(z_{n,k})(z - z_{n,k})} = \frac{1 - \overline{B_n(0)}B_n(z)}{(1 - \overline{B_n(0)}\,)(B_n(z) - 1)}.$$

Letting $z = 0$ gives us

$$\sum_{k=0}^{n} \frac{1}{z_{n,k}B_n'(z_{n,k})} = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2},$$

which, by Lemma 1, is the identity we need to verify. □

Lemma 3. *The linear span of* $\mathcal{R} \cup \mathcal{R}^*$ *is dense in* $C(\mathbf{T})$ *if and only if* $\lim_{n\to\infty} \sigma_n = \infty$.

Proof. This follows from an application of a result in [1, Addendum A, §2, p. 244]. It can be proved by the same method. □

Lemma 4. *For* $k = 0, 1, \ldots, n-1$, *there exist numbers* $\xi_k, \zeta_k \in (\theta_{n,k}, \theta_{n,k+1})$ *such that*

$$\theta_{n,k+1} - \theta_{n,k} = \frac{2\pi}{|B_n'(e^{i\xi_k})|} \tag{7}$$

and

$$2\pi - (\theta_{n,k+1} - \theta_{n,k})|B_n'(z_{n,k})| = \frac{2\pi\gamma_n''(\zeta_k)(\theta_{n,k+1} - \zeta_k)}{\gamma_n'(\zeta_k)} \tag{8}$$

where $\gamma_n(\theta)$ *is defined as a continuous function satisfying*

$$B_n(e^{i\theta}) = e^{i\gamma_n(\theta)}, \qquad \theta \in [0, 2\pi].$$

It turns out that, such a function γ_n is uniquely determined (up to an additional constant) and continuously differentiable on $[0, 2\pi]$. Indeed, we have (see, for example, [13])

$$\gamma_n'(\theta) = |B_n'(e^{i\theta})| = \sum_{k=0}^{n} \frac{1 - |\alpha_k|^2}{|e^{i\theta} - \alpha_k|^2} = \sum_{k=0}^{n} \frac{1 - r_k^2}{1 - 2r_k\cos(\theta - \omega_k) + r_k^2} \tag{9}$$

where $\alpha_k = r_k e^{i\omega_k}$ with $r_k > 0$, $k = 0, 1, \ldots$. From (9), we can estimate γ' as follows:

$$\gamma'(\theta) = |B_n'(e^{i\theta})| \geq \sum_{k=0}^{n} \frac{1 - r_k}{1 + r_k} > \frac{1}{2}\sum_{k=0}^{n}(1 - r_k) = \frac{1}{2}\sigma_n. \tag{10}$$

Proof of Lemma 4. Note that $\gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k}) = 2\pi$. Now, by the mean value theorem, we can write, for some $\xi_k \in (\theta_{n,k}, \theta_{n,k+1})$,

$$2\pi = \gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k}) = \gamma_n'(\xi_k)(\theta_{n,k+1} - \theta_{n,k}),$$

which, by (9), implies (7).

To verify (8), let

$$g(\theta) := -\gamma_n(\theta_{n,k+1}) + \gamma_n(\theta) + (\theta_{n,k+1} - \theta)\gamma_n'(\theta).$$

Then,

$$g(\theta_{n,k+1}) - g(\theta_{n,k}) = 2\pi - (\theta_{n,k+1} - \theta_{n,k})|B_n'(z_{n,k})|$$

and

$$g'(\theta) = \gamma_n''(\theta)(\theta_{n,k+1} - \theta).$$

Then, by the generalized Cauchy mean value theorem, there exists a number $\zeta_k \in (\theta_{n,k}, \theta_{n,k+1})$ such that

$$\frac{g(\theta_{n,k+1}) - g(\theta_{n,k})}{\gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k})} = \frac{g'(\zeta_k)}{\gamma_n'(\zeta_k)}.$$

