A LIOUVILLE TYPE THEOREM FOR MINIMIZING MAPS

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Abstract. Here we establish a Liouville type theorem for minimizing maps from $\mathbb{R}^2$ (or in general, from $\mathbb{R}^m$) into a compact Riemannian manifold $N$. As a consequence of this, we prove a local gradient estimate for minimal solutions to a variational problem arise from planar ferromagnetism and anti-ferromagnetism. The latter can be applied to study the asymptotic behavior of entire solutions.

1. Introduction. In [HnL] we studied the following simplified mathematical model for the planar ferromagnetism and anti-ferromagnetism. Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected smooth subset, $S^2 \subset \mathbb{R}^3$ be the standard 2-sphere, $S^1$ be the horizontal great circle on $S^2$, and $g : \partial \Omega \rightarrow S^1$ be a smooth map. For any $\varepsilon > 0$ and $u \in H^1_g(\Omega, S^2)$, we define

$$I_\varepsilon(u) = \int_{\Omega} \frac{1}{2} \left[ |\nabla u|^2 + \left( \frac{u^3}{\varepsilon^2} \right)^2 \right] dx.$$  

We analyzed the asymptotic behavior of the minimizers of $I_\varepsilon$ over $H^1_g(\Omega, S^2)$ as $\varepsilon \rightarrow 0^+$. One of the crucial step in our proof is gradient estimates for minimizers (see Theorem 1.3 in [HnL]), which was proved by combining a blow-up argument with some Liouville type theorems. The main theme relies on the fact that minimizers of such boundary value problems always lie in a half sphere. In order to study the asymptotic behavior of minimizing solutions or to understand the behavior of general minimizers of (1.1) (without the Dirichlet boundary condition), we lead to the following:

**Theorem 1.1.** Assume $1 < p < \infty$, $N$ is a connected compact Riemannian manifold such that either $1 < p < 2$ or $p \geq 2$ but $\pi_1(N)$ is finite and $\pi_i(N) = 0$ for $2 \leq i \leq \lfloor p \rfloor - 1$. $m \in \mathbb{N}$, $u \in W^{1,p}_{loc}(\mathbb{R}^m, N)$ is a locally minimizing $p$-harmonic map.

- If $1 < p < m$, then $|du|^p \leq c(m, p, N)r^{m-p}$ for any $r > 0$.
- If $m \leq p < \infty$, then $u$ must be a constant map.

From this Liouville type theorem we may deduce the following gradient estimates for minimizing $p$ harmonic maps.

**Theorem 1.2.** Let $m, p$ and $N$ be the same as in Theorem 1.1, $\Omega \subset \mathbb{R}^m$ be an open subset, $u \in W^{1,p}_{loc}(\Omega, N)$ be a locally minimizing $p$-harmonic map.

- If $1 < p < m$, then for any $x \in \Omega$, $0 < r < d(x, \mathbb{R}^m \setminus \Omega)$ we have

$$\int_{B_r(x)} |du|^p \leq c(m, p, N) \left( 1 - \frac{r^{m-p}}{d(x, \mathbb{R}^m \setminus \Omega)^{p-1}} \right).$$

- If $m \leq p < \infty$, then $u \in C^1(\Omega, N)$ and

$$|du(x)| \leq c(m, p, N) \frac{d(x, \mathbb{R}^m \setminus \Omega)}{d(x, \mathbb{R}^m \setminus \Omega)}$$

for any $x \in \Omega$.  

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An interesting consequence of Theorem 1.2 is the compactness for p-energy minimizing maps.

**Corollary 1.1.** Let \( m, p, \Omega \) and \( N \) be the same as in Theorem 1.2, \( u_i \in W^{1,p}_loc(\Omega, N) \) be a sequence of locally minimizing p-harmonic maps, then there exists a subsequence \( u_{i_k} \) and a locally minimizing p-harmonic map \( u \in W^{1,p}_loc(\Omega, N) \) such that \( u_{i_k} \rightarrow u \) in \( W^{1,p}_loc(\Omega, N) \).

The main point in Corollary 1.1 is that, under the topological condition on \( N \), one may drop the condition that the p-energy of the sequence of maps is uniformly bounded as in the Luckhaus compactness theorem for minimizing p-harmonic maps (see [Lu1] and [Lu2]). This fact has already been observed in [HKL] in a special case. We also have the following

**Theorem 1.3.** Let \( m, p \) and \( N \) be the same as in Theorem 1.1, \( H_0 = \{ x^m : x^m > 0 \} \) be the upper half space, and \( u \in W^{1,p}_loc(\overline{H}_0, N) \) be a locally minimizing p-harmonic map such that \( u|_{\partial H_0} \) is a constant map, then \( u \) is a constant map.

When \( p = 2 \), the topological condition on the target stated in Theorem 1.1 simply says the fundamental group is finite, or equivalently, the universal covering space is compact. Typical examples of Riemannian manifolds with finite fundamental group are compact Riemannian manifolds with strictly positive Ricci curvature. When the fundamental group of the target manifold is infinite, we may have nonconstant minimizing harmonic maps with arbitrary growth rates for the energy. Indeed a lifting argument tells us if \( N \) is a complete Riemannian manifold with non-positive sectional curvature, then for \( m \geq 2 \), any harmonic map from \( \mathbb{R}^m \) to \( N \) is minimizing. A typical example is the case \( N = T^n = S^1 \times \cdots \times S^1 \) (\( n \) factors). A map \( u : \mathbb{R}^m \rightarrow T^n \) is a harmonic map if and only if \( u = (e^{ih_1}, \ldots, e^{ih_n}) \) and \( h_1, \ldots, h_n \) are harmonic functions on \( \mathbb{R}^m \). This shows Theorem 1.1, Theorem 1.2 and Theorem 1.3 can not be true if we drop the topological condition.

We also would like to point out a few known facts related to our results. It was proved in [SU] that for \( n \geq 3 \), every stable harmonic map from \( \mathbb{R}^2 \) to \( S^n \) is a constant map (see Theorem 2.9 in [SU]). Note that in Theorem 1.1 one could have \( N = S^2 \) or \( N = S^n \), \( n \geq 2 \) but with arbitrary smooth Riemannian metric. It is well-known that holomorphic or anti-holomorphic maps from \( \mathbb{R}^2 \) to \( S^2 \) are stable. In fact, a theorem of A. Lichnerowicz says every holomorphic or anti-holomorphic map from a compact Kähler manifold to another Kähler manifold is energy minimizing in its homotopy class (see Theorem 4.2 in [Xii]). If one looks at the proof closely, one can easily show that without the compactness condition on the domain manifold, any holomorphic or anti-holomorphic map is energy minimizing in its homotopy class if only those homotopies supported in compact subsets are considered. In particular, it shows holomorphic or anti-holomorphic maps between Kähler manifolds are always stable harmonic maps. We also note that it was proved in Corollary 6 of [So] that any minimizing harmonic map from \( \mathbb{R}^2 \) to \( S^2 \) which misses a nonempty open subset of \( S^2 \) is a constant map. On the other hand, for \( m \geq 7 \), there exists a nonconstant smooth harmonic map \( u : \mathbb{R}^m \rightarrow S^m \) with image lying in open upper half sphere (see Example 2.2 in [SU]), and hence it is a minimizing harmonic map by Lemma 2.1 in [SU]. For general p-harmonic maps, we note that if \( m - 1 \leq p < m \) or \( 1 \leq p \leq m - 1 \) but \( p \in \mathbb{Z} \), then \( x/|x| : \mathbb{R}^m \rightarrow S^{m-1} \) is a minimizing p-harmonic map. See [AL], [CG], [HLW] and the references therein.

The key concept related to Theorem 1.1, Theorem 1.2 and Theorem 1.3 is the so called p-\textit{extension property} for \( 1 < p < \infty \) (see Definition 2.1). Based on an
important lemma and some techniques from [HrL] (see Section 6 of [HrL]), we may show that a compact Riemannian manifold satisfies $p$-extension property if and only if it is $(|p| - 1)$-simply connected (see Theorem 2.1).

Once we show every minimizing harmonic map from $\mathbb{R}^2$ to $S^2$ is a constant map, we are able to classify blow-up limits of local minimizers of $I_\varepsilon$, $\varepsilon \to 0^+$. We have the following

**Theorem 1.4.** Suppose $u \in C^\infty(\mathbb{R}^2, S^2)$ satisfies

$$-\Delta u = \left( |\nabla u|^2 + (u^3)^2 \right) u - u^3 e_3$$

on $\mathbb{R}^2$, also assume $u$ locally minimizes $I_1$, then the image of $u$ lies in upper half sphere or lower half sphere and it satisfies

$$|u^3(x)| \leq c(u) e^{-\frac{|x|}{16}}, \quad |\nabla u(x)| \leq c(u) |x|.$$  

In addition, either $u$ is a constant in $S^1$ or the degree of $\left( \frac{u^1, u^2}{|u^3|} \right)$ is $+1$ or $-1$. In the latter case we have $\int_{\mathbb{R}^2} (u^3)^2 = \pi$.

