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ON THE DYNAMICS OF GINZBURG-LANDAU VORTICES IN INHOMOGENEOUS SUPERCONDUCTORS*

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Abstract. In this paper, the authors study the dynamical behavior for a system of parabolic equations from Ginzburg-Landau inhomogeneous superconductors.

1. Main Results. Let m and n be positive integers. Consider the following initial-boundary value problem of Ginzburg-Landau system for $U_\varepsilon = (u_\varepsilon^1, u_\varepsilon^2, \cdots, u_\varepsilon^m)$: $\Omega \longrightarrow R^m$ with a smooth bounded domain $\Omega \subset R^n$:

$$(1.1) \quad \begin{cases} \dfrac{\partial U_\varepsilon}{\partial t} - \Delta U_\varepsilon &= \dfrac{U_\varepsilon}{\varepsilon^2}(a(x) - |U_\varepsilon|^2) + f(x, U_\varepsilon)U_\varepsilon \quad \text{in } \Omega \times (0,\infty) \\ U_\varepsilon(x,t) &= g_1(x) \quad \text{on } \partial\Omega \times (0,\infty) \\ U_\varepsilon(x,0) &= U_\varepsilon^0(x) \quad \text{in } \Omega, \end{cases}$$

where a and f are known functions, while ε is a very small positive parameter. The equation in (1.1) is a simple model which simulates inhomogeneous superconducting materials with $a(x)$ being its equilibrium density of superconducting electrons. Here we do not intend to discuss its physical background, since there are detailed discussions in [1, 2, 3].

As $\varepsilon \to 0$, Lin in [4], independently Mete and Soner in [5], studied the dynamical law for the vortices of $U_\varepsilon(x,t)$ solving the initial-boundary value problem (1.1) for the case $a \equiv 1$ and $f \equiv 0$. Their dynamical law is described by an ODE, $\frac{d}{dt}y(t) = -\nabla w(y(t)$. Here w is the renormalized energy functional given by [6, p.21]. The results in [4] and in [5] were generalized to the Neumann boundary condition by Lin in [7].

However, the situation is completely different in the case where $a(x)$ is not a constant. Chapman and Richardson in [1] used a matched asymptotic method to predict a new phenomenon, i.e., the vortices for problem (1.1) (more generally, for a more complicated equation involving magnetic field and electric field), are attracted to the the minimum points of $a(x)$. In this paper we will prove this dynamical phenomenon rigorously.

For this purpose, we made the following assumptions:

(H$_1$) $g_1 : \partial\Omega \longrightarrow R^m$ is smooth, $|g_1(x)| = \sqrt{a(x)}$ on $\partial\Omega$ and $deg(g_1, \partial\Omega) > 0$;

(H$_2$) $a \in C^{2,\alpha}(\bar\Omega)$ $(\alpha > 0)$, and $a(x) > 0$ for all $x \in \bar\Omega$;

(H$_3$) $f \in C^\alpha(\bar\Omega \times R^m)$ $(\alpha > 0)$ is a bounded function;

(H$_4$) the initial data $U_\varepsilon^0 \in C^2(\bar\Omega; R^m)$ $(\varepsilon > 0)$ satisfy $U_\varepsilon^0(x) = g_1(x)$ on $\partial\Omega$ and

$$||U_\varepsilon^0||_{C(\bar\Omega)} \le K, \quad \int_\Omega \rho^2(x)[|\nabla U_\varepsilon^0|^2 + \frac{1}{2\varepsilon^2}(|U_\varepsilon^0|^2 - a(x))^2]dx \le K$$

for a constant K (independent of ε) and some l distinct points $b_1, b_2, \cdots, b_l$ in Ω, where $\rho(x) = min\{|x - b_j|, j = 1, 2, \cdots, l\}$.

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To describe the vortex dynamics, we need to consider the ODE system

$$\left\{\begin{array}{lcl}\frac{d}{dt}y_j(t) & = & -a^{-1}(y_j(t))\nabla a(y_j(t)), \quad 0 \le t \le T, \\ y_j(0) & = & b_j\end{array}\right. \tag{1.2}$$

for $j = 1, 2, \cdots, l$, where $b_j's$ are the same as in (H_4) and ∇a is the gradient of the function a with respect to $x = (x_1, x_2, \cdots, x_n) \in R^n$.

