THE ROLE OF THE DISTANCE FUNCTION IN SOME SINGULAR PERTURBATION PROBLEM

ANGELA PISTOIA

0. Introduction. This paper deals with the study of solutions to a class of nonlinear singularly perturbed problems of the form

\[
\begin{cases}
-\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( \Omega \) is a bounded smooth domain of \( \mathbb{R}^N \), \( N \geq 2 \), \( \varepsilon > 0 \), \( 1 < p < \frac{N+2}{N-2} \) if \( N \geq 3 \) or \( p > 1 \) if \( N = 2 \) and \( \nu \) is the unit outward normal at the boundary of \( \Omega \).

A solution of the Dirichlet problem can be interpreted as a steady state of the corresponding reaction-diffusion equation \( u_t = \varepsilon^2 \Delta u - u + u^p \), which arises in a numbers of problems, such as dynamic population and pattern formation theories and chemical reactor theory. The Neumann problem is known as the stationary equation of Keller-Segal system in chemotaxis. It can also be seen as the limiting stationary equation of the Gierer-Heinhardt system in biological pattern formation.

Neumann problem

In the pioneering papers [29], [31] and [32] Lin, Ni and Takagi established the existence of least energy solutions and showed that for \( \varepsilon \) small enough the least energy solution has only one local maximum point \( x_0 \) which belongs to \( \partial \Omega \). Moreover the limit point \( x_0 = \lim_{\varepsilon \to 0} x_\varepsilon \) satisfies \( H(x_0) = \max_{x \in \partial \Omega} H(x) \), where \( H \) denotes the mean curvature of \( x \) at \( \partial \Omega \). In [33] Ni and Takagi constructed boundary spike solutions for axially symmetric domains. In [39] Wei studied the general domain case and proved that for single boundary spike solutions the boundary spike must approach a critical point of the mean curvature. He also proved that for any nondegenerate critical point of the mean curvature one can construct boundary spike solutions whose spike approaches such a point.

In [22] Gui constructed multiple boundary spike layer solutions at multiple local maximum points. In [44] Wei and Winter constructed multiple boundary spike layer solutions at multiple nondegenerate critical points of \( H \). In [24] the authors proved that for any fixed integers \( K \) there exist boundary \( K \)-peaks solutions at a local minimum point of \( H \).

In [40] and in [41] Wei proved the existence of single interior spike solutions of (0.1) under some restricted geometric conditions on \( \Omega \). In [42] and [20] the authors constructed single interior spike solutions by using the distance function \( \text{dist}(x, \partial \Omega) \). More precisely in [42] Wei proved that for any local maximum point \( x_0 \) of the distance function there exists a family of solutions with a single maximum point which approaches \( x_0 \). In [35] the author proved the existence of a symmetric single interior spike solution in symmetric domains, by using a degree argument.

*Dipartimento Me.Mo.Mat., via A.Scarpa 16, 00100 Roma, Italy. (pistoia@dmmm.uniroma1.it).
In [23] Gui constructed multiple interior peak solutions. It was shown that for any fixed positive integer $K$ there exists a solution of (0.1) which has exactly $K$ maximum points $x_1^1, \ldots, x_K^K$ such that $D_K(x_1^1, \ldots, x_K^K)$ converges to $\max(D_K(x_1^1, \ldots, x_K^K) \mid x^i \in \Omega, \ i = 1, \ldots, K}$ as $\varepsilon$ tends to zero, where

$$D_K(x^1, \ldots, x^K) = \min \left\{ \text{dist}(x^i, \partial \Omega), \frac{|x^j - x^l|}{2} \mid i, j, l = 1, \ldots, K, j \neq l \right\}.$$ 

Cerami and Wei in [9] and Yan in [37] some multiplicity results are obtained by using Ljusternik-Schnirelman category. In [25] Kowalczyk proved that any “nondegenerate stationary lattice” supports a multiple spike layer solution.

We would like to point out that Bates and Fusco in [5] got similar results for the Cahn-Hilliard equation. By using a “quasi-invariant” manifold approach they established the existence of a stationary solution with many nuclei and they also gave a criteria for the asymptotic location of those nuclei as $\varepsilon \to 0$ in terms of the geometry of the domain.

**Dirichlet problem**

Multiplicity results about the Dirichlet problem were firstly obtained by Benci, Cerami and Passaseo in [2] and [3], by using the Ljusternik-Schnirelman category. Successively Ni and Wei in [34] established the existence of a least energy solution. They proved that as $\varepsilon \to 0$ the least energy solution has exactly one local maximum point and this local maximum point tends to a point which attains the global maximum of the distance function $\text{dist}(x, \partial \Omega)$. In [39] Wei proved that for any local maximum $x_0$ of the distance function there exists a family of solutions with a single global maximum point which approaches $x_0$. In [16] Del Pino, Felmer and Wei proved the existence of single-peaked solutions at any “suitable” critical point of the distance function. In [28] Li and Nirenberg proved another result which involves the critical points of the distance function. More precisely they show that if the Brower degree $\text{deg}(\nabla \text{dist}(\cdot, \partial \Omega), V, 0) \neq 0$ where $V$ is a suitable subset of $\Omega$, then there exist a family of solutions with a unique local maximum point which converges to a critical point of the distance function.

In [7] and [8] Cao, Dancer, Noussair and Yan constructed $K-$peak solutions with the peaks near the local maximum points or saddle points $x_1, \ldots, x_K$ of $\text{dist}(\cdot, \partial \Omega)$, provided $\text{dist}(x_i, \partial \Omega) = \text{dist}(x_j, \partial \Omega)$ for any $i$ and $j$. In [17] Del Pino, Felmer and Wei used a variational method to construct a $K-$peak solution with its peaks close to some local maximum points $x_1, \ldots, x_K$ of $\text{dist}(\cdot, \partial \Omega)$, provided $\max_i \text{dist}(x_i, \partial \Omega)$ is small when compared with the distance between $x_1, \ldots, x_K$. In [15] Dancer and Wei proved the existence of two-peak solutions. Concerning the effect of the domain topology on the existence of multipeak solution Dancer and Yan in [13] proved that if the homology of the domain is nontrivial, then for any positive integer $K$ problem (0.1) has at least one $K-$peak solution. In [14] Dancer and Yan assumed that the distance function has $K$ isolated compact connected critical sets $T_1, \ldots, T_K$ satisfying $\text{dist}(x, \partial \Omega) = c_j = \text{constant}$ for all $x \in T_j$, $\min_{i \neq j} \text{d}(T_j, T_i) > 2 \max_{1 \leq j \leq K} \text{d}(T_j, \partial \Omega)$ and the critical group of each critical set $T_i$ is nontrivial. They constructed a solution which has exactly one local maximum point in a small neighbourhood of $T_i$ for $i = 1, \ldots, K$. Moreover they proved that if $\Omega$ is strictly convex problem (0.1) does not have any $K-$peak solution.

**Main results**
In this paper we describe some results obtained by Grossi, Wei and the author in [20] and in [21].

In [20] the authors proved that any critical point “topologically non trivial” $x_0$ of the distance function generates a family of single interior spike solutions.

**Theorem 0.1.** Let $x_0$ be a critical point of $\text{dist} (\cdot , \partial \Omega)$. Assume $c = \text{dist} (x_0 , \partial \Omega)$ is a critical value topologically nontrivial (see Definition (2.4)). Then for $\varepsilon$ small enough there exists a family of solutions $u_\varepsilon$ of (0.1), whose maximum point tends to a critical point $x'_0$ of the distance function with $\text{dist} (x'_0 , \partial \Omega) = \text{dist} (x_0 , \partial \Omega)$.

Moreover they proved that the peak of any single solution must converge to a critical point of the distance function.

**Theorem 0.2.** Let $u_\varepsilon$ be a solution of (0.1) with exactly one local interior maximum point $x_\varepsilon$. If $x_0 = \lim_{\varepsilon \to 0} x_\varepsilon \in \Omega$ then $x_0$ is a critical point of $\partial_\Omega$.

In [21] the authors proved that any critical point “topologically non trivial” of the function $\mathcal{D}_K$ generates a $K$—peaks solution.

