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ON TWO-ORTHOGONAL POLYNOMIALS RELATED TO THE BATEMAN'S $J_n^{u,v}$-FUNCTION*

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Abstract. The purpose of the present paper is to study the sequence of hypergeometric polynomials $\{B_n^{\alpha,\beta}\}_{n\geqslant 0}$ defined by $B_n^{\alpha,\beta}(x) = {}_1F_2(-n; 1+\alpha, 1+\beta; x)$, where $\alpha, \beta \notin -\mathbb{N}^*$, which are (with the change of notations $u = \alpha$, $v + u/2 = \beta$) related to the Bateman's function $J_n^{u,v}$. We show that it constitutes a two-parameter family of 2-orthogonal classical polynomials of Laguerre type. Among other things, we give their recurrence relation and the third order differential equation satisfied by each polynomial. When $\alpha \geqslant \beta > -1$, we obtain that these polynomials are 2-orthogonal on the interval $(0, +\infty)$ with respect to a pair of weight functions $\mathcal{W}_{\alpha,\beta}$ and $(x\mathcal{W}_{\alpha,\beta})'$, where $\mathcal{W}_{\alpha,\beta}$ is a positive function involving the MacDonald function K_ν ($\nu = \alpha - \beta$). For a suitable choice of the parameters α and β, we obtain that $\mathcal{W}_{\alpha,\beta}$ is the ultra-exponential function introduced by Voronoï and then encounter an open problem posed by Prudnikov for which we give an answer. In addition, we obtain by means of the technique of cubic decomposition that the polynomials in question are connected to another kind of 2-orthogonal polynomials of Hermite type.

1. Introduction and basic notations. Before introducing our problem, let us recall the definition of the d-orthogonality. We say that the monic sequence $\{P_n\}_{n\geqslant 0}$ is d-orthogonal polynomial sequence (d-OPS) with respect to the vector of linear functional $\mathbf{U} = {}^t(u_0, u_1, \cdots, u_{d-1})$, if it satisfies the following orthogonality relations [16,22]:

$$\langle u_\nu \, , \, P_m P_n \rangle = 0 \, , \; n \geqslant md + \nu + 1 \, , \; m \geqslant 0, \tag{1.1}$$

$$\langle u_\nu \, , \, P_m P_{md+\nu} \rangle \neq 0 \, , \; m \geqslant 0, \tag{1.2}$$

for each integer ν with $0 \leqslant \nu \leqslant d-1$ ($d \geqslant 1$). The functionals $u_0, u_1, \dots, u_{d-1}$ are the d first elements of the dual sequence $\{u_n\}_{n\geqslant 0}$ associated to the sequence of polynomials $\{P_n\}_{n\geqslant 0}$ and defined by $\langle u_m \, , \, P_n \rangle = \delta_{m,n}$; $m \geqslant 0$, $n \geqslant 0$. In particular, we denote by $(u)_n := \langle u \, , \, x^n \rangle$, $n \geqslant 0$, the moments of the linear functional u.
Also, we denote by $\mathcal{P}$ the vector space of polynomials with complex coefficients and $\mathcal{P}'$ its algebraic dual.

The sequences of polynomials satisfying the conditions (1.1)-(1.2) are characterized by the fact that they satisfy a $(d+1)th$ order recurrence relation, that is a relation between $d+2$ consecutive polynomials [22]. Now, let us introduce the sequence of monic polynomials $\{Q_n\}_{n\geqslant 0}$ defined by $Q_n := (n+1)^{-1} D_x P_{n+1}$, $n \geqslant 0$ where D_x denotes the derivative operator d/dx. We denote by $\{\widetilde{u}_n\}_{n\geqslant 0}$ its corresponding dual sequence. If the sequence $\{Q_n\}_{n\geqslant 0}$ is also d-OPS, the sequence $\{P_n\}_{n\geqslant 0}$ is called *classical* (in the sense of having the Hahn property [13]).

Recently, in this context, the second author introduced in [6] a one-parameter family, denoted by $\{P_n(.,\alpha)\}_{n\geqslant 0}$, of classical 2-orthogonal polynomials of Laguerre type. Some of their interesting properties which are analogous of those satisfied by

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the classical Laguerre polynomials as well as the connection between the two families of polynomials are given. Furthermore, by investigating the d-orthogonal polynomials which are at the same time Appell polynomials [7], the author introduced the classical d-orthogonal (monic) polynomials $\{P_n\}_{n\geqslant 0}$ which are another generalization of the classical Hermite (monic) polynomials. These polynomials satisfy $P_n = Q_n$ and are defined by the generating function $A(t)e^{xt}$ where $A(t) = \exp\left(\zeta_0 + \zeta_1 t + \cdots + \zeta_{d+1}t^{d+1}\right)$, with $\zeta_{d+1} \neq 0$. In particular, when $\zeta_k = 0$ for $k = 0, 1, \ldots, d$ (the *d-symmetric* case, see below) this sequence is reduced (apart from a linear transformation) to the polynomials known in the literature as the Gould-Hopper polynomials [12].
For $d = 1$, $\{P_n\}_{n\geqslant 0}$ is reduced to the classical Hermite polynomials $\{\widehat{H}_n\}_{n\geqslant 0}$.
For $d = 2$, $\{P_n\}_{n\geqslant 0}$ is the " classical" 2-orthogonal sequence of Hermite type satisfying the third-order recurrence relation (see [9,7])

$$\begin{aligned} &P_{n+3}(x) = xP_{n+2}(x) - \rho(n+2)(n+1)P_n(x)\ ,\ n \geqslant 0\,,\ (\rho \neq 0), \\ &P_0(x) = 1\ ,\ P_1(x) = x\ ,\ P_2(x) = x^2. \end{aligned} \tag{1.3}$$

In this case, $\{P_n\}_{n\geqslant 0}$ is 2-orthogonal with respect to the vector functional $\mathbf{U} = {}^t(u_0, u_1)$.

On the other hand and in another framework, Bateman [1] introduced a set of functions $\{J_n^{u,v}\}_{n\geqslant 0}$ when he was interested in constructing inverse Laplace transforms by studying the solution of the following integral equation:

$$f(x) = \int_0^\infty e^{-xt} g(t)dt, \tag{1.4}$$

where the variable x takes positive values, and so the function $g(t)$ is to be derived from the values of x for $x > 0$. For solving the integral equation (1.4), some methods are developed and used by several authors (see the references in [1]). For instance, Murphy gave a method in which $xf(x)$ is expanded in a series of power of $1 - x^{-1}$ and $g(t)$ in a series of polynomials of Laguerre. In order to make use of integrals involving $f(x)$, Bateman adopted the Murphy's idea of using an expansion in a series of orthogonal functions, but to apply it to $f(x)$ instead of $g(t)$. For this purpose, he introduced the functions $J_n^{u,v}$ and made use of the Laguerre polynomials L_n^α, $n \geqslant 0$ and the equation

$$x^{-u-1} L_n^{v+u/2}(x^{-1}) = \int_0^\infty e^{-xt} t^{u/2} J_n^{u,v}(t^{1/2})dt, \tag{1.5}$$

where $x^{-u}J_n^{u,v}(x)$ is a polynomial defined as follows

$$x^{-u}J_n^{u,v}(x) = k_n^{-1}(u,v)\,{}_1F_2\left(-n; 1+u, 1+v+\frac{1}{2}u; x^2\right)\ ,\ n \geqslant 0 \tag{1.6}$$

with

$$k_n(u,v) = n!\frac{\Gamma(1+u)\Gamma(1+v+\frac{1}{2}u)}{\Gamma\left(1+n+v+\frac{1}{2}u\right)}. \tag{1.7}$$

Now, we shall adopt the following notations: we put $u = \alpha$, $v + u/2 = \beta$, and denote by $B_n^{\alpha,\beta}$, $n \geqslant 0$, the polynomials defined as

$$B_n^{\alpha,\beta}(x) = {}_1F_2\left(-n; 1+\alpha, 1+\beta; x\right),\quad \alpha, \beta \neq -1, -2, \ldots . \tag{1.8}$$

From this, it may be seen that $B_n^{\alpha,\beta}$ has the explicit formula

$$B_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k,\ n \geqslant 0, \tag{1.9}$$

where

$$\lambda_m = \lambda_m(\alpha,\beta) = \frac{(-1)^m}{(1+\alpha)_m(1+\beta)_m},\ m \geqslant 0, \tag{1.10}$$

with $(\mu)_n$ is Pochhammer's symbol defined by

$$(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)} = \begin{cases} \mu(\mu+1)\dots(\mu+n-1)\,, & n = 1,2,\dots\,, \\ 1, & n = 0. \end{cases} \tag{1.11}$$

According to (1.6), we have

$$B_n^{\alpha,\beta}(x) = k_n(\alpha,\beta) x^{-\alpha/2} J_n^{\alpha,\beta-\alpha/2}(\sqrt{x}), \tag{1.12}$$

where

$$k_n(\alpha,\beta) = n!\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+n+\beta)}. \tag{1.13}$$

We also denote by $\{v_n\}_{n\geqslant 0}$ the dual sequence associated with $\{B_n^{\alpha,\beta}\}_{n\geqslant 0}$.

In this paper, we are interested in the sequence of polynomials $\{B_n^{\alpha,\beta}\}_{n\geqslant 0}$. We show that these polynomials are 2-orthogonal "classical" and constitute another family of Laguerre type of two parameters. The paper is organized as follows. Firstly, in Section 2, we give some properties of the polynomials $B_n^{\alpha,\beta}$ that are analogous of the ones satisfied by the classical Laguerre polynomials. In particular, the generating function and the third order differential equation for the polynomials $B_n^{\alpha,\beta}$. In Section 3, we establish that the sequence $\{B_n^{\alpha,\beta}\}_{n\geqslant 0}$ as well as the derivative sequence $\{D_x B_{n+1}^{\alpha,\beta}\}_{n\geqslant 0}$ are 2-orthogonal. Section 4, is devoted to the study of the integral representations of the two linear functionals v_0 and v_1 corresponding to $\{B_n^{\alpha,\beta}\}_{n\geqslant 0}$ by means of weight functions involving the MacDonald function K_ν ($\nu = \alpha - \beta$) and generalizing the ultra-exponential functions given in [23]. Finally, in the last section, with suitable choice of the parameters α and β, we establish the connection between the polynomials $B_n^{\alpha,\beta}$ (of Laguerre type) and the polynomials $\{P_n\}_{n\geqslant 0}$ (of Hermite type) defined by the recurrence (1.3). This link allows us to give integral representations of the two linear functionals u_0 and u_1 corresponding to $\{P_n\}_{n\geqslant 0}$ by means of the MacDonald function $K_{1/3}$ plus a Stieltjes-type functions suitably chosen.

