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ON MULTIPLICATORS IN HÖLDER SPACES WITH NONHOMOGENEOUS METRIC*

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Abstract. We give a criterion when function $\hat{m} : E^n \to \mathbb{C}$ is a multiplicator in Hölder spaces with nonhomogeneous metrics. This criterion is applied to nonstationary system of hydrodynamical type.

I know two criterions when function $\hat{m} : E^n \to C$ is a multiplicator in the space $L^\gamma(E^n)$. The first of them goes back to Marcinkievicz (see [1] — [3]). It has found numerous applications to the study of elliptic equations and systems. But it is not applicable to parabolic equations. Even for $\hat{m}(\xi, \lambda) = \frac{\xi_i \xi_k}{i\lambda + \xi^2}$ arising for heat equation this criterion does not fulfilled. For this $\hat{m}$ the criterion given by Lizorkin and Stein ([4], [5]) can be used. As it was shown in [6] the later can be also applied to functions $\hat{m}$ which we have got solving the Cauchy problem for the system

$$\partial_t u_j - D_{jk,lm} \partial^2_{x_k x_m} u_l + \partial_{x_j} q = f_j, \quad j = 1,2,3, \qquad \operatorname{div} u = 0 \tag{1}$$

with coefficients $D_{jk,lm}$ fixed at a point (x,t). System (1) is a linearization (up to lower nonsignificant terms) of the Modified Navier-Stokes system (see [7], [8], etc)

$$\partial_t v + v \cdot \nabla v - \operatorname{div} \frac{\partial D(\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon = \varepsilon(v)} + \nabla p = g, \quad \operatorname{div} v = 0, \tag{2}$$

where $D : M^{3\times 3}_{sym} \to E^1_+ = [0, \infty)$ is a smooth convex function of the symmetric tensors $\varepsilon = (\varepsilon_{ij})$ and $\varepsilon(v) = \frac{1}{2}\nabla v + \frac{1}{2}(\nabla v)^T$ (here ∇v is the gradient of v with respect to the space variables $x = (x_1, x_2, x_3) \in E^3$).

In (2) velocity field v and pressure p are unknown functions and in (1) their variations u and q are the functions under consideration. At first, we reduce the nonhomogeneous initial data $v\Big|_{t=0} = v^0$ to the data

$$v\Big|_{t=0} = 0 \tag{3}$$

and study the Cauchy problem (1), (3), supposing that $f(x,0) \equiv 0$. After that we put u, q and f equal to zero for $(x,t) \in E^3 \times E^1_- = (-\infty, 0)$ and pass to their Fourier images $\hat{u}$, $\hat{q}$ and $\hat{f}$. For them we get the system

$$i\lambda \hat{u}_j + A_{jl}(\xi)\hat{u}_l + i\xi_j \hat{q} = \hat{f}_j, \quad j = 1,2,3, \quad \xi_k \hat{u}_k = 0, \tag{4}$$

where $A_{jl}(\xi) = D_{jk,lm}\xi_k \xi_m$. The variables $\xi = (\xi_1, \xi_2, \xi_3)$ are dual to variables (x_1, x_2, x_3) and λ is dual to t. The conditions which are usually put on D (see [7]—[9], etc) generate for $D_{jk,lm}$ the restrictions

$$\nu_0 |\varkappa|^2 \le D_{jk,lm} \varkappa_{jk} \varkappa_{lm} \le \mu_0 |\varkappa|^2, \tag{5}$$

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where ν_0 and μ_0 are positive constants, $\varkappa$ is arbitrary symmetric tensor and $|\varkappa|^2 = \sum_{j,k} \varkappa_{jk}\varkappa_{jk}$.

The relations

$$D_{jk,lm} = D_{kj,lm} = D_{kj,ml} = D_{jk,ml} = D_{lm,jk} \tag{6}$$

are also fulfilled. According to (5) and (6) matrixes $A(\xi) = (A_{jl}(\xi))$ are symmetric and they satisfy the inequalities

$$A_{jl}(\xi)\eta_j\eta_l \geq \nu_0|\xi|^2|\eta|^2$$

for arbitrary vectors ξ, $\eta \in E^3$.