**Theorem 0.3.** Let $X_0 = (x_0^1, \ldots, x_0^K)$ be a critical point of $\mathcal{D}_K$. Assume $\mathcal{D}_K (X_0) > 0$ is a critical value topologically nontrivial (see Definition (2.4)). Then for $\varepsilon$ small enough there exists a family of solutions $u_\varepsilon$ of (0.1), with Neumann boundary condition, whose $K$ maximum points $x_\varepsilon^1, \ldots, x_\varepsilon^K$ tend to a point $\hat{X}_0 = (\hat{x}_0^1, \ldots, \hat{x}_0^K)$ such that $\mathcal{D}_K (\hat{X}_0) = \mathcal{D}_K (X_0)$, $\hat{x}_0^i \in \Omega$, $\hat{x}_0^i \neq x_0^j$ for $i \neq j$ and $\hat{X}_0$ is a critical point of $\mathcal{D}_K$.

Moreover they proved that the $K$ peaks of any single solution must converge to a critical point of the function $\mathcal{D}_K$.

**Theorem 0.4.** Let $u_\varepsilon$ be a solution of (0.1), with Neumann boundary condition, with exactly $K$ local interior maximum points $x_\varepsilon^1, \ldots, x_\varepsilon^K$ and let $x_\varepsilon^i = \lim_{\varepsilon \to 0} x_\varepsilon^i$ for $i = 1, \ldots, K$. If $x_0^i \in \Omega$ then $x_0^i \neq x_0^j$ for $i \neq j$ and $(x_0^1, \ldots, x_0^K)$ is a critical point of $\mathcal{D}_K$.

The method used to prove the results relies on an idea of Bahri (see [1]).

Firstly for $\varepsilon$ small enough we reduce the problem of finding a single-peak or a multi-peak solution for (0.1) to that of finding a critical point for a function $K_\varepsilon$ defined in a finite dimensional domain.

Secondly we compute the asymptotic expansion of the function $K_\varepsilon$, in order to point out the connection between $K_\varepsilon$ and function $\mathcal{D}_K$. Such an expansion allows us to prove that any “topologically nontrivial” critical point of the function $\mathcal{D}_K$ generates a $K$—peak solution.

Finally we compute the asymptotic expansion of the function $\nabla K_\varepsilon$, in order to point out the connection between $\nabla K_\varepsilon$ and $\nabla \mathcal{D}_K$. Such an expansion allows us to prove that the $K$ peaks of any single solution must converge to a critical point of the function $\mathcal{D}_K$.

We would like to emphasize that $\mathcal{D}_K$ is a Lipschitz continuous function which may be not smooth. So a suitable notion of critical points for non-smooth functions is needed. The generalized gradient introduced by Clarke (see [11]) becomes our main tool. The new idea in [20] and in [21] is to evaluate the gradient of $K_\varepsilon$ in terms of the generalized gradient of Clarke of the function $\mathcal{D}_K$. By this result, we were able to get some new results and also to clarify many results that were previously known.
The paper is organized as follows. In Section 1 we recall some properties of the generalized gradient of Clarke. In Section 2 we introduce the notion of "topologically nontrivial" critical values for locally Lipschitz continuous function. In Section 3 we study the distance function and the function $D_K$ and we give a criteria to localize critical points of $D_K$. In Section 4 we recall some results obtained by Ni and Wei in [34]. In Section 5 we study the one-peak solutions. In Section 6 we study the multi-peak solutions. In Section 7 we give some examples.

1. The generalized gradient. Let $D$ be a smooth bounded domain of $\mathbb{R}^N$. Let $f : D \rightarrow \mathbb{R}$ be a Lipschitz continuous function. We recall the following definition due to Clarke (see [11]).

**Definition 1.1.** The generalized gradient of $f$ at $x \in D$ is the set:

$$\partial f(x) = \{ \alpha \in \mathbb{R}^N \mid f^\alpha(x, v) \geq \alpha \cdot v \ \forall \ v \in \mathbb{R}^N \}$$

where the generalized directional derivative $f^\alpha(x, v)$ is defined by

$$f^\alpha(x; v) = \lim_{\lambda \to 0^+} \frac{f(x + h + \lambda v) - f(x + h)}{\lambda}.$$

If $f$ is continuously differentiable at $x$ then $\partial f(x) = \{ \nabla f(x) \}$. If $f$ is only differentiable at $x$, $\partial f(x)$ can contain points other than $\nabla f(x)$. For example, if $f(x) = x^2 \sin \frac{1}{x}$ then it is easy to show that $f^\alpha(0; v) = |v|$ and so $\partial f(0) = [-1, 1]$, which contains the derivative $f'(0) = 0$.

**Definition 1.2.** The function $f$ is said to be regular at $x \in D$ provided that for any $v \in \mathbb{R}^N$ there exists the usual one-sided directional derivative $f'(x; v) = \lim_{t \to 0^+} \frac{f(x + tu) - f(x)}{t}$ and $f'(x; v) = f^\alpha(x; v)$.

By ([11], Proposition 2.2.4) and ([11], (b) of Proposition 2.3.6) we deduce

**Proposition 1.3.** If $\partial f(x)$ reduces to a singleton $\{ \alpha \}$ then $f$ is differentiable at $x$ and $\nabla f(x) = \alpha$. Conversely, if $f$ is differentiable and regular at $x$ then $\partial f(x) = \{ \nabla f(x) \}$.

It is useful to point out the following property (see [11], Proposition 2.1.5).

**Remark 1.4.** Let $x_n$ and $\alpha_n$ be sequences in $\mathbb{R}^N$ such that $x_n \in D$ and $\alpha_n \in \partial f(x_n)$. Suppose that $x_n$ converges to $x$ and $\alpha_n$ converges to $\alpha$. Then $\alpha \in \partial f(x)$.

Now let us suppose $x = (x_1, x_2)$. We denote by $\partial_1 f(x_1, x_2)$ the (partial) generalized gradient of $f(\cdot, x_2)$ at $x_1$ and by $\partial_2 f(x_1, x_2)$ that of $f(x_1, \cdot)$ at $x_2$. The following result holds (see [11], Proposition 2.3.15).

**Remark 1.5.** If $f$ is regular at $(x_1, x_2)$ then

$$\partial f(x_1, x_2) \subset \partial_1 f(x_1, x_2) \times \partial_2 f(x_1, x_2).$$

Let us recall another useful result. Assume that $\{f_i\}_{i \in I}$ is a finite collection of Lipschitz continuous functions defined on $D$. The function

$$f(x) = \min \{ f_i(x) \mid i \in I \}$$
is easily seen to be a Lipschitz continuous function. For any \( x \in D \) we let \( \mathcal{I}(x) \) denote the set of indices \( i \) for which \( f(x) = f_i(x) \) (i.e. the indices at which the minimum defining \( f \) is attained). Then the following result holds (see [11], Proposition (2.3.12)).

**Proposition 1.6.** If \( f_i \) is regular at \( x \) for any \( i \in \mathcal{I}(x) \) then \( f \) is regular at \( x \) and

\[
\partial f(x) = \text{co} \{ \partial f_i(x) \mid i \in \mathcal{I}(x) \}.
\]

Finally we give the definition of a critical point for a nonsmooth function.

**Definition 1.7.** A point \( x_0 \) in \( D \) is said to be a critical point of \( f \) if \( 0 \in \partial f(x_0) \). A real number \( c \) is said to be a critical value of \( f \) if there exists a critical point \( x_0 \) of \( f \) such that \( f(x_0) = c \).

By Definition (1.1) we easily deduce that if \( x_0 \) is a minimum point or a maximum point for a Lipschitz continuous function \( f \) then \( 0 \in \partial f(x_0) \).

### 2. Critical values topologically nontrivial.

In this section we recall a result of the critical point theory. The following one is given by Ramos in [36] and it is a jointed version of the classical linking theorem and the local saddle point proved in [30]. Although it concerns \( C^1 \)-function, it is possible to extend such a result to Lipschitz continuous function, by using deformation lemma proved by Chang in [10].

