1. Introduction. A number of proofs have been offered of the fact that Burgers’ equation, with Brownian external force, settles down, with time, into a statistically steady state: see, for instance, Sinai [1996], E-Khanin-Mazel-Sinai [2000], and Kuksin-Shirikyan [2001]. I propose a simple proof based on ideas of Doblin [1940] and Feller [1966]. The equation in question:

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \epsilon \frac{db}{dt}
\]

represents an \(\infty\)-dimensional diffusion in the space of functions \(v(x): 0 \leq x < 1\) of period 1 say, with mean value \(\int_0^1 v = 0\). The external force \(\epsilon dB/dt\) is a sum of “modes” \(e_n(x) \equiv \epsilon_n \times \sqrt{2} \sin (\cos(2\pi nx))/n\), indexed by \(n \geq 1\), multiplied each by the differential of its private 1-dimensional standard Brownian motion \(b_n(t): 0 \leq t < \infty\).

It is assumed for the present proof that all modes are active, i.e. \(e_n \neq 0\) for any \(n \geq 1\), and that the force is smooth in respect to \(0 \leq x < 1\), i.e. that \(e_n\) vanishes rapidly; the second proviso permits you to realize the diffusion in the space \(C^\infty[0,1)\). The force competes with the restoring drift \((1/2)\partial^2 v/\partial x^2\), pulling back towards the origin as per \(\int_0^1 v_v'' = \int_0^1 (v')^2 \leq 0\), and with the twist \(v_v/\partial x\), so-called because \(\int_0^1 v(v') = 0\), the outcome being the statistical steady state cited at the start. The simplicity of the present method has its price: in particular, it does not yield the exponentially fast convergence of \(F_t(v) \equiv E_v[F(v_t)]\) to the invariant mean \(\int F(v)dM(v)\), which must be a consequence of the rapid return of the diffusion to the vicinity of \(v \equiv 0\). Observe, in this connection,

\[
d \int_0^1 v^2 = -\int_0^1 (v')^2 dt + 2 \int_0^1 ev db + \int_0^1 e^2 dt
\]

\[
\leq -4\pi^2 \int_0^1 v^2 dt + 2 \int_0^1 ev db + \int_0^1 e^2 dt
\]

with the obvious result that, up to the passage time \(T = \min(t: \int_0^1 v^2 = r^2)\),

\[
e^{4\pi^2 t} \leq e^{4\pi^2 t} \int_0^1 e^2 dt \leq \int_0^1 e^2 + 2 \int_0^1 e^{4\pi^2 s} \int_0^1 e^2 \sqrt{\int_0^1 e^2 - \frac{1}{4\pi^2},}
\]

which yields

\[
E_v(e^{4\pi^2 T}) \leq \frac{R^2}{r^2 - (1/4\pi^2)} \int_0^1 e^2 \quad \text{for } R^2 = \int_0^1 v^2 > r^2 > \frac{1}{4\pi^2} \int_0^1 e^2.
\]
2. The Diffusion. The equation can be solved with the help of the Cole-Hopf substitution: if \( w = \exp\left[-\int_0^1 dt \int_0^t v(\eta)d\eta\right]\), then
\[
\frac{\partial w}{\partial t} = \frac{1}{2} \frac{\partial^2 w}{\partial x^2} + w f dB + \frac{w}{2}\left[f^2 - \int_0^1 \left(\frac{w'}{w}\right)^2\right]
\]
with \(-f' = e, w > 0\) and \(\int_0^1 \ell nw = 0\), and this equation yields to the Feynman-Kač formula: \(w(t, x) = Z^{-1} E_x[\delta w(0, x_t)]\), in which \(x(t) : t \geq 0\) is an auxiliary 1-dimensional standard Brownian motion,
\[
Z = \exp\left[\int_0^t f(x_{t-})db_s\right], \text{ and } Z = \exp\left[\int_0^1 \ell nE_x(\delta w)\right]
\]
is a normalizer to keep \(\int_0^1 \ell nw = 0\). The recipe may be re-expressed in terms of the auxiliary Brownian motion tied at \(x(0) = 0\) and \(x(t) = 0\). Then a simple application of Kolmogorov-Centsov shows that the path \(w\) (and so also \(v\)) can be realized in the space of functions jointly of class \(C[0, \infty)\) in respect to \(t \geq 0\) and of class \(C^\infty[0,1]\) in respect to \(0 \leq x < 1\). In this way the diffusion is constructed: \(v = -w'/w\). The aim is now to prove the existence of the limit \(F(v) \lim_{t \to \infty} E_v[F(v_t)]\) and to identify it as the invariant mean \(\int F(v) dM(v)\). Naturally, it is essential that the mass of the distribution of \(v\) not run out to \(\infty\). I dispose of this once by the estimate employed at the end of Section 1 which yields
\[
E\left(\int_0^1 v^2\right) \leq e^{-4\pi^2 t} \int_0^1 v_0^2 + \int_0^1 e^{2} \frac{1}{4\pi^2} (1 - e^{-4\pi^2 t})
\]
whence
\[
P\left(\int_0^1 v^2 > R^2\right) \leq R^{-2} \left[e^{-4\pi^2 t} r^2 + \frac{1}{4\pi^2} \int_0^1 e^2\right] \text{ with } r^2 = \int_0^1 v_0^2.
\]