2. Some properties of the polynomials $B_n^{\alpha,\beta}$. Before proceeding to give the properties of the polynomials $B_n^{\alpha,\beta}$ in question, we wish to give the following known results and recall some properties of the classical Laguerre polynomials L_n^α, $n \geqslant 0$:

LEMMA 2.1. *Let $\{B_n\}_{n\geqslant 0}$ the sequence of polynomials defined by the generating function*

$$e^t A(xt) = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{2.1}$$

where $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \neq 0$. *The following statements are equivalent:*

(*i*) *The polynomials* B_n , $n \geqslant 0$, *are generated by (2.1).*

(*ii*) *The polynomials* B_n , $n \geqslant 0$, *are generated by*

$$(2.2) \qquad (1-t)^{-\kappa} F\left(\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} (\kappa)_n B_n(x) \frac{t^n}{n!} , \quad |t| < 1,$$

where

$$(2.3) \qquad F(z) = \sum_{n=0}^{\infty} (\kappa)_n a_n z^n \quad ; \quad a_n \neq 0.$$

(*iii*) *The sequence* $\{R_n\}_{n \geqslant 0}$ *defined by* $R_n(x) = x^n B_n\left(\frac{1}{x}\right)$ *is an Appell (such polynomials are called the reversed polynomials with reference to the original polynomials* B_n , $n \geqslant 0$*).*

(*iv*) *The polynomials* B_n , $n \geqslant 1$, *satisfy the differential-recurrence relation*

$$(2.4) \qquad xB_n'(x) = nB_n(x) - nB_{n-1}(x) , \; n \geqslant 1.$$

(*v*) *The polynomials* B_n , $n \geqslant 0$, *possess a multiplication formula of the form*

$$(2.5) \qquad B_n(xy) = \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} B_k(x).$$

Proof.

- The equivalence (*i*) $\Longleftrightarrow$ (*ii*) was given by Rainville [18].
- If we replace in (2.1) x by $1/x$ and t by xt, we get

$$A(t)e^{xt} = \sum_{n=0}^{\infty} R_n(x) \frac{t^n}{n!}$$

which leads to the equivalence (*i*) $\Longleftrightarrow$ (*iii*).

- The equivalence (*iii*) $\Longleftrightarrow$ (*iv*) is trivial.
- The equivalence (*iii*) $\Longleftrightarrow$ (*v*) was given by Carlitz [3]. □

Note that these polynomials are particular cases of the Brenke polynomials which are defined by the generating function [4]

$$G(x,t) = B(t)C(xt),$$

where $B(z) = \sum_{n=0}^{\infty} b_n z^n$; $b_0 \neq 0$, and $C(z) = \sum_{n=0}^{\infty} c_n z^n$; $c_n \neq 0$.

Otherwise, the Laguerre polynomials L_n^{α}, $n \geqslant 0$, are defined as (see, e.g.,[4]):

$$(2.6) \qquad L_n^{\alpha}(x) = \binom{n+\alpha}{n} {}_1F_1(-n; 1+\alpha; x),$$

where $\binom{\mu}{\nu}$ denotes the generalized binomial coefficient

$$(2.7) \qquad \binom{\mu}{\nu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\nu)\Gamma(1+\mu-\nu)}.$$

They are generated by the function

$$(1-t)^{-\alpha-1}e^{-xt/(1-t)} = \sum_{n=0}^{\infty} L_n^\alpha(x)t^n. \tag{2.8}$$

Their corresponding monic polynomials $\{\widehat{L}_n^\alpha\}_{n\geqslant 0}$ are defined by $\widehat{L}_n^\alpha = (-1)^n n! L_n^\alpha$, $n \geqslant 0$, and satisfy the following relations:

$$D_x \widehat{L}_{n+1}^\alpha = (n+1)\widehat{L}_n^{\alpha+1}\,,\ n \geqslant 0\,, \tag{2.9}$$

$$x\widehat{L}_n^{\alpha+1} = \widehat{L}_{n+1}^\alpha + (n+\alpha+1)\widehat{L}_n^\alpha\,,\ n \geqslant 0, \tag{2.10}$$

$$\widehat{L}_{n+1}^\alpha = \widehat{L}_{n+1}^{\alpha+1} + (n+1)\widehat{L}_n^{\alpha+1}\,,\ n \geqslant 0\,, \tag{2.11}$$

and the second-order recurrence relation:

$$\begin{aligned} &\widehat{L}_{n+2}^\alpha(x) = \big(x-(2n+\alpha+3)\big)\widehat{L}_{n+1}^\alpha(x) - (n+1)(n+\alpha+1)\widehat{L}_n^\alpha(x)\,,\ n \geqslant 0,\\ &\widehat{L}_0^\alpha(x) = 1\,,\quad \widehat{L}_1^\alpha(x) = x-(\alpha+1). \end{aligned} \tag{2.12}$$

Finally, it is well known that these polynomials L_n^α, $n \geqslant 0$ are orthogonal, for $\alpha > -1$, with respect to the weight function $w_\alpha(x) = x^\alpha e^{-x}$ on the interval $0 \leqslant x < +\infty$, that is

$$\int_0^\infty L_m^\alpha(x)L_n^\alpha(x)w_\alpha(x)dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{m,n}\ ,\ m,n \geqslant 0. \tag{2.13}$$

In the sequel, we return to the polynomials $B_n^{\alpha,\beta}$ and give some of their properties that are analogous of the above ones. Let us denote by $\{\widehat{B}_n^{\alpha,\beta}\}_{n\geqslant 0}$ the sequence of monic polynomials corresponding to $\{B_n^{\alpha,\beta}\}_{n\geqslant 0}$. By virtue of the definition of the polynomials $B_n^{\alpha,\beta}$, it is clear that the leading coefficient of the polynomial $B_n^{\alpha,\beta}$ is

$$\lambda_n = \frac{(-1)^n}{(1+\alpha)_n(1+\beta)_n}\ ,\ n \geqslant 0. \tag{2.14}$$

Thus, we have the relation $B_n^{\alpha,\beta} = \lambda_n \widehat{B}_n^{\alpha,\beta}$, $n \geqslant 0$.

2.1. The generating functions. From the Brafman identity (see [2], p.947):

$$e^t\,{}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ 1+b_1,\dots,1+b_q\end{matrix};-xt\right) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,{}_{p+1}F_q\left(\begin{matrix}-n, & a_1,\dots,a_p\\ & 1+b_1,\dots,1+b_q\end{matrix};x\right), \tag{2.15}$$

for $p=0, q=2, b_1=\alpha$ and $b_2=\beta$, we obtain that the polynomials $B_n^{\alpha,\beta}$; $n \geqslant 0$ are defined by the generating function

$$e^t{}_0F_2\big(1+\alpha,1+\beta;-xt\big) = \sum_{n=0}^{\infty} B_n^{\alpha,\beta}(x)\frac{t^n}{n!}. \tag{2.16}$$

Otherwise, owing to Lemma 2.1, with

$$A(z) = {}_0F_2\big(1+\alpha,1+\beta;-z\big) = \sum_{n=0}^{\infty}\lambda_n z^n \tag{2.17}$$

and

$$F(z) = {}_1F_2\big(\kappa; 1+\alpha, 1+\beta; -z\big) = \sum_{n=0}^{\infty} (\kappa)_n \lambda_n z^n,$$

we also obtain another generating function for the polynomials $B_n^{\alpha,\beta}$, $n \geqslant 0$,

$$(2.18) \qquad (1-t)^{-\kappa}\, {}_1F_2\big(\kappa; 1+\alpha, 1+\beta; -\frac{xt}{1-t}\big) = \sum_{n=0}^{\infty} (\kappa)_n\, B_n^{\alpha,\beta}(x) \frac{t^n}{n!}, \quad |t| < 1.$$

Now, let us introduce the hyper-Bessel function $\mathcal{J}_{(\alpha_d)}$ defined as follows (see, e.g., [5,14]):

$$\mathcal{J}_{(\alpha_d)}(z) = {}_0F_d \left(\begin{matrix} - \\ 1+\alpha_1, \dots, 1+\alpha_d \end{matrix}; -\big(\frac{z}{d+1}\big)^{d+1} \right),$$

where (α_d) abbreviates a set of d parameters: $\alpha_1, \alpha_2, \dots, \alpha_d$.
The generating function (2.16) can be also written as

$$(2.19) \qquad e^t \mathcal{J}_{\alpha,\beta}\big(3(xt)^{1/3}\big) = \sum_{n=0}^{\infty} B_n^{\alpha,\beta}(x) \frac{t^n}{n!}.$$

Recall that the Laguerre polynomials L_n^{α} , $n \geqslant 0$, are also defined by the classical generating function [15]

$$(2.20) \qquad e^t J_\alpha\big(2\sqrt{xt}\big) = \sum_{n=0}^{\infty} \frac{1}{(1+\alpha)_n} L_n^{\alpha}(x) \frac{t^n}{n!},$$

where J_ν is the Bessel function of the first kind of order ν.