Now, equation (8) follows. □

Lemma 5. *For $\zeta, z \in \mathbf{C}$, we have*

$$\sum_{j=0}^{n} \frac{B_n(z)}{B_n'(\alpha_j)(z-\alpha_j)(1-\overline{\zeta}\alpha_j)} = \begin{cases} \dfrac{zB_n'(z)}{B_n(z)}, & \text{if } z = \dfrac{1}{\overline{\zeta}}, \\[2ex] \dfrac{1-\overline{B_n(\zeta)}B_n(z)}{1-\overline{\zeta}z}, & \text{otherwise.} \end{cases}$$

Proof. First, assume $\zeta \in \mathbf{D}$. Then, note that

$$\frac{1-\overline{B_n(\zeta)}B_n(z)}{1-\overline{\zeta}z}$$

is the rational function from $\mathcal{R}_n$ that interpolates $f_\zeta(z) = 1/(1-\overline{\zeta}z)$ at $z = \alpha_0, \alpha_1, \ldots, \alpha_n$. Thus, by Lagrange's interpolation formula, we have

$$\frac{1-\overline{B_n(\zeta)}B_n(z)}{1-\overline{\zeta}z} = \sum_{k=0}^{n} \frac{B_n(z)}{B_n'(z_{n,k})(z-\alpha_k)(1-\overline{\zeta}\alpha_k)}. \tag{11}$$

Observe that both sides of (11) are rational functions of the same type in z and, after taking the complex conjugate of both sides in (11), in ζ. So, in general, the equality in (11) holds as long as $\overline{\zeta}z \neq 1$.

Finally, the case when $z = 1/\overline{\zeta}$ can be handled by first taking the limit as $z \to 1/\overline{\zeta}$ in (11) and then write ζ as $1/\overline{z}$. □

We will need the system of functions from $\mathcal{R}$ given by

$$\varphi_n(z) = \frac{\sqrt{1-|\alpha_n|^2}}{z-\alpha_n}B_n(z), \qquad n = 0, 1, 2, \ldots.$$

This system was probably introduced (with a constant multiplier of modulus 1) first by Takenaka in [23] and Malmquist in [16]. See also [8] and [25, p. 224]. It is orthogonal with respect to the uniform measure $d\theta/(2\pi)$ on the unit circle $\mathbf{T}$ (with $z = e^{i\theta}$). We collect some properties of $\{\varphi_n\}$ in the following lemma.

Lemma 6. (i) *For* $n = 0, 1, \ldots$, $\{\varphi_k(z)\}_{k=0}^n \subseteq \mathcal{R}_n$ *and,* $j, k = 0, 1, 2, \ldots$,

$$\frac{1}{2\pi} \int_{z\in \mathbf{T}} \varphi_j(z)\overline{\varphi_k(z)}\,|dz| = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases} \tag{12}$$

(ii) (*Christoffel-Darboux formula for* $\{\varphi_n\}$) *For all* $\zeta, z \in \mathbf{C}$, *we have*

$$\sum_{k=0}^{n} \overline{\varphi_k(\zeta)}\varphi_k(z) = \frac{1 - \overline{B_n(\zeta)}B_n(z)}{1 - \overline{\zeta}z}. \tag{13}$$

Proof. A proof of equation (12) can be found in [25, p. 227]. The identity (13) is well known in the literature (see [8]). It is a special case of the more general Christoffel-Darboux formula established for rational functions orthogonal on the unit circle. See, for example, [2]. □

Our next result is interesting in its own right. It generalizes a relation first observed by Walsh and Sharma [26, Formula (16)] on polynomials.