When the base points of blow-ups are somewhat close to the boundary, we get blow-up limits defined on a half plane. Then we have the boundary version of Theorem 1.4, which in some sense corresponds to the fact that the vortices should “stay inside” $\Omega$ in Theorem 1.2 of [HnL].

**Theorem 1.5.** Let $H_0 = \{ x \mid x \in \mathbb{R}^2, x^2 > 0 \}$ be the open upper half plane. Assume $u \in C^\infty(\overline{H_0}, S^2)$ satisfies (1.2) in $H_0$ and locally minimizes $I_1$ in $\overline{H_0}$, $u|_{\partial H_0} \equiv e$, $e \in S^1$ is a constant, then $u \equiv e$ in $\overline{H_0}$.

The ingredients in proving Theorem 1.4 and Theorem 1.5 are the gradient estimate, which follows from a blowing up argument, and energy comparison maps from [Sa2] and Section 6 of [HrL]. We have just learned from Sylvia Serfaty that in [AS] and [Sa1], the authors made a similar investigation as our previous work [HnL]. However, [AS] seems to have missed this key gradient estimate (see page 677 of [AS]). It is also necessary to have this gradient estimates to understand the fine properties of minimizers. An interesting point in Theorem 1.4 and Theorem 1.5 is that we do not have any growth condition on solutions to start with. It remains as an open problem if after translation, rotation, reflection with respect to $x^1$ axis on $\mathbb{R}^2$ and reflection with respect to the horizontal plane on $S^2$, a minimizer in Theorem 1.4 is either a constant or the degree 1 radial solution in Proposition 5.2 of [HnL]. For the Ginzburg-Landau model case, the corresponding problem was solved in [Mi].

The paper is written as follows. In Section 2, we study the relation between minimizing $p$-harmonic maps and the topology of the target manifolds and prove Theorem 1.1, Theorem 1.2 and Theorem 1.3. In Section 3, we classify the blow-up limits of minimizers of $I_\varepsilon$ as $\varepsilon \to 0^+$ and prove Theorem 1.4 and Theorem 1.5.

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2. **Minimizing $p$-harmonic maps.** In this section we shall study the relations between minimizing $p$-harmonic maps and the topology of the target manifolds. As
mentioned in the introduction, the key concept related to Theorem 1.1 and Theorem 1.2 is the following

**Definition 2.1** (*p*-Extension property). Assume $1 < p < \infty$, $N$ is a smooth compact manifold. If for any Riemannian metric $g$ on $N$, any $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^m$ open, bounded and piecewisely smooth, there exists a constant $c = c(p, g, \Omega, N)$ such that for any $f \in W^{1-\frac{1}{p}, p}(\partial \Omega, (N, g))$, there exists a $u \in W^{1,p}(\Omega, (N, g))$ such that

$$u|_{\partial \Omega} = f \quad \text{and} \quad \int_{\Omega} |du|^p \leq c(p, g, \Omega, N) \left[ f \right]_{W^{1-\frac{1}{p}, p}(\partial \Omega, (N, g))}^p,$$

then we say $N$ satisfies the $p$-extension property.

It is easy to see that once there exists a Riemannian metric $g_0$ on $N$ such that we may do extensions satisfying (2.1), then $N$ has the $p$-extension property. We may also define $(m, p)$-extension property by putting the dimension $m$ in, but we don’t need this here. The $p$-extension property is a topological property, in fact one has the following

**Theorem 2.1.** If $N$ is a smooth connected compact manifold, $1 < p < \infty$, then it has the $p$-extension property if and only if $\pi_i(N) = 0$ for $1 \leq i \leq [p] - 1$.

To prove this theorem, we need Lemma 6.1 in [HrL], which is stated below for reader’s convenience.

**Lemma 2.1** (Lemma 6.1 in [HrL]). Let $N^n \subset \mathbb{R}^k$ be a smooth connected compact submanifold, $l \in \mathbb{Z}$, $l \geq 0$. If for any $1 \leq i \leq l$, $\pi_i(N) = 0$, then there exists a compact $(k - l - 2)$-dimensional Lipschitz polyhedron $X \subset \mathbb{R}^k$ and a locally Lipschitz retraction $P : \mathbb{R}^k \setminus X \rightarrow N$ such that

$$\int_{B_R} |dP(x)|^p dx < \infty \quad \text{for any } 1 \leq p < l + 2 \text{ and } R > 0.$$

Moreover, $P$ is smooth in an open neighborhood of $N$.

Proof of Theorem 2.1. If $N$ satisfies $p$-extension property, then, for any $1 \leq i \leq [p] - 1$, any smooth map $f : S^i = \partial B^{i+1}_1 \rightarrow N$, there exists a $u \in W^{1,p}(B^{i+1}_1, N)$ such that $u|_{\partial B^1} = f$. If $p$ is not an integer or $p$ is an integer but $i \neq [p] - 1$, then by Sobolev embedding theorem, $u$ is continuous, hence $f$ is homotopic to a constant. If $p$ is an integer and $i = p - 1$, then it follows from [BN] that $f$ is still homotopic to a constant. In any case, $\pi_i(N) = 0$.

Let $N$ be such that $\pi_i(N) = 0$ for $1 \leq i \leq [p] - 1$. First of all, we may assume there is an embedding $N \subset \mathbb{R}^k$ for some $k$. From Lemma 2.1 we may find a compact $(k - [p] - 1)$-dimensional Lipschitz polyhedron $X \subset \mathbb{R}^k$ and a local Lipschitz retraction $P : \mathbb{R}^k \setminus X \rightarrow N$ such that

$$\int_{B_R} |dP(x)|^q dx < \infty \quad \text{for } 1 \leq q < [p] + 1 \text{ and } R > 0.$$

Moreover, $P$ is smooth in an open neighborhood of $N$. We may find a $\delta \in (0, 1)$ such that for any $a \in B_\delta$, the map $P_a : N \rightarrow N$, which is defined by $P_a(y) = P(y - a)$, is a diffeomorphism with $|dP_{a^{-1}}(y)| \leq c(N)$. Now given any open bounded piecewisely smooth subset $\Omega \subset \mathbb{R}^m$ and any $f \in W^{1-\frac{1}{p}, p}(\partial \Omega, N)$, let $v : \Omega \rightarrow \mathbb{R}^k$ be the harmonic extension of $f$, then if we denote $R_0 = \sup_{y \in \Omega} |y|$, we have

$$|v(x)| \leq R_0 \quad \text{and} \quad \int_{\Omega} |dv|^p \leq c(p, \Omega, N) \left[ f \right]_{W^{1-\frac{1}{p}, p}(\partial \Omega)}^p.$$
For any $a \in B^k_\delta$, denote $v_a(x) = P(v(x) - a)$, then
\begin{equation}
\int_{B^k_\delta} da \int_\Omega |d\nu_a(x)|^p dx \leq \int_{B^k_\delta} da \int_\Omega |dP(v(x) - a)|^p |dv(x)|^p dx
\end{equation}
\[\leq \int_\Omega dx \int_{B_{\delta,\alpha+1}} |dP(y)|^p |dv(x)|^p dy \leq c(p, \Omega, N)[f]^p_{W^{1,\frac{1}{p}}(\partial \Omega)}.
\]

Here we used (2.3) with $q = p$ and (2.4). From (2.5) we may find an $a \in B_\delta$ such that
\[\int_\Omega |d\nu_a(x)|^p dx \leq c(p, \Omega, N)[f]^p_{W^{1,\frac{1}{p}}(\partial \Omega)},
\]
then $u = P^{-1}_\alpha \circ v_a$ is the needed extension. \qed

We note the extension problem without energy estimate was considered in [BD]. In fact, Theorem 5 in [BD] is in the same spirit as the necessary part of Theorem 2.1. To prove Theorem 1.1 and 1.2 we need some technical lemmas.

**Definition 2.2.** Let $X$ be a metric space, $k \in \mathbb{Z}$, $k \geq 0$, $E \subset X$. If there exists a sequence of bounded subsets, namely $A_i \subset \mathbb{R}^k$ and a sequence of Lipschitz maps, namely $\phi_i : A_i \to X$ such that $E = \bigcup_{i=1}^\infty \phi_i(A_i)$, then we say $E$ is countably $k$-rectifiable.

**Lemma 2.2.** Let $X$ and $Y$ be metric spaces, $s \geq 0$, $k \in \mathbb{Z}$, $k \geq 0$. If $A \subset X$ satisfies $\mathcal{H}^s(A) = 0$, $B \subset Y$ is countably $k$-rectifiable, then $\mathcal{H}^{k+s}(A \times B) = 0$.

