1. Introduction. In this paper we show that in two-body scattering the scattering matrix at a fixed energy determines real-valued exponentially decreasing potentials. This result has been proved by Novikov previously [3], see also [2], using a $\mathcal{D}$-equation. We present a different method, which combines a density argument and real analyticity in part of the complex momentum. The latter has been noted in [2]; here we give a short proof using contour deformations, similarly to [1, Section 1.5].

We thus prove:

**Theorem 1.1.** Suppose that $n \geq 3$, $V, V' \in e^{-\gamma_0 |w|}L^\infty(\mathbb{R}^{n};\mathbb{R})$ for some $\gamma_0 > 0$, and $\lambda > 0$. If $S_+^{\prime}(\lambda) = S_+^{\prime}(\lambda)$, then $V = V'$. Here $S_+^{\prime}(\lambda)$, resp. $S_+^{\prime}(\lambda)$ are the scattering matrices of $H = \Delta + V$ and $H' = \Delta + V'$ at energy $\lambda$.

Theorem 1.1 for compactly supported potentials follows from an analogous result in [4] for the corresponding Dirichlet-to-Neumann map. See [5, Section 12], and the references given in these papers for a review of the relation between the Dirichlet-to-Neumann map and the fixed energy problem.

The general method follows [4], as discussed in [1]. We thus recall the construction of complex exponential solutions $u_\rho$, $\rho \in \mathbb{C}^n$ of $(H - \lambda)u_\rho = 0$, where $u_\rho(w) = e^{i\rho \cdot w}(1 + v_\rho(w))$, $\rho \cdot \rho = \lambda$, and $v_\rho \to 0$ in an appropriate sense as $\rho \to \infty$. These solutions exist for $\rho$ outside an ‘exceptional set’ which is discrete in $\rho$. We also show that if we write $\rho = z\nu + \rho_{\perp}$, $\nu \in \mathbb{S}^{n-1}$, $\rho_{\perp} \in \mathbb{R}^n$ perpendicular to $\nu$, and $z \in \mathbb{C} \setminus \mathbb{R}$, then for fixed $\nu$, $u_\rho$ is analytic in $z$ and real analytic in $\rho_{\perp}$, hence extends to be analytic in a neighborhood of $\mathbb{R}^{n-1} \setminus \{0\}$ in $\mathbb{C}_{\rho_{\perp}}$. The exceptional set is then given by the zeros of an analytic function of $z$ and $\rho_{\perp}$. We caution the reader that the extension of $u_\rho$ to complex $\rho_{\perp}$ does not agree with $u_{2z\nu + \rho_{\perp}}$ where $\rho_{\perp}$ is allowed to be complex; indeed $u_\rho$ will merely lie in $e^{\gamma |w|}L^2(\mathbb{R}^n)$ for some $\gamma > 0$.

We use this in the inverse problem as follows. Let $u_\rho$, $u_\rho'$ be exponential eigenfunctions of $H$, resp. $H'$, as above. Now consider the pairing

$$\int_{\mathbb{R}^n} u_\rho(V - V')u_\rho'$$

where $\rho = z\nu + \rho_{\perp}$, $\rho' = z'\nu + \rho'_{\perp}$, and $\nu$ is fixed. If $u_\rho$, $u_\rho'$ are replaced by tempered distributional eigenfunctions of $H$ and $H'$, then a standard argument shows that $S_+^{\prime}(\lambda) = S_+^{\prime}(\lambda)$ implies that the corresponding pairing vanishes. We employ a density argument to deduce that the pairing also vanishes for the complex exponentials provided that $|\text{Im } z + \text{Im } z'|$ is small and $\rho \cdot \rho = \lambda = \rho' \cdot \rho'$. We then let $\rho, \rho' \to \infty$. By analyticity, the pairing still vanishes. On the other hand, $v_\rho, v_\rho' \to 0$, so for $\zeta = \rho - \rho' \in \mathbb{R}^n$ we deduce that $\int_{\mathbb{R}^n} e^{i\zeta \cdot w}(V - V') = 0$, i.e. the Fourier transform
of $V - V'$, hence $V - V'$, vanish. In fact, this step will be slightly more complicated, because the density argument imposes restrictions on $\zeta$, and we first deduce vanishing of the Fourier transform of $V - V'$ in a spherical shell of finite 'thickness', and then use the exponential decay of $V - V'$ to conclude that it is in fact identically zero.