2.2. The third-order differential equation. Now, the use of the identity (see [18, p.107]):

$$(2.21) \qquad D_x\, {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x \right) = \frac{\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_j}\, {}_pF_q \left(\begin{matrix} 1+a_1, \dots, 1+a_p \\ 1+b_1, \dots, 1+b_q \end{matrix}; x \right),$$

with $p = 1$, $q = 2$, $a_1 = -n$, $b_1 = 1+\alpha$ and $b_2 = 1+\beta$, leads to

$$(2.22) \qquad D_x B_{n+1}^{\alpha,\beta} = \lambda_1 (n+1) B_n^{\alpha+1,\beta+1}, \ n \geqslant 0,$$

with $\lambda_1 = -1/(1+\alpha)(1+\beta)$, which gives for the monic polynomials $\widehat{B}_n^{\alpha,\beta}$:

$$(2.23) \qquad D_x \widehat{B}_{n+1}^{\alpha,\beta} = (n+1) \widehat{B}_n^{\alpha+1,\beta+1}, \ n \geqslant 0.$$

Lastly, from the differential equation satisfied by the hypergeometric ${}_pF_q$ (see [18, p.75]), we obtain, for $p = 1$ and $q = 2$, that the polynomials $B_n^{\alpha,\beta}$, $n \geqslant 0$, fulfilled the third order differential equation

$$(2.24) \qquad \Big[x(\theta - n) - \theta(\theta+\alpha)(\theta+\beta)\Big] B_n^{\alpha,\beta}(x) = 0,$$

where $\theta = x\frac{d}{dx}$. By using the formulas

$$\theta(x^k) = kx^k, \ \theta^2 = x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} \ \text{ and } \ \theta^3 = x^3 \frac{d^3}{dx^3} + 3x^2 \frac{d^2}{dx^2} + x\frac{d}{dx},$$

the above differential equation becomes

$$x^2Y''' + (3+\alpha+\beta)xY'' + [(1+\alpha)(1+\beta) - x]Y' + nY = 0, \tag{2.25}$$

where $Y = B_n^{\alpha,\beta}(x)$.

2.3. Other properties.

i) By taking into account of the dependence on the parameters α, β , we obtain the symmetry property

$$\widehat{B}_n^{\alpha,\beta} = \widehat{B}_n^{\beta,\alpha}. \tag{2.26}$$

ii) The differential-recurrence relation (2.4), leads to

$$x\widehat{B}_n^{\alpha+1,\beta+1} = \widehat{B}_{n+1}^{\alpha,\beta} + (n+\alpha+1)(n+\beta+1)\widehat{B}_n^{\alpha,\beta}\ ,\ n \geqslant 0. \tag{2.27}$$

iii) From the relation (15) in [17, p.82], we obtain

$$\widehat{B}_{n+1}^{\alpha,\beta} = \widehat{B}_{n+1}^{\alpha+1,\beta} + (n+1)(n+\beta+1)\widehat{B}_n^{\alpha+1,\beta}\ ,\ n \geqslant 0, \tag{2.28}$$

and by exchanging α and β and using the symmetry property, the above leads to

$$\widehat{B}_{n+1}^{\alpha,\beta} = \widehat{B}_{n+1}^{\alpha,\beta+1} + (n+1)(n+\alpha+1)\widehat{B}_n^{\alpha,\beta+1}\ ,\ n \geqslant 0. \tag{2.29}$$

iv) From (2.5), we have the multiplication formula

$$B_n^{\alpha,\beta}(xy) = \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} B_k^{\alpha,\beta}(x). \tag{2.30}$$

v) On the other hand, by virtue of Srivastava identity [19], we also have

$$B_n^{\alpha,\beta}(xy) = \frac{n!}{(1+\alpha)_n} \sum_{k=0}^{n} \frac{1}{(1+\beta)_k} y^k L_{n-k}^{\alpha+k}(y) L_k^{\beta}(x). \tag{2.31}$$

vi) Finally, by virtue of the inversion formula given in ([18, p.243]), we obtain:

$$x^n = (1+\alpha)_n (1+\beta)_n \sum_{k=0}^{n} (-1)^k \binom{n}{k} B_k^{\alpha,\beta}(x). \tag{2.32}$$

3. The 2-orthogonality of the sequence $\{B_n^{\alpha,\beta}\}_{n\geqslant 0}$. Following the characterization given in [22], to prove the 2-orthogonality of the polynomials $B_n^{\alpha,\beta}$, $n \geqslant 0$, it suffices to show that they satisfy a third order (standard) recurrence relation.
Indeed, since the generating function (2.16) is of type (2.1), then the polynomials $B_n^{\alpha,\beta}$, $n \geqslant 0$, satisfy (2.4) which by reiteration leads to the relation [18]:

$$\frac{B_{n-k}^{\alpha,\beta}(x)}{(n-k)!} = (-1)^k (\theta - n)_k \frac{B_n^{\alpha,\beta}(x)}{n!}\ ,\ \ 0 \leqslant k \leqslant n, \tag{3.1}$$

where the operator $(T)_k$ denotes

$$(T)_k = (T)(T+1)\cdots(T+k-1)\ ;\quad (T)_0 = 1.$$

On the other hand, using the following relations, which one can easily verify:

$$\begin{cases} \theta = (\theta - n) + n, \\ \theta^2 = (\theta - n)_2 + (2n-1)(\theta - n) + n^2, \\ \theta^3 = (\theta - n)_3 + 3(n-1)(\theta-n)_2 + (3n^2 - 3n + 1)(\theta - n) + n^3, \end{cases}$$

to rewrite the differential equation (2.24) under the form

$$\Big[n(\alpha+n)(\beta+n) - \big\{3n^2 - 3n + 1 + (2n-1)(\alpha+\beta) + \alpha\beta - x\big\}(\theta - n) \\ + (3n - 3 + \alpha + \beta)(\theta - n)_2 + (\theta - n)_3\Big] B_n^{\alpha,\beta}(x) = 0, \tag{3.2}$$

which, combined with (3.1), leads to the following recurrence relation:

$$\begin{aligned} &(n+\alpha)(n+\beta)B_n^{\alpha,\beta}(x) \\ &= \big[3n^2 + (2\alpha + 2\beta - 3)n + (1-\alpha)(1-\beta) - x\big] B_{n-1}^{\alpha,\beta}(x) \\ &- (n-1)\big(3n - 3 + \alpha + \beta\big) B_{n-2}^{\alpha,\beta}(x) + (n-1)(n-2) B_{n-3}^{\alpha,\beta}(x)\ ,\ n \geqslant 3. \end{aligned} \tag{3.3}$$

The initial conditions are

$$\begin{aligned} &B_0^{\alpha,\beta}(x) = 1\ ,\ B_1^{\alpha,\beta}(x) = \lambda_1 x + 1, \\ &B_2^{\alpha,\beta}(x) = \lambda_2 x^2 + 2\lambda_1 x + 1, \end{aligned} \tag{3.4}$$

where λ_1 and λ_2 are given by (1.10) for $m = 1$ and $m = 2$, respectively.

The recurrence relation (3.3) was also obtained by Rainville from a pure recurrence relation satisfied by the Bateman functions $J_n^{u,v}$ (see [17,18]).
It is readily seen that the above recurrence relation may be rewritten in the monic form as

$$\begin{aligned} &\widehat{B}_{n+3}^{\alpha,\beta}(x) = (x - \beta_{n+2})\widehat{B}_{n+2}^{\alpha,\beta}(x) - \alpha_{n+2}\widehat{B}_{n+1}^{\alpha,\beta}(x) - \gamma_{n+1}\widehat{B}_n^{\alpha,\beta}(x)\ ,\ n \geqslant 0, \\ &\widehat{B}_0^{\alpha,\beta}(x) = 1\ ,\ \widehat{B}_1^{\alpha,\beta}(x) = x - \beta_0\ ,\ \widehat{B}_2^{\alpha,\beta}(x) = (x - \beta_1)\widehat{B}_1^{\alpha,\beta}(x) - \alpha_1, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\beta_n = \beta_n(\alpha,\beta) = 3n^2 + (2\alpha + 2\beta + 3)n + (1+\alpha)(1+\beta)\ ,\ n \geqslant 0, \\ &\alpha_n = \alpha_n(\alpha,\beta) = n(3n + \alpha + \beta)(n+\alpha)(n+\beta)\ ,\ n \geqslant 1, \\ &\gamma_n = \gamma_n(\alpha,\beta) = n(n+1)(n+\alpha)(n+\alpha+1)(n+\beta)(n+\beta+1)\ ,\ n \geqslant 1. \end{aligned} \tag{3.6}$$

Then, taking into account of the identity (2.23), the above recurrence relation (3.5) leads to

$$\begin{aligned} &Q_{n+3}^{\alpha,\beta}(x) = (x - \widetilde{\beta}_{n+2})Q_{n+2}^{\alpha,\beta}(x) - \widetilde{\alpha}_{n+2}Q_{n+1}^{\alpha,\beta}(x) - \widetilde{\gamma}_{n+1}Q_n^{\alpha,\beta}(x)\ ,\ n \geqslant 0, \\ &Q_0^{\alpha,\beta}(x) = 1\ ,\ Q_1^{\alpha,\beta}(x) = x - \widetilde{\beta}_0\ ,\ Q_2^{\alpha,\beta}(x) = (x - \widetilde{\beta}_1)Q_1^{\alpha,\beta}(x) - \widetilde{\alpha}_1, \end{aligned} \tag{3.7}$$

where $Q_n^{\alpha,\beta} = (n+1)^{-1}D_x\widehat{B}_{n+1}^{\alpha,\beta} = \widehat{B}_n^{\alpha+1,\beta+1}\ ,\ n \geqslant 0,$ and
(3.8)

$$\begin{aligned} &\widetilde{\beta}_n = \beta_n(\alpha+1,\beta+1) = 3n^2 + (2\alpha + 2\beta + 7)n + (2+\alpha)(2+\beta)\ ,\ n \geqslant 0, \\ &\widetilde{\alpha}_n = \alpha_n(\alpha+1,\beta+1) = n(3n + \alpha + \beta + 2)(n+\alpha+1)(n+\beta+1)\ ,\ n \geqslant 1, \\ &\widetilde{\gamma}_n = \gamma_n(\alpha+1,\beta+1) = n(n+1)(n+\alpha+1)(n+\alpha+2)(n+\beta+1)(n+\beta+2)\ ,\ n \geqslant 1. \end{aligned}$$

Therefore, we obtain that the sequence of derivatives $\{Q_n^{\alpha,\beta}\}_{n\geqslant 0}$ also satisfies a standard third-order recurrence relation. Thus $\{\widehat{B}_n^{\alpha,\beta}\}_{n\geqslant 0}$ and $\{Q_n^{\alpha,\beta}\}_{n\geqslant 0}$ are simultaneously 2-orthogonal, that is $\{\widehat{B}_n^{\alpha,\beta}\}_{n\geqslant 0}$ is classical. In this case, the polynomials $\widehat{B}_n^{\alpha,\beta}$, $n\geqslant 0$, will be called the two-parameter family of 2-orthogonal polynomials of Laguerre type. Recall that the polynomials $\{P_n(.;\alpha)\}_{n\geqslant 0}$, studied in [6], constitutes a one-parameter family of 2-orthogonal polynomials of Laguerre type of other kind. Finally, by differentiating (3.5) and making use of (3.7), we obtain after the shift $n\to n-2$

$$\begin{aligned}\widehat{B}_n^{\alpha,\beta} =&\widehat{B}_n^{\alpha+1,\beta+1} + (n+1)(2n+3+\alpha+\beta)\widehat{B}_{n-1}^{\alpha+1,\beta+1}\\ &+n(n+1)(n+\alpha+1)(n+\beta+1)\widehat{B}_{n-2}^{\alpha+1,\beta+1}\,,\ n\geqslant 0,\end{aligned}\tag{3.9}$$

with $\widehat{B}_{-1}^{\alpha,\beta}=\widehat{B}_{-2}^{\alpha,\beta}=0$.
This relation is the analogous of the relation (2.11) satisfied by the Laguerre polynomials .