**Proof.** We may assume $k > 0$, $s > 0$ and $B = \phi(E)$, where $E \subset [0, 1]^k$ and $\phi$ is a map from $E$ to $X$ with $\text{Lip}(\phi) \leq L$. Given any $0 < \varepsilon < 1$, we may find $(A_i)_{i=1}^\infty$ such that
\[A \subset \bigcup_{i=1}^\infty A_i, \quad \sum_{i=1}^\infty d(A_i)^s < \varepsilon.
\]
Choose $\alpha_i > d(A_i)$ such that $\sum_{i=1}^\infty \alpha_i^s < \varepsilon$, then $0 < \alpha_i < 1$. Set $l_i = [1/\alpha_i] + 1$, then
\[[0, 1]^k = \bigcup_{j=1}^{l_i^k} C_{ij}, \quad E_{ij} = C_{ij} \cap E, \quad E = \bigcup_{j=1}^{l_i^k} E_{ij},
\]
here $C_{ij}$ is a cube with side length $1/l_i$. We have
\[A \times B \subset \bigcup_{i=1}^\infty A_i \times B = \bigcup_{i=1}^\infty \bigcup_{j=1}^{l_i^k} A_i \times \phi(E_{ij}),
\]
which shows
\[d(A_i \times \phi(E_{ij})) \leq d(A_i) + d(\phi(E_{ij})) \leq c(k, L)\alpha_i,
\]
which shows
\[\sum_{i=1}^\infty \sum_{j=1}^{l_i^k} d(A_i \times \phi(E_{ij}))^{k+s} \leq \sum_{i=1}^\infty c(k, s, L)l_i^k \alpha_i^{k+s} \leq c(k, s, L) \sum_{i=1}^\infty \alpha_i^s \leq c(k, s, L)\varepsilon.
\]
This implies $H^{k+1}(A \times B) = 0$. \[ \square \]

**Lemma 2.3.** Assume $m \geq 3$, $F \subset \mathbb{R}^m$ is a closed subset such that $H^{m-2}(F) = 0$, then $\mathbb{R}^m \setminus F$ is simply connected.

**Proof.** First we want to show $\mathbb{R}^m \setminus F$ is path connected. In fact, given any two points $x_0$, $x_1$ in $\mathbb{R}^m \setminus F$, let $\gamma(t) = (1 - t)x_0 + tx_1$ for $0 \leq t \leq 1$. Since $F$ is closed, we may find a $\delta > 0$ such that $B_\delta(x_i) \subset \mathbb{R}^m \setminus F$ for $i = 0, 1$. From Lemma 2.2 we know $H^{m-1}(F - \gamma([0, 1])) = 0$, hence we may find a point $\xi \in B_\delta^m$ such that $\xi \notin F - \gamma([0, 1])$.

Clearly for any $0 \leq t \leq 1$, $\gamma(t) + \xi \notin F$, this means $x_0 + \xi$ can be connected to $x_1 + \xi$ in $\mathbb{R}^m \setminus F$. On the other hand, it is clear that $x_i$ can be connected to $x_i + \xi$ in $\mathbb{R}^m \setminus F$ by the line segment connecting them for $i = 0, 1$. Hence $x_0$ can be connected to $x_1$ in $\mathbb{R}^m \setminus F$.

Since $\mathbb{R}^m \setminus F$ is open and connected, to show it is simply connected, it suffices to show for any Lipschitz map $f : \partial B^m_1 \to \mathbb{R}^m \setminus F$, there exists a Lipschitz map $\tilde{f} : \bar{B}^m_1 \to \mathbb{R}^m \setminus F$ such that $\tilde{f}|_{\partial B^m_1} = f$. In fact for any $f \in Lip(\partial B^m_1, \mathbb{R}^m \setminus F)$, we may find a $\delta > 0$ such that for any $x \in \partial B^m_1$, $B_\delta^m(f(x)) \subset \mathbb{R}^m \setminus F$. On the other hand, we may find a $\tilde{f} \in Lip(\bar{B}^m_1, \mathbb{R}^m)$ such that $\tilde{f}$ is an extension of $f$. Indeed one may take $\tilde{f}(x) = |x|f(x/|x|)$ for any $x \in \partial B^m_1$. Via Lemma 2.2 we know $H^m(F - \tilde{f}(\bar{B}^m_1)) = 0$.

Hence we may find a $\xi \in \partial B^m_\delta$ such that $\xi \notin F - \tilde{f}(\bar{B}^m_1)$. This implies $\tilde{f}(x) + \xi \notin F$ for any $x \in \bar{B}^m_1$. Define

$$\tilde{f}(x) = \begin{cases} \tilde{f}(2x) + \xi, & \text{for } x \in \bar{B}^m_1/2; \\ f(x/|x|) + 2(1 - |x|)\xi, & \text{for } x \in \bar{B}^m_1 \setminus \bar{B}^m_1/2. \end{cases}$$

Clearly $\tilde{f} \in Lip(\bar{B}^m_1, \mathbb{R}^m \setminus F)$ is the needed extension of $f$. \[ \square \]

**Proof of Theorem 1.1.** Let us first consider the special case when $\pi_i(N) = 0$ for $1 \leq i \leq |p| - 1$. From Theorem 2.1 we know $N$ satisfies the $p$-extension property, hence for any $f \in W^{1 - \frac{1}{p}}(\partial B^m_1, N)$, there exists a $v \in W^{1, p}(B^m_1, N)$ such that

$$v|_{\partial B_1} = f \quad \text{and} \quad \int_{B_1} |dv|^p \leq c(m, p, N)[f]^p_{W^{1 - \frac{1}{p}}(\partial B_1)}.$$ 

A scaling argument shows for any $r > 0$, any $f \in W^{1 - \frac{1}{p}}(\partial B^m_1, N)$, there exists a $v \in W^{1, p}(B^m_r, N)$ such that

$$v|_{\partial B_r} = f \quad \text{and} \quad \int_{B_r} |dv|^p \leq c(m, p, N)[f]^p_{W^{1 - \frac{1}{p}}(\partial B_r)}.$$ 

The point here is that the constant $c(m, p, N)$ doesn’t depend on $r$. Suppose $u : \mathbb{R}^m \to N$ is a minimizing $p$-harmonic map, for $r \geq 0$, let $\phi(r) = \int_{B_r} |du|^p$. For any $r > 0$, let $f = u|_{\partial B_r}$ in (2.7), from the minimality of $u$ and (2.7) we have

$$\phi(r) = \int_{B_r} |du|^p \leq c(m, p, N)[u|_{\partial B_r}]^p_{W^{1 - \frac{1}{p}}(\partial B_r)} \leq c(m, p, N) \left( \int_{\partial B_r} |u|^p dH^{m-1} \right)^{\frac{1}{p}} \left( \int_{\partial B_r} |d(u|_{\partial B_r})|^p dH^{m-1} \right)^{1 - \frac{1}{p}}.$$
\[ \leq c(m, p, N) R^{-\frac{m-1}{p+1}} \left( \int_{\partial B_r} |du|^p d\mathcal{H}^{m-1} \right)^{1-\frac{1}{p}} = c(m, p, N) R^{-\frac{m}{p+1}} \phi'(r)^{1-\frac{1}{p}}. \]

Assume for some \( R > 0 \), we have \( \phi(R) > 0 \), then for any \( r \geq R \),

\[ \frac{1}{c(m, p, N) R^{\frac{m-1}{p+1}}} \leq \frac{\phi'(r)}{\phi(r)^{\frac{1}{p+1}}}. \]

If \( p > m \), then integrating (2.9), we obtain

\[ \frac{1}{c(m, p, N) (R')^{-\frac{m-1}{p+1}}} \leq \frac{1}{\phi(R')^{-\frac{1}{p+1}}} \leq \frac{1}{\phi(R)^{-\frac{1}{p+1}}}, \]

for any \( R' \geq R \). Let \( R' \to \infty \) in (2.10), we lead to a contradiction. Hence \( \phi \equiv 0 \), that is \( u \) must be a constant map.

If \( p = m \), then integrating (2.9) one gets

\[ \frac{1}{c(m, p, N)} \log \frac{R'}{R} \leq \frac{1}{\phi(R')^{-\frac{1}{p+1}}} \leq \frac{1}{\phi(R)^{-\frac{1}{p+1}}}, \]

for any \( R' \geq R \). Let \( R' \to \infty \) in (2.11), we obtain again a contradiction. Hence \( u \) is a constant.

If \( 1 < p < m \), then integrating (2.9), one has

\[ \frac{1}{c(m, p, N)} (R' - R) \leq \frac{1}{\phi(R')^{-\frac{1}{p+1}}} \leq \frac{1}{\phi(R)^{-\frac{1}{p+1}}}, \]

for any \( R' \geq R \). Let \( R' \to \infty \) in (2.12), we thus conclude

\[ \phi(R) \leq c(m, p, N) R^{m-p} \quad \text{whenever} \quad \phi(R) > 0. \]

Now let us prove Theorem 1.1 in its full generality. If \( 1 < p < 2 \), this has been proved above because \([p] - 1 = 0\). If \( p \geq 2 \), then since \( \pi_1(N) \) is a finite group, the universal covering space of \( N \), namely \( \tilde{N} \), is compact. Denote \( \pi \) as the natural projection map from \( \tilde{N} \) to \( N \), and let \( \tilde{N} \) be endowed with the induced Riemannian metric \( \pi_*^* \). Note that \( \tilde{N} \) satisfies \( \pi_1(\tilde{N}) = 0 \) for \( 1 \leq i \leq |p| - 1 \).