We consider three compact subsets \( \partial Q, Q \) and \( A \) of \( D \) such that

\[(2.1) \quad \partial Q \subset Q \quad \text{and} \quad Q \cap A = \emptyset.\]

\( \partial Q \) is not necessarily the topological boundary of \( Q \) and \( A \) can be the empty set. We define the class:

\[
\Gamma = \{ \gamma \in C^0([0,1] \times Q, D \setminus A) \mid \gamma_0 \equiv \text{Id}, \quad \gamma_t|_{\partial Q} \equiv \text{Id} \quad \forall t \in [0,1] \},
\]

where \( \text{Id} \) is the identity map. We note that \( \Gamma \neq \emptyset \) because \( \text{Id} \in \Gamma \).

**Definition 2.1.** Let \( S \) be a subset of \( D \). We say that \( S \) links \( Q \) via \( \partial Q \) by homotopy in \( D \setminus A \) if

\[(2.2) \quad S \cap \partial Q = \emptyset \quad \text{and} \quad \gamma_1(Q) \cap S \neq \emptyset \quad \forall \gamma \in \Gamma.\]

It is useful to point out the following fact.

**Remark 2.2.** Assume \( \partial Q_1, Q_1, A_1 \) and \( S_1 \) and \( \partial Q_2, Q_2, A_2 \) and \( S_2 \) are two families of subset of \( D \) which satisfy (2.1) and (2.2). Then \( \partial Q = (\partial Q_1 \times Q_2) \cup (Q_1 \times \partial Q_2) \), \( Q = Q_1 \times Q_2 \), \( A = (A_1 \times S_2) \cup (S_1 \times A_2) \) and \( S = S_1 \times S_2 \) are subsets of \( D \times D \) which satisfy (2.1) and (2.2).

The following result holds.

**Theorem 2.3.** Let \( f : D \to \mathbb{R} \) be a Lipschitz continuous function. Assume \( S \) links \( Q \) via \( \partial Q \) by homotopy in \( D \setminus A \) and

\[(2.3) \quad \max_{\partial Q} f < \min_{S} f \leq \max_{Q} f < \min_{A} f.\]

Let

\[(2.4) \quad c = \inf_{\gamma \in \Gamma} \max_{u \in Q} f(\gamma_1(u)).\]
If \( c \in \mathbb{R} \) and the set \( \{ x \in D \text{ s.t. } c - \varepsilon \leq f(x) \leq c + \varepsilon \} \) is complete for some \( \varepsilon > 0 \) then \( c \) is a critical value of \( f \).

If \( A = \emptyset \) we get the classical linking theorem. The "local saddle point" of [30] is a particular case of the previous theorem when \( A \neq \emptyset \).

In the following definition we introduce the notion of critical values of a Lipschitz continuous function \( f : D \rightarrow \mathbb{R} \) which are "stable" with respect to suitable perturbations (see [20], Definition (1.7) and [21], Definition (1.11))

**Definition 2.4.** We say that \( c \) is a critical value topologically nontrivial of \( f \) if there exists a family of subsets \( \partial Q_\delta, Q_\delta, A_\delta \) and \( S_\delta \) of \( D \) which satisfy (2.1), (2.2) and (2.3), with the properties

\[
\max_{\partial Q_\delta} f < \min_{S_\delta} f \leq c < \max_{A_\delta} f < \min_{S_\delta} f
\]

and

\[
\lim_{\delta \to 0} \min_{S_\delta} f = \lim_{\delta \to 0} \max_{Q_\delta} f = c.
\]

We point out that if we assume that the sets \( \{ x \in D \text{ s.t. } c' - \varepsilon \leq f(x) \leq c' + \varepsilon \} \) are complete for any \( c' \) close enough to \( c \) and for some \( \varepsilon > 0 \) then by Theorem (2.3) we deduce that \( c \) is a critical value of \( f \).

**3. The distance function and the function \( D_K \).** Let \( \Omega \) be a smooth open bounded domain of \( \mathbb{R}^N \).

**Definition 3.1.** Let \( d_{\partial \Omega} : \Omega \rightarrow \mathbb{R} \) be the distance function defined by \( d_{\partial \Omega}(x) = \text{dist}(x, \partial \Omega) = \min_{y \in \partial \Omega} |x - y| \).

It is well known that \( d_{\partial \Omega} \) is a Lipschitz continuous function. By using (see [11], Corollary 2, p. 87) we can compute the generalized gradient of the distance function.

**Remark 3.2.** For any \( x \in \Omega \) we have

\[
\partial d_{\partial \Omega}(x) = \left\{ \int_{\partial \Omega} \nu^{(i)}(y) d\mu_x(y) \mid \begin{array}{l}
d\mu_x(y) \text{ is a bounded Borel measure on } \partial \Omega, \\
\int_{\partial \Omega} d\mu_x(y) = 1, \supp(d\mu_x(y)) \subset \Pi_{\partial \Omega}(x) \end{array} \right\},
\]

where

\[
\Pi_{\partial \Omega}(x) = \{ y \in \partial \Omega \mid |y - x| = d_{\partial \Omega}(x) \}
\]

and \( \nu^{(i)}(y) \) denotes the unit inward normal at the point \( y \) of \( \partial \Omega \).

By ([11], Corollary 2, p. 87) we deduce that the distance function is regular at any \( x \in \Omega \). Therefore by Proposition (1.3) we get

**Remark 3.3.** \( d_{\partial \Omega} \) is differentiable at \( x \) if and only if \( \Pi_{\partial \Omega}(x) \) reduces to a singleton \( \{ \pi(x) \} \) and \( \nabla d_{\partial \Omega}(x) = \nu^{(i)}(\pi(x)) \), where \( \nu^{(i)}(\pi(x)) \) denotes the unit inward normal at \( \pi(x) \).
Finally since $\Omega$ is smooth we have the following property.

**Proposition 3.4.** There exists a neighbourhood $U$ of the boundary of $\Omega$ such that $0 \notin \partial \delta_\Omega(x)$ for any $x \in U \cap \Omega$.

Now let us introduce the function $D_K$ which will play a crucial role in the next sections.

**Definition 3.5.** Let $K \geq 1$ be an integer. Set $\Omega^K = \Omega \times \ldots \times \Omega$. Let $D_K : \Omega^K \to \mathbb{R}$ be defined by

$$D_K(X) = \min \left\{ d_\Omega(x^i), \frac{|x^j - x^l|}{2} \mid i, j, l = 1, \ldots, K, j \neq l \right\}. \quad (3.3)$$

Let us point out that

$$D_1(x) = d_\Omega(x) \quad \forall x \in \Omega.$$

Set

$$\mathcal{M}_K(\Omega) = \left\{ X = (x^1, \ldots, x^K) \in \Omega^K \mid x^i \neq x^j, i \neq j, i, j = 1, \ldots, K \right\}. \quad (3.4)$$

By using the regularity of the distance function and Proposition (1.6) we can compute the generalized gradient of $D_K$.

**Lemma 3.6.** For any $X \in \mathcal{M}_K(\Omega)$ we have that $\beta(X) \in \partial D_K(X)$ if and only if

$$\beta(X) = \left( a_1 \alpha(x^1) + \frac{1}{2} \sum_{j=1}^{K} b_{1j} \frac{x^1 - x^j}{|x^1 - x^j|}, \ldots, a_K \alpha(x^K) + \frac{1}{2} \sum_{j \neq K}^{K} b_{1j} \frac{x^K - x^j}{|x^K - x^j|} \right),$$

with $\alpha(x^i) \in \partial \delta_\Omega(x^i)$, $a_i, b_{ij} \geq 0$, $b_{jj} = b_{jj}$, $\sum_{i=1}^{K} a_i + \frac{1}{2} \sum_{j \neq K}^{K} b_{jj} = 1$.

In particular by Lemma (3.6) we deduce that if $x^1, \ldots, x^K$ are $K$ different critical points of the distance function then $X = (x^1, \ldots, x^K)$ is a critical point of $D_K$.

Next results generalizes Proposition (3.4). More precisely we prove that there is not any critical point of $D_K$ close to the boundary of $\mathcal{M}_K(\Omega)$.