Lemma 7. *For* $p, q = 0, 1, \ldots, n$, *we have*

$$\frac{1}{2\pi}\int_{\mathbf{T}} \frac{B_n(z)-1}{z-z_{n,p}} \overline{\left(\frac{B_n(z)-1}{z-z_{n,q}}\right)}\,|dz| = \begin{cases} 0, & p \neq q, \\ |B_n'(z_{n,p})|, & p = q. \end{cases}$$

Proof. Let

$$I(p,q) := \frac{1}{2\pi}\int_{\mathbf{T}} \frac{B_n(z)-1}{z-z_{n,p}} \overline{\left(\frac{B_n(z)-1}{z-z_{n,q}}\right)}\,|dz|.$$

We claim that

$$I(p,q) = z_{n,q}\overline{z_{n,p}}\,\overline{\sum_{j=0}^{n} \frac{B_n(z_{n,p})}{B_n'(\alpha_j)(z_{n,p}-\alpha_j)(1-\overline{z_{n,q}}\alpha_j)}}. \tag{14}$$

Assuming the truth of (14), we see that Lemma 5 implies $I(p,q) = 0$ when $p \neq q$, and Lemmas 1 and 5 infer that $I(p,q) = |B_n'(z_{n,p})|$. Therefore, the lemma follows from (14).

Now, let us verify (14). We need the orthonormal basis $\{\varphi_k\}_{k=0}^n$ in $\mathcal{R}_n$ introduced above. Since

$$\frac{B_n(z)-1}{z-z_{n,p}}, \qquad \frac{B_n(z)-1}{z-z_{n,q}} \in \mathcal{R}_n,$$

we can write

$$\frac{B_n(z)-1}{z-z_{n,p}} = \sum_{k=0}^{n} a_k\varphi_k(z) \qquad \text{and} \qquad \frac{B_n(z)-1}{z-z_{n,q}} = \sum_{k=0}^{n} b_k\varphi_k(z)$$

for some $\{a_k\}_{k=0}^n$ and $\{b_k\}_{k=0}^n$. Then, for $p, q = 0, 1, \ldots, n$, by (12) in Lemma 6, we have $I(p,q) = \sum_{k=0}^n a_k\overline{b_k}$. Now,

$$\begin{aligned} \overline{a_k} &= \overline{\frac{1}{2\pi}\int_{\mathbf{T}} \frac{B_n(z)-1}{z-z_{n,p}}\overline{\varphi_k(z)}\,|dz|} = \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{(1-B_n(z))\varphi_k(z)}{B_n(z)(1-\overline{z_{n,p}}z)}\,dz \\ &= \sum_{j=0}^{n} \frac{\varphi_k(\alpha_j)}{B_n'(\alpha_j)(1-\overline{z_{n,p}}\alpha_j)}, \end{aligned}$$

by the residue theorem. Similarly,

$$\overline{b_k} = \sum_{j=0}^{n} \frac{\varphi_k(\alpha_j)}{B_n'(\alpha_j)(1-\overline{z_{n,q}}\alpha_j)}.$$

Hence,

$$\begin{aligned} I(p,q) &= \sum_{k=0}^{n}\sum_{j_1=0}^{n} \frac{\overline{\varphi_k(\alpha_{j_1})}}{\overline{B_n'(\alpha_{j_1})}(1-z_{n,p}\overline{\alpha_{j_1}})} \sum_{j_2=0}^{n} \frac{\varphi_k(\alpha_{j_2})}{B_n'(\alpha_{j_2})(1-\overline{z_{n,q}}\alpha_{j_2})} \\ &= \sum_{j_1,j_2=0}^{n} \frac{1}{\overline{B_n'(\alpha_{j_1})}B_n'(\alpha_{j_2})(1-z_{n,p}\overline{\alpha_{j_1}})(1-\overline{z_{n,q}}\alpha_{j_2})} \sum_{k=0}^{n} \overline{\varphi_k(\alpha_{j_1})}\varphi_k(\alpha_{j_2}) \\ &= \sum_{j_1,j_2=0}^{n} \frac{1}{\overline{B_n'(\alpha_{j_1})}B_n'(\alpha_{j_2})(1-z_{n,p}\overline{\alpha_{j_1}})(1-\overline{z_{n,q}}\alpha_{j_2})} \frac{1-\overline{B_n(\alpha_{j_1})}B_n(\alpha_{j_2})}{1-\overline{\alpha_{j_1}}\alpha_{j_2}}, \end{aligned}$$