**Claim 2.1.** If \( p \geq 2 \), then there exists a minimizing \( p \)-harmonic map \( \tilde{u} \in W^{1,p}(\mathbb{R}^m, \tilde{N}) \) such that \( \pi \circ \tilde{u} = u \).

**Proof of Claim 2.1.** If \( p \geq m \), then from Corollary 2.6 of [HrL] we know \( u \in C(\mathbb{R}^m, N) \). Since \( \mathbb{R}^m \) is simply connected, we may find a \( \tilde{u} \in C(\mathbb{R}^m, \tilde{N}) \) such that \( \pi \circ \tilde{u} = u \). It is clear that \( \tilde{u} \in W^{1,p}(\mathbb{R}^m, \tilde{N}) \).

If \( 2 \leq p < m \), then from Corollary 2.6 of [HrL] we may find a closed subset \( S_u \subset \mathbb{R}^m \) such that \( u|_{\mathbb{R}^m \setminus S_u} \) is locally Hölder continuous and \( \mathcal{H}^{m-p}(S_u) = 0 \). From Lemma 2.3 we know \( \mathbb{R}^m \setminus S_u \) is simply connected, hence we may find a \( \tilde{u} \in C(\mathbb{R}^m \setminus S_u, \tilde{N}) \) such that \( \pi \circ \tilde{u} = u \). It is then clear that \( \tilde{u} \in W^{1,p}(\mathbb{R}^m, \tilde{N}) \).

For any \( r > 0 \), any \( \tilde{v} \in W^{1,p}(B_r, \tilde{N}) \) such that \( \tilde{v}|_{\partial B_r} = \tilde{u}|_{\partial B_r} \), then \( v = \pi \circ \tilde{v} \in W^{1,p}(B_r, N) \) and \( v|_{\partial B_r} = u|_{\partial B_r} \). From the minimality of \( u \) and the fact \( \pi \) is a local isometry we know

\[ \int_{B_r} |d\tilde{u}|^p = \int_{B_r} |d(\pi \circ \tilde{u})|^p = \int_{B_r} |du|^p \leq \int_{B_r} |dv|^p = \int_{B_r} |d\tilde{v}|^p. \]
Hence \( \bar{u} \) is also a minimizing \( p \)-harmonic map. This proves Claim 2.1.

This reduces the general case to the special case we have treated, and hence completes the proof of Theorem 1.1. \( \Box \)

**Proof of Theorem 1.2.** We consider first the case \( 1 < p < m \).

**Claim 2.2.** If \( \Omega = B^m_1 \), then \( \int_{B_1^r} |du|^p \leq \frac{c(m,p,N)r^{m-p}}{(1-r)^{p-1}} \) for \( 0 < r < 1 \).

**Proof of Claim 2.2.** First look at the case \( \sigma_i(N) = 0 \) for \( 1 \leq i \leq [p] - 1 \). Denote \( \phi(r) = \int_{B^r_1} |du|^p \) for \( 0 < r < 1 \), then the arguments in the proof of Theorem 1.1 gives us (see (2.12)) that, if \( \phi(r) > 0 \), then for any \( r < s < 1 \), we have

\[
\frac{1}{c(m,p,N)} (r^{m-p} - s^{m-p}) \leq \frac{1}{\phi(r)^{p-1}}.
\]

Let \( s \to 1^- \), we get

\[
\phi(r) \leq c(m,p,N) \frac{r^{m-p}}{(1-r)^{p-1}} \leq \frac{c(m,p,N)r^{m-p}}{(1-r)^{p-1}}.
\]

Hence Claim 2.2 is true under the assumption that the target is \( [p] - 1 \) simply connected. The general case can be proved by the lifting argument presented above.

When \( \Omega \) is an arbitrary open subset, the conclusion in Theorem 1.2 follows from Claim 2.2 by a simple scaling.

Next let us look at the case \( m \leq p < \infty \). In this case it follows from Corollary 2.6 of [HrL] that \( u \in C^1(\Omega, N) \). Again by scalings, to prove the gradient estimate, it suffices to show the following

**Claim 2.3.** If \( u \in C^1(B^m_1, N) \) is a minimizing \( p \)-harmonic map, then \( |du(x)| \leq \frac{c(m,p,N)}{1-|x|} \) for any \( x \in B^m_1 \).

**Proof of Claim 2.3.** If the conclusion of Claim 2.3 were false, then we would find a sequence \( u_i \in C(B^m_1, N) \) such that \( u_i \) is a minimizing \( p \)-harmonic map and

\[
K_i = \max_{x \in B^m_1} (1 - |x|)|du_i(x)| \to \infty \quad \text{as } i \to \infty.
\]

Let \( x_i \in B_1 \) be such that \( K_i = (1 - |x_i|)|du_i(x_i)| \). Denote \( \sigma_i = 1 - |x_i| \). Define \( v_i(x) = u_i(x_i + \frac{x-x_i}{\sigma_i}) \) for \( x \in B_{K_i} \), then \( v_i \) is a minimizing \( p \)-harmonic map with \( |dv_i(x)| \leq \frac{1}{\sigma_i K_i} \) and \( |dv_i(0)| = 1 \). It follows from Theorem 3.1 of [HrL] that for any \( r > 0 \), \( |v_i|_{C^{1,\alpha}(B^m_r)} \leq c(m,p,r,N) \) for \( i \) large enough, here \( \alpha = \alpha(m,p,N) \in (0,1) \). Hence after passing to a subsequence, we may find a \( v \in C^{1,\alpha}_{loc}(\mathbb{R}^m, N) \) such that \( v_i \to v \) in \( C^{1,\alpha}_{loc}(\mathbb{R}^m, N) \). It is clear that \( v \) is still a locally minimizing \( p \)-harmonic map. By Theorem 1.1, \( v \) is a constant map. On the other hand, \( |dv(0)| = 1 \) because \( |dv_i(0)| = 1 \) for any \( i \). This gives us a contradiction. We finish the proof of Claim 2.3 and hence also the Theorem 1.2. \( \Box \)

**Proof of Theorem 1.3.** Again we consider first the case \( \sigma_i(N) = 0 \) for \( 1 \leq i \leq [p] - 1 \). For any \( r \geq 0 \), we denote the open upper half ball as \( B^+_r = B_r \cap H_0 \), and let \( \phi_i(r) = \int_{B^+_r} |du|^p \). Replacing \( B_r \) in the proof of Theorem 1.1 by \( B^+_r \) and we observe that

\[
\int_{\partial B^+_r} |d(u|_{\partial B^+_r})|^p d\mathcal{H}^{m-1} = \int_{\partial B_r \cap H_0} |d(u|_{\partial B^+_r})|^p d\mathcal{H}^{m-1}
\]

(2.14)
(2.9) remains true if we replace $\phi$ by $\phi_+$. When $m \leq p < \infty$, we prove in the same way as before that $\phi_+ \equiv 0$, that is $u$ is a constant map. If $1 < p < m$, via the proof of Theorem 1.1 we get

\begin{equation}
\int_{B_r^c} |du|^p \leq c(m, p, N)r^{m-p} \quad \text{for } r > 0.
\end{equation}

To show $u$ is a constant map, we need the monotonicity formula.

**Claim 2.4 (Monotonicity identity).** For almost every $r > 0$, we have

\begin{equation}
\frac{d}{dr} \left( r^{p-m} \int_{B_r^c} |du|^p dH^m \right) = pr^{p-m} \int_{\partial B_r \cap H_0} |\partial_r u|^2 |du|^{p-2} dH^{m-1}.
\end{equation}

**Proof of Claim 2.4.** See Lemma 4.1 in [HrL].

Define a function $\rho_u$ by $\rho_u(r) = r^{p-m} \int_{B_r^c} |du|^p$ for $r > 0$. From Claim 2.4 and (2.15) we know $\rho_u$ is a bounded increasing function. Hence there is a limit $\rho_u(\infty) \in \mathbb{R}$. For any $\lambda > 0$, we denote $u_\lambda(x) = u(x/\lambda)$ for $x \in H_0$. Then $\rho_u(\lambda) = \rho_u(\lambda/\lambda)$. From the proof of Corollary 2.8 and Theorem 6.4 in [HrL] or [Lu1], [Lu2] we know there exists a $v \in W^{1,p}_{\text{loc}}(H_0, N)$ and a sequence of positive numbers $\lambda_j \to 0$ such that $u_{\lambda_j} \to v$ in $W^{1,p}_{\text{loc}}(H_0, N)$ and $v$ is a minimizing $p$-harmonic map. By the strong convergence, one has $\rho_u(\lambda) \equiv \rho_u(\lambda/\lambda)$, and hence by (2.16) we get $\partial_r v = 0$. Since $v$ is a constant map on $\partial H_0$, it follows from Theorem 5.7 of [HrL] that $v$ itself is a constant map. The latter implies $\rho_u(\infty) \equiv \rho_u(0) = 0$, and therefore $u$ is a constant map.