THEOREM 1.1. *Suppose that $n, m \ge 2$ and hypothesis (H_2) is satisfied. Then there exists $T \in (0, +\infty]$ such that ODE system (1.2) has a unique C^3-solution*

$$(y_1, y_2, \cdots, y_l) : [0, T) \longrightarrow R^{lm}.$$

Moreover, if for each j there is a Lipschitz domain G_j such that

(H_5) $b_j \in G_j \subset\subset \Omega$, $\min_{x\in\partial G_j} a(x) > a(b_j), j = 1, \cdots, l,$

then $T = +\infty, y_j(t) \neq y_l(t)$ for $j \neq l$ and for all $t \in [0, +\infty)$, and $y_j(t) \in G_j$ for all $t \in [0, +\infty)$ and all $j, l = 1, \cdots, l$. Furthermore, one has that for each $j = 1, \cdots, l$, $y_j(t) \longrightarrow B_j$ for some $B_j \in \bar{G}_j$ with $\nabla a(B_j) = 0$ as $t \to \infty$ if the function $lna(x)$ is an analytic function in a neighborhood of any b in Ω with $\nabla a(b) = 0$.

When $n = m = 2$, Theorem 1.1 was proved in [8] by the first author. For general case, the proof is completely similar. So, we omit the details.

THEOREM 1.2. *Suppose that $n, m \ge 2$ and hypotheses $(H_1), (H_2), (H_3), (H_4)$ and (H_5) are satisfied. Let $y_j(t)$ $(1 \le j \le n)$ be solutions to problem (1.2) and set*

$$\Omega(a) = \bar{\Omega} \times (0, \infty) \setminus \cup_{j=1}^{m} \{(x, t) : x = y_j(t), 0 < t < \infty\}.$$

Then one has a positive constant ε_0 (depending only the superemun of $|f|$) such that the set $\{U_\varepsilon : \ \varepsilon \in (0, \varepsilon_0)\}$ of the classical solutions to problem (1.1) is bounded in $H^1_{loc}(\Omega(a))$. Moreover, there is a subsequence $\varepsilon_n \downarrow 0$, such that $U_{\varepsilon_n} \longrightarrow U$ weakly in $H^1_{loc}(\Omega(a))$, $|u(x,t)| = \sqrt{a(x)}$ a.e. in $\Omega(a)$, $U = g_1$ on $\partial\Omega \times (0, \infty)$ and U satisfies the equation

$$\frac{\partial U}{\partial t} - \Delta U = \frac{U}{2a}(2|\nabla U|^2 - \Delta a) \quad in \quad D'(\Omega(a)).$$

Furthermore, Suppose that $n = m = 2$ and the initial data U_ε^0 satisfy, in addition to (H_3), the following (1.3) and (1.4):

$$U_\varepsilon^0(x) \longrightarrow \sqrt{a(x)} \prod_{j=1}^{l} (\frac{x - b_j}{|x - b_j|})^{d_j} e^{ih_0(x)} \quad weakly \quad in \quad H^1_{loc}(\bar{\Omega} \setminus \{b_1, \cdots, b_l\}) \tag{1.3}$$

for some function $h_0(x)$ and some integers $d_j \neq 0$, $j = 1, 2, \cdots, l$ with

$$\sum_{k=1}^{l} d_k = d \equiv deg(g_1, \partial\Omega);$$

$$\int_\Omega [|\nabla U_\varepsilon^0|^2 + \frac{1}{\varepsilon^2}(|U_\varepsilon^0|^2 - a(x))^2] dx \le K(|\log \varepsilon| + 1). \tag{1.4}$$