**Proposition 3.7.** There exists a neighbourhood $U$ of the boundary of $\mathcal{M}_K(\Omega)$ such that $0 \notin \partial D_K(X)$ for any $X \in U \cap \mathcal{M}_K(\Omega)$.

**Proof.** We prove that if $X_\varepsilon$ is a sequence in $\mathcal{M}_K(\Omega)$ such that $\lim_{\varepsilon \to 0} X_\varepsilon = X_0$ and $X_0 \in \partial \mathcal{M}_K(\Omega)$, then there exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$|\beta_\varepsilon(X_\varepsilon)| \geq C > 0 \quad \forall \beta_\varepsilon(X_\varepsilon) \in \partial D_K(X_\varepsilon).$$

We proceed by induction on the number $K$.

Let $K = 1$ and let $x_\varepsilon$ be a sequence in $\Omega$ such that $x_0 = \lim_{\varepsilon} x_\varepsilon \in \partial \Omega$. By Remark (3.3) and Remark (3.4) it follows that for $\varepsilon$ small enough $\partial D_1(x_\varepsilon) = \{ \nu(1)(\pi(x_\varepsilon)) \}$ and the claim follows.

Suppose the claim to be true for any integer $1 \leq H \leq K - 1$. Let us prove the claim for $K$. 

Let $X_\varepsilon$ be a sequence in $\mathcal{M}_K(\Omega)$ such that $\lim_{\varepsilon \to 0} X_\varepsilon = X_0$ and $X_0 \in \partial \mathcal{M}_K(\Omega)$. Then we have either

(i) $\exists i, j \in \{1, \ldots, K\}$ s.t. $x_0^i \neq x_0^j$,

or

(ii) $x_0^1 = \ldots = x_0^K \in \partial \Omega$,

or

(iii) $x_0^1 = \ldots = x_0^K \in \Omega$.

By using Lemma (3.6) and inductive assumptions the claim easily follows. □

Next results allows us to localize some special critical points of the function $\mathcal{D}_K$.

**Proposition 3.8.** Let $(x^1, \ldots, x^K) \in \mathcal{M}_K(\Omega)$ be a critical point of $\mathcal{D}_K$. Assume that for any integer $1 \leq H \leq K-1$ and for any set of indices $\{i_1, \ldots, i_H\} \subset \{1, \ldots, K\}$ $(x^{i_1}, \ldots, x^{i_H})$ is not a critical point of $\mathcal{D}_H$. Then $\mathcal{D}_H(x^i) = \frac{|x^i - x^H|}{2}$ for any $i, l, h$ and $0 \in \co\{\alpha(x^i) \mid \alpha(x^i) \in \partial \mathcal{D}_\Omega(x^i), \ i = 1, \ldots, K\}$.

**Proof.** We argue by contradiction. Then we have either

(i) $\exists i, j \in \{1, \ldots, K\}$ s.t. $\mathcal{D}_K(X) < \frac{|x^i - x^j|}{2}$,

or

(ii) $\forall l, h \in \{1, \ldots, K\}$ $\mathcal{D}_K(X) = \frac{|x^l - x^h|}{2}$ and $\exists i \in \{1, \ldots, K\}$ s.t. $\mathcal{D}_K(X) = \mathcal{D}_\Omega(x^i)$.

By using Lemma (3.6) a contradiction arises in both cases. □

In particular by Proposition (3.8) and by Remark (3.3) we deduce the following characterization of the critical points of $\mathcal{D}_2$.

**Corollary 3.9.** Let $(x^1, x^2) \in \mathcal{M}_2(\Omega)$ be a critical point of $\mathcal{D}_2$ such that the distance function is differentiable at $x^1$ and $x^2$. Then $\mathcal{D}_\Omega(x^1) = \mathcal{D}_\Omega(x^2) = \frac{|x^1 - x^2|}{2}$ and $\nu^{(i)}(\pi(x^1)) = -\nu^{(i)}(\pi(x^2)) = \frac{x^2 - x^1}{|x^2 - x^1|}$.

4. Some preliminary results. Let us introduce the ground state solution $U$. We recall the following results (see, for example, [6], [19] and [26]).

**Theorem 4.1.** The equation:

\[
\begin{cases}
-\Delta u + u = u^p & \text{in } \mathbb{R}^N \\
u(X) \to 0 & \text{for } |x| \to +\infty
\end{cases}
\]

possesses a unique non trivial regular solution $U$ with the following properties:

(i) $U(x) > 0$ $\forall$ $x \in \mathbb{R}^N$,

(ii) $U$ is spherically symmetric, i.e. $U(x) = U(r)$ where $r = |x|$, and $U$ decreases with respect to $r$,

(iii) $U \in C^2(\mathbb{R}^N)$,

(iv) $U$ together with its derivatives up to order 2 have exponential decay at infinity, that is there exist $C > 0$ and $\delta > 0$ such that $|D^\alpha U(x)| \leq Ce^{-\delta|x|} \forall$ $x \in \mathbb{R}^N$ and $|\alpha| \leq 2$.

(v) there exists $\beta > 0$ such that $\lim_{r \to +\infty} r^{\frac{n-1}{2}} e^r U(r) = \beta > 0$.

Let us introduce some notation. Set $\Omega_\varepsilon = \{y \mid \varepsilon y \in \Omega\}$ and for $x \in \Omega$ $\Omega_{\varepsilon,x} = \{y \mid \varepsilon y + x \in \Omega\}$. Of course solving problem (0.1) is equivalent to solve the rescaled
problem

\begin{equation}
\begin{cases}
-\Delta u + u = u^p & \text{in } \Omega_\varepsilon \\
u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial \Omega_\varepsilon.
\end{cases}
\end{equation}

We set $\mathcal{P}_{\Omega_\varepsilon} U$ to be the unique solution of the problem

\begin{equation}
\begin{cases}
-\Delta u + u = U^p & \text{in } \Omega_{\varepsilon,x} \\
u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial \Omega_{\varepsilon,x}.
\end{cases}
\end{equation}

$\mathcal{P}_{\Omega_\varepsilon} U$ is the projection of the ground state $U$ into $H^1_0(\Omega_{\varepsilon,x})$ in the Dirichlet case or into $H^1(\Omega_{\varepsilon,x})$ in the Neumann case. The idea of projections has been introduced in [1].

Set

$$\varphi_{\varepsilon,x}(z) = U(y) - \mathcal{P}_{\Omega_\varepsilon} U(y)$$

with $z = \varepsilon y + x$, $x \in \Omega$, $z \in \Omega$.

The following estimate plays a fundamental role (see [41], Section 2 and [34], Section 4).

**Lemma 4.2.** For $x \in \Omega$ set

\begin{equation}
\psi_{\varepsilon}(x) = -\varepsilon \log (\varphi_{\varepsilon,x}(x)) \quad \text{in the Dirichlet case,}
\end{equation}

or

\begin{equation}
\psi_{\varepsilon}(x) = -\varepsilon \log (-\varphi_{\varepsilon,x}(x)) \quad \text{in the Neumann case,}
\end{equation}

Then

\begin{equation}
\lim_{\varepsilon \to 0} \psi_{\varepsilon}(x) = 2d_{\partial \Omega}(x) \quad \text{uniformly in } \Omega.
\end{equation}

By Lemma (4.2) and by (v) of Theorem (4.1) we easily deduce that

**Lemma 4.3.** Let for $X \in \mathcal{M}_K(\Omega)$

\begin{equation}
\Phi_{\varepsilon}(X) = -\varepsilon \log \left[ -\sum_{i=1}^{K} \varphi_{\varepsilon,x^i}(x^i) + \sum_{i \neq j}^{K} U \left( \frac{|x^j - x^i|}{\varepsilon} \right) \right].
\end{equation}

Then in the Neumann case

$$\lim_{\varepsilon \to 0} \Phi_{\varepsilon}(X) = 2\mathcal{D}_K(X) \quad \text{uniformly in } \mathcal{M}_K(\Omega).$$
5. Existence of one-peak solutions. Let $H_\varepsilon$ be the Hilbert space

$$H_\varepsilon = H^2(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon)$$

in the Dirichlet case

or

$$H_\varepsilon = \left\{ u \in H^2(\Omega_\varepsilon) \mid \frac{\partial u}{\partial n_\varepsilon} = 0 \text{ on } \partial \Omega_\varepsilon \right\}$$

in the Neumann case

Define

$$S_\varepsilon(u) = \Delta u - u + (u^+)^p \quad \text{for} \quad u \in H_\varepsilon.$$ 