System (4) is uniquely solvable for all real ξ and λ except the axis λ (i.e. when $\xi = 0$ and λ is arbitrary). The principal matrix $B(\xi, \lambda)$ of the system (4) satisfies the inequalities

$$|\det B(\xi, \lambda)| \geq \nu_1|\xi|^2(|\xi|^4 + |\lambda|^2) \tag{7}$$

with a positive constant ν_1 and arbitrary $(\xi, \lambda) \in E^4$. The solution $\hat{u}$, $\hat{q}$ of (4) has the form

$$\hat{u}_j(\xi, \lambda) = \frac{1}{\det B(\xi, \lambda)} \sum_{k=1}^{3} C_{jk}(\xi, \lambda)\hat{f}_k(\xi, \lambda), \quad j = 1, 2, 3$$

$$\hat{q}(\xi, \lambda) = \frac{1}{\det B(\xi, \lambda)} \sum_{k=1}^{3} C_{4k}(\xi, \lambda)\hat{f}_k(\xi, \lambda).$$

The functions C_{jk}, $j < 4$ have the structure

$$C_{jk}(\xi, \lambda) = C_{jk,lm}(\xi, \lambda)\xi_l\xi_m \tag{8_1}$$

where $C_{jk,lm}$ are 2-homogeneous polynomials in the following sense:

$$C_{jk,lm}(a\xi, a^2\lambda) = a^2 C_{jk,lm}(\xi, \lambda) \tag{8_2}$$

Functions C_{4k} are 5-homogeneous polynomials, i.e.

$$C_{4k}(a\xi, a^2\lambda) = a^5 C_{4k}(\xi, \lambda). \tag{8_3}$$

For f satisfying the condition

$$\operatorname{div} f = 0 \tag{9_1}$$

functions $\hat{f} = F(f)$ satisfy the relations

$$\xi_k \hat{f}_k(\xi, \lambda) = 0, \tag{9_2}$$

and C_{4k} have the following structure

$$C_{4k}(\xi, \lambda) = C_{4k,lmn}(\xi, \lambda)\xi_l\xi_m\xi_n \tag{9_3}$$

with 2-homogeneous polynomials $C_{4k,lmn}$. Besides (7), function $\det B$ has the properties

$$\det B(\xi, \lambda) = b_{kl}(\xi, \lambda)\xi_k\xi_l \tag{10_1}$$

and

$$(10_2) \qquad b_{kl}(a\xi, a^2\lambda) = a^4 b_{kl}(\xi, \lambda).$$

It is not difficult to calculate that the fractions

$$(11_1) \qquad \frac{C_{jk}(\xi,\lambda)\xi_l\xi_m}{\det B(\xi,\lambda)}, \quad j,k,l,m = 1,2,3,$$

and also the fractions

$$(11_2) \qquad \frac{C_{4k}(\xi,\lambda)\xi_l}{\det B(\xi,\lambda)}, \quad k,l = 1,2,3,$$

(when C_{4k} have the form (9_3)) satisfy the conditions of Theorem 2 which we formulate and prove below. Thus, they are multiplicators in the Holder space $H^{(\gamma,\gamma/2)}(Q_T)$ with parabolic metric

$$\rho((x,t),(x',t')) = |x-x'| + |t-t'|^{1/2}$$

and any $\gamma \in (0,1)$. The norm in $H^{(\gamma,\gamma/2)}(Q_T)$, $Q_T = E^3 \times (0,T)$, is determined by the equality

$$|u|_{H^{(\gamma,\gamma/2)}}(Q_T) = \sup_{Q_T} |u| + \langle u\rangle_{Q_T}^{(\gamma,\gamma/2)}$$

where the second term is called Hölder constant and has the form

$$\langle u\rangle_{Q_T}^{(\gamma,\gamma/2)} = \sup_{\substack{(x,t),(x',t') \in Q_T \\ \rho^\gamma((x,t),(x',t')) \le 1}} \frac{|u(x,t)-u(x',t')|}{\rho^\gamma((x,t),(x',t'))}$$