Theorem 1.3 in its full generality can be proved by the same lifting argument as that in the proof of Theorem 1.1. $\square$

**Proof of Corollary 1.1.** This follows from Theorem 1.2 and the Luckhaus Compactness Theorem (see [Lu1] and [Lu2]). $\square$

### 3. Minimal solutions of a simplified Landau-Lifschitz equation

The aim of this section is to classify all blow-up limits of minimizers of $I_\varepsilon$ (see (1.1)). That is we want to study minimal solutions of the simplified Landau-Lifschitz equation

\begin{equation}
-\Delta u = \left( |\nabla u|^2 + (u^3)^2 \right) u - u^3 e_3
\end{equation}

for a $S^2$ valued $u$ defined on the entire plane.

To proceed, we need the following gradient estimate.

**Proposition 3.1.** Suppose $u \in C^\infty(\overline{B_1}, S^2)$ satisfies (3.1) in $B_1$. If $u$ minimizes $I_1$ on $\overline{B_1}$, then $|\nabla u(x)| \leq \frac{c}{1-|x|}$ on $B_1$, here $c$ is an absolute constant.

**Proof.** Otherwise, we would find a sequence $u_j \in C^\infty(\overline{B_1}, S^2)$, minimizing $I_1$ on $\overline{B_1}$ and

\[ K_j = \sup_{x \in \overline{B_1}} (1-|x|)|\nabla u_j(x)| \to \infty. \]

Choose $x_j \in B_1$ such that $(1-|x_j|)|\nabla u_j(x_j)| = K_j$, put $\sigma_j = 1 - |x_j|$, and define $v_j(x) = u_j(x + \frac{\sigma_j}{K_j} x)$ for $x \in B_{K_j}$. Then

\[ -\Delta v_j = \left( |\nabla v_j|^2 + \frac{\sigma_j^2 (v_j^3)^2}{K_j^2} \right) v_j - \frac{\sigma_j^2 v_j^3}{K_j} e_3 \text{ on } B_{K_j}, \quad |\nabla v_j(x)| \leq \frac{1}{1 - \frac{\sigma_j}{K_j}}, |\nabla v_j(0)| = 1, \]

where
and $v_j$ minimizes $I_{ij}$. Hence $|v_j|_{C^{1,\alpha}(\overline{B_r})} \leq c(\alpha, r)$. After passing to a subsequence we may assume $v_j \to v$ in $C^\infty(\mathbb{R}^2)$, then $v \in C^\infty(\mathbb{R}^2, S^2)$ and

$$-\Delta v = |\nabla v|^2 v \text{ on } \mathbb{R}^2, \quad |\nabla v(0)| = 1, \quad |\nabla v(x)| \leq 1.$$ 

Moreover, $v$ is a locally minimizing harmonic map. It follows from Theorem 1.1 that $v$ is a constant, we obtain a contradiction. □

We also need the following edition of Theorem 2.1. The key point here is that the constant doesn’t depend on domain $\Omega$.

**Lemma 3.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with Lipschitz boundary, $n \geq 2$, $u \in H^1(\Omega, \mathbb{R}^{n+1})$ such that $u|_{\partial \Omega} \in S^n$, then there exists a $\tilde{u} \in H^1(\Omega, S^n)$ such that

$$\tilde{u}|_{\partial \Omega} = u|_{\partial \Omega} \quad \text{and} \quad \int_{\Omega} |\nabla \tilde{u}|^2 \leq c(n) \int_{\Omega} |\nabla u|^2. \quad (3.2)$$

**Proof.** For any $a \in B^{n+1}_2$, let $u_a(x) = \frac{u(x)-a}{|u(x)-a|}$. Since

$$|\nabla u|^2 = |\nabla u-a|^2 + |u-a|^2 |\nabla u_a|^2,$$

we have $|\nabla u_a|^2 \leq \frac{|\nabla u|^2}{|u-a|^2}$. Integrating both $a$ and $x$, we get

$$\int_{B^{n+1}_2} da \int_{\Omega} |\nabla u_a(x)|^2 dx \leq \int_{B^{n+1}_2} da \int_{\Omega} \frac{|\nabla u|^2}{|u-a|^2} dx = \int_{\Omega} dx \int_{B^{n+1}_2} \frac{|\nabla u(x)|^2}{|u-a|^2} da \leq c(n) \int_{\Omega} |\nabla u(x)|^2 dx.$$

Hence we may find a $b \in B^{n+1}_2$ such that $\int_{\Omega} |\nabla u_b(x)|^2 dx \leq c(n) \int_{\Omega} |\nabla u(x)|^2 dx$. For any $a \in B^{n+1}_2$, define $P_a : S^n \to S^n$ by $P_a(y) = \frac{y-a}{|y-a|}$, then $P_a$ is a diffeomorphism with $|\nabla S^* P_a^{-1}(y)| + |\nabla S^* P_a(y)| \leq c(n)$ for $y \in S^n$. Let $\tilde{u}(x) = P_b^{-1}(u_b(x))$, then $\tilde{u}$ is the needed map. □

We note the method above was introduced in Section 6 of [HrL]. Now we may turn to Theorem 1.4.

**Proof of Theorem 1.4.** From Proposition 3.1 we deduce

$$|\nabla u(x)| \leq c \quad \text{for } x \in \mathbb{R}^2. \quad (3.3)$$

Next we will combine Lemma 3.1 together with the comparison method in [Sa2] to show $u$ has nice decay properties.

**Claim 3.1.** $I_1(u, B_R) \leq cR$ for $R \geq 0$.

**Proof of Claim 3.1.** We may assume $R \geq 2$, define

$$u_R(x) = \begin{cases} (R-|x|)c_1 + (|x|-R+1)u(x), & \text{if } R-1 \leq |x| \leq R, \\ c_1, & \text{if } |x| \leq R-1, \end{cases}$$

here $c_1 = (1,0,0)$. From (3.3) we know $|\nabla u_R(x)| \leq c$. Hence $\int_{B_R \setminus B_{R-1}} |\nabla u_R|^2 \leq cR$. By Lemma 3.1 we may find a $\tilde{u}_R \in H^1(B_R \setminus B_{R-1}, S^2)$ such that $\tilde{u}_R|_{\partial B_R \cup \partial B_{R-1}} = \tilde{u}_R|_{\partial B_{R-1}}$, then

$$\tilde{u}_R \in H^1(B_R \setminus B_{R-1}, S^2)$$
\[ u_R \mid \partial B_{R_0} \cup \partial B_{R-1} \text{ and } \int_{B_R \setminus B_{R-1}} \|
abla u_R\|^2 \leq cR. \] Let \( \tilde{u}_R = e_1 \) in \( B_{R-1} \), then \( I_1(u, B_R) \leq I_1(\tilde{u}_R, B_R) \leq cR \). This proves Claim 3.1.

In the next step we want to show indeed the growth of \( I_1(u, B_R) \) is sublinear in \( R \).

**Claim 3.2.** \( I_1(u, B_R) \leq cR^\frac{1}{4} \) for \( R \geq 0 \).

**Proof of Claim 3.2.** We may assume \( R \geq 4 \). Via Claim 3.1 we have \( I_1(u, B_{2R}) \leq cR \). Hence we may find \( R_1 \in [R, 2R] \) such that

\[ (3.4) \quad \int_{\partial B_{R_1}} (|\nabla u|^2 + (u^3)^2) \, ds \leq c. \]

Let \( B = \{ x \in \partial B_{R_1}, |u^3(x)| \geq \frac{1}{2} \} \). By estimates (3.3) and (3.4), and by a covering argument we may find a finite number of unit length arcs on \( \partial B_{R_1} \), namely \( I_1, \ldots, I_m \) such that \( \cup_{j=1}^m I_j \supset B \) and \( m \leq c \). Denote \( \hat{I} = \partial B_{R_1} \setminus \cup_j I_j \). Let \( v = \Gamma^{-1} \circ u \) on \( \hat{I} \), \( \Gamma \) is the stereographic projection, that is

\[ (3.5) \quad \Gamma : \mathbb{R}^2 \rightarrow S^2 \setminus \{(0,0,-1)\}, \quad \Gamma(y^1, y^2) = \left( \frac{2y^1}{1 + |y|^2}, \frac{2y^2}{1 + |y|^2}, \frac{1 - |y|^2}{1 + |y|^2} \right). \]

From co-area formula we have

\[ (3.6) \quad \int_{S^1} \# \left( \left( \frac{v}{|v|} \right)^{-1} (\{\xi\}) \right) \, d\mathcal{H}^1(\xi) = \int_I |\partial_\tau \left( \frac{v}{|v|} \right) | \, d\mathcal{H}^1 \leq c \int_I |\partial_\tau v| \, d\mathcal{H}^1 \]

\[ \leq c\sqrt{R} \left( \int_{\partial B_{R_1}} |\nabla v|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}} \leq c\sqrt{R} \left( \int_{\partial B_{R_1}} |\nabla u|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}} \leq c\sqrt{R}. \]