by (13) in Lemma 6. So,

$$\begin{aligned} I(p,q) &= \sum_{j_1,j_2=0}^{n} \frac{1}{\overline{B_n'(\alpha_{j_1})}B_n'(\alpha_{j_2})(1-z_{n,p}\overline{\alpha_{j_1}})(1-\overline{z_{n,q}}\alpha_{j_2})} \frac{1}{1-\overline{\alpha_{j_1}}\alpha_{j_2}} \\ &= \sum_{j_1=0}^{n} \frac{1}{\overline{B_n'(\alpha_{j_1})}(1-z_{n,p}\overline{\alpha_{j_1}})} \left(\sum_{j_2=0}^{n} \frac{1}{B_n'(\alpha_{j_2})(1-\overline{z_{n,q}}\alpha_{j_2})(1-\overline{\alpha_{j_1}}\alpha_{j_2})} \right). \end{aligned}$$

Now, note that

$$\sum_{j_2=0}^{n} \frac{1}{B_n'(\alpha_{j_2})(1-\overline{z_{n,q}}\alpha_{j_2})(1-\overline{\alpha_{j_1}}\alpha_{j_2})} = z_{n,q} \sum_{j_2=0}^{n} \frac{B_n(z_{n,q})}{B_n'(\alpha_{j_2})(z_{n,q}-\alpha_{j_2})(1-\overline{\alpha_{j_1}}\alpha_{j_2})},$$

which, by using Lemma 5 with $z = z_{n,q}$ and $\zeta = \alpha_{j_1}$, is equal to

$$z_{n,q} \frac{1-\overline{B_n(\alpha_{j_1})}B_n(z_{n,q})}{1-\overline{\alpha_{j_1}}z_{n,q}} = \frac{z_{n,q}}{1-\overline{\alpha_{j_1}}z_{n,q}}.$$

Thus,

$$\begin{aligned} I(p,q) &= \sum_{j_1=0}^{n} \frac{1}{\overline{B_n'(\alpha_{j_1})}(1-z_{n,p}\overline{\alpha_{j_1}})} \frac{z_{n,q}}{1-\overline{\alpha_{j_1}}z_{n,q}} \\ &= z_{n,q}\overline{z_{n,p}}\, \overline{\sum_{j=0}^{n} \frac{B_n(z_{n,p})}{B_n'(\alpha_j)(z_{n,p}-\alpha_j)(1-\overline{z_{n,q}}\alpha_j)}}, \end{aligned}$$

which is (14). This completes the proof. □

5. Proofs of the main theorems

We prove Theorem 2 first since our proof of Theorem 1 is based on it.

Proof of Theorem 2. We first claim that

$$\lim_{n\to\infty} \int_{\mathbf{T}} r\, d\nu_n = \frac{1}{2\pi} \int_{\mathbf{T}} r(z)\, |dz| \tag{15}$$

for every $r \in \mathcal{R} \cup \mathcal{R}^*$. This is a consequence of the following identity.

$$\sum_{k=0}^{n} \frac{f(z_{n,k})}{|B_n'(z_{n,k})|} = \frac{1}{2\pi(1 - B_n(0))} \int_{\mathbf{T}} f(z)\, |dz| \tag{16}$$

for every $f \in \mathcal{R}_n$. Indeed, for $f \in \mathcal{R}_n$, we have $f(z) = R_n(f; z)$. So, by Lemma 1,

$$f(0) = R_n(f; 0) = \sum_{k=0}^{n} \frac{f(z_{n,k})(1 - B_n(0))}{z_{n,k} B_n'(z_{n,k})} = \sum_{k=0}^{n} \frac{f(z_{n,k})(1 - B_n(0))}{|B_n'(z_{n,k})|}.$$