So, with the help of Theorem 2 we get the statement

THEOREM 1. *For solutions u, q of the Cauchy problem* (1), (3) *with* $f \in L^2(Q_T) \cap H^{(\gamma,\gamma/2)}(Q_T)$ *the following estimates*

$$(12_1) \qquad \|\partial_t u, \partial_{xx}^2 u, \partial_x u, u\|_{L^2(Q_T)} \le C\|f\|_{L^2(Q_T)}$$

and

$$(12_2) \qquad \langle \partial_{xx}^2 u\rangle_{Q_T}^{(\gamma,\gamma/2)} \le C(\gamma)\langle f\rangle_{Q_T}^{(\gamma,\gamma/2)}$$

hold. If, additionally, f *satisfy* (9_1) *and* $f(x,0) = 0$, *then*

$$(12_3) \qquad \langle \nabla q\rangle_{Q_T}^{(\gamma,\gamma/2)} \le C(\gamma)\langle f\rangle_{Q_T}^{(\gamma,\gamma/2)}$$

is also true.

Basing on this theorem the theorem on unique solvability of the Cauchy problem for system (1) in space $W_2^{2,1}(Q_T) \cap H^{2+\gamma,1+\gamma/2}(Q_T)$ for u and space $L^2(Q_T) \cap H^{\gamma,\gamma/2}(Q_T)$ for ∇q is proved by known methods. Let us point out that in the work [6] of G. Seregin and myself the estimate (12_1) in the space $L^\gamma(Q_T)$ with arbitrary $\gamma \in (1,\infty)$ is proved. To do this we have convinced that the functions $\xi^2(\det B(\xi,\lambda))^{-1}C_{jk}(\xi,\lambda)$, $j,k = 1,2,3$, satisfy Lizorkin-Stein criterion and hence they are multiplicators in the spaces $L^\gamma(E^4)$, $\gamma \in (1,\infty)$. In [6] we have proved also the estimate (12_2) (but not (12_3)). We did this by special technique developed for the study of parabolic systems (see Campanato [10] and others). Here inequalities (12_k), $k = 2,3$, are proved with the help of following criterion.

THEOREM 2. *Let $\hat{m}$ be a bounded function of variables $\xi = (\xi', \xi_n) \in E^n$, $\xi' = (\xi_1, \dots, \xi_{n-1})$ satisfying inequalities*

$$\|\check{m}_j\|_{W_2^s(\omega)} \le \mu, \quad \forall j \in \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}, \tag{13}$$

with a $\mu < \infty$ and $s > n/2$. In (13) $\check{m}_j(\xi) = \hat{m}(2^j \xi', 2^{j/\alpha}\xi_n)$ with $\alpha \in (0,1]$ and $\omega = \{\xi \mid \frac{1}{4} \le \rho(\xi) \le 4\}$ where $\rho(\xi) = |\xi'| + |\xi_n|^\alpha$ and $|\xi'| = \sum_{k=1}^{n-1} |\xi_k|$. Such $\hat{m}$ is a multiplicator in Hölder spaces $H^{(\gamma,\alpha\gamma)}(E^n)$, $\gamma \in (0,1)$, i.e.

$$\langle m * u\rangle_{E^n}^{(\gamma,\alpha\gamma)} \le C(\gamma,\alpha,n)\mu\langle u\rangle_{E^n}^{(\gamma,\alpha\gamma)} \tag{14}$$

where $m = F^{-1}(\hat{m})$ and u is an arbitrary function from $C_0^1(E^n)$.

Here we have used the notations

$$\langle u\rangle_{E^n}^{(\gamma,\alpha\gamma)} = \sup_{\substack{(x',x_n),(y',y_n) \in E^n \\ \rho(x-y) \le 1}} \frac{|u(x',x_n) - u(y',y_n)|}{\rho^\gamma(x-y)} \tag{15}$$

and $m = F^{-1}(\widehat{m})$ is the inverse Fourier image of $\widehat{m}$.

Above we have used this theorem for $\alpha = 1/2$ and 0-homogeneous functions, so that for them $\check{m}_j(\xi) = \widehat{m}(\xi)$ and we have examine (13) only for function $\widehat{m}(\xi)$ itself.