Then for any given $\varepsilon_n \downarrow 0$, there is a subsequence such that

$$U_{\varepsilon_n} \longrightarrow \sqrt{a(x)} \prod_{j=1}^{l} (\frac{x - y_j(t)}{|x - y_j(t)|})^{d_j} e^{ih(x,t)} \quad strongly \quad in \quad H^1_{loc}(\Omega(a)) \tag{1.5}$$

and for any $t \in (0, \infty)$,

$$U_{\varepsilon_n}(x,t) \longrightarrow \sqrt{a(x)} \prod_{j=1}^{l} \left(\frac{x - y_j(t)}{|x - y_j(t)|}\right)^{d_j} e^{ih(x,t)} \tag{1.6}$$

$$\textit{strongly in } H^1_{loc}(\bar{\Omega} \setminus \{y_1(t), \cdots, y_l(t)\}).$$

Here each limit $h(x,t)$ *satisfies a linear parabolic equation in the set* $\Omega(a)$.

We will prove Theorem 1.2 in the next section.

2. Proof of Theorem 1.2. Throughout this section, we use the letter C to denote various constants independent of ε but maybe depending on Ω, a, g_1, f, K and other known quantities.

we assume $(H_1), (H_2), (H_3), (H_4)$ and (H_5), although some conclusions below need only part of these assumptions.

LEMMA 2.1. *Let* U_ε *be classical solutions to (1.1). Then there is a positive constant* $\varepsilon_0 > 0$ *such that*

$$|U_\varepsilon(x,t)|^2 \le C, \quad \forall (x,t) \in \bar{\Omega} \times [0, \infty) \tag{2.1}$$

and

$$|\nabla U_\varepsilon(x,t)|^2 + \left|\frac{\partial U_\varepsilon}{\partial t}\right| \le \frac{C}{\varepsilon^2}, \forall (x,t) \in \bar{\Omega} \times [\varepsilon^2, \infty). \tag{2.2}$$

Proof. Let $W = |U_\varepsilon|^2$. Dropping the subscript ε, we see that the equation in (1.1) reads as

$$\partial_t W - \Delta W + 2|\nabla U|^2 = \frac{2W}{\varepsilon^2}(a - W) + 2f(x,U)W. \tag{2.3}$$

We use a contradiction arguement. If (2.1) were not true, one could use (H_1) and (H_3), and employ the usual arguements for maximum principle to find a point $(x_\varepsilon, t_\varepsilon) \in \Omega \times (0, \infty)$ (for each ε) at which

$$W > 1 + a(x_\varepsilon, t_\varepsilon), \quad \nabla W = 0, \quad \partial_t W \ge 0 \;\; \textit{and} \;\; \Delta W \le 0.$$

Moreover, (2.3) gives us

$$\partial_t W \le 2W(f(x, U_\varepsilon) - \frac{1}{\varepsilon^2})$$

at $(x_\varepsilon, t_\varepsilon)$. This yields a contradiction if

$$\varepsilon < \varepsilon_0 \equiv (sup_{(x,v) \in \Omega \times R^n} \sqrt{|f(x,v)| + 1})^{-1}.$$

(see (H_3)).

By a scaling arguement, considering the equation for $V_\varepsilon(x,t) = U_\varepsilon(\varepsilon x, \varepsilon^2 t)$ and using the equation in (1.1) and standard local parabolic estimates, we immediately obtain (2.2).

For classical solutions U_ε to problem (1.1), define

$$F(x,t) = f(x, U(x,t))U(x,t)$$

and

$$A(x) = \sqrt{a(x)}, \quad V_\varepsilon(x,t) = \frac{U_\varepsilon(x,t)}{A(x)}, \quad g = \frac{g_1}{A}, \quad V_\varepsilon^0 = \frac{u_\varepsilon^0}{A}. \tag{2.4}$$

Then V_ε satisfies

$$\begin{cases} \partial_t V - A^{-1}\Delta(AV) &= \frac{A^2}{\varepsilon^2}V(1-|V|^2) + F(x,t) \quad in \ \Omega \times (0,\infty) \\ V &= g, \quad on \ \partial\Omega \times (0,\infty) \\ V(x,0) &= V_\varepsilon^0(x), \quad in \ \Omega. \end{cases} \tag{2.5}$$