REMARK. If we put

$$\begin{aligned}\widetilde{\beta}_n &= \beta_{n+1}+\delta_n\,,\ n\geqslant 0,\\ \widetilde{\alpha}_n &= \frac{n}{n+1}\alpha_{n+1}\rho_n\,,\ n\geqslant 1\quad (\rho_n\neq 0),\\ \widetilde{\gamma}_n &= \frac{n}{n+2}\gamma_{n+1}\varepsilon_n\,,\ n\geqslant 1\quad (\varepsilon_n\neq 0),\end{aligned}\tag{3.10}$$

then, it follows from the formulas (3.6) and (3.8) that δ_n, ρ_n and ε_n are given by
(3.11)

$$\delta_n=-(2n+3+\alpha+\beta)\,,\ n\geqslant 0\,,\qquad \rho_n=\frac{3n+2+\alpha+\beta}{3n+3+\alpha+\beta}\,,\ n\geqslant 1 \text{ and } \varepsilon_n=1\,,\ n\geqslant 1.$$

Recall that the two sequences of 2-OPS of Hermite type [7,8] $\{P_n\}_{n\geqslant 0}$ and the one-parameter family of Laguerre type [6] $\{P_n(.;\alpha)\}_{n\geqslant 0}$ are obtained by solving a nonlinear system satisfied by the recurrence coefficients. In such cases, we have obtained the two sets of solutions: $\delta_n=0\,,\ n\geqslant 0\,,\ \rho_n=1\,,\ n\geqslant 1$ and $\varepsilon_n=1\,,\ n\geqslant 1$, give the 2-OPS of Hermite type, and $\delta_n=-1\,,\ n\geqslant 0\,,\ \rho_n=1\,,\ n\geqslant 1$ and $\varepsilon_n=1\,,\ n\geqslant 1$, give the 2-OPS of Laguerre type with one parameter.

4. Integral representations. We denote by $\mathbf{V}={}^t(v_0,v_1)$, respectively, $\widetilde{\mathbf{V}}={}^t(\widetilde{v}_0,\widetilde{v}_1)$, the vector functional with respect to which $\{\widehat{B}_n^{\alpha,\beta}\}_{n\geqslant 0}$, respectively, $\{Q_n^{\alpha,\beta}\}_{n\geqslant 0}$ are 2-orthogonal. Following the characterization given in [10], since $\{\widehat{B}_n^{\alpha,\beta}\}_{n\geqslant 0}$ is classical 2-OPS, then the vector functional $\mathbf{V}$ satisfy the (distributional) equation

$$(\mathbf{\Phi V})'+\mathbf{\Psi V}=0,\tag{4.1}$$

where $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are two 2×2 matrices defined as

$$\mathbf{\Phi}(x)=\begin{pmatrix}\phi_{11}(x) & \phi_{12}(x)\\ \phi_{21}(x) & \phi_{22}(x)\end{pmatrix}\quad,\quad \mathbf{\Psi}(x)=\begin{pmatrix}0 & 1\\ \psi(x) & \zeta\end{pmatrix}\tag{4.2}$$

with $\deg\phi_{ij}\leqslant 1$, $(i,j)\neq(2,1)$, $\deg\phi_{21}\leqslant 2$ and ψ, ζ are given by $\psi(x)=$

$2\gamma_1^{-1}\widehat{B}_1^{\alpha,\beta}(x) = \lambda_2(x+\lambda_1^{-1})$, $\zeta = -2\gamma_1^{-1}\alpha_1 = -\lambda_2\alpha_1$, where

$$\alpha_1 = (1+\alpha)(1+\beta)(3+\alpha+\beta)\,,\ \gamma_1 = 2(1+\alpha)(1+\beta)(2+\alpha)(2+\beta),$$
$$\lambda_1 = -1/(1+\alpha)(1+\beta)\,,\ \lambda_2 = 1/(1+\alpha)(1+\beta)(2+\alpha)(2+\beta).$$

Moreover, the derivative sequence $\{Q_n^{\alpha,\beta}\}_{n\geqslant 0}$ is 2-OPS with respect to the vector functional $\mathbf{\Phi V}$, that is, $\widetilde{\mathbf{V}} = \mathbf{\Phi V}$. Thus, $\widetilde{v}_0 = \phi_{11}(x)v_0+\phi_{12}(x)v_1$ and $\widetilde{v}_1 = \phi_{21}(x)v_0+\phi_{22}(x)v_1$.
Recall that for a linear functional u and a polynomial h the left-multiplication hu is defined by $\langle hu, f\rangle = \langle u, hf\rangle$ for all polynomial f and the derivative u' of u is defined by $\langle u', f\rangle = -\langle u, f'\rangle$.

Now, to determine the explicit expression of the polynomials ϕ_{ij} we write

$$(4.3)\qquad \begin{aligned} \phi_{11}(x) &= a_{11}^1 x + a_{11}^0,\\ \phi_{12}(x) &= a_{12}^1 x + a_{12}^0,\\ \phi_{22}(x) &= a_{22}^1 x + a_{22}^0,\\ \phi_{21}(x) &= a_{21}^2 x^2 + a_{21}^1 x + a_{21}^0. \end{aligned}$$

Firstly, we write (3.9) as

$$\widehat{B}_n^{\alpha,\beta} = Q_n^{\alpha,\beta} + (n+1)(2n+3+\alpha+\beta)Q_{n-1}^{\alpha,\beta} + n(n+1)(n+\alpha+1)(n+\beta+1)Q_{n-2}^{\alpha,\beta}\,,\ n\geqslant 0.$$

Next, applying the functional $\widetilde{v}_0$, respectively, $\widetilde{v}_1$ to the last relation we, respectively, obtain

$$\begin{aligned} \langle \widetilde{v}_0\,,\, \widehat{B}_0^{\alpha,\beta}\rangle &= 1,\\ \langle \widetilde{v}_0\,,\, \widehat{B}_1^{\alpha,\beta}\rangle &= 2(5+\alpha+\beta),\\ \langle \widetilde{v}_0\,,\, \widehat{B}_2^{\alpha,\beta}\rangle &= 6(\alpha+3)(\beta+3)\,,\\ \langle \widetilde{v}_0\,,\, \widehat{B}_n^{\alpha,\beta}\rangle &= 0\,,\ n\geqslant 3, \end{aligned}$$

respectively,

$$\begin{aligned} \langle \widetilde{v}_1\,,\, \widehat{B}_0^{\alpha,\beta}\rangle &= 0,\\ \langle \widetilde{v}_1\,,\, \widehat{B}_1^{\alpha,\beta}\rangle &= 1,\\ \langle \widetilde{v}_1\,,\, \widehat{B}_2^{\alpha,\beta}\rangle &= 3(7+\alpha+\beta),\\ \langle \widetilde{v}_1\,,\, \widehat{B}_3^{\alpha,\beta}\rangle &= 12(\alpha+4)(\beta+4),\\ \langle \widetilde{v}_1\,,\, \widehat{B}_n^{\alpha,\beta}\rangle &= 0\,,\ n\geqslant 4. \end{aligned}$$

By using the same technique as in [6], Section 4, we obtain after some calculations, two linear systems, easy to solve, in which the unknowns are the coefficients a_{ij}^k. Thus, we get

$$(4.4)\qquad \begin{aligned} a_{11}^1 &= -\lambda_1\,,\ a_{11}^0 = 0\,,\\ a_{12}^1 &= a_{12}^0 = 0\,,\\ a_{21}^1 &= -\lambda_2\,,\ a_{21}^2 = a_{21}^0 = 0\,,\\ a_{22}^1 &= -\lambda_2\lambda_1^{-1}\,,\ a_{22}^0 = 0. \end{aligned}$$

Consequently, we have

$$\Phi(x) = \begin{pmatrix} -\lambda_1 x & 0 \\ -\lambda_2 x & -\lambda_2\lambda_1^{-1}x \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} 0 & 1 \\ \lambda_2(x+\lambda_1^{-1}) & -\lambda_2\alpha_1 \end{pmatrix}. \tag{4.5}$$

We obtain that the linear functionals v_0 and v_1 satisfy, in $\mathcal{P}'$, the following system:

$$\begin{cases} v_1 = \lambda_1(xv_0)', \\ \lambda_1(xv_0)' + (xv_1)' - (\lambda_1 x + 1)v_0 + \lambda_1\alpha_1 v_1 = 0, \end{cases} \tag{4.6}$$

which gives

$$\begin{cases} x^2v_0'' - (\alpha+\beta-1)xv_0' - (x-\alpha\beta)v_0 = 0\,, \\ (1+\alpha)(1+\beta)v_1 = -(xv_0)', \end{cases} \tag{4.7}$$

that is, each of the functionals v_0 and v_1 satisfies a second order (distributional) differential equation.