We proof Theorem 2 following to the same plan that was applied by Hörmander for the proof of Theorem 7.9.6 from [3], but we work with other auxiliary functions. Thus, we take a smooth function $\psi : E_+^1 \to [0,1]$ which is equal 1 on the interval $[0,1]$ and zero on $[2,\infty)$ and the function $\varphi : E_+^1 \to [0,1]$ determining by equality $\varphi(\rho) = \psi(\rho) - \psi(2\rho)$, $\rho \in E_+^1$. The later is equal zero on the sets $[0,\frac{1}{2}]$ and $[2,\infty)$. Let us introduce also the functions $\varphi_j : E_+^1 \to [0,1]$ by equalities $\varphi_j(\rho) = \varphi(\frac{\rho}{2^j})$. It is easy to see that

$$\sum_{j=-\infty}^{\infty} \varphi_j(\rho) = 1, \qquad \rho \in (0,\infty). \tag{16_1}$$

Now, using φ and φ_j, we determine the functions $\widehat{\phi}$ and $\widehat{\phi}_j : E^n \to [0,1]$ in the following way:

$$\widehat{\phi} = \varphi \circ \rho, \qquad i.e. \;\; \widehat{\phi}(\xi) = \varphi(\rho(\xi)) \tag{16_2}$$

$$\begin{gathered} \widehat{\phi}_j = \varphi_j \circ \rho, \qquad i.e. \;\; \widehat{\phi}_j(\xi) = \varphi_j(\rho(\xi)) = \\ \varphi(\frac{\rho(\xi)}{2^j}) = \varphi(\rho(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}})) = \widehat{\phi}(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}}). \end{gathered} \tag{16_3}$$

The equality (16_1) generate the following

$$\sum_{j=-\infty}^{\infty} \widehat{\phi}_j(\xi) = 1, \qquad \xi \in E^n \setminus \{0\}. \tag{16_4}$$

According to (16_k), $k = 1, \dots, 4$ we represent $\widehat{u}(\xi) = F(u)(\xi)$ as the sum $\widehat{u}(\xi) = \sum_j \widehat{u}_j(\xi)$ where $\widehat{u}_j(\xi) = \widehat{u}(\xi)\widehat{\phi}_j(\xi)$. Let ϕ_j be the inverse Fourier transform of $\widehat{\phi}_j$, i.e.

$\phi_j = F^{-1}(\widehat{\phi}_j)$. Then using a change of variables we get

$$(17)\quad \begin{aligned}\phi_j(x) = F^{-1}(\widehat{\phi}_j)(x) &= \int e^{ix\xi}\widehat{\phi}(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}})d\xi \\ &= 2^{(n-1)j+j/\alpha}\int e^{i2^j x'\eta'+i2^{j/\alpha}x_n\eta_n}\widehat{\phi}(\eta',\eta_n)d\eta = 2^{(n-1)j+j/\alpha}\phi(2^j x', 2^{j/\alpha}x_n)\end{aligned}$$

where $\phi(x) = F^{-1}(\widehat{\phi})(x) = \int e^{ix\xi}\widehat{\phi}(\xi)d\xi$

Let us introduce a smooth function $\theta : E^1_+ \to [0,1]$, which is equal to 1 for $\rho \in [\frac{1}{2}, 2]$, to zero for $\rho \in [0, \frac{1}{4}]$ and $\rho \geq 4$, and the function $\chi : E^n \to [0,1]$ determined by equalities $\chi(\xi) = \theta(\rho(\xi))$, $\xi \in E^n$. It is clear that $\widehat{\phi}(\xi) = \widehat{\phi}(\xi)\chi(\xi)$ and $\widehat{\phi}_j(\xi) = \widehat{\phi}_j(\xi)\chi(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}})$ and therefore

$$\widehat{u}_j(\xi) = \widehat{u}(\xi)\widehat{\phi}_j(\xi) = \widehat{u}(\xi)\widehat{\phi}(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}})\chi(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}}) = \widehat{u}_j(\xi)\chi(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}}).$$

Let

$$(18_1)\quad v \equiv m * u.$$

Then for its Fourier image $\widehat{v}$ we have

$$(18_2)\quad \widehat{v}(\xi) = \widehat{m}(\xi)\widehat{u}(\xi) = \sum_j \widehat{m}(\xi)\widehat{u}_j(\xi) = \sum_j \widehat{v}_j(\xi),$$

where $\widehat{v}_j(\xi) = \widehat{m}_j(\xi)\widehat{u}_j(\xi)$ and $\widehat{m}_j(\xi) = \widehat{m}(\xi)\chi(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}})$.