This, on writing $f(0) = \int_{\mathbf{T}} f(z)|dz|$, implies equation (16). Since $\mathcal{R}_n \subset \mathcal{R}_{n+1}$, we let $n \to \infty$ in (16) to obtain (15) for $r \in \mathcal{R}$ by using the fact that $\lim_{n\to\infty} B_n(0) = 0$. Next, by taking the complex conjugate of both sides of (15), we see that (15) holds for $f \in \mathcal{R}_n^*$ as well.

Now, from Lemma 2, it follows that

$$\int_{\mathbf{T}} d\nu_n = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2} \le \frac{2}{1 - |B_n(0)|} \le \frac{2}{1 - |\alpha_0|},$$

for $n = 0, 1, 2, \ldots$. Thus $\{d\nu_n\}$ is compact in the weak-star topology. Let $d\nu$ be any weak-star limit of $\{d\nu_n\}$ and

$$\lim_{\substack{n\to\infty \\ n\in\Gamma}} \int_{\mathbf{T}} f\, d\nu_n = \int_{\mathbf{T}} f\, d\nu,$$

for every $f \in C(\mathbf{T})$. In view of (15), this means

$$\int_{\mathbf{T}} r\, d\nu = \frac{1}{2\pi} \int_{\mathbf{T}} r(z)\, |dz|$$

for every $r \in \mathcal{R} \cup \mathcal{R}^*$. Since, by Lemma 3, the linear span of $\mathcal{R} \cup \mathcal{R}^*$ is dense in $C(\mathbf{T})$, then $d\nu(\theta) = d\theta/(2\pi)$. Therefore, the whole sequence converges to $d\theta/(2\pi)$. □

Proof of Theorem 1. Let $f \in C(\mathbf{T})$. By using Lemma 1, we can write

$$R_n(f; z) = \sum_{k=0}^{n} \frac{f(z_{n,k})(B_n(z) - 1)}{B_n'(z_{n,k})(z - z_{n,k})} = (1 - B_n(z)) \sum_{k=0}^{n} \frac{z_{n,k} f(z_{n,k})}{z_{n,k} - z} \frac{1}{|B_n'(z_{n,k})|}.$$

For each fixed $z \in \mathbf{D}$, we have $\lim_{n\to\infty} B_n(z) = 0$ and that

$$\frac{tf(t)}{t - z}$$

is continuous for $t \in [0, 2\pi]$. So, using Theorem 2, we obtain

$$\lim_{n\to\infty} R_n(f; z) = \frac{1}{2\pi} \int_{\mathbf{T}} \frac{tf(t)}{t - z}\, |dt| = \frac{1}{2\pi} \int_{\mathbf{T}} \frac{f(t)}{t - z} dt,$$

which is $f_{\mathbf{D}}(z)$. □

Proof of Theorem 3. We show that

$$\left|\frac{1}{|B_n'(z_{n,k})|} - \frac{\theta_{n,k+1}-\theta_{n,k}}{2\pi}\right| \le \frac{8\pi}{|B_n'(z_{n,k})|\sigma_n\delta_n}. \tag{17}$$

Dividing both sides of (8) by $2\pi|B_n'(z_{n,k})|$, we have

$$\left|\frac{1}{|B_n'(z_{n,k})|} - \frac{\theta_{n,k+1}-\theta_{n,k}}{2\pi}\right| = \frac{|\gamma''(\zeta_k)(\theta_{n,k+1}-\zeta_k)|}{|B_n'(z_{n,k})|\gamma'(\zeta_k)}. \tag{18}$$