3. Equicontinuity. Let \(v^*(t, x)\) be the functional gradient \(\partial v(t, x)/\partial v(0, y)\) for fixed \(0 \leq y < 1\). You have \(\partial^2 v^* \partial t = (1/2)\partial^2 v \partial x^2 - (\partial / \partial x^2)(v^*)\) with \(v^*(0, x) dx = \text{the unit mass at } x = y\), and this may be solved by a combination of Cameron-Martin and Feynman-Kač: to wit,
\[
v^*(t, x) = E_x[e^{-\int_0^t v(t-s, x_u)dx_u - \frac{1}{2} \int_0^t v^2(t-s, x_u)dx_u - \int_0^t v'(t-s, x_u)dx_u}, x_t = y]^{1}
\]
which reduces to
\[
E_y[e^{-\int_0^t v(s, x_u)dx_u - \frac{1}{2} \int_0^t v^2(s, x_u)dx_u}, x_t = x] \equiv E_y[v, x_t = x]
\]
upon reversal of the auxiliary Brownian path as per \(x(s) \to x(t-s) (s \leq t)\). Now the chain rule in function space applied to \(F_t(v) = E_v[F(v_t)] = 2BM[F(v_t)]\) with \(F\) of class \(C^1[0, 1] \to \mathbb{R}\) and \(v + \theta \Delta v\) in place of \(v\), plain, yields
\[
F_t(v + \Delta v) - F_t(v) = \int_0^1 d\theta \int_0^1 \Delta v(y)dy BM \int_0^1 \text{grad} F E_y[v, x_t = x]dx
\]
\(^1E[I, x_t = y]\) is short for the density \((\partial / \partial y)E[I, x_t \leq y]\).
\(^2BM\) is the Brownian mean over the individual motions \(b_n : n \geq 1\).
with \( \text{grad} \, F \) taken at \( v \), so that

\[
|F_t(v + \Delta v) - F_t(v)| \leq |\text{grad} \, F|_\infty \int_0^1 |\Delta v| dy E_y(v) \leq |\text{grad} \, F|_\infty |\Delta v|_\infty
\]

in view of \( E(v) \leq 1 \). This provides compactness, permitting you to choose \( \alpha = \alpha_1 > \alpha_2 > \text{etc.} \). \( \lim_0 \) so as to make \( G_\alpha(v) = \alpha \int_0^\infty e^{-\alpha t} F_t(v) dt \) converge to a function \( G_0(v) \) of class \( C[0,1] \rightarrow \mathbb{R} \), uniformly on compact figures such as \( K = (v : \int_0^1 (v')^2 \leq R^2) \).

I prefer this mode of convergence to the plain \( \lim_\infty F_t(v) \) as it avoids a difficulty with the non-compactness of \( C[0,1] \).