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2. Exponential eigenfunctions. In this section we recall the construction of exponential solutions of $(H - \lambda)u = 0$ from [4]. First, for $\rho \in \mathbb{C}^n$, let

$$u^0_\rho(w) = e^{i\rho \cdot w}.$$

Thus, $u^0_\rho$ is an 'exponential eigenfunction' of $\Delta$, namely

$$(\Delta - \lambda)u^0_\rho = 0, \quad \rho \cdot \rho = \lambda.$$ 

We assume everywhere that $n \geq 3$.

For the Hamiltonian $H$, we then seek exponential solutions $u$ of the form

$$u = u_\rho = e^{i\rho \cdot w}(1 + v_\rho), \quad \rho \cdot \rho = \lambda, \quad \rho \in \mathbb{C}^n,$$

where $v_\rho$ is considered a perturbation. In fact, we will have $v_\rho \in L^2_r(\mathbb{R}^n)$ for all $r < 0$ (when $v_\rho$ exists). Here $L^2_r(\mathbb{R}^n)$ denotes the $L^2(\mathbb{R}^n, \langle w \rangle^r dw)$, $\langle w \rangle^s = (1 + |w|^2)^{s/2}$. Substituting $u$ into $(H - \lambda)u = 0$, we obtain

$$(\Delta + 2\rho \cdot D_w + V)v_\rho = -V.$$ 

The construction given below works under power-decay assumptions on $V$, but we state it for exponentially decaying $V$, since $v_\rho$ is real analytic in the appropriate components of $\rho$ only in that case. So we assume that

$$V \in e^{-\gamma_0|w|}L^\infty(\mathbb{R}^n), \quad \gamma_0 > 0.$$ 

Thus, we need to construct a right inverse $G(\rho)$ to

$$P(\rho) = \Delta + 2\rho \cdot D_w + V$$

that can be applied to rapidly decreasing functions. Once this is done,

$$u_\rho = e^{i\rho \cdot w}(1 - G(\rho)V)$$

is the solution to the original problem. Below we write

$$P_0(\rho) = \Delta + 2\rho \cdot D_w.$$ 

Since a right inverse $G_0(\rho)$ of $P_0(\rho)$ can be constructed explicitly, perturbation theory will give the existence of $G(\rho)$.

Namely, let

$$G_0(\rho) = \mathcal{F}^{-1}((|\xi|^2 + 2\rho \cdot \xi)^{-1} \mathcal{F},$$

so $P_0(\rho)G_0(\rho) = \text{Id}$ e.g. on Schwartz functions. Thus, on the Fourier transform side $G_0(\rho)$ acts via multiplication by $(|\xi|^2 + 2\rho \cdot \xi)^{-1}$ which is in $L^1(\mathbb{R}^n)$. It is convenient to represent $\rho$ as

$$\rho = z\nu + \rho_\perp, \quad \rho_\perp \in \mathbb{R}^n, \quad \nu \in S^{n-1}, \quad z \in \mathbb{C}, \quad \rho_\perp \cdot \nu = 0.$$
We often identify $\text{span}[\nu]\perp$ with $\mathbb{R}^{n-1}$. For $z \notin \mathbb{R}$, this distribution is conormal to

\begin{equation}
S(\rho) = \{ \xi \in \mathbb{R}^n : |\xi|^2 + 2 \Re \rho \cdot \xi = 0, \Im \rho \cdot \xi = 0 \} \\
= \{ \xi \in \mathbb{R}^n : (\xi + \rho\perp)^2 = \rho\perp^2, \nu \cdot \xi = 0 \}.
\end{equation}

(2.5)

Note that $S(\rho)$ actually depends only on $\rho\perp$ and $\nu$, not on $z$.

Below we write $e^{\gamma(w)} L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n ; e^{-2\gamma(w)} dw)$, and if $m$ is an integer,

\[ e^{\gamma(w)} H^m(\mathbb{R}^n) = \{ u \in e^{\gamma(w)} L^2(\mathbb{R}^n) : D^\alpha u \in e^{\gamma(w)} L^2(\mathbb{R}^n), |\alpha| \leq m \} \]

The latter is equivalent to $e^{-\gamma(w)} u \in H^m(\mathbb{R}^n)$, hence the notation.