Then solving equation (0.1) or equation (4.2) is equivalent to solve the following one

$$S_\varepsilon(u) = 0, \quad u \in H_\varepsilon.$$ 

Let us consider the linearized operator $L_\varepsilon : H_\varepsilon \rightarrow L^2(\Omega_\varepsilon)$ given by

$$L_\varepsilon(v) = \Delta v - v + pP_{\Omega_\varepsilon,U}U^{p-1}v.$$ 

It is easy to see that the cokernel of $L_\varepsilon$ coincides with its kernel. Choose approximate cokernel and kernel as

$$K_{\varepsilon,x} = \text{span} \left\{ \frac{\partial P_{\Omega_\varepsilon,U}}{\partial x_i} \mid i = 1, \ldots, N \right\} \subset H^2(\Omega_\varepsilon),$$

$$C_{\varepsilon,x} = \text{span} \left\{ \frac{\partial P_{\Omega_\varepsilon,U}}{\partial x_i} \mid i = 1, \ldots, N \right\} \subset L^2(\Omega_\varepsilon).$$

Now we state the following lemmas, which allow us to reduce problem (4.2) to a finite dimensional problem.

**Lemma 5.1.** For any compact set $K \subset \Omega$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $x \in K$ there exists a unique $e_\varepsilon \in K^\perp_{\varepsilon,x}$ such that

$$S_\varepsilon \left( P_{\Omega_\varepsilon,U} + e_{\varepsilon,x} \right) \in C_{\varepsilon,x}.$$ 

Moreover $e_{\varepsilon,x}$ is $C^1$ in $x$ and

$$\|e_{\varepsilon,x}\|_{H^2(\Omega_\varepsilon)} \leq Ce^{-(1+\sigma)\frac{d_\Omega}{\varepsilon}},$$

where $C$ is a positive constant and $\sigma = \min\{1, p-1\}$.

**Proof.** The proof relies on a contraction mapping argument. The claim can be proved by collecting some results obtained in [41] and [42].

Now we define the function $K_\varepsilon : \Omega \rightarrow \mathbb{R}$

$$K_\varepsilon(x) = J_\varepsilon(P_{\Omega_\varepsilon,U} + e_{\varepsilon,x}),$$

where the “rescaled” energy functional $J_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow \mathbb{R}$ is defined by

$$J_\varepsilon(u) = \left[ \frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2) - \frac{1}{p+1} \int_{\Omega_\varepsilon} (u^+)^{p+1} \right].$$
THE ROLE OF THE DISTANCE FUNCTION IN SOME ELLIPTIC PROBLEMS 311

Now we evaluate the asymptotic expansion of $K_\varepsilon$.

**Proposition 5.2.** $x_\varepsilon$ is a critical point of $K_\varepsilon$ if and only if $u_\varepsilon = \mathcal{P}_{\Omega, x_\varepsilon} U + \Phi_\varepsilon(x_\varepsilon)$ is a solution of (4.2). Moreover the following estimates hold uniformly on compact sets of $\Omega$

\begin{equation}
K_\varepsilon(x) = A + \frac{1}{2} \gamma e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}} + o\left(e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}}\right) \quad \text{in the Dirichlet case}
\end{equation}

or

\begin{equation}
K_\varepsilon(x) = A - \frac{1}{2} \gamma e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}} + o\left(e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}}\right) \quad \text{in the Neumann case},
\end{equation}

where

\[ A = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + U^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} U^{p+1}, \quad \gamma = \int_{\mathbb{R}^N} U^p(y)e^{-y_1}dy. \]

**Proof.** See [20], [23], [41] and [42]. \hfill \Box

The next results play a crucial role in connecting the topological structure of the sublevels of the distance function with the topological structure of the sublevels of the function $K_\varepsilon$.

**Lemma 5.3.** Let $x_1^\varepsilon, x_2^\varepsilon$ be sequences in $\Omega$ be such that $\lim_{\varepsilon \to 0} x_1^\varepsilon = x_1 \in \Omega$, $\lim_{\varepsilon \to 0} x_2^\varepsilon = x_2 \in \Omega$ and $d_{\partial \Omega}(x_1) < d_{\partial \Omega}(x_2)$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

\begin{equation}
K_\varepsilon(x_1^\varepsilon) > K_\varepsilon(x_2^\varepsilon) \quad \text{in the Dirichlet case}
\end{equation}

or

\begin{equation}
K_\varepsilon(x_1^\varepsilon) < K_\varepsilon(x_2^\varepsilon) \quad \text{in the Neumann case}.
\end{equation}

**Proof.** We prove (5.7). The proof of (5.6) is the same. By the expansion of $K_\varepsilon$ given in (5.5) of Proposition (5.2) we have

\begin{equation}
k_\varepsilon(x_2^\varepsilon) - k_\varepsilon(x_1^\varepsilon) = \frac{1}{2} \gamma \left(e^{-\frac{\psi_\varepsilon(x_1^\varepsilon)}{\varepsilon}} - e^{-\frac{\psi_\varepsilon(x_2^\varepsilon)}{\varepsilon}}\right)
\end{equation}

or

\begin{equation}
k_\varepsilon(x_1^\varepsilon) - k_\varepsilon(x_2^\varepsilon) = \frac{1}{2} \gamma \left(e^{-\frac{\psi_\varepsilon(x_2^\varepsilon)}{\varepsilon}} - e^{-\frac{\psi_\varepsilon(x_1^\varepsilon)}{\varepsilon}}\right).
\end{equation}

Since $d_{\partial \Omega}(x_1) < d_{\partial \Omega}(x_2)$, by Lemma (4.2) we deduce that for $\varepsilon$ small enough $\psi_\varepsilon(x_1^\varepsilon) < \psi_\varepsilon(x_2^\varepsilon)$. Then by (5.8) we get

\[ e^{-\frac{\psi_\varepsilon(x_1^\varepsilon)}{\varepsilon}} [K_\varepsilon(x_2^\varepsilon) - K_\varepsilon(x_1^\varepsilon)] = \frac{1}{2} \gamma \left[1 - e^{-\frac{\psi_\varepsilon(x_1^\varepsilon) - \psi_\varepsilon(x_2^\varepsilon)}{\varepsilon}}\right] + o(1) \]

and the claim follows. \hfill \Box

**Lemma 5.4.** Let $C_1, C_2$ be two compact subsets of $\Omega$. If

\[ \min_{x \in C_1} d_{\partial \Omega}(x) > \max_{x \in C_2} d_{\partial \Omega}(x) \]

then
then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

\begin{equation}
\min_{x \in C_1} (-K_\varepsilon)(x) > \max_{x \in C_2} (-K_\varepsilon)(x) \quad \text{in the Dirichlet case,}
\end{equation}

or

\begin{equation}
\min_{x \in C_1} K_\varepsilon(x) > \max_{x \in C_2} K_\varepsilon(x) \quad \text{in the Neumann case.}
\end{equation}

Now we prove that a suitable critical point of the distance function generates a critical point of $K_\varepsilon$.

**Theorem 5.5.** Let $c$ be a critical value topologically nontrivial of the distance function (see Definition (2.4)). Then there exists a sequence $(x_\varepsilon)$ of critical points of $K_\varepsilon$ such that $\lim_{\varepsilon \to 0} x_\varepsilon = x_0$ and $d_{\partial \Omega}(x_0) = c$.