Now, note that

$$\left|\frac{\gamma_n''(\zeta_k)}{\gamma_n'(\zeta_k)}\right| \le \frac{\displaystyle\sum_{j=0}^{n} c_j \frac{2r_j|\sin(\zeta_k-\omega_j)|}{1-2r_j\cos(\zeta_k-\omega_j)+r_j^2}}{\displaystyle\sum_{j=0}^{n} c_j}$$

with

$$c_j := \frac{1-r_j^2}{1-2r_j\cos(\zeta_k-\omega_j)+r_j^2}, \qquad j=0,1,\ldots,n.$$

Note that, $c_j > 0$, $j=0,1,\ldots,n$, and, for $0 \le r < 1$,

$$\max_{\theta\in[0,2\pi]} \frac{2r|\sin(\theta)|}{1-2r\cos(\theta)+r^2} = \frac{2r}{1-r^2}.$$

So, we have

$$\left|\frac{\gamma_n''(\zeta_k)}{\gamma_n'(\zeta_k)}\right| \le \max_{0\le j\le n} \frac{2r_j}{1-r_j^2} < \frac{2}{\delta_n}. \tag{19}$$

On the other hand, by (7) in Lemma 4 and (10), we have

$$|\theta_{n,k+1}-\zeta_k| \le |\theta_{n,k+1}-\theta_{n,k}| = \frac{2\pi}{|B_n'(e^{i\xi_k})|} \le \frac{2\pi}{\frac{1}{2}\sigma_n} = \frac{4\pi}{\sigma_n}. \tag{20}$$

Using (19) and (20) in (18), we obtain (17).

To prove (4), we use (17) and compare the sum

$$\sum_{k=0}^{n} \frac{f(z_{n,k})}{|B_n'(z_{n,k})|}$$

with

$$\frac{1}{2\pi}\sum_{k=0}^{n} f(e^{i\theta_{n,k}})(\theta_{n,k+1}-\theta_{n,k}),$$

a Riemann sum of f on $\mathbf{T}$. We have, with M_f denoting a upper bound of f on $\mathbf{T}$, by using (17) and Lemma 2,

$$\begin{aligned}\left|\sum_{k=0}^{n} \frac{f(z_{n,k})}{|B_n'(z_{n,k})|} - \frac{1}{2\pi}\sum_{k=0}^{n} f(e^{i\theta_{n,k}})(\theta_{n,k+1}-\theta_{n,k})\right| &\le \sum_{k=0}^{n} |f(z_{n,k})| \left|\frac{1}{|B_n'(z_{n,k})|} - \frac{\theta_{n,k+1}-\theta_{n,k}}{2\pi}\right| \\ &\le M_f \sum_{k=0}^{n} \frac{8\pi}{|B_n'(z_{n,k})|\sigma_n\delta_n} = \frac{8\pi M_f(1-|B_n(0)|^2)}{\sigma_n\delta_n(|1-B_n(0)|^2)},\end{aligned}$$

which goes to the limit zero by the assumption (3). This proves (4).

The proof of (5) is obtained from (4) in the same way as that of Theorem 1 is obtained from Theorem 2. □

Proof of Theorem 4. Since $f \in A(\mathbf{D})$, by Mergelyan's theorem ([10, Theorem 1, p. 97]), there exists a sequence of polynomials (the arithmetic means of the partial sums of the Taylor series of f about $z = 0$) that converges uniformly to f on $\overline{\mathbf{D}}$. By using the argument in [1, Addendum A, pp. 243–246], we can see that each monomial z^k ($k \ge 0$) can be approximated by rational functions from $\mathcal{R}$ as closely as we please. Therefore, for $\varepsilon > 0$, there exists a function $r \in \mathcal{R}$ such that

$$\max_{z\in\overline{\mathbf{D}}} |f(z) - r(z)| < \varepsilon.$$

Note that $R_n(r; z) = r(z)$ for all n large enough. Now we have, for n large enough,

$$\begin{aligned}\int_{\mathbf{T}} |R_n(f;z) - f(z)|^2|dz| &= \int_{\mathbf{T}} |R_n(f-r;z) + r(z) - f(z)|^2|dz| \\ &\le 2\left\{\int_{\mathbf{T}} |R_n(f-r;z)|^2|dz| + \int_{\mathbf{T}} |r(z)-f(z)|^2|dz|\right\} \\ &\le 2\int_{\mathbf{T}} |R_n(f-r;z)|^2|dz| + 4\pi\varepsilon^2.\end{aligned}$$