We first recall:

**PROPOSITION 2.1.** [4, Proposition 3.1], [6, Theorem 1.1] $G_0(\rho) : L^2_p \to L^2_r$ is bounded for $p > 0$, $r < 0$, $r < p - 1$. Moreover, the norm of $G_0(\rho)$ as a bounded operator between these spaces goes to 0 as $|\rho| \to \infty$.

Our central result is the following proposition.

**PROPOSITION 2.2.** Suppose that $\gamma > 0$ and fix $\nu \in \mathbb{S}^{n-1}$. Then there exists a neighborhood $U$ of $\mathbb{R}^{n-1} \setminus \{0\}$ in $\mathbb{C}^{n-1}$ and an operator

\[ G_0(z, \rho\perp) : e^{-\gamma(w)} L^2(\mathbb{R}^n) \to e^{\gamma(w)} H^2(\mathbb{R}^n) \]

defined on $(\mathbb{C} \setminus \mathbb{R}) \times U$ such that $G_0$ is analytic on $(\mathbb{C} \setminus \mathbb{R}) \times U$ as a bounded operator between these spaces, and its restriction to $(\mathbb{C} \setminus \mathbb{R}) \times (\mathbb{R}^{n-1} \setminus \{0\})$ is $G_0(\rho)$, $\rho = z\nu + \rho\perp$. Thus, for $z \in \mathbb{C} \setminus \mathbb{R}$, $\rho\perp \in \mathbb{R}^{n-1} \setminus \{0\}$, the operator $G_0(\rho) : e^{-\gamma(w)} L^2 \to e^{\gamma(w)} L^2$ is complex-analytic in $z$, real analytic in $\rho\perp$. Moreover, $G_0(\rho) \to 0$ as a bounded operator on this space as $|\rho| \to \infty$.

**Proof.** We fix some $(z^0, \rho\perp^0)$, and show that $G_0(\rho)$ extends to be complex analytic in a neighborhood of this in $\mathbb{C}_z \times \mathbb{C}_{\rho\perp}$. In fact, it is convenient to consider

\[ R_0(\rho) = e^{-i\rho\perp \cdot w} G_0(\rho) e^{i\rho\perp \cdot w}. \]

Since the multipliers are holomorphic as maps

\[ e^{\gamma(w)} H^m(\mathbb{R}^n) \to e^{\gamma'(w)} H^m(\mathbb{R}^n), \gamma < \gamma', \]

for $|\Im \rho\perp|$ sufficiently small, and unitary for $\rho\perp$ real, the original statement follows after we show that $R_0(\rho)$ extends analytically.

We do so by contour deformation on the Fourier transform side. Let $\xi = (\xi\parallel, \xi\perp)$ be the decomposition of $\xi$ according to the decomposition $\text{span}\{\nu\} \oplus \text{span}\{\nu\}^\perp$ of $\mathbb{R}^n$. Thus, $\mathcal{F} R_0(\rho) \mathcal{F}^{-1}$ is a multiplication operator by $F^{-1}$ where

\begin{equation}
F(\xi, z, \rho\perp) = |\xi - \rho\perp|^2 + 2(\xi - \rho\perp) \cdot \rho = \xi\parallel^2 + 2z \xi\parallel + \xi\perp^2 - \rho\perp^2.
\end{equation}

(2.6)

Then

\[ \Im F = 2 \Im z \xi\parallel, \Re F = \xi\parallel^2 + 2 \Re z \xi\parallel + \xi\perp^2 - \rho\perp^2. \]

Thus the multiplication operator by $F^{-1}$ is singular where $F = 0$, i.e. at

\[ \tilde{S}(\rho) = \{ \xi : \xi\parallel = 0, \xi\perp^2 = \rho\perp^2 \}, \]
which is a sphere in the hyperplane $\xi = 0$.

It is convenient to break up $G_0(\rho)$ into two pieces by introducing a cutoff $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ that is identically 1 near $S(\rho_0)$. For instance, we may take

$$\psi(\xi) = \phi(\xi_\parallel, \xi_\perp)$$

with $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, identically 1 near $(0, |\rho_0|^2)$. Then

$$R_0(\rho) = R_0^r(\rho) + R_0^s(\rho), \quad R_0^s(\rho) = \mathcal{F}^{-1}(|\xi|^2 + 2\zeta \cdot \rho - \rho_\perp^2)^{-1}\psi(\xi)\mathcal{F}.$$

Then $R_0^s(\rho)$ is a Fourier multiplier by the function $(1 - \psi(\xi))F(\xi, z, \rho_\perp)^{-1}$, which is in fact a symbol of order $-2$, analytic in $z$ and $\rho_\perp$ for $\text{Im} \rho_\perp$ small, hence $R_0^s(\rho)$ is analytic as a map $L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$.