Now, we are ready to look for integral representations of the linear functionals v_0 and v_1 with respect to which $\{\widehat{B}_n^{\alpha,\beta}\}_{n\geqslant 0}$ is 2-OPS. Since v_1 depends on v_0, it suffices to find integral representation of v_0 to obtain a v_1 one. Thus, the problem is to determine a weight function $\mathcal{W}_{\alpha,\beta}$ such that, for all polynomial $f \in \mathcal{P}$, we have

$$\langle v_0\ ,\, f\rangle = \int_C f(x)\mathcal{W}_{\alpha,\beta}(x)dx, \tag{4.8}$$

where we suppose that the function $\mathcal{W}_{\alpha,\beta}$ is regular as far as it is necessary and the path C consists of a part of the complex plane. However, here we only consider the representations in terms of integrals evaluated along an interval (a, b) of the real axis with $-\infty \leqslant a < b \leqslant +\infty$.

We have the following result:

THEOREM 4.1. *When* $\alpha \geqslant \beta > -1$ *and* $(a,b) = (0\,, +\infty)$*, the pair of functionals* v_0 *and* v_1 *have the integral representations*

$$\langle v_0\ ,\, f\rangle = \int_0^\infty f(x)\mathcal{W}_{\alpha,\beta}(x)dx, \tag{4.9}$$

$$\langle v_1\ ,\, f\rangle = \lambda_1 \int_0^\infty f(x)\big(x\mathcal{W}_{\alpha,\beta}(x)\big)'dx, \tag{4.10}$$

where the weight function $\mathcal{W}_{\alpha,\beta}$ *is given by*

$$\mathcal{W}_{\alpha,\beta}(x) = \frac{2}{\Gamma(1+\alpha)\Gamma(1+\beta)} x^{(\alpha+\beta)/2} K_{\alpha-\beta}\big(2\sqrt{x}\big), \tag{4.11}$$

with K_ν *is the MacDonald function (modified Bessel function of the third kind) and the parameter* $\lambda_1 = -1/(1+\alpha)(1+\beta)$.

Proof. From (4.7) - (4.8), by integration by parts two times it follows that

$$\int_a^b \left\{x^2\mathcal{W}''_{\alpha,\beta}(x) - (\alpha+\beta-1)x\mathcal{W}'_{\alpha,\beta}(x) - \big(x-\alpha\beta\big)\mathcal{W}_{\alpha,\beta}(x)\right\} f(x)dx$$
$$+ x^2\mathcal{W}_{\alpha,\beta}(x)f'(x) - \left\{x^2\mathcal{W}'_{\alpha,\beta}(x) - (\alpha+\beta+1)x\mathcal{W}_{\alpha,\beta}(x)\right\}f(x)\Big]_a^b = 0.$$

If the following condition holds:

$$x^2\mathcal{W}_{\alpha,\beta}(x)f'(x)-\left\{x^2\mathcal{W}'_{\alpha,\beta}(x)-(\alpha+\beta+1)x\mathcal{W}_{\alpha,\beta}(x)\right\}f(x)\Big]_a^b=0, \tag{4.12}$$

then, we obtain that the function $\mathcal{W}_{\alpha,\beta}$ satisfies the second-order differential equation

$$x^2\mathcal{W}''_{\alpha,\beta}(x)-(\alpha+\beta-1)x\mathcal{W}'_{\alpha,\beta}(x)-\big(x-\alpha\beta\big)\mathcal{W}_{\alpha,\beta}(x)=\eta\mathcal{N}(x), \tag{4.13}$$

where η is an arbitrary constant and $\mathcal{N}$ is a function representing the null functional over the interval (a,b)

$$\int_a^b x^n\mathcal{N}(x)dx=0\,,\ n\geqslant 0.$$

In the sequel, we choose $\eta=0$. Hence (4.13) leads to the Bessel differential equation

$$x^2\mathcal{W}''_{\alpha,\beta}(x)-(\alpha+\beta-1)x\mathcal{W}'_{\alpha,\beta}(x)-\big(x-\alpha\beta\big)\mathcal{W}_{\alpha,\beta}(x)=0. \tag{4.14}$$

Under the transformation $s=2\sqrt{x}\,,\ \nu=\alpha-\beta$, and setting $\mathcal{W}_{\alpha,\beta}(x)=x^{(\alpha+\beta)/2}\mathcal{W}(s)$, the last equation becomes

$$s^2\mathcal{W}''(s)+s\mathcal{W}'(s)-\big(s^2+\nu^2\big)\mathcal{W}(s)=0 \tag{4.15}$$

which is a modified Bessel equation.
A solution of (4.14) is to be found satisfying the condition (4.12). One of the solutions of the last equation is the function K_ν known as a modified Bessel function of the third kind; or also a Macdonald's function. Thus, by choosing $(a,b)=(0,+\infty)$, a solution of Eq. (4.15) can be taken as

$$\mathcal{W}(s)=\lambda K_{\alpha-\beta}(s), \tag{4.16}$$

where $\lambda=\lambda(\alpha,\beta)$ is the normalization constant which we determine below.
Then, a solution of (4.14) is given by

$$\mathcal{W}_{\alpha,\beta}(x)=\lambda x^{(\alpha+\beta)/2}K_{\alpha-\beta}\big(2\sqrt{x}\big). \tag{4.17}$$

The function $K_\nu(z)$, for arbitrary order ν, is an analytic function of the complex variable z for $z\neq 0$, and is an even entire analytic function of ν for each $z\neq 0$, that is, $K_{-\nu}=K_\nu$. It satisfies some simple recurrence relations, see, e.g., [15]

$$-2K'_\nu(z)=K_{\nu-1}(z)+K_{\nu+1}(z), \tag{4.18}$$

$$-2\nu z^{-1}K_\nu(z)=K_{\nu-1}(z)-K_{\nu+1}(z). \tag{4.19}$$

One evident reason of the choice of the Macdonald's function as solution of the equation (4.15) is that, for $x>0$ and $\nu\geqslant 0$, the function K_ν is a positive function which decreases monotonically as $x\longrightarrow\infty$.
The asymptotic behavior of K_ν, as $x\longrightarrow\infty$, is given by

$$K_\nu(x)\sim\sqrt{\frac{\pi}{2x}}e^{-x}\quad,\quad\text{as}\quad x\longrightarrow\infty. \tag{4.20}$$

For small x, we have the asymptotic formula.

$$\begin{aligned}&K_\nu(x)\sim 2^{\nu-1}\Gamma(\nu)\,x^{-\nu}\ ,\ \nu\neq 0\quad,\quad\text{as}\quad x\longrightarrow 0,\\&K_0(x)\sim\log\frac{2}{x}\quad,\quad\text{as}\quad x\longrightarrow 0.\end{aligned} \tag{4.21}$$

Also, we have

$$K_\nu(z) = \frac{2^{-\nu}\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})} z^\nu \int_1^\infty e^{-zt}\big(t^2-1\big)^{\nu-1/2} dt\ , \ \Re z > 0 \text{ and } \Re\nu > -1/2, \tag{4.22}$$

and

$$K_\nu(x) = \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}} x^{-\nu} \int_0^\infty \frac{\cos(xt)}{\big(t^2+1\big)^{\nu+\frac{1}{2}}} dt\ , \ x > 0 \text{ and } \Re\nu > -1/2. \tag{4.23}$$

Finally, we terminate these properties of the function K_ν by the following formula:
(4.24)

$$\int_0^\infty t^\mu K_\nu(ct)dt = 2^{\mu-1}c^{-(\mu+1)}\Gamma\Big(\frac{1+\mu+\nu}{2}\Big)\Gamma\Big(\frac{1+\mu-\nu}{2}\Big),\ \Re(1+\mu\pm\nu)>0,\ \Re c>0.$$

In particular, for $c = 1$ and $\mu = n \in \mathbb{N}$, we obtain the moments of the function K_ν.

Now, we are ready to determine the constant λ. Indeed, the normalization condition

$$\langle v_0, 1\rangle = \int_0^\infty \mathcal{W}_{\alpha,\beta}(x)dx = 1,$$

leads to

$$\lambda \int_0^\infty x^{(\alpha+\beta)/2} K_{\alpha-\beta}\big(2\sqrt{x}\big)dx = 1. \tag{4.25}$$

Under the transformation $t = \sqrt{x}$ with $c = 2$, we obtain from the formula (4.24) that

$$\int_0^\infty x^{(\alpha+\beta)/2} K_{\alpha-\beta}\big(2\sqrt{x}\big)dx = 2\int_0^\infty t^{\alpha+\beta+1}K_{\alpha-\beta}(2t)dt = \frac{1}{2}\Gamma(1+\alpha)\Gamma(1+\beta).$$

Whence (4.25) gives

$$\lambda = \frac{2}{\Gamma(1+\alpha)\Gamma(1+\beta)}. \tag{4.26}$$

Hence, the weight function $\mathcal{W}_{\alpha,\beta}$ is given by (4.11).
Now, it remains to verify that the condition (4.12) holds.
Since $\mathcal{W}_{\alpha,\beta} = \mathcal{W}_{\beta,\alpha}$, in all the sequel we put $\alpha \geqslant \beta$.

Next, from (4.11) and making use of the asymptotic formula (4.20), we get, as $x \longrightarrow \infty$,

$$x\mathcal{W}_{\alpha,\beta}(x) \sim \frac{\sqrt{\pi}}{2}\lambda\, x^{(2\alpha+2\beta+3)/4}e^{-2\sqrt{x}} \longrightarrow 0,$$

$$x^2\mathcal{W}_{\alpha,\beta}(x) \sim \frac{\sqrt{\pi}}{2}\lambda\, x^{(2\alpha+2\beta+7)/4}e^{-2\sqrt{x}} \longrightarrow 0.$$

Otherwise, by differentiating $\mathcal{W}_{\alpha,\beta}$ and making use of the formulas (4.18)-(4.19), we get

$$\mathcal{W}'_{\alpha,\beta}(x) = \lambda\beta x^{(\alpha+\beta-2)/2}K_{\alpha-\beta}\big(2\sqrt{x}\big) - \lambda x^{(\alpha+\beta-1)/2}K_{\alpha-\beta-1}\big(2\sqrt{x}\big).$$

Then,

$$x^2\mathcal{W}'_{\alpha,\beta}(x) = \lambda\beta x^{(\alpha+\beta+2)/2}K_{\alpha-\beta}\big(2\sqrt{x}\big) - \lambda x^{(\alpha+\beta+3)/2}K_{\alpha-\beta-1}\big(2\sqrt{x}\big).$$

It is easy to verify that $x^2\mathcal{W}'_{\alpha,\beta}(x) \longrightarrow 0\ ,\quad$ as $x \longrightarrow \infty$.