The integral in the last expression can be expanded as

$$\sum_{p,q=0}^{n} \frac{f(z_{n,p}) - r(z_{n,p})}{B_n'(z_{n,p})} \overline{\left[\frac{f(z_{n,q}) - r(z_{n,q})}{B_n'(z_{n,q})}\right]} \int_{\mathbf{T}} \frac{B_n(z)-1}{z - z_{n,p}} \overline{\left(\frac{B_n(z)-1}{z-z_{n,q}}\right)} |dz|,$$

which, according to Lemma 7, is equal to

$$\sum_{k=0}^{n} \frac{2\pi|f(z_{n,k}) - r(z_{n,k})|^2}{|B_n'(z_{n,k})|}.$$

Now, combining the above estimates and using Lemma 2, we have

$$\int_{\mathbf{T}} |R_n(f;z) - f(z)|^2|dz| \le \frac{4\pi\varepsilon^2(1-|B_n(0)|^2)}{|1-B_n(0)|^2} + 4\pi\varepsilon^2 \le \frac{12\pi\varepsilon^2}{1-|B_n(0)|}.$$

Thus, by letting $n \to \infty$, we obtain

$$0 \le \limsup_{n\to\infty} \int_{\mathbf{T}} |R_n(f;z) - f(z)|^2 \; |dz| \le 12\pi\varepsilon^2.$$

This implies the desired result. □

Proof of Theorem 5. By Hermite's formula, for each $n = 1, 2, \dots,$

$$R_n(f;z) = \frac{1}{2\pi}\int_\gamma \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{(B_n(\zeta)-1)(\zeta - z)}d\zeta \tag{21}$$

for all $z \in \mathrm{Int}(\gamma)$, the interior of γ where γ is a Jordan curve in $\{z : \psi(z) < \rho\}$ that surrounds $\{z_{n,0}, z_{n,1}, \dots, z_{n,n}\} \subseteq \mathbf{T}$. On the other hand, by [25, Lemmas I and II on p. 225, Theorem 2 on p. 227], the least square approximation $r_n(f;z)$ is the same as the rational function from $\mathcal{R}_n$ that interpolates f at $\alpha_0, \alpha_1, \dots, \alpha_n$. (For repeated points, r_n interpolates f and its derivatives.) Again, by Hermite's formula,

$$r_n(f;z) = \frac{1}{2\pi}\int_\gamma \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{B_n(\zeta)(\zeta - z)}d\zeta, \qquad z \in \mathrm{Int}(\gamma), \tag{22}$$

where γ is as in (21). Therefore, by (21) and (22), we obtain, for $z \in \mathrm{Int}(\gamma)$,

$$R_n(f;z) - r_n(f;z) = \frac{1}{2\pi}\int_\gamma \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{(\zeta - z)}\frac{d\zeta}{(B_n(\zeta)-1)B_n(\zeta)}.$$