To analyze $R_0^r(\rho)$, it is convenient to introduce polar coordinates in $\xi_\perp$: $\xi_\perp = r\omega$, $|\omega| = 1$, $r \geq 0$. Then

$$F = \xi_\parallel^2 + 2z\xi_\parallel + r^2 - \rho_\perp^2.$$

Now, by Fubini’s theorem $R_0^r(\rho)f = \mathcal{F}^{-1}\mathcal{F}Ff$ can be written as

$$(R_0^r(\rho)f)(w) = \mathcal{F}^{-1}\mathcal{F}Ff = (2\pi)^{-n} \int_{\mathbb{R}} \int_{S^{n-2}} \int_{0}^{\infty} e^{irw \cdot \omega} \psi(\xi_\parallel, r\omega)\mathcal{F}(\xi_\parallel, r\omega) dr d\omega d\xi_\parallel.$$

We divide the $\xi_\parallel$ integral into two pieces, corresponding to $\xi_\parallel \geq 0$ and $\xi_\parallel \leq 0$. In each piece, we then deform the contour of the $r$ integral in a compact set disjoint from supp$(1 - \psi)$ near $r_0 = |\rho_0^\parallel|$ in such a way that $\text{Im} r^2 = 2\text{Re } r \text{ and } \text{Im } z \xi_\parallel$ have the same sign on the contour. Note that the integrand is analytic in $r$ for $\text{Im } r$ small and $\xi_\parallel \neq 0$.

Thus, suppose that $\text{Im } z > 0$. For $\xi_\parallel > 0$, we deform the contour $[0, +\infty)$, near $r_0$ to a curve $\Gamma_+$, so that $\text{Im } r \geq 0$ on $\Gamma_+$ and $r_0$ does not lie on the $\Gamma_+$. Now, $F$ never vanishes along $\Gamma_+$, provided that $\rho_\perp$ is close to $\rho_0^\parallel$. Thus, extending $\psi$ to be 1 on $\Gamma_+ \setminus [0, +\infty)$, and using that $\mathcal{F}F$ extends to be analytic in a tube $\{\xi \in \mathcal{C}_c^n : |\text{Im } \xi| < \gamma\}$,

$$= (2\pi)^{-n} \int_{\Gamma_+} \int_{S^{n-2}} \int_{0}^{\infty} e^{irw \cdot \omega} \psi(\xi_\parallel, r\omega)\mathcal{F}(\xi_\parallel, r\omega) dr d\omega d\xi_\parallel,$$

and on the right hand side we can allow $\rho_\perp$ to become complex, proving real analyticity of $R_0^r(\rho)$ in $\rho_\perp$, and extending it as an analytic family of operators $R_0^r(z, \rho_\perp)$. This argument parallels the analytic continuation argument of [1, Chapter 1]. It is now easy to see that $R_0^r(z, \rho_\perp)$ maps into $e^{i\gamma w}H^2(\mathbb{R}^n)$; indeed, it maps into $e^{i\gamma w}C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ is the space of smooth functions which are bounded with all derivatives.

For $\xi_\parallel < 0$ we proceed similarly, deforming the contour $[0, +\infty)$, near $r_0$ to a curve $\Gamma_-$ so that $\text{Im } r \geq 0$ on $\Gamma_-$ and $r_0$ does not lie on the $\Gamma_-$. Again, we deduce real analyticity in $\rho_\perp$. 
The last part follows from the preceding proposition since $e^{-\gamma|w|}L^2 \subset L^2_p \subset L^2_r \subset e^{\gamma|w|}L^2$.

Instead of the explicit contour deformation, we could have used the partial Fourier transform in $w_1$, to deduce that
\[
G_0(\rho)f = e^{i\rho \cdot w}F^{-1}(\Delta_{\perp} + \xi_{\parallel}^2 + 2\xi_{\parallel} - \rho_{\perp}^2)^{-1}F_{\parallel}e^{-i\rho \cdot w}f
\]
is real analytic in $\rho_{\perp}$ and analytic in $z$ by inserting step functions $1 = H(\xi_{\parallel}) + H(-\xi_{\parallel})$, and using the analyticity of
\[
(\Delta_{\perp} - \sigma)^{-1} : e^{-\gamma|w_\perp|L^2} (\text{span}\{\nu\}^\perp) \rightarrow e^{\gamma|w_\perp|L^2} (\text{span}\{\nu\}^\perp)
\]
in $\sigma$. \[\square\]

**Corollary 2.3.** Suppose that $\gamma, \gamma_0 > 0$. Then the operator
\[
e^{-\gamma_0\cdot w}G_0(z, \rho_{\perp}) \in \mathcal{B}(e^{-\gamma|w|}L^2, e^{-\gamma_0(\gamma)|w|}H^2)
\]
is analytic in $z$ and in $\rho_{\perp}$ as a bounded operator between these spaces.