Finally, at the origin, using (4.20), we obtain:

- When $\alpha \neq \beta$

$$x\mathcal{W}_{\alpha,\beta}(x) \sim \lambda\Gamma(\alpha-\beta)x^{\beta+1} \longrightarrow 0 \quad \text{if } \beta > -1,$$
$$x^2\mathcal{W}_{\alpha,\beta}(x) \sim \lambda\Gamma(\alpha-\beta)x^{\beta+2} \longrightarrow 0 \quad \text{if } \beta > -2,$$

and

$$x^2\mathcal{W}'_{\alpha,\beta}(x) = \frac{\lambda}{2}\beta\Gamma(\alpha-\beta)x^{\beta+1} - \frac{\lambda}{2}\Gamma(\alpha-\beta-1)x^{\beta+2} \longrightarrow 0 \quad \text{if } \beta > -1.$$

- When $\alpha = \beta$, we have

$$\mathcal{W}_{\alpha,\alpha}(x) = \lambda x^{\alpha} K_0\left(2\sqrt{x}\right),$$

so that, taking into account of (4.21), we get

$$x\mathcal{W}_{\alpha,\alpha}(x) \sim -\frac{\lambda}{2}x^{\alpha+1}\log x \longrightarrow 0 \quad \text{if } \alpha > -1,$$
$$x^2\mathcal{W}_{\alpha,\alpha}(x) \sim -\frac{\lambda}{2}x^{\alpha+2}\log x \longrightarrow 0 \quad \text{if } \alpha > -2,$$

and

$$x^2\mathcal{W}'_{\alpha,\alpha}(x) = \lambda\alpha x^{\alpha+1}K_0\left(2\sqrt{x}\right) - \lambda\alpha x^{\alpha+2}K_1\left(2\sqrt{x}\right) \longrightarrow 0 \quad \text{if } \alpha > -3/2.$$

We will now proceed to look for integral representation for the functional v_1. From (4.6), we have

$$\langle v_1, f\rangle = \lambda_1\langle (xv_0)', f\rangle = -\lambda_1\langle xv_0, f'\rangle = -\lambda_1\langle v_0, xf'\rangle = -\lambda_1\int_0^\infty x\mathcal{W}_{\alpha,\beta}(x)f'(x)dx.$$

Integration by parts gives

$$\int_0^\infty x\mathcal{W}_{\alpha,\beta}(x)f'(x)dx = x\mathcal{W}_{\alpha,\beta}(x)f(x)\Big]_0^\infty - \int_0^\infty \left(x\mathcal{W}_{\alpha,\beta}(x)\right)' f(x)dx.$$

Since $x\mathcal{W}_{\alpha,\beta}(x)f(x)\Big]_0^\infty = 0$, then

$$\langle v_1, f\rangle = \lambda_1\int_0^\infty \left(x\mathcal{W}_{\alpha,\beta}(x)\right)' f(x)dx.$$

□

Thus, we have the following trivial result:

COROLLARY 4.1. *The moments of the functionals v_0 and v_1 are given by*

$$(v_0)_n = \int_0^\infty x^n\mathcal{W}_{\alpha,\beta}(x)dx = (1+\alpha)_n(1+\beta)_n\,,\ n \geqslant 0, \tag{4.27}$$

and

$$(v_1)_n = -\lambda_1 n(v_0)_n = -\lambda_1 n(1+\alpha)_n(1+\beta)_n\,,\ n \geqslant 0, \tag{4.28}$$

where $(v_0)_n := \langle v_0, x^n\rangle$, $n \geqslant 0$, respectively, $(v_1)_n := \langle v_1, x^n\rangle$, $n \geqslant 0$, denote the moments of the functional v_0, respectively, v_1.

REMARK. It is clear that the weight function $\mathcal{W}_{\alpha,\beta}$ is positive on the interval $0 \leqslant x < \infty$. Then, the linear functional v_0 is positive definite. Thus, there exists a sequence of monic polynomials $\{\widehat{V}_n^{\alpha,\beta}\}_{n\geqslant 0}$ orthogonal (in the ordinary sense) on the interval $0 \leqslant x < \infty$ with respect to the weight function $\mathcal{W}_{\alpha,\beta}$, that is to say,

$$\int_0^\infty \widehat{V}_m^{\alpha,\beta}(x)\widehat{V}_n^{\alpha,\beta}(x)\mathcal{W}_{\alpha,\beta}(x)dx = k_n\delta_{m,n}\,,\ m,\, n \geqslant 0,$$

where $k_n \neq 0$ is the normalization coefficient given by

$$k_n = \left\langle v_0, (\widehat{V}_n^{\alpha,\beta})^2 \right\rangle = \int_0^\infty \left(\widehat{V}_n^{\alpha,\beta}(x)\right)^2 \mathcal{W}_{\alpha,\beta}(x)dx. \tag{4.29}$$

If we denote by $\{w_n\}_{n\geqslant 0}$ the dual sequence associated to the polynomials $\{\widehat{V}_n^{\alpha,\beta}\}_{n\geqslant 0}$, it is clear that the first (canonical) functional $w_0 = v_0$, so that $w_n = k_n^{-1}\widehat{V}_n^{\alpha,\beta}\, w_0 = k_n^{-1}\widehat{V}_n^{\alpha,\beta}\, v_0\,,\ n \geqslant 0$.

Now, writing the recurrence relation for the sequence $\{\widehat{V}_n^{\alpha,\beta}\}_{n\geqslant 0}$ as

$$\begin{aligned} &\widehat{V}_{n+2}^{\alpha,\beta}(x) = (x - \zeta_{n+1})\widehat{V}_{n+1}^{\alpha,\beta}(x) - \varrho_{n+1}\widehat{V}_n^{\alpha,\beta}(x)\,;\ n \geqslant 0\,,\ \big(\varrho_n \neq 0\,,\ n \geqslant 1\big), \\ &\widehat{V}_0^{\alpha,\beta}(x) = 1\,,\ \widehat{V}_1^{\alpha,\beta}(x) = x - \zeta_0. \end{aligned} \tag{4.30}$$

With the aid of the formulas

$$\zeta_n = k_n^{-1} \int_0^\infty x\left(\widehat{V}_n^{\alpha,\beta}(x)\right)^2 \mathcal{W}_{\alpha,\beta}(x)dx\,,\ n \geqslant 0,$$

and

$$\varrho_{n+1} = k_n^{-1} \int_0^\infty x\widehat{V}_n^{\alpha,\beta}(x)\widehat{V}_{n+1}^{\alpha,\beta}(x)\mathcal{W}_{\alpha,\beta}(x)dx\,,\ n \geqslant 0,$$

we can compute the following first coefficients:

$$\begin{aligned} \zeta_0 &= \beta_0 = (1+\alpha)(1+\beta), \\ \zeta_1 &= -1 + \frac{2(2+\alpha)(2+\beta)(4+\alpha+\beta)}{(1+\alpha)(1+\beta)(3+\alpha+\beta)}, \\ \varrho_1 &= \alpha_1 = (1+\alpha)(1+\beta)(3+\alpha+\beta). \end{aligned}$$

Also, upon writing

$$\widehat{V}_n^{\alpha,\beta}(x) = \sum_{m=0}^{n} c_{n,m}\widehat{B}_m^{\alpha,\beta}(x)\,,\ n \geqslant 0,$$

it is simple to verify that $c_{n,n} = 1\,,\ n \geqslant 0$ and $c_{n,0} = 0\,,\ n > 1$.
For example, we obtain

$$\begin{aligned} \widehat{V}_0^{\alpha,\beta}(x) &= \widehat{B}_0^{\alpha,\beta}(x) = 1, \\ \widehat{V}_1^{\alpha,\beta}(x) &= \widehat{B}_1^{\alpha,\beta}(x) = x - \beta_0, \\ \widehat{V}_2^{\alpha,\beta}(x) &= \widehat{B}_2^{\alpha,\beta}(x) - \frac{2(2+\alpha)(2+\beta)}{(3+\alpha+\beta)}\widehat{B}_1^{\alpha,\beta}(x). \end{aligned}$$

PROBLEM. Find all the properties satisfied by the polynomials $\widehat{V}_n^{\alpha,\beta}\,,\ n \geqslant 0$. For instance, the recurrence relation, the differential equation, the generating function, the connection between $\{\widehat{V}_n^{\alpha,\beta}\}_{n\geqslant 0}$ and $\{\widehat{B}_n^{\alpha,\beta}\}_{n\geqslant 0}, \ldots$.

Particular cases.

(**1**) For $\beta = 0$ and $\alpha = k$, $k \in \mathbb{N}$ (or $\alpha = 0$, $\beta = k$ because $\mathcal{W}_{\alpha,\beta} = \mathcal{W}_{\beta,\alpha}$), we obtain that

$$\mathcal{W}_{k,0}(x) = \frac{2}{k!} x^{k/2} K_k(2\sqrt{x}) = \frac{1}{k!}\xi_k(x), \tag{4.31}$$

where ξ_k, $k = 0, 1, \dots$, are the ultra-exponential weight functions introduced by Voronoï [23] and defined as

$$\xi_k(x) = \frac{2\sqrt{\pi}}{\Gamma(k+\frac{1}{2})} x^k \int_1^\infty e^{-2t\sqrt{x}}(t^2-1)^{k-1/2} dt\,,\ x > 0 \quad (k = 0, 1, \dots). \tag{4.32}$$

In this case, the moments of v_0 are

$$(v_0)_n = \int_0^\infty x^n \mathcal{W}_{k,0}(x) dx = \frac{(n+k)!n!}{k!},$$

so that

$$\int_0^\infty x^n \xi_k(x) dx = (n+k)!n!\,,\ n \geqslant 0\,,\ k = 0, 1, \dots .$$

Note that, these functions are also linked with another set of functions denoted by $\xi(x,k)$, $k = 1, 2, \dots$, and defined as follows [23]

$$\Gamma^k(s) = \int_0^\infty x^{s-1}\xi(x,k) dx\,,\ \Re s > 0\,,\ k = 1, 2, \dots,$$

or by inversion

$$\xi(x,k) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma^k(s)}{x^s} ds\,,\ c, x, \Re s > 0.$$

• When $k = 1$, then $\xi(x,1) = e^{-x}$, so that the corresponding orthogonal polynomials are the Laguerre $\widehat{L}_n$, $n \geqslant 0$.
• When $k = 2$, we obtain that $\xi(x,2) = \xi_0(x) = \mathcal{W}_{0,0}(x)$, where

$$\mathcal{W}_{0,0}(x) = 2K_0(2\sqrt{x}) = 2\int_0^\infty \frac{\cos(2t\sqrt{x})}{\sqrt{t^2+1}} dt\,,\ x > 0. \tag{4.33}$$

In this case, we have

$$(v_0)_n = 2\int_0^\infty x^n K_0(2\sqrt{x}) dx = (n!)^2\,,\ n \geqslant 0. \tag{4.34}$$

Thus, we encounter the open problem posed by Prudnikov (see [20, pp.239-241]) in the Seventh Spanish Symposium on Orthogonal Polynomials and their Applications (VII SPOA) (Granada, September 23-27, 1991). This problem has been recently studied by Van Assche and Yakubovich [21]. Hence the two papers bring, independently, an answer to that open problem.