4. \( G_0(v) \) is Constant in Respect to \( v \). The point is that the diffusion comes close to the origin \( v \equiv 0 \) so that the path emanating from that place is typical; this is the idea of Dobšin [1940]. Let a small number \( r \) and a big number \( R \) be fixed, let \( K \) be the compact figure \( (v : \int_0^1 (v')^2 \leq r^2 \& \int_0^1 (v')^2 \leq R^2) \), and let \( T \) be the smaller of the passage time to \( K \) and an adjustable integer \( N = 1, 2, 3 \text{ etc.} \). Then

\[
G_\alpha(v) = \alpha E_v \int_0^T e^{-\alpha t} F_t(v) dt + E_v [e^{-\alpha T} G_\alpha(v_T)]
\]

implies 1) \( G_0(v) = E_v[G_0(v_T)] \) since \( T \leq N ; 2 \) the same with \( T \) now equal to the passage time to \( K \), by making \( N \uparrow \infty \); and 3) \( G_0(v) = G_0(0) \) by making \( r \downarrow 0 \) so that \( K \) shrinks to the origin. It is here that the proviso \( e_n \neq 0 (n \geq 1) \) is used. Of course 2) is correct only if the passage time to \( K \) is finite with probability one. This is so provided \( R \) is big enough.

**Proof.** If, for some small \( r \) and big \( R \), the passage time \( T \) is infinite, then for every \( t \geq 0 \), either \( \int_0^1 ev > r^2 \) or \( \int_0^1 (v')^2 > R^2 \). Let \( E \) be the set of times \( s \leq t \) when \( \int_0^1 ev > r^2 \) and \( E' \) its complement, on which you must have \( \int_0^1 (v')^2 > R^2 \). Two cases arise.

Case 1: \( \int_0^\infty e^{4\pi^2 t} \left( \int_0^1 ev \right)^2 dt < \infty \). Then

\[
d \int_0^1 v^2 = -\int_0^1 v^2 \, dt + 2 \int_0^1 ev \, db + \int_0^1 e^2 \, dt,
\]

\[
\leq -\frac{1}{2} \int_0^1 (v')^2 \, dt - 2\pi^2 \int_0^1 v^2 \, dt + 2 \int_0^1 ev \, db + \int_0^1 e^2 \, dt,
\]

and the resulting estimate

\[
e^{2\pi^2 t} \int_0^1 v^2 \leq \int_0^1 v_0^2 - \frac{1}{2} \int_0^t e^{2\pi^2 s} \int_0^1 (v')^2 \, ds + 2 \int_0^t e^{2\pi^2 s} \int_0^1 ev \, db + \int_0^1 e^2 \times \frac{e^{2\pi^2 t}}{2\pi^2}
\]
implies\(^3\)

\[
\int_0^t ds e^{2\pi^2 s} \int_0^1 (v')^2 \leq \int_0^1 e^s \times e^{2\pi^2 s} \text{ for } t \uparrow \infty.
\]

But now

\[
2e^{2\pi^2 t} \geq \int_0^1 e^{2\pi^2 s} (\int_0^1 ev)^2 + \int_0^t e^{2\pi^2 s} \int_0^1 (v')^2
\]

\[
\geq r^2 \int_E e^{4\pi^2 s} + R^2 \int_{E'} e^{2\pi^2 s}
\]

cannot be balanced as \( t \uparrow \infty \) if \( R \) is too big in comparison to \( \int_0^1 e^2 \), no matter how small the fixed number \( r > 0 \) may be.