Let us show that $F^{-1}(\widehat{m}_j)$ have the uniformly (with respect to j) bounded norms in L^1. For this purpose we remark that

$$\begin{aligned}||F^{-1}\widehat{m}_j||_{L^1(E^n)} &= \int_{E^n} dx\left|\int e^{ix\xi}\widehat{m}(\xi)\chi(\frac{\xi'}{2^j}, \frac{\xi_n}{2^{j/\alpha}})d\xi\right| \\ &= \int dy\left|\int e^{i\eta y}\widehat{m}(2^j\eta', 2^{j/\alpha}\eta_n)\chi(\eta)d\eta\right| = ||F^{-1}M_j||_{L^1},\end{aligned}$$

where $M_j(\eta) = \widehat{m}(2^j\eta', 2^{j/\alpha}\eta_n)\chi(\eta) = \check{m}_j(\eta)\chi(\eta)$.

In virtue of well known inequalities, for any M_j

$$(19)\quad ||F^{-1}M_j||_{L^1(E^n)} \leq C(n,s)||M_j||_{W_2^s(E^n)}, \quad s > \frac{n}{2}.$$

The right hand side of (19) is uniformly bounded due to our condition (13) and our choice of χ. Thus we have

$$(20)\quad ||F^{-1}\widehat{m}_j||_{L^1} \leq c_0\mu \quad \text{for any } j \in \mathbb{Z}$$

Due to (17) we have

$$(21_1)\quad \begin{aligned}u_j(x) &= F^{-1}(\hat{u}_j) = F^{-1}(\hat{u}\hat{\phi}_j) \\ &= u * F^{-1}(\hat{\phi}_j) = u * \phi_j = 2^{(n-1)j+j/\alpha}\int u(x-y)\phi(2^j y', 2^{j/\alpha}y_n)dy'dy_n \\ &= \int \Big(u(x' - \frac{z'}{2^j}, x_n - \frac{z_n}{2^{j/\alpha}}) - u(x)\Big)\phi(z)dz\end{aligned}$$

Here we have used $\hat{\phi}(0) = 0 = \int \phi(z)dz$. Analogously we get

$$(21_2) \qquad \partial_{x_k} u_j(x) = 2^j \int \Big(u(x' - \frac{z'}{2^j}, x_n - \frac{z_n}{2^{j/\alpha}}) - u(x)\Big)\partial_{z_k}\phi(z)dz, \quad k < n$$

and

$$(21_3) \qquad \partial_{x_n} u_j(x) = 2^{j/\alpha} \int \Big(u(x' - \frac{z'}{2^j}, x_n - \frac{z_n}{2^{j/\alpha}}) - u(x)\Big)\partial_{z_n}\phi(z)dz.$$

The following inequalities we deduce from (21_k):

$$(22_1) \qquad \sup_{x\in E^n} |u_j(x)| \le \langle u\rangle^{(\gamma,\alpha\gamma)} \int (\frac{|z'|}{2^j} + \frac{|z_n|^\alpha}{2^j})^\gamma \phi(z)dz \equiv c_1(\gamma,\alpha)2^{-j\gamma}\langle u\rangle^{(\gamma,\alpha\gamma)},$$

$$(22_2) \qquad \sup_{x\in E^n} |\partial_{x_k} u_j(x)| \le \langle u\rangle^{(\gamma,\alpha\gamma)} 2^{j-j\gamma} \int (|z'| + |z_n|^\alpha)^\gamma |\partial_{z_k}\phi(z)|dz \equiv c_2(\gamma,\alpha)2^{j-j\gamma}\langle u\rangle^{(\gamma,\alpha\gamma)},$$

for $k < n$ and

$$(22_3) \qquad \sup_{x\in E^n} |\partial_{x_n} u_j(x)| \le \langle u\rangle^{(\gamma,\alpha\gamma)} 2^{j/\alpha-j\gamma} \int (|z'| + |z_n|^\alpha)^\gamma |\partial_{z_n}\phi(z)|dz \equiv c_3(\gamma,\alpha)2^{j/\alpha-j\gamma}\langle u\rangle^{(\gamma,\alpha\gamma)}.$$