(**2**) For $\beta = \alpha$, we have

$$\mathcal{W}_{\alpha,\alpha}(x) = \lambda x^\alpha K_0(2\sqrt{x}) = \frac{2}{\Gamma^2(1+\alpha)} x^\alpha \int_0^\infty \frac{\cos(2t\sqrt{x})}{\sqrt{t^2+1}} dt\,,\ x > 0 \tag{4.35}$$

and

$$(4.36) \qquad (v_0)_n = \int_0^\infty x^n \mathcal{W}_{\alpha,\alpha}(x)dx = \big((1+\alpha)_n\big)^2 \,,\; n \geqslant 0.$$

In particular, when $\alpha = 0$, we meet again

$$\mathcal{W}_{0,0}(x) = 2K_0\big(2\sqrt{x}\big).$$

(**3**) For $\beta = \alpha + 1/2$, we have

$$(4.37) \qquad \mathcal{W}_{\alpha,\alpha+1/2}(x) = \frac{2^{2\alpha+1}}{\Gamma(2+2\alpha)} x^{\alpha+1/4} K_{1/2}\big(2\sqrt{x}\big).$$

But, from the formula

$$K_{n-1/2}(z) = \sqrt{\frac{\pi}{2}} z^{n-1/2}\Big(-\frac{1}{z}\frac{d}{dz}\Big)^n e^{-z} \,,\; n = 0,1,2,\ldots\,,$$

we obtain $K_{1/2}(z) = \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z}$, then

$$(4.38) \qquad \mathcal{W}_{\alpha,\alpha+1/2}(x) = \frac{\sqrt{\pi} 2^{2\alpha+1}}{\Gamma(2+2\alpha)} x^{\alpha} e^{-2\sqrt{x}}.$$

5. The relation between the 2-OPSs of Hermite type and Laguerre type. It is well known that the Hermite and Laguerre polynomials are related by

$$(5.1) \qquad \begin{aligned} \widehat{H}_{2n}(x) &= \widehat{L}_n^{-\frac{1}{2}}(x^2), \\ \widehat{H}_{2n+1}(x) &= x\widehat{L}_n^{\frac{1}{2}}(x^2). \end{aligned}$$

In this section, we are concerned in the connection between the sequences $\{P_n\}_{n\geqslant 0}$ defined by the recurrence (1.3) and $\{\widehat{B}_n^{\alpha,\beta}\}_{n\geqslant 0}$ and in the integral representations of the pair of linear functionals u_0, u_1 with respect to which $\{P_n\}_{n\geqslant 0}$ is 2-orthogonal. These polynomials satisfy the 2-symmetric property, that is

$$P_n(\omega x) = \omega^n P_n(x) \,,\; n \geqslant 0 \,,\; \text{where } \omega = e^{i\frac{2\pi}{3}}.$$

Consequently, we obtain the following cubic decomposition of the sequence $\{P_n\}_{n\geqslant 0}$ [9]:

$$(5.2) \qquad P_{3n+\tau}(x) = x^\tau P_n^\tau(x^3) \,,\; n \geqslant 0 \,,\; \tau = 0,1,2, \Longleftrightarrow \begin{cases} P_{3n}(x) = P_n^0(x^3), \\ P_{3n+1}(x) = xP_n^1(x^3), \\ P_{3n+2}(x) = x^2 P_n^2(x^3), \end{cases}$$

where the sequences $\{P_n^\tau\}_{n\geqslant 0}$ $(\tau = 0,1,2)$ are the cubic components of $\{P_n\}_{n\geqslant 0}$. For each τ, the corresponding component is also 2-orthogonal with respect to the pair of linear functionals u_0^τ, u_1^τ (see [9]), with

$$(5.3) \qquad \begin{cases} u_0^\tau = \sigma_3\big(x^\tau u_\tau\big) \\ u_1^\tau = \sigma_3\big(x^\tau u_{\tau+3}\big) \end{cases} ;\; \tau = 0,1,2,$$

where the operator σ_3 is defined by

$$\begin{aligned}\sigma_3 :\ \mathcal{P} &\longrightarrow \mathcal{P}\\ f(x) &\longmapsto f(x^3).\end{aligned}$$

By transposition, we define ${}^t\sigma_3 :\ \mathcal{P}' \longrightarrow \mathcal{P}'$ by

$$\langle {}^t\sigma_3(u)\ ,\ f\rangle = \langle u\ ,\ \sigma_3 f\rangle\ ,\ \forall u \in \mathcal{P}'\ ,\ \forall f \in \mathcal{P}.$$

Therefore, from now on, we set ${}^t\sigma_3 = \sigma_3$.

Each sequence $\{P_n^\tau\}_{n\geqslant 0}$ $(\tau = 0,1,2)$ satisfy a third order recurrence relation
(5.4)
$$\begin{cases} P_{n+3}^\tau(x) = (x-\beta_{n+2}^\tau)P_{n+2}^\tau(x) - \alpha_{n+2}^\tau P_{n+1}^\tau(x) - \gamma_{n+1}^\tau P_n^\tau(x), \quad (\gamma_{n+1}^\tau \neq 0,\, n \geqslant 0),\\ P_0^\tau(x) = 1 \quad , \quad P_1^\tau(x) = x - \beta_0^\tau \quad , \quad P_2^\tau(x) = (x-\beta_1^\tau)P_1^\tau(x) - \alpha_1^\tau. \end{cases}$$

In this case, the coefficients β_n^τ , α_{n+1}^τ and γ_{n+1}^τ , $n \geqslant 0$, are given as follows (see [9] for the formulas).

- For $\tau = 0$:

$$\begin{aligned}
\beta_n^0 &= \rho(27n^2+9n+2)\ ,\ n\geqslant 0,\\
\alpha_n^0 &= 3\rho^2(3n)(3n-1)^2(3n-2)\,,\ n\geqslant 1,\\
\gamma_n^0 &= \rho^3(3n+3)(3n+2)(3n+1)(3n)(3n-1)(3n-2)\,,\ n\geqslant 1.
\end{aligned}$$

- For $\tau = 1$:

$$\begin{aligned}
\beta_n^1 &= \rho(3n+1)(3n)^2(3n-1)\,,\ n\geqslant 0,\\
\alpha_n^1 &= 3\rho^2(3n+1)(3n)^2(3n-1)\,,\ n\geqslant 1,\\
\gamma_n^1 &= \rho^3(3n+4)(3n+3)(3n+2)(3n+1)(3n)(3n-1)\,,\ n\geqslant 1.
\end{aligned}$$

- For $\tau = 2$:

$$\begin{aligned}
\beta_n^2 &= \rho(27n^2+45n+20)\ ,\ n\geqslant 0,\\
\alpha_n^2 &= 3\rho^2(3n+2)(3n+1)^2(3n)\,,\ n\geqslant 1,\\
\gamma_n^2 &= \rho^3(3n+5)(3n+4)(3n+3)(3n+2)(3n+1)(3n)\,,\ n\geqslant 1.
\end{aligned}$$

From the above formulas, with $\rho = 1/9$, we obtain that

$$P_n^0 = \widehat{B}_n^{-\frac{2}{3},-\frac{1}{3}}\ ,\ P_n^1 = \widehat{B}_n^{-\frac{1}{3},\frac{1}{3}} \text{ and } \quad P_n^2 = \widehat{B}_n^{\frac{1}{3},\frac{2}{3}},$$

that is to say,

$$\begin{aligned}
P_{3n}(x) &= \widehat{B}_n^{-\frac{2}{3},-\frac{1}{3}}(x^3),\\
P_{3n+1}(x) &= x\widehat{B}_n^{-\frac{1}{3},\frac{1}{3}}(x^3),\\
P_{3n+2}(x) &= x^2\widehat{B}_n^{\frac{1}{3},\frac{2}{3}}(x^3).
\end{aligned}$$

From (5.3), the functionals u_0^τ , u_1^τ $(\tau = 0,1,2)$ are given by
- $u_0^0 = \sigma_3(u_0)$, $\quad u_1^0 = \sigma_3(u_3)$,
- $u_0^1 = \sigma_3(xu_1)$, $\quad u_1^1 = \sigma_3(xu_4)$,
- $u_0^2 = \sigma_3(x^2u_2)$, $\quad u_1^2 = \sigma_3(x^2u_5)$.

The functionals u_τ, $0 \leqslant \tau \leqslant 5$ are the first five elements of the dual sequence associated to the sequence of polynomials $\{P_n\}_{n\geqslant 0}$, where u_τ, $2 \leqslant \tau \leqslant 5$ can only be expressed in terms of u_0 and u_1.