Set

$$E_\varepsilon(V) = \frac{1}{2}[|\nabla(AV)|^2 + \frac{A^4}{2\varepsilon^2}(1-|V|^2)^2]. \tag{2.6}$$

LEMMA 2.2. *For any $T > 0$, there exists a positive constant $\sigma(T)$ (depending on T) such that for all $\varepsilon \in (0, \varepsilon_0)$ all $\delta \in (0, \sigma(T))$ and all $t \in [0,T]$, one has*

$$B_\delta(y_i(t)) \subset \Omega, \quad B_\delta(y_i(t)) \cap B_\delta(y_j(t)) = \emptyset \quad for \quad i \neq j,$$

$$\int_0^T \int_{\Omega_{\delta/4}(t)} |\frac{\partial V_\varepsilon}{\partial t}|^2 dxdt + \sup_{0\le t\le T} \int_{\Omega_{\delta/4}(t)} [|\nabla V_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|V_\varepsilon|^2)^2]dx \le C(\delta,T)$$

and

$$\int_0^T \int_{\Omega_{\delta/4}(t)} |\frac{\partial U_\varepsilon}{\partial t}|^2 dxdt + \sup_{0\le t\le T} \int_{\Omega_{\delta/4}(t)} [|\nabla U_\varepsilon|^2 + \frac{1}{\varepsilon^2}(a(x)-|U_\varepsilon|^2)^2]dx \le C(\delta,T).$$

Proof. For each $T > 0$, by Theorem 1.1 we can find a $\sigma = \sigma(T) > 0$ such that

$$\sigma \le \frac{m}{2}(1 + \sup_{x\in\Omega} |\nabla \ln a(x)|)^{-1} \tag{2.7}$$

and for all $t \in [0,T]$,

$$\min_{1\le l,j\le m} \{dist(y_j(t), \partial\Omega), \ |y_j(t) - y_l(t)| \ for \ l \neq j\} \ge 4\sigma. \tag{2.8}$$

As in [7; p.392], we choose a smooth monotone function $\phi : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\phi(r) = \begin{cases} r^2, & if \ r \le \sigma \\ \sigma^2, & if \ r \ge 2\sigma. \end{cases} \tag{2.9}$$

Let

$$\rho(x,t) = \min_{1\le j\le m} |x - y_j(t)|.$$

Then $\phi(\rho(x,t))$ is smooth both in x and in t for all $(x,t) \in \bar{\Omega} \times [0,T]$ by (2.8). Neglecting the subscript ε, using integration by partts and the fact $\partial_t V = \partial_t g = 0$ on

$\partial\Omega \times (0, \infty)$, we obtain, by (2.5) and (2.6), that

$$\frac{d}{dt}\int_\Omega \phi(\rho(x,t)E(V)dx \tag{2.10}$$
$$= \int_\Omega \frac{d\phi(\rho)}{dt}E(V) + \int_\Omega \phi(\rho))[\nabla(AV)\nabla(AV_t) - \varepsilon^{-2}A^4V(1-|V|^2)V_t]$$
$$= \int_\Omega \frac{d\phi(\rho)}{dt}E(V) - \int_\Omega AV_t\nabla(AV)\nabla\phi(\rho) - \int_\Omega \phi(\rho)[A\Delta(AV) + \varepsilon^{-2}A^4V(1-|V|^2)]V_t$$
$$= \int_\Omega \frac{d\phi(\rho)}{dt}E(V) - \int_\Omega [A^2|V_t|^2\phi + AV_t(V\nabla A\nabla\phi - \phi F) + A^2V_t\nabla V\nabla\phi]dx$$
$$\le \int_\Omega \frac{d\phi(\rho)}{dt}E(V) - \frac{1}{2}\int_\Omega \phi(\rho)A^2|V_t|^2 + \int_\Omega [4(|F|^2 + |V|^2|\nabla A\nabla\sqrt{\phi}|^2) - A^2V_t\nabla V\nabla\phi]dx$$
$$\le C(K,\sigma) - \frac{1}{2}\int_\Omega \phi(\rho)A^2|V_t|^2 + \int_\Omega [\frac{d\phi(\rho)}{dt}E(V) - A^2V_t\nabla V\nabla\phi(\rho)]dx,$$

where we have used (2.1), (H_3) and (2.9).