Hence we may find a \( \xi_0 \in S^1 \) such that

\[ (3.7) \quad \# \left( \left( \frac{v}{|v|} \right)^{-1} (\{\xi_0\}) \right) \leq c\sqrt{R}. \]

For simplicity we assume \( \xi_0 = -1 \), then let \( J_1, \ldots, J_n \) be those unit length arcs centered at points in \( (\frac{v}{|v|})^{-1}(\{-1\}) \). Let \( G = \partial B_{R_1} \setminus (\cup_j I_j \cup (\cup_k J_k)) \). On \( G \) we write \( v(x) = \rho(x)e^{i\alpha(x)} \) with \( |\alpha| < \pi \). Fix a \( \delta \in (0, \frac{R_1}{4}) \) to be determined later, let \( V = \{ x \in B_{R_1} \setminus B_{R_1-\delta}, \frac{R_1}{|x|} \in G \} \), then for \( x \in V \), we set

\[ v_R(x) = \left( \frac{R_1 - |x|}{\delta} \right) + \frac{|x| - R_1 + \delta}{\delta} \rho \left( \frac{R_1}{|x|} \right) e^{i\frac{|x|^2 - R_1 + \delta}{\delta\rho |x|}}. \]

Let \( u_R(x) = \Gamma(v_R(x)) \) for \( x \in V \), \( u_R(x) = u(x) \) for \( x \in \partial B_{R_1} \) and \( u_R(x) = e_1 \) for \( x \in B_{R_1-\delta} \). Then we check that

\[ \text{Lip}(u_R, V \cup \partial B_{R_1} \cup \partial B_{R_1-\delta}) \leq c. \]

Hence we may extend \( u_R \) to \( B_{R_1} \setminus B_{R_1-\delta} \) such that

\[ (3.8) \quad \text{Lip}(u_R, B_{R_1} \setminus B_{R_1-\delta}) \leq c. \]
A computation using the polar coordinates and the stereographic coordinates yields

\[(3.9) \quad \int_V |\nabla u_R|^2 + (u_R^3)^2 \leq c \frac{R}{\delta}.\]

By (3.7) and (3.8) we get

\[\int_{(B_{R_1} \setminus B_{R_1 - \delta}) \cap V} |\nabla u_R|^2 \leq c \sqrt{R}\delta.\]

From Lemma 3.1 we may find a \( \tilde{u}_R \in H^1((B_{R_1} \setminus B_{R_1 - \delta}) \cap V, \Omega) \) such that

\[(3.10) \quad I_1(u, B_{R_1}) \leq I_1(u, B_{R_1}) \leq I_1(\tilde{u}_R, B_{R_1}) \leq c(\frac{R}{\delta} + \sqrt{R}\delta).\]

By taking \( \delta = R^{\frac{1}{4}} \) in (3.10), we obtain the Claim 3.2.

Now we proceed to show the growth of \( I_1(u, B_{R_1}) \) is at most of order \( \log R \). That is

**Claim 3.3.** For \( R \) large enough, \( I_1(u, B_{R_1}) \leq c \log R \), here \( c \) is an absolute constant.

**Proof of Claim 3.3.** Denote \( \phi(R) = I_1(u, B_{R_1}) \). Given \( R > 0 \), choose \( R_1 \in [R, 2R] \) such that

\[(3.11) \quad \phi'(R_1) = \min_{R \leq r \leq 2R} \phi'(r).\]

From Claim 3.2 we know

\[\int_R^{2R} \phi'(r)dr \leq \phi(2R) \leq cR^\frac{3}{4}.\]

This and (3.11) imply that

\[(3.12) \quad \int_{\partial B_{R_1}} (|\nabla u|^2 + (u^3)^2) \frac{ds}{\delta} = 2\phi'(R_1) \leq \frac{c}{R^\frac{3}{4}}.\]

Combining (3.12) and (3.3), one has \( |u^3| \leq \frac{1}{2} \) on \( \partial B_{R_1} \) when \( R \) is large enough. Let \( v_R(x) = \Gamma^{-1}(u(x)) \) for \( x \in \partial B_{R_1} \). For each \( x \in B_{R_1} \setminus B_{R_1 - 1} \), set

\[v_R(x) = (R_1 - |x|) \left( \frac{R_1 x}{|x|} \right) + (|x| - R_1 + 1) v_R \left( \frac{R_1 x}{|x|} \right).\]

We have

\[(3.13) \quad \int_{\partial B_{R_1 - 1}} |\partial_r v_R| ds \leq c \sqrt{R} \left( \int_{\partial B_{R_1 - 1}} |\partial_r v_R|^2 ds \right)^\frac{1}{2} \leq c(R \phi'(R_1))^\frac{1}{2} \leq c(R \phi'(R))^\frac{1}{2}.\]
Define a continuous function $\alpha : [0, 2\pi] \to \mathbb{R}$ such that $v_R((R_1 - 1)e^{i\theta}) = e^{i\alpha(\theta)}$ for $0 \leq \theta \leq 2\pi$. Let $\theta_0 = 0$, choose $\theta_1 \in [0, 2\pi]$ be such that $e^{i\alpha(\theta_1)} = e^{i\alpha(0)}$, $|\alpha(\theta_1) - \alpha(0)| = 2\pi$ and $|\alpha(\theta) - \alpha(0)| < 2\pi$ for any $0 \leq \theta < \theta_1$. Starting from $\theta_1$, we may inductively define $\theta_2, \theta_3, \ldots, \theta_n$, until we come back to $\theta = 2\pi$. From (3.13) we get a rough bound $n \leq c(R\phi'(R))^{\frac{1}{2}}$. $(R_1 - 1)e^{i\theta_1}, \ldots, (R_1 - 1)e^{i\theta_n}$ breaks $\partial B_{R_1} = \Gamma$ into arcs. First we assume there are some arcs on which the degree of $v_R$ is +1 or -1, then after combining neighbored arcs, we may assume on each arc (which could be a union of several original arcs) $v_R$ has degree +1 or -1. For every such resulting arc $I_j$, we let $J_j$ be the circular arc (with the circle’s center at the intersection of two tangent lines at $\partial I_j$) lie inside the $\partial B_{R_1}$ and orthogonal to $I_j$ at $\partial I_j$. $I_j$ and $J_j$ together encloses a domain called $\Omega_j$. Set $v_R|_{\partial B_{R_1} \setminus \cup \Omega_j} = e^{i\alpha(0)}$. Choose $a_j \in \Omega_j$ such that $B_{2r_0}(a_j) \subset \Omega_j$ for some $r_0 > 0$, an absolute constant. Suppose the degree of $v_R$ on $I_j$ is +1, then let

$$v_R|_{\partial I_j} = \frac{x-a_j}{|x-a_j|}e^{i\varphi_j(x)}.$$  

Set $\varphi_j$ in $\Omega_j$ as the harmonic extension of the boundary function. Let

$$v_R|_{\Omega_j \setminus B(a_j)} = \frac{x-a_j}{|x-a_j|}e^{i\varphi_j(x)},$$  

and $v_R|_{B(a_j)}$ be the harmonic extension of $v_R|_{\partial B(a_j)}$. We may proceed similarly for the degree -1 case. If no arc has nonzero degree, then we have $v_R|_{\partial B_{R_1}} = e^{i\varphi}$. Then using the harmonic extension to define $\varphi$ inside $B_{R_1}$, and let $v_R|_{B_{R_1}} = e^{i\varphi}$. Let $u_R = \Gamma \circ v_R$, by a careful computation as in [Sa2], we have

$$\phi(R) \leq \phi(R_1) = I_1(u, B_{R_1}) \leq I_1(u_R, B_{R_1})$$

$$\leq c(R\phi'(R))^{\frac{1}{2}} \log R + c \log R \quad \text{for } R \text{ large enough},$$

If we put $\tilde{\phi}(R) = \phi(R) + \log R$, then (3.14) implies

$$\tilde{\phi}(R) \leq c(R\tilde{\phi}'(R))^{\frac{1}{2}} \log R.$$

In other words

$$\frac{1}{cR \log^2 R} \leq \frac{\tilde{\phi}'(R)^2}{\tilde{\phi}(R)}.$$  

By integrating on both sides we get for any $\tilde{R} > R$,

$$\frac{1}{\phi(R)} \geq \frac{1}{\phi(R)} - \frac{1}{\tilde{\phi}(R)} \geq \frac{1}{c} \left( \frac{1}{\log R} - \frac{1}{\log \tilde{R}} \right).$$

Let $\tilde{R} \to \infty$, we get $\tilde{\phi}(R) \leq c \log R$ for $R$ large. This implies $I_1(u, B_R) \leq c \log R$ and Claim 3.3 is proved.