**Corollary 2.4.** Suppose that $V \in e^{-\gamma_0|w|}L^\infty$ and $\gamma_0 > 2\gamma$, and let $U$ be as in Proposition 2.2. Then there exists a set
\[
\mathcal{E} \subset (\mathbb{C} \setminus \mathbb{R})_z \times U,
\]
which is given by the zeros of an analytic function and whose intersection with
\[
(\mathbb{C} \setminus \mathbb{R})_z \times (\mathbb{R}^{n-1} \setminus \{0\})_{\rho_{\perp}}
\]
is bounded, such that $(\text{Id} + V G_0(z, \rho_{\perp}))^{-1}$ exists in the complement of $\mathcal{E}$, and in a neighborhood of every point where it exists, $(\text{Id} + V G_0(z, \rho_{\perp}))^{-1}$ is analytic with values in compact operators on $e^{-\gamma|w|}L^2$.

**Proof.** By the preceding corollary, $e^{-\gamma_0\cdot w}G_0(z, \rho_{\perp}) : e^{-\gamma|w|}L^2 \rightarrow e^{-\gamma_0(\gamma)|w|}H^2$ is analytic in $z$ and $\rho_{\perp}$. But the inclusion $e^{-\gamma_0(\gamma)|w|}H^2 \hookrightarrow e^{-\gamma|w|}L^2$ is compact, and $e^{-\gamma_0\cdot w}V \in L^\infty$, so $V G_0(z, \rho_{\perp})$ is an analytic family of compact operators on $e^{-\gamma|w|}L^2$. Moreover, as $|z| \rightarrow \infty$ or $|\rho_{\perp}| \rightarrow \infty$, $\rho_{\perp}$ real, $V G_0(z, \rho_{\perp}) = V G_0(\rho) \rightarrow 0$ in norm. Thus, the conclusion follows by analytic Fredholm theory. \[\square\]

We write
\[
G(z, \rho_{\perp}) = G_0(z, \rho_{\perp})(\text{Id} + V G_0(z, \rho_{\perp}))^{-1}, \ G(\rho) = G_0(\rho)(\text{Id} + V G_0(\rho))^{-1}.
\]

We immediately deduce the following result.

**Proposition 2.5.** Suppose that $V \in e^{-\gamma_0|w|}L^\infty$,
\[
v_{z, \rho_{\perp}} = -G(z, \rho_{\perp})V.
\]
Then
\[
(((\mathbb{C} \setminus \mathbb{R}) \times U) \setminus \mathcal{E}) \ni (z, \rho_{\perp}) \mapsto v_{\rho}
\]
is an analytic function, with values in $e^{\gamma|w|}L^2$, for any $\gamma > 0$. 

Corollary 2.6. Let $\nu \in S^{n-1}$. Suppose that $V, V' \in e^{-\gamma_0|w|}L^\infty$, and let $\mathcal{E}, \mathcal{E}'$ be the exceptional sets of these two potentials. Then for $(z, \rho_\perp) \notin \mathcal{E}, (z', \rho_\perp') \notin \mathcal{E}'$ the pairing

$$
\int_{\mathbb{R}^n} u_\nu(V - V')u'_{\nu'}
$$

converges if $\text{Im} z + \text{Im} z' < \gamma_0$, and is analytic in $z, z', \rho_\perp, \rho_\perp'$.