Using the notation $\omega_i = \frac{\partial\omega}{\partial x_i}$ and the summation convention and repeating the arguements of Theorem 1.2 in [9], one can obtain that

$$\frac{d}{dt}\int_\Omega \phi(\rho(x,t))E(V)dx \le C(K,\sigma) - \frac{1}{2}\int_\Omega \phi(\rho)A^2|V_t|^2$$
$$+ \int_\Omega \{E(V)[(\phi(\rho)_t - \nabla \ln a\nabla\phi] + I_1(V)\}dx + I_2(V), \tag{2.11}$$

where

$$I_1(V) = (AV)_i(AV)_j\phi_{ij} - \Delta\phi e(V) - (4\varepsilon^2)^{-1}A^4(1-|V|^2)^2\nabla\ln a\nabla\phi$$

and

$$I_2(V) = -\frac{1}{2}\int_\Omega \phi_i(\ln a)_{ij}(AV)_jAV - \int_\Omega A_iV(AV)_j\phi_{ij}$$
$$\le C(\sigma,a)[1 + \int_\Omega \phi|\nabla(AV)|^2]. \tag{2.12}$$

If $\rho(x,t) \ge \sigma$, by (2.8) and (2.9) one has that

$$E(V)|\phi(\rho)_t - \nabla\ln a\nabla\phi| + |I_1(V)| \le C(\sigma,a)\phi(\rho)E(V). \tag{2.13}$$

If $\rho(x,t) \le \sigma$, on the other hand, then $\phi(\rho(x,t)) = |x - y_j(t)|^2$ for some j. Hence

$$\phi_{ij} = \delta_{ij}$$

and

$$I_1(V) = |\nabla A|^2(1 - \frac{n}{2}) - \frac{A^4(1-|V|^2)^2}{2\varepsilon^2}[\frac{n}{2} + (x - y_l(t))\cdot\nabla\ln a] \le 0 \tag{2.14}$$

by (2.7). Moreover, (1.2) yields

$$\phi(\rho)_t - \nabla\ln a\nabla\phi = 2(x - y_l(t))(\nabla\ln a(y_l(t)) - \nabla\ln a(x))$$
$$= 2||\ln a||_{C^2(\bar\Omega)}\phi(\rho(x,t)). \tag{2.15}$$

Combing (2.10)-(2.13), one gets that

$$\frac{d}{dt}\int_\Omega \phi(\rho(x,t))E_\varepsilon(V_\varepsilon)dx + \frac{1}{2}\int_\Omega \phi(\rho(x,t))A^2|(V_\varepsilon)_t|^2 \le C[1+\int_\Omega \phi(\rho(x,t))E_\varepsilon(V_\varepsilon)dx]$$

for all $t \in [0,T]$. Hence, by Gronwall's inequality and (H_4), we deduce that

$$\int_\Omega \phi(\rho(x,t))E_\varepsilon(V_\varepsilon)dx + \frac{1}{2}\int_0^t\int_\Omega e^{C(t-s)}\phi(\rho(x,t))A^2|\partial_t V_\varepsilon|^2 dxds \le C(\sigma,T,K).$$

This result, together with (H_2), immediately implies the desired conclusions of Lemma 2.2.

Proof of Theorem 1.2. Using Lemmas 2.1 and 2.2 and applying a diagnonal method for $\delta \downarrow 0$ and $T \uparrow \infty$, we can prove the first conclusion of Theorem 1.2. Since the details are similar to those in [9], we omit them.

Finally, if one suppose that $m = 2$ and (1.3) and (1.4) are satisfied, then by virtue of Lemmas 2.1 and 2.2 again, the proof of (1.5) and (1.6) can be completed by the arguements similar to those in [7, p.394-395]. In this way, we have proved Theorem 1.2.

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