**Proof.** We prove the claim in Neumann case. In the Dirichlet case we consider the function $-K_\varepsilon$ and we argue in the same way. By Definition (2.4) there exist a family of subsets $\partial Q_\delta$, $Q_\delta$, $A_\delta$ and $S_\delta$ of $\Omega$ which satisfy (2.1), (2.2), (2.5) and (2.6), that is:

\begin{equation}
\max_{x \in \partial Q_\delta} d_{\partial \Omega}(x) < \min_{x \in S_\delta} d_{\partial \Omega}(x) \leq c \leq \max_{x \in Q_\delta} d_{\partial \Omega}(x) < \min_{x \in A_\delta} d_{\partial \Omega}(x).
\end{equation}

Then by (5.9) of Lemma (5.4) there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

\begin{equation}
a_{\varepsilon, \delta} = \max_{x \in \partial Q_\delta} K_\varepsilon(x) < \min_{x \in S_\delta} K_\varepsilon(x) \leq \max_{x \in Q_\delta} K_\varepsilon(x) < \min_{x \in A_\delta} K_\varepsilon(x) = b_{\varepsilon, \delta}.
\end{equation}

It is not difficult to prove that for $\varepsilon$ and $\delta$ small enough the set \{ $x \in \Omega \mid a_{\varepsilon, \delta} \leq K_\varepsilon(x) \leq b_{\varepsilon, \delta}$ \} is complete. Now by (5.12) and Theorem (2.3) there exists $x_{\varepsilon, \delta}$ critical point of $K_\varepsilon$ in $\Omega$ such that:

\begin{equation}
\min_{x \in S_\delta} K_\varepsilon(x) \leq K_\varepsilon(x_{\varepsilon, \delta}) \leq \max_{x \in Q_\delta} K_\varepsilon(x).
\end{equation}

Up to a subsequence we can assume that $x_{\varepsilon, \delta}$ goes to $x_0$ as $\varepsilon$ and $\delta$ go to 0. It is easy to show that $d_{\partial \Omega}(x_0) = c > 0$. Therefore the claim is proved. □

Finally we want to show that a family of critical points of $K_\varepsilon$ converges to a critical point of the distance function. Firstly we have to compute the asymptotic expansion of the gradient of $K_\varepsilon$.

**Proposition 5.6.** Let $x_\varepsilon$ be a sequence in $\Omega$ such that $\lim_{\varepsilon \to 0} x_\varepsilon = x_0 \in \Omega$. Then

\begin{equation}
\nabla K_\varepsilon(x_\varepsilon) = -\frac{1}{\varepsilon} \gamma \alpha(x_0) e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}} + o\left(\frac{1}{\varepsilon} e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}}\right), \quad \text{in the Dirichlet case,}
\end{equation}

or

\begin{equation}
\nabla K_\varepsilon(x_\varepsilon) = \frac{1}{\varepsilon} \gamma \alpha(x_0) e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}} + o\left(\frac{1}{\varepsilon} e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}}\right), \quad \text{in the Neumann case,}
\end{equation}

where $\alpha(x_0) \in \partial d_{\partial \Omega}(x_0)$ (see (3.1)) and $\gamma$ is a positive constant (see Proposition 5.2).

**Proof.** See Lemma (4.1) of [20]. □
THEOREM 5.7. Let \( x_\varepsilon \) be a critical point of \( K_\varepsilon \) such that \( x_0 = \lim_{\varepsilon \to 0} x_\varepsilon \in \Omega \). Then \( x_0 \) is a critical point of the distance function.

Proof. Since \( x_\varepsilon \) is a critical point of \( K_\varepsilon \) by Proposition (5.6) we get

\[
0 = \nabla K_\varepsilon(x_\varepsilon) = \frac{1}{\varepsilon} \gamma \alpha(x_0) e^{-\frac{\phi(x_\varepsilon)}{\varepsilon}} + o \left( \frac{1}{\varepsilon} e^{-\frac{\phi(x_\varepsilon)}{\varepsilon}} \right),
\]

where \( \alpha(x_0) \in \partial d_{\partial \Omega}(x_0) \) and \( \gamma \) is a positive constant. By (5.16) we deduce

\[
\gamma \alpha(x_0) + o(1) = 0,
\]

which implies \( \alpha(x_0) = 0 \). Then \( x_0 \) is a critical point of the distance function since \( 0 \in \partial d_{\partial \Omega}(x_0) \) (see Definition (1.7)). □

Proof of Theorem (0.1). It follow by Proposition (5.2) and Theorem (5.5). □

Proof of Theorem (0.2). It follow by Proposition (5.2) and Theorem (5.7). □

6. Existence of multi-peak solutions. Let \( H_\varepsilon \) be the Hilbert space

\[
H_\varepsilon = \left\{ u \in H^2(\Omega_\varepsilon) \mid \frac{\partial u}{\partial \nu_\varepsilon} = 0 \text{ on } \partial \Omega_\varepsilon \right\}
\]

Define

\[
S_\varepsilon(u) = \Delta u - u + (u^+)^p \quad \text{for} \quad u \in H_\varepsilon.
\]

Then solving equation (0.1) or equation (4.2) is equivalent to solve the following one

\[
S_\varepsilon(u) = 0, \quad u \in H_\varepsilon.
\]

Fix \( X = (x^1, \ldots, x^K) \in M_K(\Omega) \). Let us consider the linearized operator \( L_\varepsilon : H_\varepsilon \to L^2(\Omega_\varepsilon) \) given by

\[
L_\varepsilon(v) = \Delta v - v + p \left( \sum_{i=1}^{K} \mathcal{P}_{\Omega_\varepsilon,x_i} U \right)^{p-1} v.
\]

It is easy to see that the cokernel of \( L_\varepsilon \) coincides with its kernel. Choose approximate cokernel and kernel as

\[
K_{\varepsilon,X} = \text{span} \left\{ \frac{\partial \mathcal{P}_{\Omega_\varepsilon,x_i} U}{\partial x_j^i} \mid i = 1, \ldots, K, \ j = 1, \ldots, N \right\} \subset H_\varepsilon,
\]

\[
C_{\varepsilon,X} = \text{span} \left\{ \frac{\partial \mathcal{P}_{\Omega_\varepsilon,x_i} U}{\partial x_j^i} \mid i = 1, \ldots, K, \ j = 1, \ldots, N \right\} \subset L^2(\Omega_\varepsilon).
\]

Now we state the following lemmas, which allow us to reduce problem (4.2) to a finite dimensional problem.

**Lemma 6.1.** For any compact set \( C \subset M_K(\Omega) \) there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( X \in C \) there exists a unique \( \Phi_{\varepsilon,X} \in K_{\varepsilon,X}^+ \) such that

\[
S_\varepsilon \left( \sum_{i=1}^{K} \mathcal{P}_{\Omega_\varepsilon,x_i} U + \Phi_{\varepsilon,X} \right) \in C_{\varepsilon,X}.
\]
Moreover $\Phi_{\varepsilon,X}$ is $C^1$ in $X$ and

\begin{equation}
\|\Phi_{\varepsilon,X}\|_{H^2(\Omega_\varepsilon)} \leq C e^{-(1+\sigma)\frac{D_K(x)}{\varepsilon}},
\end{equation}

where $C$ is a positive constant, $\sigma = \min\{1, p - 1\}$ and $D_K$ is defined in (3.3).

\textbf{Proof.} The proof relies on a contraction mapping argument. The claim can be proved by collecting some results obtained in [9] and [23]. □

We now define the function $K_\varepsilon : M_K(\Omega) \rightarrow \mathbb{R}$ by

\begin{equation}
K_\varepsilon(x^1, \ldots, x^K) = J_\varepsilon(\sum_{i=1}^{K} \mathcal{P}_{\varepsilon,x_i} U + \Phi_{\varepsilon,X}),
\end{equation}

where the "rescaled" energy functional $J_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow \mathbb{R}$ is defined in (5.3).

Firstly we compute the asymptotic expansion of $K_\varepsilon$.