Claim 3.3 along with the Pohozaev’s identity (which follows from multiplying (3.1) by $x^2 \partial_x u$ and integrating by parts, one may see Lemma 4.4 in [HnL])

$$\int_{B_r} (u^2)^2 + \frac{r}{2} \int_{\partial B_r} |\partial_r u|^2 ds = \frac{r}{2} \int_{\partial B_r} |\partial_r u|^2 ds + \frac{r}{2} \int_{\partial B_r} (u^2)^2 ds,$$
yields \( \int_{\mathbb{R}^2} (u^3)^2 \, dx < \infty \). Combining the last fact with (3.3), we get \( u^3 \to 0 \) as \( |x| \to \infty \).

Further estimates of \( u^3 \) and \( |\nabla u| \) follow from Proposition 6.1 in [HnL]. To obtain the degree of the map at \( \infty \), we assume for \( |x| \geq R_0, |u^3(x)| \leq \frac{1}{2} \), then \( \Gamma^{-1} \circ u = \rho e^{i(d\theta + \psi)} \), \( \Gamma \) is the stereographic projection defined in (3.5), \( d \) is the degree of \( \frac{u^3}{|x|}\) at \( \infty \). For \( R \geq 2R_0 \), from the Annulus Lemma (see [BMR] or Lemma 4.1 in [HnL]) we have

\[
(3.19) \quad \frac{1}{2} \int_{B_R \setminus B_{R_0}} |\nabla u|^2 \geq \pi d^2 \log \frac{R}{R_0} - c(u).
\]

On the other hand, if we set \( \tilde{\nu}(x) = \tilde{\rho} e^{i(d\theta + \psi)} \) on \( B_R \setminus B_\frac{R}{2} \), where

\[
\tilde{\rho}(x) = 1 + \frac{|x| - R}{R/2} (\rho(x) - 1), \quad \tilde{\psi}(x) = \psi_R + \frac{|x| - R}{R/2} (\psi - \psi_R), \quad \psi_R = \int_{B_R \setminus B_\frac{R}{2}} \psi,
\]

and \( \tilde{u} = \Gamma \circ \tilde{\nu} \), then

\[
(3.20) \quad \frac{1}{2} \int_{B_R \setminus B_\frac{R}{2}} |\nabla \tilde{u}|^2 + (\tilde{u}^3)^2 = \int_{B_R \setminus B_\frac{R}{2}} 2(\nabla \tilde{\nu})^2 + \frac{(1 - |\tilde{\nu}|^2)^2}{(1 + |\tilde{\nu}|^2)^2} \leq c(u).
\]

By the fact that \( |\rho - 1|, |\nabla \rho| \) decay exponentially at \( \infty \), one has via Poincare’s inequality that \( |\nabla \psi(x)| = O(|x|^{-2}) \) (see Proposition 6.1 in [HnL]). We note that \( \tilde{u}(x) = e^{i(d\theta + \psi_R)} \) on \( \partial B_\frac{R}{2} \), from Lemma 4.3 in [HnL] we may choose \( \tilde{u} \) on \( B_\frac{R}{2} \) such that

\[
(3.21) \quad \frac{1}{2} \int_{B_\frac{R}{2}} |\nabla \tilde{u}|^2 + (\tilde{u}^3)^2 \leq \pi |d| \log R + c,
\]

where \( c \) is an absolute constant. Combining (3.20) and (3.21), and energy minimizing property of \( u \), we conclude

\[
(3.22) \quad \frac{1}{2} \int_{B_R} |\nabla u|^2 + (u^3)^2 \leq \pi |d| \log R + c(u).
\]

Applying (3.19) and (3.22), and letting \( R \to \infty \), we get \( d^2 \leq |d| \). Hence \( d = 0, +1 \) or \(-1 \). If \( d = 0 \), from Proposition 6.1 in [HnL] we know \( u^3 \equiv 0 \). An estimate for harmonic function and \( |\nabla u(x)| \leq \frac{c(u)}{|x|} \) tells us \( u \equiv const. \)

**Claim 3.4.** \( \tilde{u} = (u^1, u^2, |u^3|) \) is locally minimizing \( \mathcal{I}_1 \).

**Proof of Claim 3.4.** For \( R \geq R_0 \), define

\[
w_R(x) = \Pi(u(x) + (R + 1 - |x|)(\tilde{u}(x) - u(x))) \quad \text{for } x \in B_{R+1} \setminus B_R.
\]

Here \( \Pi(\xi) = \frac{\xi}{|\xi|} \) for \( \xi \in \mathbb{R}^3 \setminus \{0\} \). From the estimates for \( u \), one easily verifies

\[
I_1(w_R, B_{R+1} \setminus B_R) = o(1), \quad I_1(u, B_{R+1} \setminus B_R) = o(1) \quad \text{as } R \to \infty.
\]

For any \( v \in H^1(B_{R_1}, S^2), v|\partial B_{R_1} = \tilde{u}|\partial B_{R_1}, R_1 \geq R_0 \), pick up a \( R > R_1 \), extend \( v \) to \( B_{R+1} \) by setting \( v|_{B_R \setminus B_{R_1}} = \tilde{u}, v|_{B_{R+1} \setminus B_R} = w_R \). Via minimizing property of \( u \) we know \( I_1(v, B_{R+1}) \geq I_1(u, B_{R+1}) \). But

\[
I_1(v, B_{R+1}) - I_1(u, B_{R+1}) = I_1(v, B_{R_1}) - I_1(u, B_{R_1}) + I_1(w_R, B_{R+1} \setminus B_R) - I_1(u, B_{R+1} \setminus B_R)
\]
= I_1(v, B_R) - I_1(\tilde{u}, B_R) + o(1).

Let \( R \to \infty \), we get \( I_1(v, B_R) \geq I_1(\tilde{u}, B_R) \), hence Claim 3.4 is proved.

From Claim 3.4 we know \( \tilde{u} \) is smooth and satisfies (3.1). Since \( \tilde{u}^3 \geq 0 \), from the equation of third component we know either \( \tilde{u}^3 > 0 \) or \( \tilde{u}^3 \equiv 0 \). The first case implies \( u^3 > 0 \) or \( u^3 < 0 \). The second case implies \( u \equiv \text{const} \). □

Remark 3.1. For any \( c \in \mathbb{R}, |c| \leq 1 \), \( u(x) = (\sqrt{1-c^2} \cos x^1, \sqrt{1-c^2} \sin x^1, c) \) is a solution to (3.1). Clearly these are not local minimizers.

Remark 3.2. It is of interest to prove that under translation, rotation, and reflection with respect to the \( x^1 \) axis and the horizontal plane, the degree 1 radial solution in Proposition 5.2 in [HnL] is the unique nonconstant local minimizer. In the Ginzburg-Landau model case, the corresponding problem was solved in [Mi].

To prove Theorem 1.5, we need the following boundary version of Proposition 3.1.

**Proposition 3.2.** Denote \( B^+_1 = B_1 \cap H_0, L_1 = B_1 \cap \partial H_0 \), where \( H_0 \) is the open upper half plane. Suppose \( u \in C^\infty(B^+_1, S^2) \) satisfies (3.1) and it locally minimizes \( I_1 \) in \( B^+_1 \), \( u|_{L_1} \equiv \text{const} \), then

\[
|\nabla u(x)| \leq c_1 - |x|,
\]

here \( c_1 \) is an absolute constant.

**Proof.** The proof goes almost the same as the one for Proposition 3.1, except in case we get half plane in the blow-up limit, we use Theorem 1.3 to find a contradiction. One may refer to the proofs of Theorem 3.1 and Proposition 6.3 in [HnL]. □

**Proof of Theorem 1.5.** Without losing of generality we may assume \( e = e_1 = (1, 0, 0) \). From Proposition 3.1 and Proposition 3.2 we get

\[
|\nabla u(x)| \leq c \quad \text{for any } x \in \overline{H_0}.
\]

Here \( c \) is an absolute constant. Denote \( B^+_R = B_R \cap H_0 \), we may show as for Theorem 1.4 that, for \( R \) large enough,

\[
I_1(u, B^+_R) \leq c \log R
\]

for some absolute constant \( c \).