Proof. We consider the strip $|\text{Im} z + \text{Im} z'| < \gamma_1 < \gamma_0, \gamma_1 > 0$. Let $\gamma \in (0, (\gamma_0 - \gamma_1)/2)$. Then $1 + v_\rho, 1 + v'_{\rho'}$ are analytic in $(z, z', \rho_\perp, \rho_\perp')$ with values in $e^{\gamma|w|}L^2$. Hence,

$$
u_\nu(V - V')u'_{\nu'} = e^{i(p + \rho') \cdot w}(V - V')(1 + v_\rho)(1 + v'_{\rho'})
$$

is analytic in $(z, z', \rho_\perp, \rho_\perp')$ with values in $L^1(\mathbb{R}^n)$. Integration preserves analyticity and proves the result. $\square$

3. Density of generalized eigenfunctions. In this section we relate tempered distributional eigenfunctions of $H = \Delta + V$ to its exponential eigenfunctions, constructed in the previous section.

We first introduce some notation. For $\lambda > 0$, the free incoming Poisson operator is given by

$$
P_+(\lambda)g = c \int_{S^{n-1}} e^{-i\sqrt{\lambda}w \cdot \omega} g d\omega, \quad g \in C^\infty(S^{n-1}), \quad c = \lambda^{n-1} e^{-\frac{n-1}{2} \pi i} (2\pi)^{\frac{n-1}{2}}.
$$

The Poisson operator of $H$ is then

$$
P_+(\lambda)g = \tilde{P}_+(\lambda)g - R(\lambda + i0)((H - \lambda)\tilde{P}_+(\lambda)g) = \tilde{P}_+(\lambda)g - R(\lambda + i0)V\tilde{P}_+(\lambda)g.
$$

Note that for $g \in C^\infty(S^{n-1}), V\tilde{P}_+(\lambda)g$ is Schwartz, in fact decays exponentially, hence $R(\lambda + i0)$ can be applied to it. For $g \in C^\infty(S^{n-1})$,

$$
P_+(\lambda)g = e^{-i\sqrt{\lambda}|w|}g_+ + e^{i\sqrt{\lambda}|w|}g_+ + L^2(\mathbb{R}^n), \quad g_+, g_- \in C^\infty(S^{n-1}), \quad g_- = g.
$$

For such $g$, $P_+(\lambda)g$ is characterized by the property that it is the unique solution $u$ of $(H - \lambda)u = 0$ which is of the form (3.1). The scattering matrix is then the operator

$$
S_+(\lambda) : C^\infty(S^{n-1}) \rightarrow C^\infty(S^{n-1}), \quad S_+(\lambda)g_- = g_+.
$$

There is also an incoming Poisson operator $P_-(\lambda)$ which is characterized by the fact that for $g \in C^\infty(S^{n-1}), P_-(\lambda)g$ is the unique solution $u$ of $(H - \lambda)u = 0$ of the form

$$
P_-(\lambda)g = e^{-i\sqrt{\lambda}|w|}g_- + e^{i\sqrt{\lambda}|w|}g_+ + L^2(\mathbb{R}^n), \quad g_+, g_- \in C^\infty(S^{n-1}), \quad g_+ = g.
$$

In particular, for $g \in C^\infty(S^{n-1}),$

$$
P_-(\lambda)g = P_+(\lambda)\overline{g}.
$$

The S-matrix is related to the Poisson operator via the following boundary pairing.
Proposition 3.1. [1, Lemma 2.2] Suppose that $\lambda > 0$, and $V \in e^{-\gamma_0|w|}L^\infty$, $\gamma_0 > 0$. Suppose that $(H - \lambda)u_+ \in L^2_s$, $(H - \lambda)u_- \in L^2_s$, $s > 1/2$, and
\[
\begin{align*}
  u_+ &= e^{-i\sqrt{|w|}g_{+-}} + e^{i\sqrt{|w|}g_{++}} + L^2, \\
  u_- &= e^{-i\sqrt{|w|}g_{--}} + e^{i\sqrt{|w|}g_{-+}} + L^2,
\end{align*}
\]
$g_{\pm\pm} \in C^\infty(S^{n-1})$. Then
\[(3.4) \quad \langle u_+, (H - \lambda)u_- \rangle = \langle (H - \lambda)u_+, u_- \rangle = 2i\sqrt{\lambda}(\langle g_{++}, g_{--} \rangle - \langle g_{-+}, g_{-+} \rangle).
\]

Remark 3.2. This is stated for $V \in C_c^\infty(\mathbb{R}^n)$ in [1]. However, if $u_+, u_-$ are as above, then $Vu_+ u_- \in e^{-\gamma_0|w|}L^1$ for $\gamma < \gamma_0$, hence the conclusion is equivalent to the corresponding statement with $H - \lambda$ replaced by $\Delta - \lambda$.