\textbf{PROPOSITION 6.2.} $X_\varepsilon = (x_1^\varepsilon, \ldots, x_K^\varepsilon)$ is a critical point of $K_\varepsilon$ if and only if $u_\varepsilon = \sum_{i=1}^{K} \mathcal{P}_{\varepsilon,x_i} U + \Phi_{\varepsilon,X}$ is a solution of (4.2). Moreover the following estimate holds uniformly on compact sets of $M_K(\Omega)$

\begin{equation}
K_\varepsilon(x) = KA - \frac{1}{2} \gamma e^{-\frac{\Psi_\varepsilon(x)}{\varepsilon}} + o \left( e^{-\frac{\Psi_\varepsilon(x)}{\varepsilon}} \right),
\end{equation}

where

\begin{equation}
A = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + U^2) - \frac{1}{p + 1} \int_{\mathbb{R}^N} U^{p+1}, \quad \gamma = \int_{\mathbb{R}^N} U^p(y)e^{-\Psi_\varepsilon(y)}dy.
\end{equation}

\textbf{Proof.} See [9] and [23]). □

The next results play a crucial role in connecting the topological structure of the sublevels of the function $D_K$ with the topological structure of the sublevels of the function $K_\varepsilon$.

\textbf{LEMMA 6.3.} Let $X_1^\varepsilon, X_2^\varepsilon$ be sequences in $M_K(\Omega)$ such that $\lim_{\varepsilon \to 0} X_1^\varepsilon = X_1 \in M_K(\Omega)$, $\lim_{\varepsilon \to 0} X_2^\varepsilon = X_2 \in M_K(\Omega)$ and $D_K(X_1) < D_K(X_2)$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

\begin{equation}
K_\varepsilon(X_1^\varepsilon) < K_\varepsilon(X_2^\varepsilon).
\end{equation}

\textbf{Proof.} We argue as in the proof of Lemma (5.3) using asymptotic expansion (6.3).

\textbf{LEMMA 6.4.} Let $C_1, C_2$ be two compact subsets of $M_K(\Omega)$. If

\[ \min_{X \in C_1} D_K(X) > \max_{X \in C_2} D_K(X) \]

then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

\begin{equation}
\min_{X \in C_1} K_\varepsilon(X) > \max_{X \in C_2} K_\varepsilon(X).
\end{equation}
THE ROLE OF THE DISTANCE FUNCTION IN SOME ELLIPTIC PROBLEMS

Now we prove that a suitable critical point of the function \( D_K \) generates a critical point of \( K_\varepsilon \).

**Theorem 6.5.** Let \( c \) be a critical value topologically nontrivial of the function \( D_K \) (see Definition (2.4)). Then there exists a sequence \( (X_\varepsilon) \) of critical points of \( K_\varepsilon \) such that \( \lim_{\varepsilon \to 0} X_\varepsilon = X_0, \, D_K(X_0) = c \) and \( X_0 \in M_K(\Omega) \).

**Proof.** By definition (2.4) there exist a family of \( Q_\delta, A_\delta, S_\delta \) of \( M_K(\Omega) \), which satisfy (2.1), (2.2), (2.3) and (2.6), namely:

\[
\max_{X \in \partial Q_\delta} D_K(X) < \min_{X \in S_\delta} D_K(X) \leq c \leq \max_{X \in Q_\delta} D_K(X) < \min_{X \in A_\delta} D_K(X)
\]

and

\[
\lim_{\delta \to 0} \min_{X \in S_\delta} D_K(X) = \lim_{\delta \to 0} \max_{X \in Q_\delta} D_K(X) = c.
\]

Then by (6.5) of Lemma (6.4) for any \( \delta \) small enough there exists \( \varepsilon_0(\delta) > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \)

\[
\max_{X \in \partial Q_\delta} K_\varepsilon(X) < \min_{X \in S_\delta} K_\varepsilon(X) \leq \max_{X \in Q_\delta} K_\varepsilon(X) < \min_{X \in A_\delta} K_\varepsilon(X).
\]

Finally we want to show that a family of critical points of \( K_\varepsilon \) converges to a critical point of the function \( D_K \). Firstly we have to compute the asymptotic expansion of the gradient of \( K_\varepsilon \).

First of all we have to compute the expansion of the gradient of \( K_\varepsilon \).

**Proposition 6.6.** For any \( X \in M_K(\Omega) \)

\[
\nabla K_\varepsilon(X) = \frac{\gamma}{\varepsilon} \beta_\varepsilon(X) e^{-\frac{\Phi_\varepsilon(X)}{\varepsilon}} + o\left(\frac{1}{\varepsilon}\right),
\]

where \( \beta_\varepsilon(X) \in \partial D_K(X) \) (see Lemma sottodiffik)) and \( \gamma \) is a positive constant (see Theorem (6.2)).

**Proof.** See Lemma (5.1) of [21].

**Theorem 6.7.** Let \( X_\varepsilon = (x_1^\varepsilon, \ldots, x_K^\varepsilon) \) be a critical point of \( K_\varepsilon \) such that for \( i = 1, \ldots, K \) \( x_0^i = \lim_{\varepsilon \to 0} x_\varepsilon^i \in \Omega \). Then \( X_0 = (x_0^1, \ldots, x_0^K) \in M_K(\Omega) \) and \( X_0 \) is a critical point of the function \( D_K \).

**Proof.** First of all we prove that \( (x_0^1, \ldots, x_0^K) \in M_K(\Omega) \), namely \( x_0^i \neq x_0^j \) if \( i \neq j \) (see [21], Theorem (6.1)).

Secondly we show that \( X_0 \) is a critical point of the function \( D_K \). Since \( X_\varepsilon \) is a critical point of \( K_\varepsilon \) by Proposition (6.6) we get

\[
0 = \nabla K_\varepsilon(X_\varepsilon) = \frac{1}{\varepsilon} \gamma \beta_\varepsilon(X_\varepsilon) e^{-\frac{\Phi_\varepsilon(X_\varepsilon)}{\varepsilon}} + o\left(\frac{1}{\varepsilon}\right),
\]

where \( \beta(X_\varepsilon) \in \partial D_K(X_\varepsilon) \) and \( \gamma \) is a positive constant. By (6.10) we deduce

\[
\beta(X_\varepsilon) + o(1) = 0.
\]

Let \( X_0 = \lim_{\varepsilon \to 0} X_\varepsilon \). By using Remark (1.4) we get \( \lim_{\varepsilon \to 0} \beta(X_\varepsilon) = \beta(X_0) \in \partial D_K(X_0) \) and by (6.11) we deduce that \( \beta(X_0) = 0 \). Then \( X_0 \) is a critical point of the function \( D_K \) since \( 0 \in \partial D_K(X_0) \) (see Definition (1.7)).

**Proof of Theorem (0.3).** It follow by Proposition (6.2) and Theorem (6.5).

**Proof of Theorem (0.4).** It follow by Proposition (6.2) and Theorem (6.7).
7. Examples.

**Example 7.1.** (A domain with one hole) Let \( \Omega = \Sigma \setminus \sigma \) where \( \sigma \subset \Sigma \) are open sets. Assume \( \max_\Omega \omega \geq \frac{1}{2} \text{dist} (\partial \sigma, \partial \Sigma) \). Then \( c_1 = \omega(x_1) = \max_\Omega \omega \) and \( c_2 = \omega(x_2) = \frac{1}{2} \text{dist} (\partial \sigma, \partial \Sigma) \) are two critical values topologically nontrivial of the distance function.

**Proof.** The existence of \( c_1 \) is trivial. Let us prove the existence of \( c_2 \). Let \( y_0 \in \partial \Sigma \) and \( z_0 \in \partial \sigma \) such that \( |y_0 - z_0| = \text{dist} (\partial \sigma, \partial \Sigma) \). Set \( x_0 = \frac{y_0 + z_0}{2} \). Then \( \omega(x_0) = \frac{1}{2}|y_0 - z_0| \). Let:

\[
S = \{ x \in \Sigma \mid \text{dist} (x, \partial \sigma) = \omega(x_0) \}
\]

and

\[
Q = \{ ty_0 + (1 - t)z_0 \mid t \in [\delta, 1 - \delta] \}
\]

for some \( \delta > 0 \).