Assume $1 < \rho_1 < \rho_2 < \rho$. Let $\gamma(r) = \{z : \psi(z) = r\}$, $r > 1$. (For the properties of $\gamma(r)$, see the remarks after the statement of Theorem 5 in Section 3.) Then, for $z \in \mathrm{Int}(\gamma(\rho_1))$ and $\varepsilon \in (0, \rho - 1)$, when n is large enough, we have

$$|R_n(f;z) - r_n(f;z)| \le \frac{1}{2\pi}\frac{\max_{\zeta\in\gamma(\rho_2)}|f(\zeta)|((\rho_2+\varepsilon)^n + (\rho_1+\varepsilon)^n)l(\gamma(\rho_2))}{\mathrm{dist}(\gamma(\rho_1),\gamma(\rho_2))((\rho_2-\varepsilon)^n - 1)(\rho_2-\varepsilon)^n}$$

where $l(\gamma(\rho_2))$ denotes the length of $\gamma(\rho_2)$ and $\mathrm{dist}(\gamma(\rho_1), \gamma(\rho_2))$ is the distance between $\gamma(\rho_1)$ and $\gamma(\rho_2)$. It follows that

$$\limsup_{n\to\infty}\max_{z\in\gamma(\rho_1)}|R_n(f;z) - r_n(f;z)|^{1/n} \le \frac{1}{\rho_2}.$$

Letting $\rho_2 \nearrow \rho$ first and then letting $\rho_1 \nearrow \rho$ yield

$$\limsup_{n\to\infty}\max_{z\in\gamma(\rho)}|R_n(f;z) - r_n(f;z)|^{1/n} \le \frac{1}{\rho}. \tag{23}$$

Now, from the fact that

$$\frac{R_n(f;z) - r_n(f;z)}{B_n(z)}$$

is analytic in $\mathbf{C}\setminus\mathbf{D}$ (including ∞), by using the maximum modulus principle, we have

$$\max_{z\in\gamma(\rho_2)}\left|\frac{R_n(f;z) - r_n(f;z)}{B_n(z)}\right| \le \max_{z\in\gamma(\rho_1)}\left|\frac{R_n(f;z) - r_n(f;z)}{B_n(z)}\right|.$$

Thus, for $1 < \rho_1 < \rho_2$,

$$\begin{aligned}
&\limsup_{n\to\infty}\max_{z\in\gamma(\rho_2)}\left|\frac{R_n(f;z) - r_n(f;z)}{B_n(z)}\right|^{1/n} \\
&\quad = \limsup_{n\to\infty}\max_{z\in\gamma(\rho_2)}\frac{|R_n(f;z) - r_n(f;z)|^{1/n}}{\rho_2} \\
&\quad \le \max_{z\in\gamma(\rho_1)}\left|\frac{R_n(f;z) - r_n(f;z)}{B_n(z)}\right|^{1/n} \\
&\quad = \limsup_{n\to\infty}\max_{z\in\gamma(\rho_1)}\frac{|R_n(f;z) - r_n(f;z)|^{1/n}}{\rho_1},
\end{aligned}$$

and so,

$$\begin{aligned}\limsup_{n\to\infty} \max_{z\in\gamma(\rho_2)} &|R_n(f;z) - r_n(f;z)|^{1/n} \\ &\le \frac{\rho_2}{\rho_1} \limsup_{n\to\infty} \max_{z\in\gamma(\rho_1)} |R_n(f;z) - r_n(f;z)|^{1/n}.\end{aligned}$$

This and (23) imply that, for $1 < \rho_* < \rho$,

$$\begin{aligned}\limsup_{n\to\infty} \max_{z\in\gamma(\rho_*^2)} &|R_n(f;z) - r_n(f;z)|^{1/n} \\ &\le \rho_* \limsup_{n\to\infty} \max_{z\in\gamma(\rho_*)} |R_n(f;z) - r_n(f;z)|^{1/n} \\ &\le \rho_* \limsup_{n\to\infty} \max_{z\in\gamma(\rho)} |R_n(f;z) - r_n(f;z)|^{1/n} \\ &\le \frac{\rho_*}{\rho} < 1,\end{aligned}$$

which implies that $R_n(f;z) - r_n(f;z)$ converges to the limit zero locally and uniformly in $\mathrm{Int}(\gamma(\rho^2))$ at a geometric rate. This finishes the proof of Theorem 5. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816