On the other hand, if we take into account the dependence on the parameters α and β, and put $v_0 = v_0(\alpha,\beta)$, $v_1 = v_1(\alpha,\beta)$, we have

- for $\alpha = -\frac{1}{3}$, $\beta = -\frac{2}{3}$: $u_0^0 = v_0\left(-\frac{1}{3}, -\frac{2}{3}\right)$, $u_1^0 = v_1\left(-\frac{1}{3}, -\frac{2}{3}\right)$,
- for $\alpha = \frac{1}{3}$, $\beta = -\frac{1}{3}$: $u_0^1 = v_0\left(\frac{1}{3}, -\frac{1}{3}\right)$, $u_1^1 = v_1\left(\frac{1}{3}, -\frac{1}{3}\right)$,
- for $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$: $u_0^2 = v_0\left(\frac{2}{3}, \frac{1}{3}\right)$, $u_1^2 = v_1\left(\frac{2}{3}, \frac{1}{3}\right)$.

For example, the integral representations of the linear functionals u_0^0, u_1^0 are

$$\langle u_0^0 , f\rangle = \int_0^\infty f(x)\mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x)dx, \tag{5.5}$$

$$\langle u_1^0 , f\rangle = -\frac{9}{2}\int_0^\infty f(x)\big(x\mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x)\big)'dx, \tag{5.6}$$

where

$$\mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x) = \frac{\sqrt{3}}{\pi}x^{-1/2}K_{1/3}(2\sqrt{x}). \tag{5.7}$$

5.1. Integral representation of the functionals u_0 and u_1 corresponding to $\{P_n\}_{n\geqslant 0}$. Recall that $\{P\}_{n\geqslant 0}$ is an Appell polynomials sequence if and only if $P'_{n+1}(x) = (n+1)P_n(x)$. Such a character is translated in $\mathcal{P}'$ by the fact that the dual sequence $\{u_n\}_{n\geqslant 0}$ satisfy

$$Du_n = -(n+1)u_{n+1} \iff u_n = \frac{(-1)^n}{n!}D^n u_0 \quad n \geqslant 0, \tag{5.8}$$

which is another characteristic of Appell sequences and which gives all the elements of the dual sequence $\{u_n\}_{n\geqslant 0}$ in terms of the successive derivatives of the (canonical) linear functional u_0. Then, we have $u_1 = -u_0'$. Consequently, it suffices to determine an integral representation of u_0 to obtain the u_1 one.

This is given in the following theorem:

THEOREM 5.1. *The two linear functionals u_0 and u_1 have the integral representations*

$$\langle u_0, f\rangle = \int_0^\infty \mathcal{U}(x)f(x)dx, \tag{5.9}$$

$$\langle u_1, f\rangle = -\int_0^\infty \mathcal{U}'(x)f(x)dx, \tag{5.10}$$

where the weight function $\mathcal{U}$ is given by

(5.11)

$$\mathcal{U}(x) = \frac{3\sqrt{3}}{\pi}x^{1/2}K_{1/3}(2x^{3/2}) + 3\eta_1 x^{1/2}e^{-x}\cos(\sqrt{3}\,x) + 3\eta_2 x^2 e^{-x}\sin(\sqrt{3}\,x)\ ,\ x \in \mathbb{R}_+,$$

with

$$\eta_1 = \frac{2^7 3^{-4}}{\pi\sqrt{2\pi}}\{6\Gamma(2/3) - \Gamma(1/3)\} \textit{ and } \eta_1 = \frac{2^3 3^{-4}}{\pi}\{4\Gamma(1/3) - 15\Gamma(2/3)\}. \tag{5.12}$$

Proof. To prove this result, we use the same technique explained in [8]. We obtained in the above subsection the representation of the form $u_0^0 = \sigma_3(u_0)$:

$$\langle \sigma_3(u_0), f \rangle = \int_{-\infty}^{+\infty} Y(x) \mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x) f(x) dx \ , \ f \in \mathcal{P}, \tag{5.13}$$

where $\mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}$ is given by (5.7) and Y is the Heaviside function defined as

$$Y(x) = \begin{cases} 1 \ , & \text{if } x > 0, \\ 0 \ , & \text{otherwise.} \end{cases} \tag{5.14}$$

We now look for a weight function $\mathcal{U}$ such that

$$\langle u_0, f \rangle = \int_{-\infty}^{+\infty} \mathcal{U}(x) f(x) dx. \tag{5.15}$$

From which, we have, in particular

$$\langle u_0, \sigma_3 f \rangle = \int_{-\infty}^{+\infty} \mathcal{U}(x) f(x^3) dx = \frac{1}{3} \int_{-\infty}^{+\infty} x^{-2/3} \mathcal{U}(x^{1/3}) f(x) dx. \tag{5.16}$$

Consequently, since $\langle \sigma_3(u_0), f \rangle = \langle u_0, \sigma_3 f \rangle$, from (5.13) and (5.16), it follows that

$$\int_{-\infty}^{+\infty} Y(x) \mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x) f(x) dx = \frac{1}{3} \int_{-\infty}^{+\infty} x^{-2/3} \mathcal{U}(x^{1/3}) f(x) dx. \tag{5.17}$$

Whence,

$$\frac{1}{3} x^{-2/3} \mathcal{U}(x^{1/3}) - Y(x) \mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x) = \mathcal{N}(x), \tag{5.18}$$

where $\mathcal{N}$ is a representation of the null functional. Here, we may take $\mathcal{N}$ in the form

$$\mathcal{N}(x) = \eta_1 \mathcal{N}_1(x) + \eta_2 \mathcal{N}_2(x) \tag{5.19}$$

with , for example,

$$\begin{aligned} \mathcal{N}_1(x) &= Y(x) x^{-1/2} e^{-x^{1/3}} \cos\left(\sqrt{3} x^{1/3}\right), \\ \mathcal{N}_2(x) &= Y(x) e^{-x^{1/3}} \sin\left(\sqrt{3} x^{1/3}\right), \end{aligned} \tag{5.20}$$

and η_1, η_2 two constants which we determine below.
Then, by virtue of (5.18) and changing the independent variable x by x^3, we have

$$\mathcal{U}(x) = 3x^2 \Big\{ Y(x) \mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x^3) + \eta_1 \mathcal{N}_1(x^3) + \eta_2 \mathcal{N}_2(x^3) \Big\}. \tag{5.21}$$

The 2-symmetric character of the functional u_0 leads necessarily to $(u_0)_0 = 1$, $(u_0)_1 = (u_0)_2 = 0$, i.e.,

$$(u_0)_\nu = \int_{-\infty}^{+\infty} x^\nu \mathcal{U}(x) dx = \begin{cases} 1 \ , & \nu = 0, \\ 0 \ , & \nu = 1, 2. \end{cases} \tag{5.22}$$

It is easily seen that the first condition $\int_{-\infty}^{+\infty} \mathcal{U}(x) dx = 1$ is fulfilled for any values of the constants η_1, η_2. The other two conditions, obtained for $\nu = 1$ and $\nu = 2$, allow

us to determine the constants η_1, η_2. This is the reason of the choice of the function $\mathcal{N}$ in the form (5.19).
Indeed, from (5.22), we have the system

$$\begin{cases} \eta_1 I_{11} + \eta_2 I_{12} = -H_1, \\ \eta_1 I_{21} + \eta_2 I_{22} = -H_2, \end{cases} \tag{5.23}$$

where

$$I_{\nu\mu} = \int_0^\infty x^{\nu+2}\mathcal{N}_\mu(x^3)dx\,,\ \nu,\mu = 1,2 \text{ and } \ H_\nu = \int_0^\infty x^{\nu+2}\mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x^3)dx\,,\ \nu = 1,2.$$

First, we have

$$H_\nu = \frac{1}{3}\int_0^\infty x^{\nu/3}\mathcal{W}_{-\frac{1}{3},-\frac{2}{3}}(x)dx$$

which, under the transformation $t = \sqrt{x}$ and by virtue of (4.24), leads to

$$H_\nu = \frac{2}{\pi\sqrt{3}}\int_0^\infty t^{2\nu/3}K_{1/3}(2t)dt = \frac{1}{2\pi\sqrt{3}}\Gamma\big(\frac{\nu+2}{3}\big)\Gamma\big(\frac{\nu+1}{3}\big).$$

Therefore

$$H_1 = \frac{1}{2\pi\sqrt{3}}\Gamma(2/3) \text{ and } \ H_2 = \frac{1}{6\pi\sqrt{3}}\Gamma(1/3).$$

On the other hand, according to the two formulas [11]

$$\int_0^{+\infty} x^{p-1}e^{-ax}\cos(mx)dx = \frac{\Gamma(p)\cos(p\theta)}{\left(a^2+m^2\right)^{p/2}},$$

$$\int_0^{+\infty} x^{p-1}e^{-ax}\sin(mx)dx = \frac{\Gamma(p)\sin(p\theta)}{\left(a^2+m^2\right)^{p/2}},$$

where $p, a, m > 0$; $\sin\theta = m/r$, $\cos\theta = a/r$, $0 < \theta < \pi/2$; $r = (a^2+m^2)^{1/2}$, we obtain that

$$I_{11} = -\sqrt{2\pi}2^{-6}3^{3/2},\ \ I_{12} = -2^{-4}3^{3/2},\ \ I_{21} = -5\sqrt{2\pi}2^{-8}3^{3/2} \text{ and } I_{22} = -2^{-3}3^{3/2}.$$

Thus, the system (5.23) gives $\Delta\eta_1 = H_2 I_{12} - H_1 I_{22}$ and $\Delta\eta_2 = H_1 I_{21} - H_2 I_{11}$, where $\Delta = I_{11}I_{22} - I_{21}I_{12} = 3^4 4^{-6}\sqrt{2\pi}$. Thus (5.12) and (5.11) follow immediately.
We will now proceed to look for the representations of the functionals u_1.
At first, from (5.8), we have $u_1 = -u_0'$. Then,

$$\langle u_1, f\rangle = \langle u_0, f'\rangle = \int_0^\infty \mathcal{U}(x)f'(x)dx = \mathcal{U}(x)f(x)\big]_0^\infty - \int_0^\infty \mathcal{U}'(x)f(x)dx. \tag{5.24}$$

Since $\mathcal{U}(x)f(x)\big]_0^\infty = 0$, we obtain

$$\langle u_1, f\rangle = -\int_0^\infty \mathcal{U}'(x)f(x)dx. \tag{5.25}$$

□

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