Let $R(\lambda') = (H - \lambda')^{-1}$ for $\lambda' \in \mathbb{C} \setminus \mathbb{R}$. Let $f \in S(\mathbb{R}^n)$, $g \in C^\infty(S^{n-1})$, and apply this proposition with
\[
\begin{align*}
  u_- &= R(\lambda - i0)f = e^{-i\sqrt{|w|}g_{--}} + L^2, \\
  u_+ &= P_+(\lambda)g.
\end{align*}
\]
We deduce that
\[(3.5) \quad \langle u_+, f \rangle = -2i\sqrt{\lambda}\langle g, g_- \rangle.
\]

Our density result is the following.

Proposition 3.3. Suppose that $V \in e^{-\gamma_0|w|}L^\infty$, and let $0 < \gamma < \gamma' < \gamma_0$. Then the set
\[\mathcal{F} = \{P_+(\lambda)g_+ : g_+ \in C^\infty(S^{n-1})\}\]
is dense in the nullspace of $H - \lambda$ on $e^{\gamma|w|}L^2$ in the topology of $e^{\gamma'|w|}L^2$.

Proof. Suppose that $f \in e^{-\gamma|w|}L^2$ is orthogonal to $\mathcal{F}$. Let $u_- = R(\lambda - i0)f$. By (3.5), for all $g \in C^\infty(S^{n-1})$, $\langle f, g_+ \rangle = 0$ since $\langle f, P_+(\lambda)g \rangle$ vanishes by assumption. But $u_- = R_0(\lambda - i0)f'$, $f' = f - VR(\lambda - i0)f \in e^{-\gamma|w|}L^2$. Thus, $\mathcal{F}u_-$ is the product of an analytic function, namely $\mathcal{F}f'$, and $(|\xi|^2 - (\lambda - i0))^{-1}$. Thus, $u_- \in L^2$ implies that $\mathcal{F}f'$ vanishes on the sphere $|\xi| = \sqrt{\lambda}$. Hence $\mathcal{F}f' = (\xi^2 - \lambda)\phi$, with $\phi$ analytic in the strip $|\text{Im}\, \xi| < \gamma'$. Thus, $u_- \in e^{-\gamma|w|}L^2$ for $\gamma < \gamma'$. Thus for $v \in e^{\gamma|w|}L^2$ with $(H - \lambda)v = 0$,
\[\langle f, v \rangle = \langle (H - \lambda)u_-, v \rangle = \langle u_-, (H - \lambda)v \rangle = 0,
\]
i.e. $f$ is orthogonal to the nullspace of $H - \lambda$ on $e^{\gamma|w|}L^2$. Thus, $\mathcal{F}$ is dense in this nullspace.

Our approach to the inverse problem relies on relating the S-matrices to the pairing (2.7). Thus, we consider two operators $H$ and $H'$ induced by potentials $V$ and $V'$ respectively, and show that the equality of the S-matrices at a fixed energy $\lambda$ implies the vanishing of an analogous pairing. For this we use the following consequence of Proposition 3.1 applied with $\Delta$ in place of $H$.

Proposition 3.4. Suppose that $\lambda > 0$. Let $u_+ = P_+(\lambda)g_+$, $u_- = P_-^\ast(\lambda)g_-$. Then
\[(3.6) \quad \langle u_+, (\Delta - \lambda)u_- \rangle = \langle (\Delta - \lambda)u_+, u_- \rangle = 2i\sqrt{\lambda}(\langle S_+(\lambda)g_+, g_- \rangle - \langle g_+, S_-^\ast(\lambda)g_- \rangle).
\]
The equality of the S-matrices implies that (3.8) vanishes, hence so does (4.2), i.e. we deduce the following result.

\[
\int_{\mathbb{R}^n} (V - V') u_+ u_- = 0.
\]

Similarly, if \( u_+ = P_+(\lambda)g_+ \), \( u_- = P_-(\lambda)g_- \), then.

\[
\int_{\mathbb{R}^n} (V - V') u_+ u_- = 0.
\]

**Proof.** (3.7) follows from the preceding proposition since \( S'_-(\lambda)^* = S'_+(\lambda) \). Then (3.8) follows from (3.7) by applying the latter with \( g_- \) replaced by \( g_+ \) and using (3.3).