**Case 2:** \( \int_0^\infty e^{4\pi^2 t} \int_0^1 (ev)^2 dt = \infty \). You have

\[
e^{2\pi^2 t} \int_0^1 v'^2 \leq \int_0^t v'^2 - \frac{1}{2} \int_0^t e^{2\pi^2 s} \int_0^1 (v')^2 ds + 2 \int_0^t e^{2\pi^2 s} \int_0^1 ev db + \int_0^1 e^s \times e^{2\pi^2 t}
\]

as before, and an application of the law of the iterated logarithm to the Brownian integral produces the over-estimate of the right side by

\[
\int_0^1 v'^2 \int_0^t v'^2 - \frac{1}{2} \int_0^t e^{2\pi^2 s} \int_0^1 (v')^2 ds \times \ell n \ell n \text{ (ditto)} + \int_0^1 e^s \times e^{2\pi^2 t},
\]

valid i.o. as \( t \uparrow \infty \), so that, i.o.,

\[
N \times \int_0^t e^{4\pi^2 s} \left( \int_0^1 ev \right)^2 + \frac{1}{2} \int_0^t e^{2\pi^2 s} \int_0^1 (v')^2 + \leq \int_0^1 v'^2 + \int_0^1 e^2 \times e^{2\pi^2 t}
\]

for any \( N = 1, 2, 3 \) etc. you like, and

\[
N r \sqrt{\int_E e^{4\pi^2 s}} + \frac{R^2}{2} \int_{E'} e^{2\pi^2 s} \leq 2 \int_0^1 e^s \times e^{2\pi^2 t} \text{ i.o.}
\]

But then \( \int_{E'} e^{2\pi^2 s} \) is small compared to \( e^{2\pi^2 t} \), \( R \) being large, so that

\[
\int_E e^{4\pi^2 s} = \frac{e^{4\pi^2 t} - 1}{4\pi^2} - \int_{E'} e^{4\pi^2 s} \geq \frac{e^{4\pi^2 t} - 1}{4\pi^2} - e^{2\pi^2 t} \int_{E'} e^{2\pi^2 s}
\]

is comparable to \( (1/4\pi^2)e^{4\pi^2 t} \), and the preceding display may be unbalanced by choice of \( N \).

\(^3\)\( \int_0^\infty I db \) is finite if \( \int_0^\infty I^2 dt < \infty \) for any non-anticipating \( I \).

\(^4\)The point is that \( \int_0^t I db \) looks like a standard 1-dimensional Brownian motion run with the clock \( \int_0^t I^2 \).
5. **Identification of $G_0(0)$**. To complete the proof, it is necessary to know that $G_0(0)$ does not depend upon the mode of approach of $\alpha$ to $0^+$. Then $G_0(0) = \int F(v) dM(v)$ with invariant $M$: in fact, $G_\alpha$ formed with $F_\alpha(v) = E_v[F(v_t)]$ in place of $F$ is nothing but $E_v[G_\alpha(v_t)]$ with the old $G_\alpha$ so that

$$\int F_t(v) dM(v) = E_v[G_\alpha(v_t)] = G_0(0) = \int F(v) dM(v),$$

as advertised. The uniqueness of the invariant measure is now self evident, too. The omitted identification of $G_0(0)$ is simple. Take $F \geq 0$ an dt he compact figure $K = \{v : \int_0^1 (v')^2 \leq R^2\}$. This is harmless to the generality of $F$, $R$ being adjustable. Let $m_\alpha$ be the maximum of $G_\alpha$; obviously, $m_\alpha \downarrow m_0 \geq 0$ as $\alpha \downarrow 0$ and $G_0 \leq m_0$. It is to be proved that $G_0 \equiv m_0$.

**Proof.** Let $T$ be the passage time to $K$. Then, with the cut-off in $F$, $F(v_t) = 0$ for $t \leq T$, and $G_\alpha(v) = E_v[e^{-\alpha T}G_\alpha(v_T)];$ in particular, $G_\alpha$ peaks at some place $v_\alpha \in K$. Now, with $\alpha =$ the old $\alpha_n$ of §3 and $n \uparrow \infty$, you have $m_\alpha = G_\alpha(v_\alpha)$, and the convergence of $G_\alpha(v)$ to the constant $G_0(0)$, which is uniform on the compact $K$, implies $m_0 = G_0(v_0)$ for some $v_0 \in K$. Then $m_0 = G_0(0)$ in short, the full limit $G_\alpha(v) = m_0$ exists. This nice trick is adapted from Feller [1966].

**REFERENCES**