These inequalities and estimate (20) we use to evaluate functions v_j:

$$(23_1) \qquad v_j(x) \equiv F^{-1}(\hat{v}_j) = F^{-1}(\hat{m}_j\hat{u}_j) = m_j * u_j = \int u_j(x-y)m_j(y)dy$$

where $m_j = F^{-1}(\hat{m}_j)$ and $\hat{m}_j$ is determined after (18_2). Namely

$$(23_2) \qquad \sup_x |v_j(x)| \le \|m_j\|_{L^1} \sup_x |u_j(x)| \le c_0\mu c_1 \langle u\rangle^{(\gamma,\alpha\gamma)} 2^{-j\gamma} \equiv d_1 2^{-j\gamma},$$

$$(23_3) \qquad \sup_x |\partial_{x_k} v_j(x)| \le \|m_j\|_{L^1} \sup_x |\partial_{x_k} u_j(x)| \le c_0\mu c_2 \langle u\rangle^{(\gamma,\alpha\gamma)} 2^{j-j\gamma} \equiv d_2 2^{j-j\gamma},$$

$$(23_4) \quad \sup_x |\partial_{x_n} v_j(x)| \le \|m_j\|_{L^1} \sup_x |\partial_{x_n} u_j(x)| \le c_0\mu c_3 \langle u\rangle^{(\gamma,\alpha\gamma)} 2^{j/\alpha-j\gamma} \equiv d_3 2^{j/\alpha-j\gamma},$$

Now we estimate $|v(x) - v(y)|$ for all x, y from E^n for which $\rho(x-y) = |x'-y'| + |x_n - y_n|^\alpha \le 1$, in the following way:

$$|v(x) - v(y)| \le \sum_j |v_j(x) - v_j(y)| \le \sum_{j\ge n_0} \ldots + \sum_{j\le n_0} \ldots \equiv S_1 + S_2,$$

where $n_0 = -\log_2 \rho(x-y)$. For evaluation of S_1 we use (23_2)

$$S_1 \le \sum_{j\ge n_0} 2\sup_x |v_j| \le 2d_1 \sum_{j\ge n_0} 2^{-j\gamma} = 2d_1 2^{-n_0\gamma} \sum_{k=0}^{\infty} \frac{1}{2^{\gamma k}} \equiv e_1(\gamma,\alpha)2^{-n_0\gamma} = e_1\rho^\gamma(x-y).$$

For evaluation of S_2 we use (23_3) and (23_4):

$$\begin{aligned}S_2 &\le \sum_{j\le n_0}\Big(\sum_{k=1}^{n-1}|x_k-y_k|\sup_x|\partial_{x_k}v_j(x)|+|x_n-y_n|\sup_x|\partial_{x_n}v_j(x)|\Big)\\ &\le \sum_{j\le n_0}\Big(d_2|x'-y'|2^{j-j\gamma}+d_3|x_n-y_n|2^{j/\alpha-j\gamma}\Big)\\ &\le d_2|x'-y'|2^{(1-\gamma)n_0}\sum_{k=0}^{\infty}\frac{1}{2^{k(1-\gamma)}}+d_3|x_n-y_n|2^{n_0(1/\alpha-\gamma)}\sum_{k=0}^{\infty}\frac{1}{2^{k(1/\alpha-\gamma)}}\\ &= e_2|x'-y'|\rho^{\gamma-1}(x-y)+e_3|x_n-y_n|\rho^{\gamma-1/\alpha}(x-y)\le(e_2+e_3)\rho^{\gamma}(x-y)\end{aligned}$$

Thus we have convinced that

$$|v(x)-v(y)|\le(e_1+e_2+e_3)\rho^{\gamma}(x-y).$$

So Theorem 2 is proved.

Remark. Theorem 2 is also true if we choose

$$\rho(\xi)=\sum_{k=1}^{n}|\xi_k|^{\alpha_k},\quad \alpha_1=1,\quad \alpha_k\in(0,1]$$

Remark. In Theorem 2 instead of the condition (13) we can take the condition

$$(*)\qquad ||F^{-1}M_j||_{L^1(E^n)}\le\mu_1,\qquad \text{for all } j\in\mathbb{Z}$$

where M_j are determined just before the inequality (19). In fact, as it is seen from the proof of Theorem 2, the condition (13) was used only for to get the estimate (*) (See (19) and (20)).

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