4. Inverse results: Proof of Theorem 1.1. Let \( \lambda > 0 \), and suppose that

\[
V, V' \in e^{-\gamma_0|u|}L^\infty, \; \gamma_0 > 0.
\]

Fix \( \zeta \in \mathbb{R}^n \) such that \( |\zeta| > 2\sqrt{\lambda} \), and let \( \nu \in S^{n-1} \) be orthogonal to \( \zeta \), and let \( \mu \in S^{n-1} \) orthogonal to both \( \zeta \) and \( \nu \). For \( t \) real, \( t > \sqrt{2}|\zeta|^2 - \lambda \), let

\[
\rho = \rho(t) = \frac{\zeta}{2} + (t^2 - \frac{1}{4}|\zeta|^2 + \lambda)^{1/2} \mu + it\nu,
\]

\[
\rho' = \rho'(t) = \frac{\zeta}{2} - (t^2 - \frac{1}{4}|\zeta|^2 + \lambda)^{1/2} \mu - it\nu,
\]

so \( \rho \cdot \rho = \lambda = \rho' \cdot \rho' \). By Corollary 2.6, the integral

\[
\int_{\mathbb{R}^n} u_\rho(V - V') u_{\rho'}'
\]

converges for all \( t \), and is meromorphic in \( t \) in a neighborhood of \( (\sqrt{\frac{1}{4}}|\zeta|^2 - \lambda, +\infty) \).

We use a density argument, Proposition 3.3, and Corollary 3.5 to show that this integral actually vanishes if \( S'_+(\lambda) = S'_-(\lambda) \) and

\[
2\sqrt{\lambda} < |\zeta| < \sqrt{4\lambda + \gamma_0^2}.
\]

Indeed, for \( \sqrt{\frac{1}{4}}|\zeta|^2 - \lambda < t < \gamma < \gamma_0/2 \), \( u_{\rho'} \) can be approximated by \( P_-(\lambda)g_- \) in \( e^{-\gamma|u|}L^2 \) due to Proposition 3.3. Similarly, \( u_{\rho} \) can be approximated by \( P_+(\lambda)g_+ \) in \( e^{-\gamma_0|u|}L^2 \). On the other hand, \( V - V' \) lies in \( e^{-\gamma_0|u|}L^2 \). Hence the product can be approximated in \( L^1 \) by a product which takes the form of the integrand of (3.8). The equality of the S-matrices implies that (3.8) vanishes, hence so does (4.2), i.e. we deduce the following result.

**Proposition 4.1.** Suppose that \( \lambda > 0 \), \( V, V' \in e^{-\gamma_0|u|}L^\infty, \; S_+(\lambda) = S'_-(\lambda) \). Then for \( \zeta \) satisfying (4.3), \( \rho, \rho' \) given by (4.1) with \( (z, \rho_\perp) \notin \mathcal{E}, \; (z, \rho_\perp) \notin \mathcal{E}' \),

\[
\int_{\mathbb{R}^n} u_\rho(V - V') u_{\rho}' = 0
\]

for \( \sqrt{\frac{1}{4}}|\zeta|^2 - \lambda < t < \gamma_0/2 \).
The pairing in (4.4) is meromorphic in $t$ with $\text{Re} \ t > \sqrt{\frac{1}{4} |\zeta|^2 - \lambda}$ and $|\text{Im} \ t|$ sufficiently small. It vanishes on an interval inside this domain by the proposition. Thus, (4.4) holds for all $t > \sqrt{\frac{1}{4} |\zeta|^2 - \lambda}$. Then as $t \to \infty$, the integral on the left hand side of (4.4) converges to

\begin{equation}
\int_{\mathbb{R}^n} (V - V')u_\rho^0 u_{\rho'}^0 = \int_{\mathbb{R}^n} (V - V') e^{i\zeta \cdot w} \, dw
\end{equation}

since $v_\rho \to 0$, $v'_\rho \to 0$ in $e^{\gamma |w|} L^2(\mathbb{R}^n)$ for any $\gamma > 0$ and $V - V' \in e^{-\gamma_0 |w|} L^\infty(\mathbb{R}^n)$ with $\gamma_0 > 0$. But this is the Fourier transform of $V - V'$, evaluated at $\zeta$. Hence the vanishing of (4.5) shows that the Fourier transform of $V - V'$ vanishes on the shell (4.3). Since this Fourier transform is real analytic, as $V - V' \in e^{-\gamma_0 |w|} L^\infty$, we deduce that it vanishes everywhere, hence $V = V'$. This completes the proof of Theorem 1.1.

REFERENCES