Then it is easy to prove that the sets \( Q \) and \( S \) satisfies assumptions (2.1), (2.2) and (2.3):

\[
\max_{x \in \partial Q} \omega(x) < \min_{x \in S} \omega(x) = \omega(x_0) = \max_{x \in Q} \omega(x).
\]

That proves that \( \omega(x_0) \) is a critical value topologically nontrivial of the distance function in the sense of Definition (2.4). \( \square \)

**Example 7.2.** (A domain with two holes) Let \( \Omega = \Sigma \setminus (\overline{\sigma_1} \cup \overline{\sigma_2}) \) where \( \sigma_i \subset \Sigma \) are open sets, \( \sigma_1 \) and \( \sigma_2 \) are strictly convex and \( \sigma_1 \cap \sigma_2 = \emptyset \). Assume

\[
(7.1) \quad \text{dist} (\partial \sigma_1, \partial \Sigma) < \text{dist} (\partial \sigma_2, \partial \Sigma) < \text{dist} (\partial \sigma_1, \partial \sigma_2)
\]

Then \( c_1 = \omega(x_1) = \max_\Omega \omega, c_2 = \omega(x_2) = \frac{1}{2} \text{dist} (\partial \sigma_1, \partial \Sigma), c_3 = \omega(x_3) = \frac{1}{2} \text{dist} (\partial \sigma_2, \partial \Sigma) \) and \( c_4 = \omega(x_4) = \frac{1}{2} \text{dist} (\partial \sigma_1, \partial \sigma_2) \) are four critical values topologically nontrivial of the distance function.

**Proof.** The existence of \( c_1 \) is trivial. First of all we prove the existence of \( c_2 \) and \( c_3 \). Let \( i = 1, 2 \). Let \( y_0^i \in \partial \Sigma \) and \( z_0^i \in \partial \sigma_i \) such that \( |y_0^i - z_0^i| = \text{dist} (\partial \sigma_i, \partial \Sigma) \). Set \( x_0^i = \frac{y_0^i + z_0^i}{2} \). Then \( \omega(x_0^i) = \frac{1}{2}|y_0^i - z_0^i| \). Let:

\[
S_i = \{ x \in \Sigma \mid \text{dist} (x, \partial \sigma_i) = \omega(x_0^i) \}
\]

and

\[
Q_i = \{ ty_0^i + (1 - t)z_0^i \mid t \in [\delta, 1 - \delta] \}
\]

for some \( \delta > 0 \).

We point out that (7.1) ensures that \( \omega(x) = \text{dist} (x, \partial \sigma_i) \quad \forall x \in S_i \). Then it is easy to prove that the sets \( Q_i \) and \( S_i \) satisfies assumptions (2.1), (2.2) and (2.3):

\[
\max_{x \in \partial Q_i} \omega(x) < \min_{x \in S_i} \omega(x) = \omega(x_0^i) = \max_{x \in Q_i} \omega(x).
\]

That proves that \( \omega(x_0^i) \) is a critical value topologically nontrivial of the distance function in the sense of Definition (2.4).
Now we prove the existence of $c_4$. Since $\sigma_1$ and $\sigma_2$ are strictly convex there exist exactly two points $z_1 \in \partial \sigma_1$ and $z_2 \in \partial \sigma_2$ such that $|z_1 - z_2| = \text{dist} (\partial \sigma_1, \partial \sigma_2)$. Set $x_0 = \frac{z_1 + z_2}{2}$. Then $d_{\Omega}(x_0) = \frac{1}{2}|z_1 - z_2|$. Let for some $\delta > 0$

$$Q = \{tz_1 + (1-t)z_2 \mid t \in [\delta, 1-\delta]\},$$

$$S = \{\text{hyperplane perpendicular to } Q \text{ crossing the point } x_0 \} \cap B(x_0, \delta).$$

and

$$A = \{\text{hyperplane perpendicular to } Q \text{ crossing the point } x_0 \} \cap \partial B(x_0, \delta).$$

Then it is easy to prove that the sets $Q$, $S$ and $A$ satisfies assumptions (2.1), (2.2) and (2.3):

$$\max_{x \in \partial Q} d_{\Omega}(x) < \min_{x \in S} d_{\Omega}(x) = d_{\Omega}(x_0) = \max_{x \in Q} d_{\Omega}(x) < \min_{x \in A} d_{\Omega}(x).$$

That proves that $d_{\Omega}(x_0)$ is a critical value topologically nontrivial of the distance function in the sense of Definition (2.4). □

If the domain has a lot of holes the existence of many critical values topologically nontrivial of the distance function strongly depends on the geometry of the holes.

**Example 7.3.** *(A domain with k handles)* Let $\Omega$ be a domain with $k$ handles. Then there exist at least $2k + 1$ distinct critical values topologically nontrivial of the distance function: $k+1$ local maxima of $d_{\Omega}$ and $k$ local saddle levels.

Note that we can have more than a critical point at the same level.

**Example 7.4.** Let $\Omega$ be the dumbell. Then $d$ is a critical value topologically nontrivial of $D_2$. Moreover one can choose the dumbell so that $(0, d, 0, -d)$ is the unique critical point of $V_2$ at level $d$.

We prove that the point $(0, d, 0, -d)$ is a "local saddle point" of $D_2$. Fix $\varepsilon > 0$ and set

$$Q_\varepsilon = \{(0, x_2^1) \in \Omega \mid \|x_2^1 - d\| \leq \varepsilon\} \times \{(0, x_2^2) \in \Omega \mid \|x_2^2 + d\| \leq \varepsilon\}$$

$$\partial Q_\varepsilon = \{(0, d \pm \varepsilon)\} \times \{(0, x_2^1) \in \Omega \mid \|x_2^1 - d\| \leq \varepsilon\}$$

$$\cup \{(0, x_2^1) \in \Omega \mid \|x_2^1 - d\| \leq \varepsilon\} \times \{(0, -d \pm \varepsilon)\}$$

For $\delta > 0$ and $\rho > 0$ set

$$C_\delta = \{x \in \Omega \mid d_{\Omega}(x) = d + \delta\}$$

$$S_\delta = \left(B\left((0, d), \rho \right) \cap C_\delta\right) \times \left(B\left((0, -d), \rho \right) \cap C_\delta\right)$$

$$A_\delta = \left(\partial B\left((0, d), \rho \right) \cap C_\delta\right) \times \left(B\left((0, -d), \rho \right) \cap C_\delta\right)$$

$$\cup \left(B\left((0, d), \rho \right) \cap C_\delta\right) \times \left(\partial B\left((0, -d), \rho \right) \cap C_\delta\right).$$

Then by using Remark (2.2) it is easy to check that if we choose $\delta$ and $\rho$ small enough $\partial Q_\delta$, $Q_\delta$, $A_\delta$ and $S_\delta$ are subsets of $\Omega \times \Omega$ which satisfy (2.1), (2.2) and (2.3) and

$$\max_{\partial Q_\delta} D_2 = d - \varepsilon < \min_{S_\delta} D_2 = d - \delta < d = \max_{Q_\delta} D_2 < \min_{A_\delta} D_2 = d + \delta.$$
Moreover \( \lim_{\delta \to 0} \min S^\delta d = d. \) By Lemma (3.7) we deduce that the sets \( \{X \in \mathcal{M}_K(\Omega) \text{ s.t. } c \leq D_K(X)\} \) are complete for any \( c > 0. \) Therefore \( d \) is a critical value of \( D_2. \)

Finally by using Remark (3.4) and Remark (3.3) one can construct a dumbell in such a way the distance function is differentiable at any \( x \) with \( d(x) = d \) and by using Corollary (3.9) one can check that \( (0, d, 0, -d) \) is the unique critical point of \( D_2 \) at level \( d. \) \( \square \)

**Remark 7.5.** We note that in the dumbell the points \((a, r, a, -r)\) and \((b, R, b, -R)\) are two local maximum points of the function \( D_2 \) at different levels \( D_2(a, r, a, -r) = r \) and \( D_2(b, R, b, -R) = R. \)

However we point out that such points are not isolated critical points of \( D_2 \) at levels \( r \) and \( R, \) respectively. In fact if \( x^1 \) is a point close enough to the point \((a, r),\) which belongs to the sphere centered at \((a, 0)\) with radius \( r \) and \( x^2 \) is the point diametrically opposite, it is easy to check that \((x^1, x^2)\) is a local maximum point of the function \( D_2 \) at level \( r. \)

**References**


THE ROLE OF THE DISTANCE FUNCTION IN SOME ELLIPTIC PROBLEMS