By Pohozaev’s identity (see Lemma 4.4 in [HnL]) we have

\[
\int_{B^+_R} (u^3)^2 + \frac{R}{2} \int_{\partial B_R \cap H_0} |\partial_r u|^2 ds = \frac{R}{2} \int_{\partial B_R \cap H_0} (|\partial_r u|^2 + (u^3)^2) ds.
\]

From (3.24) one may find a sequence \( R_j \to \infty \) such that

\[
R_j \int_{\partial B_{R_j} \cap H_0} (|\nabla u|^2 + (u^3)^2) ds \leq c.
\]

(3.25) and (3.26) together imply

\[
\int_{H_0} (u^3)^2 \leq c,
\]

here \( c \) is an absolute constant. Next, using (3.23), one has \( u^3 \to 0 \) as \( |x| \to \infty \). Choose \( R_0 > 0 \) such that \( |u^3(x)| \leq \frac{1}{2} \) for \( x \in H_0 \setminus B^+_{R_0} \), then \( u' = u^1 + iu^2 = pe^{i\varphi} \) with
\[ \rho = |u'| \geq \frac{\sqrt{3}}{2}, \quad \varphi = 2d\theta + \psi. \quad d \text{ is the degree of } \frac{u'}{|u'|} \text{ on } \partial B^+_R, \quad \psi(x^1, 0) \equiv 0. \] A simple computation shows \( \text{div} (\rho^2 \nabla \varphi) = 0. \)

**Claim 3.5.** \( \int_{H_0 \setminus B^+_R} |\nabla \psi|^2 < \infty. \)

**Proof of Claim 3.5.** Denote \( A^+_R = B^+_R \setminus B^+_R \), then

\[
\int_{A^+_R} \rho^2 (2d\nabla \theta + \nabla \psi) \cdot \nabla \psi = \int_{A^+_R} \text{div}(\rho^2 \nabla \varphi)
\]

\[
= \left( \int_{\partial B^+_R \cap H_0} - \int_{\partial B^+_R \cap H_0} \right) \rho^2 \frac{\partial \varphi}{\partial \nu} ds = \int_{\partial B^+_R \cap H_0} \rho^2 \frac{\partial \psi}{\partial \nu} ds - c(u).
\]

Since \( \int_{\partial B^+_R \cap H_0} \nabla \theta \cdot \nabla \psi ds = 0 \), we have

\[
\int_{A^+_R} \rho^2 |\nabla \psi|^2 \leq \int_{\partial B^+_R \cap H_0} |\frac{\partial \psi}{\partial \nu}| ds + \int_{A^+_R} (1 - \rho^2) \frac{2|d|}{r} |\nabla \psi| + c(u).
\]

By the Poincare and Holder inequalities, we have

\[
\int_{\partial B^+_R \cap H_0} |\frac{\partial \psi}{\partial \nu}| ds \leq \frac{R}{2} \int_{\partial B^+_R \cap H_0} |\nabla \psi|^2 ds,
\]

and

\[
\int_{A^+_R} (1 - \rho^2) \frac{2|d|}{r} |\nabla \psi| \leq c(u) \int_{A^+_R} (u^3)^2 dx \leq c(u) < \infty.
\]

Here we use the fact \( |\nabla \psi| \leq c \), which follows from (3.23), also we use (3.27). We, therefore, obtain

\[
\int_{A^+_R} |\nabla \psi|^2 \leq cR \int_{H_0 \cap B^+_R} |\nabla \psi|^2 ds + c(u).
\]

Since \( \int_{A^+_R} |\nabla \psi|^2 \leq c(u) \log R \), by choosing a sequence of generic radius \( R_j \to \infty \), the right hand side with \( R = R_j \) remains bounded, we get \( \int_{H_0 \setminus B^+_R} |\nabla \psi|^2 \leq c(u) < \infty. \) This proves Claim 3.5.

Multiplying the third component’s equation by \( u^3 \) and integrating by parts we get

\[
\int_{A^+_R} |\nabla u^3|^2 + (u^3)^2 = \int_{A^+_R} (u^3)^2 (|\nabla u|^2 + (u^3)^2) + \int_{\partial B^+_R \cap H_0} u^3 \frac{\partial u^3}{\partial \nu} ds - \int_{\partial B^+_R \cap H_0} u^3 \frac{\partial u^3}{\partial \nu} ds.
\]

Combining (3.23) with (3.27) we have

\[
\int_{A^+_R} |\nabla u^3|^2 \leq c(u) + c(R \int_{\partial B^+_R \cap H_0} |\nabla u^3|^2 ds)^{\frac{1}{2}}.
\]

By choosing \( R_j \to \infty \) such that \( R_j \int_{\partial B^+_R \cap H_0} |\nabla u^3|^2 ds \leq c \), we obtain

\[
(3.28) \quad \int_{H_0} |\nabla u^3|^2 \leq c(u) < \infty.
\]
For $\rho$ we have
\begin{equation}
\int_{H_0\setminus B_{R_0}^+} |\nabla \rho|^2 = \int_{H_0\setminus B_{R_0}^+} \frac{(u^3)^2|\nabla u|^2}{1-(u^3)^2} \leq c \int_{H_0} (u^3)^2 < \infty.
\end{equation}
Using (3.25) one has
\begin{equation}
\int_{B_R^+} (u^3)^2 = 2\pi d^2 + R,
\end{equation}
where
\begin{equation*}
|R| \leq c(u) \left( R \int_{\partial B^+ \cap H_0} (|\nabla \rho|^2 + |\nabla \psi|^2 + |\nabla u|^2 + (u^3)^2)ds + R \int_{\partial B^+ \cap H_0} (|\nabla \psi|^2 + (u^3)^2)ds \right).
\end{equation*}
Since $\int_{H_0\setminus B_{R_0}^+} |\nabla \rho|^2 + |\nabla \psi|^2 + |\nabla u|^2 + (u^3)^2 < \infty$, we may find $R_j \to \infty$ such that
\begin{equation}
R_j \int_{\partial B_{R_j} \cap H_0} (|\nabla \rho|^2 + |\nabla \psi|^2 + |\nabla u|^2 + (u^3)^2)ds \to 0.
\end{equation}
Hence
\begin{equation}
\int_{H_0} (u^3)^2 = 2\pi d^2.
\end{equation}
Next we want to derive a lower bound for the energy. We have
\begin{equation}
\frac{1}{2} \int_{B_R^+ \setminus B_{R_0}^+} |\nabla u|^2 \geq \frac{1}{2} \int_{B_R^+ \setminus B_{R_0}^+} |\nabla u|^2 \geq \frac{1}{2} \int_{B_R^+ \setminus B_{R_0}^+} \rho^2 |\nabla \phi|^2 \geq \frac{1}{2} \int_{B_R^+ \setminus B_{R_0}^+} |\nabla \phi|^2 - c(u) \geq 2\pi d^2 \log \frac{R}{R_0} + 2d \int_{B_R^+ \setminus B_{R_0}^+} \partial_\nu \psi - c(u) = 2\pi d^2 \log \frac{R}{R_0} - c(u).
\end{equation}
Let $\Gamma$ be the stereographic projection defined in (3.5), we may write $\Gamma^{-1} \circ u = \rho_1 e^{(2d\theta + \psi)}$, $\rho_1 = \frac{1}{1+|x|^2}$. Set $\tilde{v} = \rho e^{(2d\theta + \psi)}$ on $B_R^+ \setminus B_{\frac{R}{2}}$, where
\begin{equation}
\tilde{\rho}(x) = 1 + \frac{|x| - \frac{R}{2}}{\frac{R}{2}}(\rho_1(x) - 1), \quad \tilde{\psi}(x) = \frac{|x| - \frac{R}{2}}{\frac{R}{2}}\psi(x),
\end{equation}
and $\tilde{u} = \Gamma \circ \tilde{v}$, then
\begin{equation}
\frac{1}{2} \int_{B_{\frac{R}{2}}^+ \setminus B_{\frac{R}{4}}^+} |\nabla \tilde{u}|^2 + (\tilde{u}^3)^2 = \int_{B_{\frac{R}{2}}^+ \setminus B_{\frac{R}{4}}^+} \frac{2 \left( |\nabla \tilde{u}|^2 + \frac{(1-|\tilde{v}|)^2}{4} \right)}{(1 + |\tilde{v}|^2)^2} \leq c(u).
\end{equation}
Here one uses the fact that $\int_{H_0\setminus B_{R_0}^+} |\nabla \rho_1|^2 < \infty$, $\int_{H_0\setminus B_{R_0}^+} (\rho_1 - 1)^2 < \infty$ and $\int_{H_0} |\nabla \psi|^2 < \infty$. Note that we have $\tilde{u}(x) = e^{2d\theta}$ on $\partial B_{\frac{R}{4}}^+ \cap H_0$, from Lemma 4.3 in [HnL] we may choose $\tilde{u}$ on $B_{\frac{R}{4}}^+$ such that
\begin{equation}
\frac{1}{2} \int_{B_{\frac{R}{4}}^+} |\nabla \tilde{u}|^2 + (\tilde{u}^3)^2 \leq \pi |d| \log R + c(d).
\end{equation}
Via (3.32) and (3.33), we get

\[
\frac{1}{2} \int_{B_R^n} |\nabla u|^2 + (u^3)^2 \leq \frac{1}{2} \int_{B_R^n} |\nabla \tilde{u}|^2 + (\tilde{u}^3)^2 \leq \pi |d| \log R + c(u). \tag{3.34}
\]

Combining (3.31) and (3.34) and letting \(R \to \infty\), we see \(2d^2 \leq |d|\). Hence \(d = 0\) and by (3.30), \(u^3 \equiv 0\). Thus \(u(x) = (e^{i \varphi(x)}, 0)\), \(\varphi\) is a harmonic function on \(H_0\) with \(\varphi|_{\partial H_0} \equiv 0\). Since \(|\nabla \varphi| \leq c\), we have \(\varphi(x) = c_2 x^2\) for some \(c_2 \in \mathbb{R}\). Now it follows from (3.24) that \(c_2 = 0\), hence \(u \equiv (1, 0, 0)\). □

REFERENCES


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